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LARGE-TIME BEHAVIOR OF SOLUTIONS FOR UNIPOLAR EULER-POISSON EQUATIONS WITH CRITICAL OVER-DAMPING

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Abstract. This paper is concerned with the large-time behavior of solutions to the Cauchy problem for the one-dimensional unipolar Euler-Poisson equations with critical time-dependent over-damping. We prove that the Cauchy problem admits a unique global smooth solution which time-asymptotically converges to the stationary solution in the logarithmic form $O(\ln^{-\frac{k}{2}}(1+t))$ for the integer $k \in [1, +\infty)$. In particular, the integer k can be large enough as the initial perturbation is small enough. This convergence rate is much better than the previous studies with critical over-damping. The proof is based on the technical time-weighted energy estimates and the mathematical induction.

1. **Introduction.** In this paper, we study the one-dimensional unipolar hydrodynamic model for semiconductors, which can be represented by the following Euler-Poisson equations

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n)\right)_x = nE - \dfrac{\mu}{(1+t)^\lambda} J, \\ E_x = n - D(x). \end{cases} \tag{1}$$

Here the unknown functions $n(x,t) > 0$, $J(x,t)$, and $E(x,t)$ denote the electron density, the current density, and the electric field, respectively. The given function $p(n)$ is the pressure-density function and $D(x)$ is the doping profile which denotes the prescribed density of positive charged background ions. The term $-\frac{\mu}{(1+t)^\lambda} J$ with physical parameter $\mu > 0$ and $\lambda \in \mathbb{R}$ is the time-dependent damping effect, which

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will have a great influence on the large-time behavior of solutions. When $\lambda < 0$, the damping effect is time-asymptotically enhancing and it is called the over-damping case. When $\lambda > 0$, the damping effect is time-asymptotically degenerate and it is called the under-damping case.

The hydrodynamic model for semiconductors is usually used in describing the motion of the charged particles, such as electrons and holes in semiconductor devices [19]. Since its firstly introduced by Bløtekjær [1], it has been one of the hot spots in mathematical physics because of its application to model hot electron effects that are not accounted for in the classical drift-diffusion model [22]. For its steady-state system, Degond and Markowich obtained the existence and uniqueness of the smooth subsonic solution in one-dimensional case [3] and in three-dimensional irrotational case [4]. The existence of subsonic solutions in two-dimensional case was considered by Markowich in [18]. Peng and Violet [21] investigated the existence and uniqueness of the supersonic solution for the one-dimensional steady-state Euler-Poisson system. For the study on the steady transonic solutions, we refer the reader to the interesting work [2] and the references therein.

When $\mu > 0$ and $\lambda = 0$, the damping in the system (1) becomes the regular damping. There are many results about the large-time behavior of the solutions to (1) with regular damping [6, 9, 14, 17, 20, 23]. Among them, Li-Markowich-Mei [14] proved that the system (1) in bounded domain $(0, 1)$ with Dirichlet boundary conditions possesses a unique global smooth solution and the solution time-exponentially converges to the corresponding steady-state. In [23], Sun-Mei-Zhang considered the system (1) on the half line with inflow/outflow/impermeable boundary conditions or the insulating boundary conditions. They found that the solutions of the inflow/outflow/impermeable problem (insulating problem) tend exponentially (exponentially/algebraically) to corresponding steady-states as $t \to +\infty$. For the Cauchy problem, Luo-Natalini-Xin [17] investigated the global existence of the solutions and obtained the solutions time-exponentially converging to the stationary solutions of the drift-diffusion equations. They required the condition $J(+\infty, 0) = J(-\infty, 0) = E(-\infty, 0) = 0$, which physically stands for the switch-off case. This stiff condition ensured that they can establish the energy estimates in L^2 framework. In [9], Huang-Mei-Wang-Yu studied the case that $J(+\infty, 0) \neq J(-\infty, 0)$, which physically stands for the switch-on case. They technically constructed some correction functions to delete the gaps between the original solutions and the stationary solutions in L^2 space. Furthermore, they proved that the solutions to the system (1) with regular damping time-exponentially converge to the stationary solutions. Regarding the multi-dimensional case, we refer to [7, 10] and the references therein. For the other interesting studies on the bipolar hydrodynamic model for semiconductors, see [5, 8] and the references therein.

When $\mu > 0$ and $\lambda \neq 0$, the effect of damping is time-asymptotically enhancing or degenerate, which makes the fantastic variety of the Euler-Poisson system, and the research results are very limit. In [24], Sun-Mei-Zhang investigated the Cauchy problem for the unipolar Euler-Poisson system (1) with $\lambda \in (-1, 0) \cup (0, 1)$. They proved that, when $\lambda \in (-1, 0)$, the Cauchy problem admits a unique global solution converging to the steady-state in the sub-exponential form as $t \to +\infty$, and when $\lambda \in (0, 1)$, the system with the completely flat doping profile possesses a unique global solution time-asymptotically converging to the constant steady-state in the sub-exponential form. The authors observed that the time-dependent damping essentially affects the large-time behavior of solutions to the unipolar Euler-Poisson system (1), and causes the decay rate to be sub-exponential, which is slower than the exponential rate in the case of $\lambda = 0$. Besides, Li-Mei-Xu [15] studied the initial boundary-value problem for the system (1) with $\lambda \in (0, 1)$. They proved that the initial boundary-value problem admits a unique global solution time-asymptotically converging to the constant steady-state in the sub-exponential form when the doping profile is completely flat. For the bipolar hydrodynamic model of semiconductors, Li-Li-Mei-Zhang [12] obtained that the one-dimensional bipolar Euler-Poisson system with time-dependent damping for $\lambda \in (-1, 1)$ possesses a unique global solution time-algebraically converging to the corresponding diffusion wave. In [16], Luan-Mei-Rubino-Zhu considered the critical case $\lambda = 1$ and $\mu > 2$, they proved that the global solution of the bipolar Euler-Poisson system time-algebraically converges to the constant steady-state. Wu [25] investigated the one-dimensional bipolar quantum Euler-Poisson system in the critical case $\lambda = -1$. The author showed that the solution of the system exists globally and time-asymptotically converges to the nonlinear diffusion wave in the logarithmic form.

Our target in this paper is to study the large-time behavior of solutions for the unipolar Euler-Poisson system (1) in the more interesting and challenging critical over-damping case, namely, $\lambda = -1$. Without loss of generality, we set $\mu = 1$, then the system (1) becomes

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n)\right)_x = nE - (1+t)J, \quad (x,t) \in \mathbb{R} \times (t_0, +\infty), \\ E_x = n - D(x), \end{cases} \tag{2}$$

where $t_0 > 0$ is a given constant. The initial data of (2) are given by

$$(n, J)(x, t_0) = (n_0, J_0)(x), \qquad x \in \mathbb{R}. \tag{3}$$

Throughout this paper, we assume that the pressure function p satisfies

$$p \in C^4((0, +\infty)), \qquad p'(s) > 0 \text{ for } s > 0, \tag{4}$$

a physical example is $p(n) = An^\gamma$ with $A > 0$ and $\gamma > 1$. And the assumptions on the doping profile $D(x)$ are

$$D(x) > 0, \qquad \lim_{x \to \pm\infty} D(x) = D_\pm, \qquad D'(x) \in C^0(\mathbb{R}) \cap H^2(\mathbb{R}). \tag{5}$$

The studies on the critical over-damping case with $\lambda = -1$ are very limited. The first studies related to this critical case were [13] for the 1-D time-dependently damped Euler equations, and [11] for the n-D Euler equations with $n \geq 7$. The similar result was then extended to the bipolar quantum Euler-Poisson system in [25]. However, for the unipolar Euler-Poisson system, we observe that there is some advantage arising from the Poisson equation. With this help, the perturbed equation around its steady-states is reduced to the well-known Klein-Gordon equation with nonlinear perturbation (see (21)). Even though the critical over-damping makes the decay very weak like $O(\ln(1+t))$ as showed in [11, 13, 25], the strong dissipative term $\bar{n}w$ in (21) will help us to expect to have a much faster decay.

As shown in [17, 24], since the doping profile $D(x)$ in (2) is nonzero, the expected asymptotic profiles of the solutions to (2)–(3) are stationary solutions. Therefore, the main task for us is to investigate the solutions of the Cauchy problem (2)–(3) converge to the stationary solutions. There are some technical issues in the proof we want to point out. In [24], the a priori estimates were established by choosing appropriate polynomial time-weights, however, this method is no longer applicable in the critical case $\lambda = -1$. Inspired by [13] for the 1-D damped Euler equations and [11] for the n-D damped compressible Euler equations, we technically construct logarithmic time-weights to establish the a priori estimates. But different from the Euler system in [13] and the bipolar quantum Euler-Poisson system in [25], we observe that the decay rates of the solutions to the unipolar Euler-Poisson system (2) can be enhanced to $\ln^{-\frac{k}{2}}(1+t)$, where $k \in [1, +\infty)$ is an integer and could be large enough as the initial perturbation is small enough. This decay rate is much better than the existing studies [11, 13, 25] in the critical over-damping case. In order to avoid repeated calculation, we adopt the mathematical induction to prove the important lemmas. Even though we use the induction in the proof, the calculation is still complicated than that in [13]. In summary, we state our main result as follows:

For the unipolar Euler-Poisson system (2) with critical over-damping, we expect that the asymptotic profile of the solution is the stationary solution $(\bar{n}, \bar{E})(x)$ to the well-known drift-diffusion equations where the current density $\bar{J}(x) = 0$, and prove that the unique solution $(n, J, E)(x, t)$ of the Cauchy problem (2)–(3) globally exists and satisfies

$$\begin{aligned} \|n(t) - \bar{n}\|_{L^\infty(\mathbb{R})} &\le C \ln^{-\frac{k}{2}}(1+t), \\ \|J(t)\|_{L^\infty(\mathbb{R})} &\le C(1+t)^{-1} \ln^{-\frac{k}{2}}(1+t), \\ \|E(t) - \bar{E}\|_{L^\infty(\mathbb{R})} &\le C \ln^{-\frac{k}{2}}(1+t), \end{aligned}$$

if the initial perturbation is sufficiently small. Here $k \in [1, +\infty)$ is an integer, as mentioned in the above, it can be large enough as the initial perturbation is small enough.

Regarding the fluid dynamics, including hydrodynamic models of semiconductors, from the mathematical point of view, the smallness of initial perturbation is usually requested for the global existence of solutions, because the Euler-Poisson system lacks the maximum principle. On the other hand, from physical point of view, the small perturbation around the subsonic initial data is also necessary for the dynamical system. Otherwise, a big initial perturbation may allow the initial data to be supersonic, which could cause the system does not hold any physical solutions, namely, the semiconductor device may not efficiently work out.

Regarding the compressible Euler equations with time-dependent damping, we refer to the relevant studies [11, 13] and the references therein.

The rest of this paper is arranged as follows. In Section 2, we introduce the well-known results of the stationary solutions and state the main results of this paper. Section 3 devotes to establish the a priori estimates, which is the crucial part of this paper.

Notations. Throughout this paper, the symbol C denotes a generic positive constant which maybe different in different lines. $C_i\,(i = 1, 2, 3, \dots)$ denotes some specific positive constant. $L^2(\mathbb{R})$ is the square integrable real-valued function space on $\mathbb{R}$ whose norm is defined by $\|\cdot\|_{L^2(\mathbb{R})} = (\int_{\mathbb{R}} |\cdot|^2 dx)^{1/2}$. $L^\infty(\mathbb{R})$ is essentially bounded measurable function space on $\mathbb{R}$ whose norm is defined by $\|\cdot\|_{L^\infty(\mathbb{R})} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |\cdot|$. For a nonnegative integer m, $H^m(\mathbb{R})$ is the Hilbert space whose norm is defined by $\|f\|^2_{H^m(\mathbb{R})} = \sum_{j=0}^m \|\partial_x^j f\|^2_{L^2(\mathbb{R})}$. For the sake of convenience, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}$ and $\|\cdot\|_m := \|\cdot\|_{H^m(\mathbb{R})}$.

