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Critical sharp front for doubly nonlinear degenerate diffusion equations with time delay

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Abstract

This paper is concerned with the critical sharp travelling wave for doubly nonlinear diffusion equation with time delay, where the doubly nonlinear degenerate diffusion is defined by $(|(u^m)_x|^{p-2}(u^m)_x)_x$ with $m > 0$ and $p > 1$. The doubly nonlinear diffusion equation is proved to admit a unique sharp type travelling wave for the degenerate case $m(p-1) > 1$, the so-called slow-diffusion case. This sharp travelling wave associated with the minimal wave speed $c^*(m, p, r)$ is monotonically increasing, where the minimal wave speed satisfies $c^*(m, p, r) < c^*(m, p, 0)$ for any time delay $r > 0$. The sharp front is C^1-smooth for $\frac{1}{p-1} < m < \frac{p}{p-1}$, and piecewise smooth for $m \geqslant \frac{p}{p-1}$. Our results indicate that time delay slows down the minimal travelling wave speed for the doubly nonlinear degenerate diffusion equations. The approach adopted for proof is the phase transform method combining the variational method. The main technical issue for the proof is to overcome the obstacle caused by the doubly nonlinear degenerate diffusion.

Keywords: doubly nonlinearity, variational approach, time delay, degenerate diffusion, sharp type wave

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(Some figures may appear in colour only in the online journal)

1. Introduction

This is a continuation of our recent study [46] on critical travelling waves for time-delayed degenerate diffusion equation. Our purpose in the present paper is to study the existence, uniqueness and regularity of the critical sharp travelling wave for the following doubly nonlinear diffusion equation with time delay

$$\frac{\partial u}{\partial t} = \left(|(u^m)_x|^{p-2}(u^m)_x\right)_x - d(u) + b(u(t-r,x)), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $p > 1, m > 0$, u is the population density, $b(u(t-r,x))$ is the birth function, $r \geqslant 0$ is the time delay, and $d(u)$ is the death rate function. The differential operator $\left(|(u^m)_x|^{p-2}(u^m)_x\right)_x$ is called 'doubly nonlinear' or non-Newtonian polytropic filtration, see [2, 19] for example. The porous medium equation ($m > 1$ and $p = 2$) and the p-Laplacian equation ($m = 1$ and $p > 2$) are two prototypical degenerate diffusion equations. Here the doubly nonlinear diffusion $(|(u^m)_x|^{p-2}(u^m)_x)_x$ has double degeneracy for $m > 1$ and $p > 2$, it degenerates at where $u = 0$ or $u_x = 0$. While for other cases with $m(p-1) > 1$ it degenerates 'singly'. We focus on the slow diffusion case $m(p-1) > 1$ such that sharp type (with semi-compact supports) travelling wave exists and the initial perturbation propagates at finite speed for the non-delayed case. The functions $d(s)$ and $b(s)$ satisfy the following conditions:

(H1) Two constant equilibria: $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0, d(u_+) = b(u_+)$, $b'(0) > d'(0) \geqslant 0$, and $d'(u_+) > b'(u_+) \geqslant 0$;

(H2) Monotonicity: $d(\cdot), b(\cdot) \in C^2([0, u_+])$, and $b'(s) > 0, d'(s) > 0$ for $s \in (0, u_+)$.

The assumptions (H_1)–(H_2) are summarized from a large number of evolution equations in ecology, such as the classical Fisher–KPP equation [13]; the well-studied Nicholson's blowflies equation [15] with the death function $d_1(u) = \delta u$ or $d_2(u) = \delta u^2$, the birth function

$$b_1(u) = pu\mathrm{e}^{-au^q}, \quad p > 0, \quad q > 0, \quad a > 0;$$

and the Mackey–Glass equation [24] with the growth function

$$b_2(u) = \frac{pu}{1 + au^q} \quad p > 0, \quad q > 0, \quad a > 0.$$

In the linear diffusion case ($p = 2$, $m = 1$), the situation is well understood. Reaction diffusion equations with time delay has been studied by Schaaf in [36], where the existence of monotone travelling waves is proved based on the sub- and super-solutions method and phase plane techniques. Since then, the study of travelling wave solutions for reaction diffusion equations with time delay has drawn considerable attentions (see, for example, [10, 12, 14, 23, 26] and references therein). Note that, the results mentioned above are all for the case that the diffusion term is the classical Laplacian. In the case of degenerate porous medium diffusion, i.e. when $p = 2$ and $m > 1$, the situation is more complicated. Here, the important feature of degenerate diffusion equation appears: travelling waves exhibit free boundaries. In [46], we found the sharp type travelling wave (with semi-compact supports) corresponding to the critical wave speed and obtained the uniqueness of these waves. Further, we proved that the initial perturbation propagates asymptotically at the same speed [48] and later sharp-oscillatory non-monotone travelling waves were found in [47].