2. **Preliminaries and main results.** In this section, we show some well-known results of the stationary solutions, and state the main results of this paper.

As shown in [17, 24], the asymptotic profiles of the solutions to (2)–(3) are stationary solutions which satisfy the following stationary equations corresponding to the well-known drift-diffusion equations

$$\begin{cases} (\bar{n}\bar{E} - p(\bar{n})_x)_x = 0, \\ \bar{E}_x = \bar{n} - D(x), \end{cases} \tag{6}$$

with the boundary value conditions

$$\lim_{x\to\pm\infty} \bar{n}(x) = D_\pm, \qquad \lim_{x\to-\infty} \bar{E}(x) = 0. \tag{7}$$

It then follows from $(6)_1$ and (7) that

$$\bar{J} := \bar{n}\bar{E} - p(\bar{n})_x = \text{constant} = \bar{n}(-\infty)\bar{E}(-\infty) = 0. \tag{8}$$

Let us denote

$$D_* = \inf_{x\in\mathbb{R}} D(x), \qquad D^* = \sup_{x\in\mathbb{R}} D(x).$$

We introduce the existence and uniqueness result of the solutions to stationary equations (6) with (7) in the following Lemma. The proof of Lemma 2.1 is omitted here and one can see [17] for details.

Lemma 2.1. *Assume that the pressure function p satisfies (4) and the doping profile $D(x)$ satisfies (5). Then there exists a unique smooth solution $(\bar{n}, \bar{E})(x)$ of (6)–(7) satisfying*

$$D_* \le \bar{n}(x) \le D^*, \quad \textit{for any } x \in \mathbb{R}, \tag{9}$$

$$\|\bar{n} - D\|_2^2 + \|\bar{n}_x\|_2^2 \le C_1\eta_0, \tag{10}$$

$$\sum_{j=1}^{3} \|\partial_x^j \bar{n}\|_\infty + \sum_{j=0}^{2} \|\partial_x^j \bar{E}\|_\infty \le C_2\delta_0, \tag{11}$$

where C_1 and C_2 are positive constants depending only on D_* and D^*, and

$$\begin{aligned} \gamma_1 &= \|D'\|^6 + \|D''\|^2, \\ \gamma_2 &= \|D'''\|^2 + \|D'\|\gamma_1^{3/2} + \|D'\|^4\gamma_1, \\ \eta_0 &= \|D'\|^2 + \gamma_1 + \gamma_2, \\ \delta_0 &= \|D'\|^{1/2}\gamma_1^{1/4} + \|D'\|\gamma_1^{1/2} + \|D'\|^{3/2}\gamma_1^{3/4} \\ &\quad + \|D'\|^{1/2}\gamma_1^{1/2}\gamma_2^{1/4} + \gamma_1^{1/4}\gamma_2^{1/4} + \|D'\|_{C^0(\mathbb{R})}. \end{aligned}$$

Based on Lemma 2.1, we are able to state the main result of this paper as follows.

Theorem 2.2. *Let $(\bar{n}, \bar{E})(x)$ be the solution of the stationary equations (6) with (7) and define*

$$\omega_0(x) := \int_{-\infty}^{x} (n_0(y) - \bar{n}(y))dy. \tag{12}$$

Suppose that (4)–(5) hold, $\omega_0(x) \in H^3(\mathbb{R})$, $J_0(x) \in H^2(\mathbb{R})$ and $\|\omega_0\|_3 + \|J_0\|_2$ is sufficiently small. For any given k, if δ_0 is chosen sufficiently small, then the

Cauchy problem (2)–(3) possesses a unique global smooth solution $(n, J, E)(x, t)$, which satisfies

$$\begin{aligned} &\ln^k(1+t)\big(\|n(t)-\bar{n}\|_2^2+\|E(t)-\bar{E}\|_3^2\big)+(1+t)^2\ln^k(1+t)\|J(t)\|_2^2 \\ &\le \hat{C}_k\big(\|\omega_0\|_3^2+\|J_0\|_2^2\big), \end{aligned} \tag{13}$$

where $k \in [1, +\infty)$ is an integer, $\hat{C}_k$ is a positive constant depending on k and satisfies $\hat{C}_k\big(\|\omega_0\|_3^2+\|J_0\|_2^2\big) < \infty$.

Remark 2.3. Actually, the integer k is closely related to the initial perturbation. In order to obtain the global existence of the solution by the continuation argument, as showed later in Proposition 2.5, we need to guarantee $(k!\sum_{j=0}^{k}\ln^j(1+t_0))^{1/2}\big(\|\omega_0\|_3+\|J_0\|_2\big) \ll 1$. Therefore, once the initial perturbation $\|\omega_0\|_3+\|J_0\|_2$ is small enough, the integer k can be large enough.

From the Sobolev inequality

$$\|f\|_\infty \le \sqrt{2}\|f\|^{1/2}\|f_x\|^{1/2}, \tag{14}$$

we have the following Corollary.

Corollary 2.4. *Under the assumptions of Theorem 2.2, it holds that*

$$\|n(t)-\bar{n}\|_\infty \le C\ln^{-\frac{k}{2}}(1+t), \tag{15}$$

$$\|J(t)\|_\infty \le C(1+t)^{-1}\ln^{-\frac{k}{2}}(1+t), \tag{16}$$

$$\|E(t)-\bar{E}\|_\infty \le C\ln^{-\frac{k}{2}}(1+t), \tag{17}$$

where $k \in [1, +\infty)$ is an integer.

Now we reformulate the original problem (2)–(3). Set

$$\varphi(x,t)=n(x,t)-\bar{n}(x), \qquad \omega(x,t)=E(x,t)-\bar{E}(x). \tag{18}$$

It then follows from (2) that $(\varphi, J, \omega)(x, t)$ satisfies

$$\begin{cases} \varphi_t+J_x=0, \\ J_t+\left(\dfrac{J^2}{\bar{n}+\varphi}\right)_x+(p(\bar{n}+\varphi)-p(\bar{n}))_x=\bar{n}\omega+(\bar{E}+\omega)\varphi-(1+t)J, \\ \omega_x=\varphi. \end{cases} \tag{19}$$

By $(19)_1$ and $(19)_3$, we obtain

$$\omega_x(x,t)=\varphi(x,t), \qquad \omega_t(x,t)=-J(x,t). \tag{20}$$

Combining $(19)_2$ and (20), one has

$$\begin{aligned} &\omega_{tt}+(1+t)\omega_t-(p'(\bar{n})\omega_x)_x+\bar{n}\omega \\ &=-(\bar{E}+\omega)\omega_x+(p(\bar{n}+\omega_x)-p(\bar{n})-p'(\bar{n})\omega_x)_x+\left(\frac{\omega_t^2}{\bar{n}+\omega_x}\right)_x. \end{aligned} \tag{21}$$

From (3), (12), (18), and (20), we can get the initial value for the perturbation system (21),

$$\omega(x,t_0)=w_0(x), \qquad \omega_t(x,t_0)=-J_0(x). \tag{22}$$

Besides, we choose

$$E(x,t_0)=\int_{-\infty}^{x}(n_0-D)(y)dy$$

$$
\begin{aligned}
&= \int_{-\infty}^{x} (n_0 - \bar{n})(y)dy + \int_{-\infty}^{x} (\bar{n} - D)(y)dy \\
&= \omega_0(x) + \bar{E}(x).
\end{aligned}
$$

Let $T \in (t_0, +\infty]$, we define the solution space for the Cauchy problem (21)–(22) as

$$
X(T) := \left\{ \omega(x,t) \mid \partial_t^j \omega \in C([t_0,T]; H^{3-j}(\mathbb{R})),\ j = 0,1,2 \right\},
$$

where its norm is defined by

$$
\begin{aligned}
N(T)^2 : = \sup_{t_0 \le t \le T} \Bigg\{ & \ln^k(1+t) \sum_{j=0}^{3} \|\partial_x^j \omega(t)\|^2 \\
& + (1+t)^2 \ln^k(1+t) \left(\sum_{j=0}^{2} \|\partial_x^j \omega_t(t)\|^2 + \sum_{j=0}^{1} \|\partial_x^j \omega_{tt}(t)\|^2 \right) \Bigg\},
\end{aligned} \tag{23}
$$

where $k \in [1, +\infty)$ is an integer.

For the Cauchy problem (21)–(22), we have the following result.

Proposition 2.5. *Under the assumptions of Theorem 2.2, if both $N(T)$ and δ_0 are sufficiently small, then, the Cauchy problem (21)–(22) admits a unique global smooth solution $\omega(x,t) \in X(T)$ satisfying*

$$
\begin{aligned}
& \ln^k(1+t) \sum_{j=0}^{3} \|\partial_x^j \omega(t)\|^2 + (1+t)^2 \ln^k(1+t) \left(\sum_{j=0}^{2} \|\partial_x^j \omega_t(t)\|^2 + \sum_{j=0}^{1} \|\partial_x^j \omega_{tt}(t)\|^2 \right) \\
& + \int_{t_0}^{t} \ln^k(1+\tau) \Bigg[(1+\tau)^{-1} \sum_{j=0}^{3} \|\partial_x^j \omega(\tau)\|^2 + (1+\tau) \sum_{j=0}^{2} \|\partial_x^j \omega_t(\tau)\|^2 \\
& + (1+\tau)^3 \sum_{j=0}^{1} \|\partial_x^j \omega_{tt}(\tau)\|^2 \Bigg] d\tau \le Ck! \sum_{j=0}^{k} \ln^j(1+t_0) \left(\|\omega_0\|_3^2 + \|J_0\|_2^2 \right),
\end{aligned} \tag{24}
$$

where $k \in [1, +\infty)$ is an integer.

Proof of Theorem 2.2. Once Proposition 2.5 is proved, from (18) and (20), we can immediately obtain Theorem 2.2. □

In what follows, we mainly focus on the proof of Proposition 2.5.

3. Proof of Proposition 2.5.

In this section, we devote to prove Proposition 2.5. In fact, the local existence of the solution to the Cauchy problem (21)–(22) can be obtained by the standard iteration method, then, from the a priori estimate (24) and the standard continuation arguments, we can extend the local solution to the global solution. Thus, the main effort in this section is to establish the a priori estimate (24).

Thanks to (23) and the Sobolev inequality (14), we have

$$
\begin{aligned}
& \sup_{t_0 \le t \le T} \left\{ \sum_{j=0}^{2} \|\partial_x^j \omega(t)\|_\infty + (1+t) \left(\sum_{j=0}^{1} \|\partial_x^j \omega_t(t)\|_\infty + \|\omega_{tt}(t)\|_\infty \right) \right\} \\
& \le C \ln^{-\frac{k}{2}}(1+t_0) N(T) =: \varepsilon,
\end{aligned} \tag{25}
$$

where $k \in [1, +\infty)$ is an integer. Since $N(T)$ is sufficiently small, we can deduce $\varepsilon \ll 1$ and further derive $\varepsilon + \delta_0$ is sufficiently small. Thus, from (9), (25) and $\varepsilon \ll 1$, we obtain, for any $(x,t) \in \mathbb{R} \times [t_0, T]$,

$$0 < D_*/2 \le \bar{n}(x) + \omega_x \le 2D^*. \tag{26}$$

The proof of Proposition 2.5 is based on several steps of energy estimates which will be stated as a sequence of Lemmas.

Lemma 3.1. *Under the assumptions of Proposition 2.5, it holds that*

$$\begin{aligned}
&\ln^k(1+t)\left(\|\omega(t)\|^2 + \|\omega_x(t)\|^2 + \|\omega_t(t)\|^2\right) + \int_{t_0}^t \ln^k(1+\tau)\left[(1+\tau)\|\omega_t(\tau)\|^2\right.\\
&\quad \left. +(1+\tau)^{-1}\left(\|\omega(\tau)\|^2 + \|\omega_x(\tau)\|^2\right)\right] d\tau \le Ck! \sum_{j=0}^{k} \ln^j(1+t_0)\left(\|\omega_0\|_1^2 + \|J_0\|^2\right),
\end{aligned} \tag{27}$$

where $k \in [1, +\infty)$ is an integer.

Proof. We use the induction to prove this Lemma and divide the proof into two steps.

Step 1. In this step, we are going to obtain the estimate (27) with $k = 1$. Firstly, multiplying (21) by $2\omega_t + (1+t)^{-1}\omega$ and integrating the resulting equation with respect to x over $\mathbb{R}$, after integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left(\omega_t^2 + p'(\bar{n})\omega_x^2 + \bar{n}\omega^2 + (1+t)^{-1}\omega\omega_t + \frac{1}{2}\omega^2 + \frac{1}{2}(1+t)^{-2}\omega^2\right) dx\\
&\quad + \int_{\mathbb{R}} [2(1+t) - (1+t)^{-1}]\omega_t^2 dx + \int_{\mathbb{R}} (1+t)^{-1} p'(\bar{n})\omega_x^2 dx\\
&\quad + \int_{\mathbb{R}} [\bar{n} + (1+t)^{-2}](1+t)^{-1}\omega^2 dx\\
&= -\int_{\mathbb{R}} (\bar{E} + \omega)\omega_x(2\omega_t + (1+t)^{-1}\omega)dx + \int_{\mathbb{R}} \left(\frac{\omega_t^2}{\bar{n} + \omega_x}\right)_x (2\omega_t + (1+t)^{-1}\omega)dx\\
&\quad + \int_{\mathbb{R}} (p(\bar{n} + \omega_x) - p(\bar{n}) - p'(\bar{n})\omega_x)_x (2\omega_t + (1+t)^{-1}\omega)dx =: I_1 + I_2 + I_3.
\end{aligned} \tag{28}$$

We estimate I_1, I_2 and I_3 as follows. From (11), (25)–(26) and Hölder inequality, I_1 and I_2 can be estimated as

$$\begin{aligned}
|I_1| &\le C(\delta_0 + \varepsilon)\int_{\mathbb{R}} |\omega_x\omega_t| dx + C(\delta_0 + \varepsilon)\int_{\mathbb{R}} (1+t)^{-1}|\omega\omega_x| dx\\
&\le C(\delta_0 + \varepsilon)\int_{\mathbb{R}} [(1+t)\omega_t^2 + (1+t)^{-1}(\omega_x^2 + \omega^2)]dx,
\end{aligned} \tag{29}$$

$$\begin{aligned}
|I_2| &= \left| -\int_{\mathbb{R}} \frac{\omega_t^2}{\bar{n} + \omega_x}(2\omega_{xt} + (1+t)^{-1}\omega_x)dx \right|\\
&\le C\int_{\mathbb{R}} |\omega_{xt}|\omega_t^2 dx + C\int_{\mathbb{R}} (1+t)^{-1}|\omega_x|\omega_t^2 dx\\
&\le C\varepsilon \int_{\mathbb{R}} (1+t)\omega_t^2 dx.
\end{aligned} \tag{30}$$

We can estimate I_3 by using (9), (25)–(26), Taylor's formula and Hölder inequality,

$$\begin{aligned}|I_3| &= \left| -\int_{\mathbb{R}} (p(\bar{n}+\omega_x) - p(\bar{n}) - p'(\bar{n})\omega_x)(2\omega_{xt} + (1+t)^{-1}\omega_x)dx \right| \\ &= \frac{1}{2}\left| \int_{\mathbb{R}} p''(\bar{n}+\theta\omega_x)\omega_x^2(2\omega_{xt} + (1+t)^{-1}\omega_x)dx \right| \\ &\le C\varepsilon \int_{\mathbb{R}} (1+t)^{-1}\omega_x^2 dx,\end{aligned} \tag{31}$$

where $\theta \in (0,1)$. Substituting (29)–(31) into (28) and employing the smallness of $\varepsilon + \delta_0$, one has

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \left(\omega_t^2 + p'(\bar{n})\omega_x^2 + \bar{n}\omega^2 + (1+t)^{-1}\omega\omega_t + \frac{1}{2}\omega^2 + \frac{1}{2}(1+t)^{-2}\omega^2 \right) dx \\ &\quad + C_3 \int_{\mathbb{R}} [(1+t)^{-1}(\omega^2 + \omega_x^2) + (1+t)\omega_t^2]dx \le 0,\end{aligned} \tag{32}$$

for some positive constant C_3. Noting that

$$|(1+t)^{-1}\omega\omega_t| \le \frac{1}{2}\omega_t^2 + \frac{1}{2}(1+t)^{-2}\omega^2. \tag{33}$$

We integrate (32) over (t_0, t) and by (4), (9) and (33) to give

$$\begin{aligned}&\int_{\mathbb{R}} (\omega^2 + \omega_x^2 + \omega_t^2)dx + \int_{t_0}^{t}\int_{\mathbb{R}} \left[(1+\tau)^{-1}(\omega^2 + \omega_x^2) + (1+\tau)\omega_t^2\right] dx d\tau \\ &\le C\big(\|\omega_0\|_1^2 + \|J_0\|^2\big).\end{aligned} \tag{34}$$

Secondly, multiplying (21) by $\ln(1+t)(2\omega_t + (1+t)^{-1}\omega)$ and integrating it over $\mathbb{R}$, and then after a similar calculation to (29)–(31), we can get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \bigg(\ln(1+t)(\omega_t^2 + p'(\bar{n})\omega_x^2 + \bar{n}\omega^2) + (1+t)^{-1}\ln(1+t)\omega\omega_t + \frac{1}{2}\ln(1+t)\omega^2 \\ &\quad + \frac{1}{2}(1+t)^{-2}\ln(1+t)\omega^2 - \frac{1}{2}(1+t)^{-2}\omega^2 \bigg) dx + \int_{\mathbb{R}} \ln(1+t)[2(1+t) \\ &\quad - (1+t)^{-1}]\omega_t^2 dx + \int_{\mathbb{R}} (1+t)^{-1}\ln(1+t)[p'(\bar{n})\omega_x^2 + (\bar{n} + (1+t)^{-2})\omega^2]dx \\ &\le \int_{\mathbb{R}} (1+t)^{-1}\left[\omega_t^2 + p'(\bar{n})\omega_x^2 + \left(\bar{n} + \frac{1}{2} + \frac{3}{2}(1+t)^{-2} \right)\omega^2 \right] dx \\ &\quad + C(\varepsilon + \delta_0)\int_{\mathbb{R}} \ln(1+t)[(1+t)\omega_t^2 + (1+t)^{-1}(\omega_x^2 + \omega^2)]dx.\end{aligned} \tag{35}$$

Applying the smallness of $\varepsilon + \delta_0$ to (35), and integrating the resultant inequality over (t_0, t), we have

$$\begin{aligned}&\int_{\mathbb{R}} \bigg[\ln(1+t)(\omega_t^2 + p'(\bar{n})\omega_x^2 + \bar{n}\omega^2) + (1+t)^{-1}\ln(1+t)\omega\omega_t + \frac{1}{2}\ln(1+t)\omega^2 \\ &\quad + \frac{1}{2}(1+t)^{-2}\ln(1+t)\omega^2 \bigg] dx \\ &\quad + C_4 \int_{t_0}^{t}\int_{\mathbb{R}} \ln(1+\tau)[(1+\tau)\omega_t^2 + (1+\tau)^{-1}(\omega_x^2 + \omega^2)]dx d\tau\end{aligned} \tag{36}$$

$$\le \int_{\mathbb{R}} \frac{1}{2}(1+t)^{-2}\omega^2 dx + C\int_{t_0}^{t}\int_{\mathbb{R}}(1+\tau)^{-1}(\omega_t^2+\omega_x^2+\omega^2)dxd\tau$$
$$+ C\ln(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right)$$
$$\le C1!\sum_{j=0}^{1}\ln^j(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right),$$

for some positive constant C_4 and we used (34) in the last inequality. Then, from (4), (9) and (33), we can obtain the estimate (27) with $k=1$,

$$\begin{aligned}&\ln(1+t)\left(\|\omega(t)\|^2+\|\omega_x(t)\|^2+\|\omega_t(t)\|^2\right)+\int_{t_0}^{t}\ln(1+\tau)\left[(1+\tau)\|\omega_t(\tau)\|^2\right.\\&\quad\left.+(1+\tau)^{-1}\left(\|\omega(\tau)\|^2+\|\omega_x(\tau)\|^2\right)\right]d\tau\\&\le C1!\sum_{j=0}^{1}\ln^j(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right).\end{aligned}\tag{37}$$

Step 2. We make the induction hypothesis that (27) holds for the integer $k=l-1$ with $l\ge 2$, i.e.,

$$\begin{aligned}&\ln^{l-1}(1+t)\left(\|\omega(t)\|^2+\|\omega_x(t)\|^2+\|\omega_t(t)\|^2\right)+\int_{t_0}^{t}\ln^{l-1}(1+\tau)\left[(1+\tau)\|\omega_t(\tau)\|^2\right.\\&\quad\left.+(1+\tau)^{-1}\left(\|\omega(\tau)\|^2+\|\omega_x(\tau)\|^2\right)\right]d\tau\\&\le C(l-1)!\sum_{j=0}^{l-1}\ln^j(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right).\end{aligned}\tag{38}$$

It suffices to prove that (27) holds for $k=l$ under the induction hypothesis (38). Multiplying (21) by $\ln^l(1+t)(2\omega_t+(1+t)^{-1}\omega)$ and integrating it over $\mathbb{R}$, then analogous to (29)–(31), one can verify that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big(\ln^l(1+t)(\omega_t^2+p'(\bar n)\omega_x^2+\bar n\omega^2)+(1+t)^{-1}\ln^l(1+t)\omega\omega_t\\&\quad+\frac{1}{2}\ln^l(1+t)\omega^2+\frac{1}{2}(1+t)^{-2}\ln^l(1+t)\omega^2-\frac{l}{2}(1+t)^{-2}\ln^{l-1}(1+t)\omega^2\Big)dx\\&\quad+\int_{\mathbb{R}}(1+t)^{-1}\ln^l(1+t)[p'(\bar n)\omega_x^2+(\bar n+(1+t)^{-2})\omega^2]dx\\&\quad+\int_{\mathbb{R}}\ln^l(1+t)[2(1+t)-(1+t)^{-1}]\omega_t^2dx\\&\quad+\frac{l}{2}(l-1)\int_{\mathbb{R}}(1+t)^{-3}\ln^{l-2}(1+t)\omega^2dx\\&\le l\int_{\mathbb{R}}(1+t)^{-1}\ln^{l-1}(1+t)\left[\omega_t^2+p'(\bar n)\omega_x^2+\left(\bar n+\frac{1}{2}+\frac{3}{2}(1+t)^{-2}\right)\omega^2\right]dx\\&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}\ln^l(1+t)[(1+t)\omega_t^2+(1+t)^{-1}(\omega_x^2+\omega^2)]dx.\end{aligned}\tag{39}$$

Employing the smallness of $\varepsilon+\delta_0$ and $l\ge 2$ to (39), and then integrating the resultant inequality over (t_0,t) gives