Nonlinear diffusion induced by the dependency on population density and gradient accounts for the appearance of clusters and cohesive swarms [11, 43] and the anticongestion effect [1, 16, 44]. Dispersal mechanism involves highly the nonlinear interactions between individuals. Classically, the presence of an external gradient (food or taxis stimulus) may underlie the formation of spatial aggregation of animals. This is the case in models for chemotactic bacteria [20]. It does not seem, however, to be true for various species, which can continue to move as a coherent swarm in the absence of food or external taxis stimulus and the only possible stimulus is the heterogeneity of the density of conspecifics [11]. We will particularly focus this paper on the doubly nonlinear degenerate diffusion of species dispersal. This generalized nonlinearity depends on population density and gradient, and the only possible stimulus is the population density itself, which accounts for self-organization of individuals to avoid overcrowding from higher to lower densities [8, 16, 43]. Ecological examples have been observed in arctic ground squirrels [9] and butterfly metapopulations [32] for positive density dependent diffusion, and in various fish and insects aggregation [21, 22, 43] for gradient dependent diffusion.

Compared with the conventional reaction diffusion models with random diffusion, the degenerate doubly nonlinear diffusion mechanism gives a realistic representation of a moving cohesive swarm of individuals with a uniform interior density and sharp edges, as observed in fish schools [7], birds flocks [29] and skeletal cell spreading [37, 38]. Our result asserts that, by the combination of density and gradient dependent dispersal processes, distribution of populations forms a sharp travelling wave pattern. Notably, linear diffusion equations and degenerate doubly nonlinear diffusion equations show distinct behaviours, with the degenerate diffusion in our model displaying a sharp migration front and the standard linear diffusion equation behaving smooth front. Our sharp travelling waves solutions exhibits phenomenologically self-organization of individuals proposed in [30], where an interior region of approximately uniform distribution, and an interface region of sharply decreasing density. These sharp travelling waves have distinct boundaries, and the population density decreases sharply to zero at a finite point, rather than tends to zero asymptotically.

Ecologically, spatial diffusion of age-structured species can be described by the population density $u(t,x)$ moves with velocity $v(t,x)$, birth function $b(u(t-r,x))$ with mature time r, and death function $d(u)$. Then $u(t,x)$ satisfies the following law of population balance arises from [16]

$$u_t + (vu)_x = -d(u) + b(u(t-r,x)). \tag{1.2}$$

Here, the nonzero maturation delay $r > 0$ represents the time required for a newborn to become matured [15]. The diffusion velocity may depend on population density and density gradients and such factors lead to an anti-crowding mechanism of the organisms [30, 42]. Correspondingly, dispersive velocities v is related to both the density u and density gradient u_x by a power-law dependence

$$v = -m^{p-1}u^{(m-1)(p-1)-1}|u_x|^{p-2}u_x. \tag{1.3}$$

Combining (1.2) and (1.3), we obtain the doubly nonlinear diffusion equation (1.1) with time delay. Such doubly nonlinear diffusion is a result of gradient perception and anti-congestion effect [42]. For the special case $p = 2$, Sengers *et al* [37] proposed a power-law density-dependent diffusivity for the *MG63* migration with $m = 2$. Sánchez-Garduño and Maini [33] considered density-dependent diffusion in a general form $(D(u)u_x)_x$ satisfies $D(0) = 0$ with $D(u) > 0$ and modelling the distribution patterns of species such as Arctic ground squirrels and ant-lions etc. Further, in [35], the authors obtained the travelling waves to describe wave invasion of the population in the ecological model with density-dependent diffusion $D(u)$ and degenerate advection. In the case of $m = 1$, $p > 2$, Kim and Shi [21, 22] considered fish schooling models involving the gradient dependent diffusion in the p-Laplacian diffusion form. For the case with $m(p-1) > 1$, the diffusion coefficient is a function of density u and density gradient u_x. Organisms detect the surrounding density fields, disperse in the opposite direction of density gradient and leave the regions of higher density [3, 7, 43]. Audrito and Vázquez considered the doubly nonlinear operator and used travelling waves to explain that the species invades all the available space with speed of propagation [2].