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\ln^l(1+t)(\omega_t^2+p'(\bar n)\omega_x^2+\bar n\omega^2)+(1+t)^{-1}\ln^l(1+t)\omega\omega_t\\
&\quad+\frac{1}{2}\ln^l(1+t)\omega^2+\frac{1}{2}(1+t)^{-2}\ln^l(1+t)\omega^2\Big)\,dx\\
&\quad+C_5\int_{t_0}^t\int_{\mathbb{R}}\ln^l(1+\tau)[(1+\tau)\omega_t^2+(1+\tau)^{-1}(\omega_x^2+\omega^2)]dxd\tau\\
&\le\frac{l}{2}\int_{\mathbb{R}}(1+t)^{-2}\ln^{l-1}(1+t)\omega^2dx+C\ln^l(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right)\\
&\quad+Cl\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^{-1}\ln^{l-1}(1+\tau)(\omega_t^2+\omega_x^2+\omega^2)dxd\tau\\
&\le Cl!\sum_{j=0}^{l}\ln^j(1+t_0)\left(\|\omega_0\|_1^2+\|J_0\|^2\right),
\end{aligned}\tag{40}$$

where we used (38) in the last inequality. Thus, by (4), (9) and (33), we have the estimate (27) with $k=l$. The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Under the assumptions of Proposition 2.5, it holds that*

$$\begin{aligned}
&\ln^k(1+t)\left(\|\omega_x(t)\|^2+\|\omega_{xx}(t)\|^2+\|\omega_{xt}(t)\|^2\right)+\int_{t_0}^t\ln^k(1+\tau)\left[(1+\tau)\|\omega_{xt}(\tau)\|^2\right.\\
&\quad\left.+(1+\tau)^{-1}\left(\|\omega_x(\tau)\|^2+\|\omega_{xx}(\tau)\|^2\right)\right]d\tau\le Ck!\sum_{j=0}^{k}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right),
\end{aligned}\tag{41}$$

where $k\in[1,+\infty)$ is an integer.

Proof. Differentiating (21) with respect to x yields

$$\begin{aligned}
&\omega_{xtt}+(1+t)\omega_{xt}-(p'(\bar n)\omega_{xx})_x+\bar n\omega_x\\
&=-\bar n_x\omega-(\bar E_x+\omega_x)\omega_x-(\bar E+\omega)\omega_{xx}\\
&\quad+[(p'(\bar n+\omega_x)-p'(\bar n))(\bar n_x+\omega_{xx})]_x+\left(\frac{\omega_t^2}{\bar n+\omega_x}\right)_{xx}.
\end{aligned}\tag{42}$$

We use induction to prove the estimate (41). The proof is divided into two steps.

Step 1. The goal of this step is to prove that the estimate (41) holds when $k=1$, i.e.,

$$\begin{aligned}
&\ln(1+t)\left(\|\omega_x(t)\|^2+\|\omega_{xx}(t)\|^2+\|\omega_{xt}(t)\|^2\right)+\int_{t_0}^t\ln(1+\tau)\left[(1+\tau)\|\omega_{xt}(\tau)\|^2\right.\\
&\quad\left.+(1+\tau)^{-1}\left(\|\omega_x(\tau)\|^2+\|\omega_{xx}(\tau)\|^2\right)\right]d\tau\\
&\le C1!\sum_{j=0}^{1}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned}\tag{43}$$

We first multiply (42) by $2\omega_{xt}+(1+t)^{-1}\omega_x$ and integrate the resultant equation with respect to x over $\mathbb{R}$ by parts to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\omega_{xt}^2+p'(\bar n)\omega_{xx}^2+\bar n\omega_x^2+(1+t)^{-1}\omega_x\omega_{xt}+\frac12\omega_x^2+\frac12(1+t)^{-2}\omega_x^2\right]dx\\
&\quad+\int_{\mathbb{R}}[2(1+t)-(1+t)^{-1}]\omega_{xt}^2dx\\
&\quad+\int_{\mathbb{R}}(1+t)^{-1}[p'(\bar n)\omega_{xx}^2+(\bar n+(1+t)^{-2})\omega_x^2]dx\\
&=-\int_{\mathbb{R}}[\bar n_x\omega+(\bar E_x+\omega_x)\omega_x+(\bar E+\omega)\omega_{xx}](2\omega_{xt}+(1+t)^{-1}\omega_x)dx\\
&\quad+\int_{\mathbb{R}}[(p'(\bar n+\omega_x)-p'(\bar n))(\bar n_x+\omega_{xx})]_x(2\omega_{xt}+(1+t)^{-1}\omega_x)dx\\
&\quad+\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\bar n+\omega_x}\right)_{xx}(2\omega_{xt}+(1+t)^{-1}\omega_x)dx=:I_4+I_5+I_6.
\end{aligned}\tag{44}$$

The right hand side of (44) can be estimated as below. It is easy to see that

$$\begin{aligned}
I_4&\le C(\varepsilon+\delta_0)\int_{\mathbb{R}}(|\omega\omega_{xt}|+|\omega_x\omega_{xt}|+|\omega_{xx}\omega_{xt}|)dx\\
&\quad+C(\varepsilon+\delta_0)(1+t)^{-1}\int_{\mathbb{R}}(|\omega\omega_x|+\omega_x^2+|\omega_x\omega_{xx}|)dx\\
&\le C(\varepsilon+\delta_0)\int_{\mathbb{R}}[(1+t)^{-1}(\omega^2+\omega_x^2+\omega_{xx}^2)+(1+t)\omega_{xt}^2]dx.
\end{aligned}\tag{45}$$

It follows from Taylor's formula, Hölder inequality, (9) and (25) that

$$\begin{aligned}
I_5&=-\int_{\mathbb{R}}(p'(\bar n+\omega_x)-p'(\bar n))(\bar n_x+\omega_{xx})(2\omega_{xxt}+(1+t)^{-1}\omega_{xx})dx\\
&=-\frac{d}{dt}\int_{\mathbb{R}}(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xx}^2dx+2\int_{\mathbb{R}}(p'(\bar n+\omega_x)-p'(\bar n))\bar n_{xx}\omega_{xt}dx\\
&\quad+\int_{\mathbb{R}}p''(\bar n+\omega_x)\omega_{xt}\omega_{xx}^2dx\\
&\quad+2\int_{\mathbb{R}}[p''(\bar n+\omega_x)(\bar n_x+\omega_{xx})-p''(\bar n)\bar n_x]\bar n_x\omega_{xt}dx\\
&\quad-\int_{\mathbb{R}}(1+t)^{-1}(p'(\bar n+\omega_x)-p'(\bar n))(\bar n_x+\omega_{xx})\omega_{xx}dx\\
&\le-\frac{d}{dt}\int_{\mathbb{R}}(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xx}^2dx\\
&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}[(1+t)^{-1}(\omega_x^2+\omega_{xx}^2)+(1+t)\omega_{xt}^2]dx.
\end{aligned}\tag{46}$$

We apply (25)–(26) and Hölder inequality to estimate I_6 as

$$\begin{aligned}
I_6&=-\int_{\mathbb{R}}\left[\frac{2\omega_t\omega_{xt}}{\bar n+\omega_x}-\frac{\omega_t^2}{(\bar n+\omega_x)^2}(\bar n_x+\omega_{xx})\right](2\omega_{xxt}+(1+t)^{-1}\omega_{xx})dx\\
&=\frac{d}{dt}\int_{\mathbb{R}}\frac{\omega_t^2\omega_{xx}^2}{(\bar n+\omega_x)^2}dx-2\int_{\mathbb{R}}\frac{\omega_t\omega_{tt}\omega_{xx}^2}{(\bar n+\omega_x)^2}dx+2\int_{\mathbb{R}}\frac{\omega_t^2\omega_{xt}\omega_{xx}^2}{(\bar n+\omega_x)^3}dx\\
&\quad-2\int_{\mathbb{R}}\frac{\omega_t\omega_{xt}^2}{(\bar n+\omega_x)^2}(\bar n_x+\omega_{xx})dx+2\int_{\mathbb{R}}\frac{\omega_{xt}^3}{\bar n+\omega_x}dx-2\int_{\mathbb{R}}\frac{\bar n_{xx}\omega_t^2\omega_{xt}}{(\bar n+\omega_x)^2}dx\\
&\quad-4\int_{\mathbb{R}}\frac{\bar n_x\omega_t\omega_{xt}^2}{(\bar n+\omega_x)^2}dx+4\int_{\mathbb{R}}\frac{\bar n_x\omega_t^2\omega_{xt}}{(\bar n+\omega_x)^3}(\bar n_x+\omega_{xx})dx
\end{aligned}\tag{47}$$

$$
\begin{aligned}
&- 2\int_{\mathbb{R}}(1+t)^{-1}\frac{\omega_t\omega_{xt}\omega_{xx}}{\bar{n}+\omega_x}dx+\int_{\mathbb{R}}(1+t)^{-1}\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2}dx\\
&+\int_{\mathbb{R}}(1+t)^{-1}\frac{\bar{n}_x\omega_t^2\omega_{xx}}{(\bar{n}+\omega_x)^2}dx\\
\le\ &\frac{d}{dt}\int_{\mathbb{R}}\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2}dx+C(\varepsilon+\delta_0)\int_{\mathbb{R}}[(1+t)^{-1}\omega_{xx}^2+(1+t)(\omega_t^2+\omega_{xt}^2)]dx.
\end{aligned}
$$

Putting (45)–(47) into (44) and using the smallness of $\varepsilon+\delta_0$, we know that there exists a positive constant C_6 such that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[\omega_{xt}^2+p'(\bar{n})\omega_{xx}^2+\bar{n}\omega_x^2+(1+t)^{-1}\omega_x\omega_{xt}+(p'(\bar{n}+\omega_x)\\
&\quad -p'(\bar{n}))\omega_{xx}^2-\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2}+\frac{1}{2}\omega_x^2+\frac{1}{2}(1+t)^{-2}\omega_x^2\Big]dx\\
&\quad +C_6\int_{\mathbb{R}}[(1+t)\omega_{xt}^2+(1+t)^{-1}(\omega_{xx}^2+\omega_x^2)]dx\\
&\le C\int_{\mathbb{R}}[(1+t)\omega_t^2+(1+t)^{-1}\omega^2]dx.
\end{aligned}
\tag{48}
$$

Integrating (48) over (t_0,t), by (34), one has

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big[\omega_{xt}^2+p'(\bar{n})\omega_{xx}^2+\bar{n}\omega_x^2+(1+t)^{-1}\omega_x\omega_{xt}+(p'(\bar{n}+\omega_x)\\
&\quad -p'(\bar{n}))\omega_{xx}^2-\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2}+\frac{1}{2}\omega_x^2+\frac{1}{2}(1+t)^{-2}\omega_x^2\Big]dx\\
&\quad +C_6\int_{t_0}^t\int_{\mathbb{R}}[(1+\tau)\omega_{xt}^2+(1+\tau)^{-1}(\omega_{xx}^2+\omega_x^2)]dxd\tau\\
&\le C\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right),
\end{aligned}
\tag{49}
$$

then, from (4), (9), (25)–(26), Cauchy-Schwartz's inequality and the smallness of ε, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}}(\omega_x^2+\omega_{xx}^2+\omega_{xt}^2)dx+\int_{t_0}^t\int_{\mathbb{R}}[(1+\tau)^{-1}(\omega_x^2+\omega_{xx}^2)+(1+\tau)\omega_{xt}^2]dxd\tau\\
&\le C\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned}
\tag{50}
$$

Next, we multiply (42) by $\ln(1+t)(2\omega_{xt}+(1+t)^{-1}\omega_x)$ and integrate it over $\mathbb{R}$, after a similar process to (45)–(47), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\big[\ln(1+t)(\omega_{xt}^2+p'(\bar{n})\omega_{xx}^2+\bar{n}\omega_x^2)+(1+t)^{-1}\ln(1+t)\omega_x\omega_{xt}\\
&\quad +\frac{1}{2}(1+t)^{-2}\ln(1+t)\omega_x^2+\frac{1}{2}\ln(1+t)\omega_x^2-\frac{1}{2}(1+t)^{-2}\omega_x^2]dx\\
&\quad +\int_{\mathbb{R}}[2(1+t)-(1+t)^{-1}]\ln(1+t)\omega_{xt}^2dx\\
&\quad +\int_{\mathbb{R}}(1+t)^{-1}\ln(1+t)[p'(\bar{n})\omega_{xx}^2+(\bar{n}+(1+t)^{-2})\omega_x^2]dx
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} (1+t)^{-1}\left[\omega_{xt}^2 + p'(\bar{n})\omega_{xx}^2 + \left(\bar{n} + \frac{1}{2} + \frac{3}{2}(1+t)^{-2}\right)\omega_x^2\right] dx \\
&\quad + C\varepsilon \int_{\mathbb{R}} (1+t)^{-1}\omega_{xx}^2 dx \\
&\quad - \frac{d}{dt}\int_{\mathbb{R}} \ln(1+t)(p'(\bar{n}+\omega_x) - p'(\bar{n}))\omega_{xx}^2 dx + \frac{d}{dt}\int_{\mathbb{R}} \ln(1+t)\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2} dx \\
&\quad + C(\varepsilon+\delta_0)\int_{\mathbb{R}} \ln(1+t)[(1+t)^{-1}(\omega^2+\omega_x^2+\omega_{xx}^2) + (1+t)(\omega_t^2+\omega_{xt}^2)]dx.
\end{aligned}
$$

Applying the smallness of $\varepsilon+\delta_0$ to (51), and integrating the inequality over (t_0, t) yields

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big[\ln(1+t)(\omega_{xt}^2 + p'(\bar{n})\omega_{xx}^2 + \bar{n}\omega_x^2) + (1+t)^{-1}\ln(1+t)\omega_x\omega_{xt} \\
&\quad + \frac{1}{2}(1+t)^{-2}\ln(1+t)\omega_x^2 + \frac{1}{2}\ln(1+t)\omega_x^2 + \ln(1+t)(p'(\bar{n}+\omega_x) \\
&\quad - p'(\bar{n}))\omega_{xx}^2 - \ln(1+t)\frac{\omega_t^2\omega_{xx}^2}{(\bar{n}+\omega_x)^2}\Big] dx \\
&\quad + C_7\int_{t_0}^t \int_{\mathbb{R}} \ln(1+\tau)[(1+\tau)\omega_{xt}^2 + (1+\tau)^{-1}(\omega_{xx}^2+\omega_x^2)]dxd\tau \\
&\le \frac{1}{2}\int_{\mathbb{R}} (1+t)^{-2}\omega_x^2 dx + C\int_{t_0}^t\int_{\mathbb{R}} (1+\tau)^{-1}(\omega_{xt}^2+\omega_{xx}^2+\omega_x^2)dxd\tau \\
&\quad + C\int_{t_0}^t\int_{\mathbb{R}} \ln(1+\tau)[(1+\tau)^{-1}\omega^2 + (1+\tau)\omega_t^2]dxd\tau \\
&\quad + C\ln(1+t_0)\left(\|\omega_0\|_2^2 + \|J_0\|_1^2\right) \\
&\le C1!\sum_{j=0}^{1}\ln^j(1+t_0)\left(\|\omega_0\|_2^2 + \|J_0\|_1^2\right),
\end{aligned} \tag{52}
$$

where we used (37) and (50) in the last inequality. Therefore, the desired estimate (43) can be deduced by using (4), (9), (25)–(26), $\varepsilon \ll 1$ and the Cauchy-Schwartz's inequality.

Step 2. We make the induction hypothesis that (41) holds when the integer $k = l-1$ with $l \ge 2$, i.e.,

$$
\begin{aligned}
&\ln^{l-1}(1+t)\left(\|\omega_x(t)\|^2 + \|\omega_{xx}(t)\|^2 + \|\omega_{xt}(t)\|^2\right) \\
&\quad + \int_{t_0}^t \ln^{l-1}(1+\tau)\left[(1+\tau)\|\omega_{xt}(\tau)\|^2 + (1+\tau)^{-1}\left(\|\omega_x(\tau)\|^2 + \|\omega_{xx}(\tau)\|^2\right)\right]d\tau \\
&\le C(l-1)!\sum_{j=0}^{l-1}\ln^j(1+t_0)\left(\|\omega_0\|_2^2 + \|J_0\|_1^2\right).
\end{aligned} \tag{53}
$$

In what follows, we only need to prove that the estimate (41) holds when $k = l$ by using the induction hypothesis (53). Multiplying (42) by $\ln^l(1+t)(2\omega_{xt}+(1+t)^{-1}\omega_x)$

and integrating it over $\mathbb{R}$, analogous to (45)–(47), one has

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[\ln^l(1+t)(\omega_{xt}^2+p'(\bar n)\omega_{xx}^2+\bar n\omega_x^2)+(1+t)^{-1}\ln^l(1+t)\omega_x\omega_{xt}\\
&\quad+\frac{1}{2}(1+t)^{-2}\ln^l(1+t)\omega_x^2+\frac{1}{2}\ln^l(1+t)\omega_x^2-\frac{l}{2}(1+t)^{-2}\ln^{l-1}(1+t)\omega_x^2\Big]dx\\
&\quad+\int_{\mathbb{R}}(1+t)^{-1}\ln^l(1+t)[p'(\bar n)\omega_{xx}^2+(\bar n+(1+t)^{-2})\omega_x^2]dx\\
&\quad+\int_{\mathbb{R}}[2(1+t)-(1+t)^{-1}]\ln^l(1+t)\omega_{xt}^2dx\\
&\quad+\frac{l}{2}(l-1)\int_{\mathbb{R}}(1+t)^{-3}\ln^{l-2}(1+t)\omega_x^2dx\\
&\le l\int_{\mathbb{R}}(1+t)^{-1}\ln^{l-1}(1+t)\left[\omega_{xt}^2+p'(\bar n)\omega_{xx}^2+\left(\bar n+\frac{1}{2}+\frac{3}{2}(1+t)^{-2}\right)\omega_x^2\right]dx\\
&\quad-\frac{d}{dt}\int_{\mathbb{R}}\ln^l(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xx}^2dx+\frac{d}{dt}\int_{\mathbb{R}}\ln^l(1+t)\frac{\omega_t^2\omega_{xx}^2}{(\bar n+\omega_x)^2}dx\\
&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}\ln^l(1+t)[(1+t)^{-1}(\omega^2+\omega_x^2+\omega_{xx}^2)+(1+t)(\omega_t^2+\omega_{xt}^2)]dx\\
&\quad+C\varepsilon l\int_{\mathbb{R}}(1+t)^{-1}\ln^{l-1}(1+t)\omega_{xx}^2dx.
\end{aligned}
\tag{54}
$$

We apply the smallness of $\varepsilon+\delta_0$ to (54), and integrate the resultant inequality over (t_0,t) to get

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big[\ln^l(1+t)(\omega_{xt}^2+p'(\bar n)\omega_{xx}^2+\bar n\omega_x^2)+(1+t)^{-1}\ln^l(1+t)\omega_x\omega_{xt}\\
&\quad+\frac{1}{2}(1+t)^{-2}\ln^l(1+t)\omega_x^2+\frac{1}{2}\ln^l(1+t)\omega_x^2\\
&\quad+\ln^l(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xx}^2-\ln^l(1+t)\frac{\omega_t^2\omega_{xx}^2}{(\bar n+\omega_x)^2}\Big]dx\\
&\quad+C_8\int_{t_0}^t\int_{\mathbb{R}}\ln^l(1+\tau)[(1+\tau)\omega_{xt}^2+(1+\tau)^{-1}(\omega_{xx}^2+\omega_x^2)]dxd\tau\\
&\le C\ln^l(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right)+\frac{l}{2}\int_{\mathbb{R}}(1+t)^{-2}\ln^{l-1}(1+t)\omega_x^2dx\\
&\quad+Cl\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^{-1}\ln^{l-1}(1+\tau)(\omega_{xt}^2+\omega_{xx}^2+\omega_x^2)dxd\tau\\
&\quad+C\int_{t_0}^t\int_{\mathbb{R}}\ln^l(1+\tau)[(1+\tau)^{-1}\omega^2+(1+\tau)\omega_t^2]dxd\tau\\
&\le Cl!\sum_{j=0}^{l}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right),
\end{aligned}
\tag{55}
$$

here we used (27) and (53) in the last inequality. Thus, by (4), (9), (25)–(26), Cauchy-Schwartz's inequality and the smallness of ε, we derive the estimate (41) with $k=l$. This finishes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Under the assumptions of Proposition 2.5, it holds that*

$$
\begin{aligned}
&(1+t)^2 \ln^k(1+t)\left(\|\omega_t(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{tt}(t)\|^2\right) \\
&\quad+\int_{t_0}^t (1+\tau)^3 \ln^k(1+\tau)\|\omega_{tt}(\tau)\|^2 d\tau \le Ck! \sum_{j=0}^k \ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right),
\end{aligned} \tag{56}
$$

where $k \in [1,+\infty)$ is an integer.

Proof. Differentiating (21) in t gives

$$
\begin{aligned}
&\omega_{ttt}+(1+t)\omega_{tt}-(p'(\bar{n})\omega_{xt})_x+(\bar{n}+1)\omega_t \\
&=-\omega_x\omega_t-(\bar{E}+\omega)\omega_{xt}+[(p'(\bar{n}+\omega_x)-p'(\bar{n}))\omega_{xt}]_x+\left(\frac{\omega_t^2}{\bar{n}+\omega_x}\right)_{xt}.
\end{aligned} \tag{57}
$$

We also use induction to prove this Lemma and divide the proof into two steps.

Step 1. This step devotes to obtain the estimate (56) with $k=1$, i.e.,

$$
\begin{aligned}
&(1+t)^2 \ln(1+t)\left(\|\omega_t(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{tt}(t)\|^2\right) \\
&\quad+\int_{t_0}^t (1+\tau)^3 \ln(1+\tau)\|\omega_{tt}(\tau)\|^2 d\tau \le C1! \sum_{j=0}^1 \ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned} \tag{58}
$$

Firstly, multiplying (57) by $2(\beta+t)^2\omega_{tt}$, where $\beta>1$ is a constant to be determined later, and integrating the resultant equality over $\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2[\omega_{tt}^2+p'(\bar{n})\omega_{xt}^2+(\bar{n}+1)\omega_t^2]dx \\
&\quad+\int_{\mathbb{R}} 2(\beta+t)[(\beta+t)(1+t)-1]\omega_{tt}^2 dx \\
&=2\int_{\mathbb{R}}(\beta+t)[p'(\bar{n})\omega_{xt}^2+(\bar{n}+1)\omega_t^2]dx \\
&\quad-2\int_{\mathbb{R}}(\beta+t)^2\omega_{tt}[\omega_x\omega_t+(\bar{E}+\omega)\omega_{xt}]dx \\
&\quad+2\int_{\mathbb{R}}(\beta+t)^2\omega_{tt}[(p'(\bar{n}+\omega_x)-p'(\bar{n}))\omega_{xt}]_x dx \\
&\quad+2\int_{\mathbb{R}}(\beta+t)^2\omega_{tt}\left(\frac{\omega_t^2}{\bar{n}+\omega_x}\right)_{xt} dx =: I_7+I_8+I_9.
\end{aligned} \tag{59}
$$

We choose $\beta=2$ to ensure that

$$(\beta+t)(1+t)-1 \ge (1+t)^2. \tag{60}$$

It is easy to verify that

$$
\begin{aligned}
I_7 &\le C(\varepsilon+\delta_0)(\beta+t)^2\int_{\mathbb{R}}(|\omega_t\omega_{tt}|+|\omega_{xt}\omega_{tt}|)dx \\
&\le C(\varepsilon+\delta_0)\int_{\mathbb{R}}[(1+t)^3\omega_{tt}^2+(1+t)(\omega_{xt}^2+\omega_t^2)]dx.
\end{aligned} \tag{61}
$$

It follows from (9), (25)–(26), Taylor's formula and Hölder inequality that

$$\begin{aligned} I_8 &= -2\int_{\mathbb{R}}(\beta+t)^2\omega_{xtt}(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}dx \\ &= -\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2dx+\int_{\mathbb{R}}(\beta+t)^2p''(\bar n+\omega_x)\omega_{xt}^3dx \\ &\quad +2\int_{\mathbb{R}}(\beta+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2dx \\ &\le -\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2dx+C\varepsilon\int_{\mathbb{R}}(1+t)\omega_{xt}^2dx. \end{aligned} \tag{62}$$

By (11), (25)–(26), and Hölder inequality, we can estimate I_9 as

$$\begin{aligned} I_9 &= -2\int_{\mathbb{R}}(\beta+t)^2\omega_{xtt}\left(\frac{2\omega_t\omega_{tt}}{\bar n+\omega_x}-\frac{\omega_t^2\omega_{xt}}{(\bar n+\omega_x)^2}\right)dx \\ &= \frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}dx-2\int_{\mathbb{R}}(\beta+t)\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}dx \\ &\quad -2\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_t\omega_{tt}\omega_{xt}^2}{(\bar n+\omega_x)^2}dx+2\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_t^2\omega_{xt}^3}{(\bar n+\omega_x)^3}dx \\ &\quad +2\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_{xt}\omega_{tt}^2}{\bar n+\omega_x}dx-2\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_t\omega_{tt}^2}{(\bar n+\omega_x)^2}(\bar n_x+\omega_{xx})dx \\ &\le \frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}dx+C(\varepsilon+\delta_0)\int_{\mathbb{R}}[(1+t)\omega_{xt}^2+(1+t)^3\omega_{tt}^2]dx. \end{aligned} \tag{63}$$

Substituting (61)–(63) into (59), by using (60) and the smallness of $\varepsilon+\delta_0$, we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\left[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2+(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2\right. \\ &\quad \left.-\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}\right]dx+\int_{\mathbb{R}}(1+t)^3\omega_{tt}^2dx\le C\int_{\mathbb{R}}(1+t)(\omega_t^2+\omega_{xt}^2)dx. \end{aligned} \tag{64}$$

We multiply the equation (21) by ω_{tt} to give

$$\omega_{tt}^2=\left[-(1+t)\omega_t-\bar n\omega-(\bar E+\omega)\omega_x+(p(\bar n+\omega_x)-p(\bar n))_x+\left(\frac{\omega_t^2}{\bar n+\omega_x}\right)_x\right]\omega_{tt}. \tag{65}$$

Integrating (65) over $\mathbb{R}$ and using (25)–(26), Hölder inequality, one has

$$\int_{\mathbb{R}}\omega_{tt}^2dx\le C\int_{\mathbb{R}}[(1+t)^2\omega_t^2+\omega^2+\omega_x^2+\omega_{xx}^2+\omega_{xt}^2]dx. \tag{66}$$

Now, integrating (64) over (t_0,t), by (34), (50) and (66), we obtain

$$\begin{aligned} &\int_{\mathbb{R}}(\beta+t)^2\left[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2+(p'(\bar n+\omega_x)\right. \\ &\qquad \left.-p'(\bar n))\omega_{xt}^2-\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}\right]dx+\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^3\omega_{tt}^2dxd\tau \\ &\le C\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)(\omega_t^2+\omega_{xt}^2)dxd\tau+C\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right) \\ &\le C\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right). \end{aligned} \tag{67}$$

Therefore, it follows from (4), (9), (25)–(26) and $\varepsilon \ll 1$ that

$$\int_{\mathbb{R}}(1+t)^2(\omega_t^2+\omega_{xt}^2+\omega_{tt}^2)dx+\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^3\omega_{tt}^2dxd\tau \le C\big(\|\omega_0\|_2^2+\|J_0\|_1^2\big). \quad (68)$$

Secondly, we multiply (57) by $2(\beta+t)^2\ln(1+t)\omega_{tt}$ and integrate it over $\mathbb{R}$, and then after a similar calculation to (61)–(63) to have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln(1+t)[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad+\int_{\mathbb{R}}2(\beta+t)\ln(1+t)[(\beta+t)(1+t)-1]\omega_{tt}^2dx\\
&\le 2\int_{\mathbb{R}}(\beta+t)\ln(1+t)[p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad+\int_{\mathbb{R}}(1+t)^{-1}(\beta+t)^2[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad-\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2dx\\
&\quad+\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln(1+t)\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}dx+C\varepsilon\int_{\mathbb{R}}(1+t)\omega_{xt}^2dx\\
&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}\ln(1+t)[(1+t)^3\omega_{tt}^2+(1+t)(\omega_{xt}^2+\omega_t^2)]dx.
\end{aligned} \quad (69)$$

Applying the smallness of $\varepsilon+\delta_0$ to (69), and integrating the inequality over (t_0,t), by (34), (37), (43), (50), (66) and (68), we get

$$\begin{aligned}
&\int_{\mathbb{R}}(\beta+t)^2\ln(1+t)\,\Big[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2+(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2\\
&\quad-\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}\Big]\,dx+\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^3\ln(1+\tau)\omega_{tt}^2dxd\tau\\
&\le C\ln(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right)+C\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)(\omega_{tt}^2+\omega_{xt}^2+\omega_t^2)dxd\tau\\
&\quad+C\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)\ln(1+\tau)(\omega_{xt}^2+\omega_t^2)dxd\tau\\
&\le C1!\sum_{j=0}^{1}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned} \quad (70)$$

Thus, the desired estimate (58) can be derived by using (4), (9), (25)–(26), and $\varepsilon \ll 1$.

Step 2. We make the induction hypothesis that (56) holds when the integer $k=l-1$ with $l\ge 2$, i.e.,

$$\begin{aligned}
&(1+t)^2\ln^{l-1}(1+t)\left(\|\omega_t(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{tt}(t)\|^2\right)\\
&\quad+\int_{t_0}^t(1+\tau)^3\ln^{l-1}(1+\tau)\|\omega_{tt}(\tau)\|^2d\tau\\
&\le C(l-1)!\sum_{j=0}^{l-1}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned} \quad (71)$$

Next, we only need to prove that (56) holds when $k = l$. Multiplying (57) by $2(\beta+t)^2 \ln^l(1+t)\omega_{tt}$ and integrating it over $\mathbb{R}$, analogous to (61)–(63), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln^l(1+t)[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad+\int_{\mathbb{R}}2(\beta+t)\ln^l(1+t)[(\beta+t)(1+t)-1]\omega_{tt}^2dx\\
&\le 2\int_{\mathbb{R}}(\beta+t)\ln^l(1+t)[p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad+l\int_{\mathbb{R}}(\beta+t)^2(1+t)^{-1}\ln^{l-1}(1+t)[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2]dx\\
&\quad-\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln^l(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2dx\\
&\quad+\frac{d}{dt}\int_{\mathbb{R}}(\beta+t)^2\ln^l(1+t)\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}dx+C\varepsilon l\int_{\mathbb{R}}(1+t)\ln^{l-1}(1+t)\omega_{xt}^2dx\\
&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}\ln^l(1+t)[(1+t)^3\omega_{tt}^2+(1+t)(\omega_{xt}^2+\omega_t^2)]dx.
\end{aligned}
\tag{72}
$$

We employ the smallness of $\varepsilon+\delta_0$ to (72) and integrate the resultant inequality over (t_0, t), by (27), (41), (66) and (71), it holds that

$$
\begin{aligned}
&\int_{\mathbb{R}}(\beta+t)^2\ln^l(1+t)\left[\omega_{tt}^2+p'(\bar n)\omega_{xt}^2+(\bar n+1)\omega_t^2+(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xt}^2\right.\\
&\quad\left.-\frac{\omega_t^2\omega_{xt}^2}{(\bar n+\omega_x)^2}\right]dx+\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^3\ln^l(1+\tau)\omega_{tt}^2dxd\tau\\
&\le C\ln^l(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right)+C\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)\ln^l(1+\tau)(\omega_{xt}^2+\omega_t^2)dxd\tau\\
&\quad+Cl\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)\ln^{l-1}(1+\tau)(\omega_{tt}^2+\omega_{xt}^2+\omega_t^2)dxd\tau\\
&\le Cl!\sum_{j=0}^{l}\ln^j(1+t_0)\left(\|\omega_0\|_2^2+\|J_0\|_1^2\right).
\end{aligned}
\tag{73}
$$

Therefore, we derive the estimate (56) with $k = l$ by using (4), (9), (25)–(26) and $\varepsilon \ll 1$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Under the assumptions of Proposition 2.5, it holds that*

$$
\begin{aligned}
&\ln^k(1+t)\left(\|\omega_{xx}(t)\|^2+\|\omega_{xxx}(t)\|^2+\|\omega_{xxt}(t)\|^2\right)\\
&\quad+\int_{t_0}^t\ln^k(1+\tau)\left[(1+\tau)\|\omega_{xxt}(\tau)\|^2+(1+\tau)^{-1}\left(\|\omega_{xx}(\tau)\|^2+\|\omega_{xxx}(\tau)\|^2\right)\right]d\tau\\
&\le Ck!\sum_{j=0}^{k}\ln^j(1+t_0)\left(\|\omega_0\|_3^2+\|J_0\|_2^2\right),
\end{aligned}
\tag{74}
$$

where $k \in [1, +\infty)$ is an integer.

Proof. Differentiating (42) with respect to x gives

$$\begin{aligned}
&\omega_{xxtt} + (1+t)\omega_{xxt} - (p'(\bar n)\omega_{xxx})_x + \bar n\omega_{xx} \\
&= -\bar n_{xx}\omega - (2\bar n_x + \bar E_{xx})\omega_x - (2\bar E_x + 3\omega_x)\omega_{xx} - (\bar E + \omega)\omega_{xxx} + \left(\frac{\omega_t^2}{\bar n + \omega_x}\right)_{xxx} \\
&\quad + [(p'(\bar n + \omega_x) - p'(\bar n))(\bar n_{xx} + \omega_{xxx}) + p''(\bar n + \omega_x)(\bar n_x + \omega_{xx})^2 - p''(\bar n)\bar n_x^2]_x.
\end{aligned} \tag{75}$$

We use induction to prove this Lemma and the proof is divided into two steps.

Step 1. In this step, we prove that the estimate (74) holds when $k = 1$. We first multiply (75) by $2\omega_{xxt} + (1+t)^{-1}\omega_{xx}$ and integrate it over $\mathbb{R}$ to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[\omega_{xxt}^2 + p'(\bar n)\omega_{xxx}^2 + \bar n\omega_{xx}^2 + (1+t)^{-1}\omega_{xx}\omega_{xxt} + \frac{1}{2}\omega_{xx}^2 \\
&\quad + \frac{1}{2}(1+t)^{-2}\omega_{xx}^2\Big]dx + \int_{\mathbb{R}}[2(1+t) - (1+t)^{-1}]\omega_{xxt}^2 dx \\
&\quad + \int_{\mathbb{R}}(1+t)^{-1}[p'(\bar n)\omega_{xxx}^2 + (\bar n + (1+t)^{-2})\omega_{xx}^2]dx \\
&= -\int_{\mathbb{R}}[\bar n_{xx}\omega + (2\bar n_x + \bar E_{xx})\omega_x + (2\bar E_x + 3\omega_x)\omega_{xx} + (\bar E + \omega)\omega_{xxx}] \\
&\quad \cdot (2\omega_{xxt} + (1+t)^{-1}\omega_{xx})dx + \int_{\mathbb{R}}[(p'(\bar n + \omega_x) - p'(\bar n))(\bar n_{xx} + \omega_{xxx}) \\
&\quad + p''(\bar n + \omega_x)(\bar n_x + \omega_{xx})^2 - p''(\bar n)\bar n_x^2]_x(2\omega_{xxt} + (1+t)^{-1}\omega_{xx})dx \\
&\quad + \int_{\mathbb{R}}\left(\frac{\omega_t^2}{\bar n + \omega_x}\right)_{xxx}(2\omega_{xxt} + (1+t)^{-1}\omega_{xx})dx \\
&=: I_{10} + I_{11} + I_{12}.
\end{aligned} \tag{76}$$

It is easy to see that

$$I_{10} \le C(\varepsilon + \delta_0)\int_{\mathbb{R}}[(1+t)^{-1}(\omega^2 + \omega_x^2 + \omega_{xx}^2 + \omega_{xxx}^2) + (1+t)\omega_{xxt}^2]dx. \tag{77}$$

It follows from (9), (11), (25)–(26), Taylor's formula and Hölder inequality that

$$\begin{aligned}
I_{11} &= -\int_{\mathbb{R}}[(p'(\bar n + \omega_x) - p'(\bar n))(\bar n_{xx} + \omega_{xxx}) + p''(\bar n + \omega_x)(\bar n_x + \omega_{xx})^2 \\
&\quad - p''(\bar n)\bar n_x^2](2\omega_{xxxt} + (1+t)^{-1}\omega_{xxx})dx \\
&= -\frac{d}{dt}\int_{\mathbb{R}}(p'(\bar n + \omega_x) - p'(\bar n))\omega_{xxx}^2 dx + \int_{\mathbb{R}} p''(\bar n + \omega_x)\omega_{xt}\omega_{xxx}^2 dx \\
&\quad + 6\int_{\mathbb{R}} p''(\bar n + \omega_x)\bar n_{xx}\omega_{xx}\omega_{xxt}dx + 2\int_{\mathbb{R}}(p'(\bar n + \omega_x) - p'(\bar n))\bar n_{xxx}\omega_{xxt}dx \\
&\quad + 6\int_{\mathbb{R}}(p''(\bar n + \omega_x) - p''(\bar n))\bar n_x\bar n_{xx}\omega_{xxt}dx + 6\int_{\mathbb{R}} p'''(\bar n + \omega_x)\bar n_x^2\omega_{xx}\omega_{xxt}dx \\
&\quad + 6\int_{\mathbb{R}} p'''(\bar n + \omega_x)\bar n_x\omega_{xx}^2\omega_{xxt}dx + 4\int_{\mathbb{R}} p''(\bar n + \omega_x)(\bar n_x + \omega_{xx})\omega_{xxx}\omega_{xxt}dx \\
&\quad + 2\int_{\mathbb{R}} p'''(\bar n + \omega_x)\omega_{xx}^3\omega_{xxt}dx + 2\int_{\mathbb{R}}(p'''(\bar n + \omega_x) - p'''(\bar n))\bar n_x^3\omega_{xxt}dx
\end{aligned} \tag{78}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} [(p'(\bar{n}+\omega_x) - p'(\bar{n}))(\bar{n}_{xx} + \omega_{xxx}) \\
&+ p''(\bar{n}+\omega_x)(\bar{n}_x + \omega_{xx})^2 - p''(\bar{n})\bar{n}_x^2](1+t)^{-1}\omega_{xxx} dx \\
\leq & -\frac{d}{dt}\int_{\mathbb{R}} (p'(\bar{n}+\omega_x) - p'(\bar{n}))\omega_{xxx}^2 dx \\
&+ C(\varepsilon+\delta_0)\int_{\mathbb{R}} [(1+t)^{-1}(\omega_x^2 + \omega_{xx}^2 + \omega_{xxx}^2) + (1+t)\omega_{xxt}^2] dx.
\end{aligned}
$$

We can estimate I_{12} by (11), (25)–(26) and Hölder inequality as

$$
\begin{aligned}
I_{12} =& -\int_{\mathbb{R}} \left(\frac{\omega_t^2}{\bar{n}+\omega_x}\right)_{xx} (2\omega_{xxxt} + (1+t)^{-1}\omega_{xxx}) dx \\
=& \frac{d}{dt}\int_{\mathbb{R}} \frac{\omega_t^2\omega_{xxx}^2}{(\bar{n}+\omega_x)^2} dx - 2\int_{\mathbb{R}} \frac{\omega_t\omega_{tt}\omega_{xxx}^2}{(\bar{n}+\omega_x)^2} dx + 2\int_{\mathbb{R}} \frac{\omega_t^2\omega_{xt}\omega_{xxx}^2}{(\bar{n}+\omega_x)^3} dx \\
&+ 10\int_{\mathbb{R}} \frac{\omega_{xt}\omega_{xxt}^2}{\bar{n}+\omega_x} dx - 12\int_{\mathbb{R}} \frac{\omega_{xt}^2\omega_{xxt}}{(\bar{n}+\omega_x)^2}(\bar{n}_x+\omega_{xx}) dx \\
&- 10\int_{\mathbb{R}} \frac{\omega_t\omega_{xxt}^2}{(\bar{n}+\omega_x)^2}(\bar{n}_x+\omega_{xx}) dx \\
&+ 24\int_{\mathbb{R}} \frac{\omega_t\omega_{xt}\omega_{xxt}}{(\bar{n}+\omega_x)^3}(\bar{n}_x+\omega_{xx})^2 dx - \int_{\mathbb{R}} \frac{\omega_t\omega_{xt}\omega_{xxt}}{(\bar{n}+\omega_x)^2}(12\bar{n}_{xx} + 8\omega_{xxx}) dx \\
&- 12\int_{\mathbb{R}} \frac{\omega_t^2\omega_{xxt}}{(\bar{n}+\omega_x)^4}(\bar{n}_x+\omega_{xx})^3 dx \\
&+ \int_{\mathbb{R}} \frac{\omega_t^2\omega_{xxt}}{(\bar{n}+\omega_x)^3}(\bar{n}_x+\omega_{xx})(12\bar{n}_{xx}+8\omega_{xxx}) dx \\
&- 2\int_{\mathbb{R}} \frac{\bar{n}_{xxx}\omega_t^2\omega_{xxt}}{(\bar{n}+\omega_x)^2} dx - \int_{\mathbb{R}} (1+t)^{-1}\omega_{xxx}\left[\frac{2\omega_{xt}^2}{\bar{n}+\omega_x} + \frac{2\omega_t\omega_{xxt}}{\bar{n}+\omega_x}\right. \\
&- \frac{4\omega_t\omega_{xt}}{(\bar{n}+\omega_x)^2}(\bar{n}_x+\omega_{xx}) + \frac{2\omega_t^2}{(\bar{n}+\omega_x)^3}(\bar{n}_x+\omega_{xx})^2 \\
&\left. - \frac{\omega_t^2}{(\bar{n}+\omega_x)^2}(\bar{n}_{xx}+\omega_{xxx})\right] dx \\
\leq& \frac{d}{dt}\int_{\mathbb{R}} \frac{\omega_t^2\omega_{xxx}^2}{(\bar{n}+\omega_x)^2} dx \\
&+ C(\varepsilon+\delta_0)\int_{\mathbb{R}} [(1+t)^{-1}\omega_{xxx}^2 + (1+t)(\omega_t^2 + \omega_{xt}^2 + \omega_{xxt}^2)] dx.
\end{aligned} \tag{79}
$$

Putting (77)–(79) into (76) and using the smallness of $\varepsilon+\delta_0$, there exists a constant $C_9 > 0$ such that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left[\omega_{xxt}^2 + p'(\bar{n})\omega_{xxx}^2 + \bar{n}\omega_{xx}^2 + (1+t)^{-1}\omega_{xx}\omega_{xxt} + \frac{1}{2}\omega_{xx}^2\right. \\
&\left. + \frac{1}{2}(1+t)^{-2}\omega_{xx}^2 - \frac{\omega_t^2\omega_{xxx}^2}{(\bar{n}+\omega_x)^2} + (p'(\bar{n}+\omega_x) - p'(\bar{n}))\omega_{xxx}^2\right] dx \\
&+ C_9\int_{\mathbb{R}} [(1+t)\omega_{xxt}^2 + (1+t)^{-1}(\omega_{xxx}^2 + \omega_{xx}^2)] dx \\
\leq& C\int_{\mathbb{R}} [(1+t)^{-1}(\omega^2 + \omega_x^2) + (1+t)(\omega_t^2 + \omega_{xt}^2)] dx.
\end{aligned} \tag{80}
$$

Integrating (80) over (t_0, t), by (34) and (50), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \Big[\omega_{xxt}^2 + p'(\bar n)\omega_{xxx}^2 + \bar n \omega_{xx}^2 + (1+t)^{-1}\omega_{xx}\omega_{xxt} + \frac{1}{2}\omega_{xx}^2 + \frac{1}{2}(1+t)^{-2}\omega_{xx}^2 \\
&\quad - \frac{\omega_t^2\omega_{xxx}^2}{(\bar n+\omega_x)^2} + (p'(\bar n+\omega_x) - p'(\bar n))\omega_{xxx}^2\Big] dx \\
&\quad + C_9 \int_{t_0}^t \int_{\mathbb{R}} [(1+\tau)\omega_{xxt}^2 + (1+\tau)^{-1}(\omega_{xxx}^2 + \omega_{xx}^2)] dx d\tau \\
&\le C \int_{t_0}^t \int_{\mathbb{R}} [(1+\tau)^{-1}(\omega^2 + \omega_x^2) + (1+\tau)(\omega_t^2 + \omega_{xt}^2)] dx d\tau + C\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right) \\
&\le C\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right).
\end{aligned} \tag{81}$$

Then, from (4), (9), (25)–(26), $\varepsilon \ll 1$ and the Cauchy-Schwartz's inequality

$$|(1+t)^{-1}\omega_{xx}\omega_{xxt}| \le \frac{1}{2}\omega_{xxt}^2 + \frac{1}{2}(1+t)^{-2}\omega_{xx}^2, \tag{82}$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} (\omega_{xx}^2 + \omega_{xxx}^2 + \omega_{xxt}^2) dx + \int_{t_0}^t \int_{\mathbb{R}} [(1+\tau)^{-1}(\omega_{xx}^2 + \omega_{xxx}^2) \\
&\quad + (1+\tau)\omega_{xxt}^2] dx d\tau \le C\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right).
\end{aligned} \tag{83}$$

Next, multiplying (75) by $\ln(1+t)(2\omega_{xxt} + (1+t)^{-1}\omega_{xx})$ and integrating it over $\mathbb{R}$, by a similar calculation to (77)–(79), one has

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \Big[\ln(1+t)(\omega_{xxt}^2 + p'(\bar n)\omega_{xxx}^2 + \bar n\omega_{xx}^2) + (1+t)^{-1}\ln(1+t)\omega_{xx}\omega_{xxt} \\
&\quad + \frac{1}{2}\ln(1+t)\omega_{xx}^2 + \frac{1}{2}\ln(1+t)(1+t)^{-2}\omega_{xx}^2 - \frac{1}{2}(1+t)^{-2}\omega_{xx}^2\Big] dx \\
&\quad + \int_{\mathbb{R}} \ln(1+t)[2(1+t) - (1+t)^{-1}]\omega_{xxt}^2 dx \\
&\quad + \int_{\mathbb{R}} (1+t)^{-1}\ln(1+t)[p'(\bar n)\omega_{xxx}^2 + (\bar n + (1+t)^{-2})\omega_{xx}^2] dx \\
&\le \int_{\mathbb{R}} (1+t)^{-1}\left[\omega_{xxt}^2 + p'(\bar n)\omega_{xxx}^2 + \left(\bar n + \frac{1}{2} + \frac{3}{2}(1+t)^{-2}\right)\omega_{xx}^2\right] dx \\
&\quad + C\varepsilon \int_{\mathbb{R}} (1+t)^{-1}\omega_{xxx}^2 dx - \frac{d}{dt}\int_{\mathbb{R}} \ln(1+t)(p'(\bar n+\omega_x) - p'(\bar n))\omega_{xxx}^2 dx \\
&\quad + \frac{d}{dt}\int_{\mathbb{R}} \ln(1+t)\frac{\omega_t^2\omega_{xxx}^2}{(\bar n+\omega_x)^2} dx + C(\varepsilon+\delta_0)\int_{\mathbb{R}} \ln(1+t)[(1+t)^{-1}(\omega^2 + \omega_x^2 \\
&\quad + \omega_{xx}^2 + \omega_{xxx}^2) + (1+t)(\omega_t^2 + \omega_{xt}^2 + \omega_{xxt}^2)] dx.
\end{aligned} \tag{84}$$

Applying the smallness of $\varepsilon + \delta_0$ to (84), and integrating the resultant inequality over (t_0, t), by (37), (43) and (83), we have

$$\begin{aligned}
&\int_{\mathbb{R}} \Big[\ln(1+t)(\omega_{xxt}^2 + p'(\bar n)\omega_{xxx}^2 + \bar n \omega_{xx}^2) + (1+t)^{-1}\ln(1+t)\omega_{xx}\omega_{xxt} \\
&\quad + \frac{1}{2}\ln(1+t)\omega_{xx}^2 + \frac{1}{2}\ln(1+t)(1+t)^{-2}\omega_{xx}^2 + \ln(1+t)(p'(\bar n + \omega_x) \\
&\quad - p'(\bar n))\omega_{xxx}^2 - \ln(1+t)\frac{\omega_t^2\omega_{xxx}^2}{(\bar n + \omega_x)^2}\Big] dx \\
&\quad + C_{10}\int_{t_0}^t \int_{\mathbb{R}} \ln(1+\tau)[(1+\tau)\omega_{xxt}^2 + (1+\tau)^{-1}(\omega_{xxx}^2 + \omega_{xx}^2)] dx d\tau \\
&\le \frac{1}{2}\int_{\mathbb{R}} (1+t)^{-2}\omega_{xx}^2 dx + C\int_{t_0}^t \int_{\mathbb{R}} (1+\tau)^{-1}(\omega_{xx}^2 + \omega_{xxx}^2 + \omega_{xxt}^2) dx d\tau \\
&\quad + C\int_{t_0}^t \int_{\mathbb{R}} \ln(1+\tau)[(1+\tau)^{-1}(\omega^2 + \omega_x^2) + (1+\tau)(\omega_t^2 + \omega_{xt}^2)] dx d\tau \\
&\quad + C\ln(1+t_0)\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right) \\
&\le C1!\sum_{j=0}^{1} \ln^j(1+t_0)\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right),
\end{aligned} \tag{85}$$

for some positive constant C_{10}. Therefore, from (4), (9), (25)–(26), (82) and the smallness of ε, we obtain the desired estimate (74) with $k = 1$, i.e.,

$$\begin{aligned}
&\ln(1+t)\left(\|\omega_{xx}(t)\|^2 + \|\omega_{xxx}(t)\|^2 + \|\omega_{xxt}(t)\|^2\right) + \int_{t_0}^t \ln(1+\tau)\left[(1+\tau)\|\omega_{xxt}(\tau)\|^2\right. \\
&\quad \left. + (1+\tau)^{-1}\left(\|\omega_{xx}(\tau)\|^2 + \|\omega_{xxx}(\tau)\|^2\right)\right] d\tau \\
&\le C1!\sum_{j=0}^{1} \ln^j(1+t_0)\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right).
\end{aligned} \tag{86}$$

Step 2. We make the induction hypothesis that the estimate (74) holds when the integer $k = l - 1$ with $l \ge 2$, i.e.,

$$\begin{aligned}
&\ln^{l-1}(1+t)\left(\|\omega_{xx}(t)\|^2 + \|\omega_{xxx}(t)\|^2 + \|\omega_{xxt}(t)\|^2\right) \\
&\quad + \int_{t_0}^t \ln^{l-1}(1+\tau)\left[(1+\tau)\|\omega_{xxt}(\tau)\|^2\right. \\
&\quad \left. + (1+\tau)^{-1}\left(\|\omega_{xx}(\tau)\|^2 + \|\omega_{xxx}(\tau)\|^2\right)\right] d\tau \\
&\le C(l-1)!\sum_{j=0}^{l-1} \ln^j(1+t_0)\left(\|\omega_0\|_3^2 + \|J_0\|_2^2\right).
\end{aligned} \tag{87}$$

Now we are going to prove that the estimate (74) holds when $k = l$ by using (87). Multiplying (75) by $\ln^l(1+t)(2\omega_{xxt} + (1+t)^{-1}\omega_{xx})$ and integrating it over $\mathbb{R}$, analogous to (77)–(79), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[\ln^l(1+t)(\omega_{xxt}^2+p'(\bar n)\omega_{xxx}^2+\bar n\omega_{xx}^2)+(1+t)^{-1}\ln^l(1+t)\omega_{xx}\omega_{xxt}\\
&\quad+\frac12\ln^l(1+t)\omega_{xx}^2+\frac12(1+t)^{-2}\ln^l(1+t)\omega_{xx}^2-\frac{l}{2}(1+t)^{-2}\ln^{l-1}(1+t)\omega_{xx}^2\Big]dx\\
&\quad+\int_{\mathbb{R}}(1+t)^{-1}\ln^l(1+t)[p'(\bar n)\omega_{xxx}^2+(\bar n+(1+t)^{-2})\omega_{xx}^2]dx\\
&\quad+\int_{\mathbb{R}}\ln^l(1+t)[2(1+t)-(1+t)^{-1}]\omega_{xxt}^2dx\\
&\quad+\frac{l}{2}(l-1)\int_{\mathbb{R}}(1+t)^{-3}\ln^{l-2}(1+t)\omega_{xx}^2dx\\
&\le l\int_{\mathbb{R}}(1+t)^{-1}\ln^{l-1}(1+t)\left[\omega_{xxt}^2+p'(\bar n)\omega_{xxx}^2+\left(\bar n+\frac12+\frac32(1+t)^{-2}\right)\omega_{xx}^2\right]dx\\
&\quad-\frac{d}{dt}\int_{\mathbb{R}}\ln^l(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xxx}^2dx+\frac{d}{dt}\int_{\mathbb{R}}\ln^l(1+t)\frac{\omega_t^2\omega_{xxx}^2}{(\bar n+\omega_x)^2}dx\\
&\quad+C(\varepsilon+\delta_0)\int_{\mathbb{R}}\ln^l(1+t)[(1+t)^{-1}(\omega^2+\omega_x^2+\omega_{xx}^2+\omega_{xxx}^2)\\
&\quad+(1+t)(\omega_t^2+\omega_{xt}^2+\omega_{xxt}^2)]dx+C\varepsilon l\int_{\mathbb{R}}(1+t)^{-1}\ln^{l-1}(1+t)\omega_{xxx}^2dx.
\end{aligned}
\tag{88}
$$

Applying the smallness of $\varepsilon+\delta_0$ to (88), and then integrating the resultant inequality over (t_0,t) yields

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big[\ln^l(1+t)(\omega_{xxt}^2+p'(\bar n)\omega_{xxx}^2+\bar n\omega_{xx}^2)+(1+t)^{-1}\ln^l(1+t)\omega_{xx}\omega_{xxt}\\
&\quad+\frac12\ln^l(1+t)\omega_{xx}^2+\frac12(1+t)^{-2}\ln^l(1+t)\omega_{xx}^2\\
&\quad+\ln^l(1+t)(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xxx}^2-\ln^l(1+t)\frac{\omega_t^2\omega_{xxx}^2}{(\bar n+\omega_x)^2}\Big]dx\\
&\quad+C_{11}\int_{t_0}^t\int_{\mathbb{R}}\ln^l(1+\tau)[(1+\tau)\omega_{xxt}^2+(1+\tau)^{-1}(\omega_{xxx}^2+\omega_{xx}^2)]dxd\tau\\
&\le C\ln^l(1+t_0)\left(\|\omega_0\|_3^2+\|J_0\|_2^2\right)+\frac{l}{2}\int_{\mathbb{R}}(1+t)^{-2}\ln^{l-1}(1+t)\omega_{xx}^2dx\\
&\quad+Cl\int_{t_0}^t\int_{\mathbb{R}}(1+\tau)^{-1}\ln^{l-1}(1+\tau)(\omega_{xxt}^2+\omega_{xxx}^2+\omega_{xx}^2)dxd\tau\\
&\quad+C\int_{t_0}^t\int_{\mathbb{R}}\ln^l(1+\tau)[(1+\tau)^{-1}(\omega^2+\omega_x^2)+(1+\tau)(\omega_t^2+\omega_{xt}^2)]dxd\tau\\
&\le Cl!\sum_{j=0}^{l}\ln^j(1+t_0)\left(\|\omega_0\|_3^2+\|J_0\|_2^2\right),
\end{aligned}
\tag{89}
$$

where we used (27), (41) and (87) in the last inequality. It then follows from (4), (9), (25)–(26), (82) and $\varepsilon\ll 1$ that the estimate (74) holds when $k=l$. The proof is finished. □

Lemma 3.5. *Under the assumptions of Proposition 2.5, it holds that*

$$(1+t)^2\ln^k(1+t)\left(\|\omega_{xt}(t)\|^2+\|\omega_{xxt}(t)\|^2+\|\omega_{xtt}(t)\|^2\right) + \int_{t_0}^t (1+\tau)^3\ln^k(1+\tau)\|\omega_{xtt}(\tau)\|^2 d\tau \le Ck!\sum_{j=0}^k \ln^j(1+t_0)\left(\|\omega_0\|_3^2+\|J_0\|_2^2\right), \tag{90}$$

where $k\in[1,+\infty)$ is an integer.

Proof. We differentiate (57) in x to obtain

$$\begin{aligned}&\omega_{xttt}+(1+t)\omega_{xtt}-(p'(\bar n)\omega_{xxt})_x+(\bar n+1)\omega_{xt}\\&=-(\bar n_x+\omega_{xx})\omega_t-(\bar E_x+2\omega_x)\omega_{xt}-(\bar E+\omega)\omega_{xxt}+\left(\frac{\omega_t^2}{\bar n+\omega_x}\right)_{xxt}\\&\quad+[(p'(\bar n+\omega_x)-p'(\bar n))\omega_{xxt}+p''(\bar n+\omega_x)(\bar n_x+\omega_{xx})\omega_{xt}]_x.\end{aligned} \tag{91}$$

We also use induction to prove this Lemma. Analogously to Lemma 3.3, multiplying (91) by $2(2+t)^2\omega_{xtt}$, $2(2+t)^2\ln(1+t)\omega_{xtt}$, and $2(2+t)^2\ln^l(1+t)\omega_{xtt}$, respectively, where the integer $l\ge 2$, and then integrating resultant equations over $\mathbb{R}$, by using (27), (34), (41), (50), (74), (83), and the equation (42), we can derive the desired estimate (90). The proof is omitted here. □

Proof of Proposition 2.5. Lemmas 3.1–3.5 imply Proposition 2.5. □

Declarations

Ethical Approval: Non-applicable for this study.

Competing interests: The authors declare that this work does not have any conflicts of interests.

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