The sharp type (with semi-compact supports) travelling wave solutions are essential in the analysis of the propagation properties of degenerate diffusion equations. In many cases, the solutions with compactly-supported initial data propagate asymptotically at the same speed of the sharp waves, which also is the minimal admissible travelling wave speed. This phenomenon was observed by Audrito and Vázquez [2] for doubly nonlinear diffusion equation (1.1) without time delay (i.e. $r = 0$), and further the speed was characterized via a variational approach by Benguria and Depassier [6]. For the investigation of sharp waves related to degenerate diffusion, see also in [25, 33, 34]. The role played by the degeneracy in the travelling wave dynamics of diffusion equations with advection was studied in [35].

Due to the complicated structure of doubly nonlinear operator, in the case $p > 1$ and $m > 0$, the problem possesses some intrinsic difficulties. Our main objective is to investigate the structure of the critical sharp waves and to estimate the corresponding critical speed using the approach of phase transform method with the help of the variational approach developed recently in our studies [45, 46]. Precisely speaking, we prove that, the doubly nonlinear diffusion equation (1.1) possesses a unique sharp type travelling wave $\phi(x + c^*t)$ (defined in definition 2.1) for the degenerate case $m(p-1) > 1$, and such a sharp travelling wave associated with the minimal wave speed $c^* = c^*(m, p, r)$ is monotonically increasing, where the minimal wave speed satisfies $c^*(m, p, r) < c^*(m, p, 0)$ for any time delay $r > 0$. Furthermore, we show the optimal regularity of the sharp front $\phi(x + c^*t)$. That is, when $\frac{p-1}{m(p-1)-1}$ is integer, then the sharp front $\phi(x + c^*t)$ is $C^{\frac{p-1}{m(p-1)-1}-1}$-smooth with ϕ and all its derivatives $\partial^j\phi$ are Lipschitz continuous for $j = 1, \ldots, \frac{p-1}{m(p-1)-1} - 1$; while, when $\frac{p-1}{m(p-1)-1}$ is non-integer, then the sharp front $\phi(x + c^*t)$ is $C^{[\frac{p-1}{m(p-1)-1}]}$-smooth, where $[\frac{p-1}{m(p-1)-1}]$ denotes the largest integer which is less then $\frac{p-1}{m(p-1)-1}$, in particular, ϕ and all its derivatives $\partial^j\phi$ for $j = 1, \ldots, [\frac{p-1}{m(p-1)-1}]$ are $C^{\alpha_{m,p}}$ Hölder continuous with the Hölder exponent $\alpha_{m,p} = \frac{p-1}{m(p-1)-1} - [\frac{p-1}{m(p-1)-1}]$. This implies that the sharp front $\phi(x + c^*t)$ is C^1-smooth for $\frac{1}{p-1} < m < \frac{p}{p-1}$, and piecewise smooth for $m \geqslant \frac{p}{p-1}$. On the other hand, we also prove that the time delay $r > 0$ slows down the minimal travelling wave speed $c^* = c^*(m, p, r)$ for the doubly nonlinear degenerate diffusion equations. Finally, let us point out a slightly unexpected phenomenon related to the doubly nonlinear operator. The main difficulty lies in the asymptotic behaviour of the phase function $\tilde{\psi}(\phi)$ defined for the sharp type travelling wave $\phi(\xi)$ by regarding $\psi(\xi) := |(\phi^m(\xi))'|^{p-2}(\phi^m(\xi))'$ as a function of ϕ. Its asymptotic behaviour near the positive equilibrium u_+ for the degenerate case $p \in (1, 2)$ is quite different from the case $p = 2$.

The paper is organized as follows. The main results are stated in section 2 and all the proofs are presented in section 3. Section 4 is a brief derivation of the models with density and gradient dependent diffusion.

2. Main results

We consider the doubly nonlinear degenerate diffusion equation with time delay (1.1). We are looking for the travelling wave solutions of sharp type that connect the two equilibria $u_- = 0$ and $u_+ =: K$. Under the hypotheses (H$_1$)–(H$_2$), the birth function $b(u)$ is monotonically increasing on $[u_-, u_+] =: [0, K]$. Let $\phi(\xi)$, where $\xi = x + ct$ and $c > 0$, be the travelling wave solution of (1.1), we get (we write ξ as t for the sake of simplicity)

$$\begin{cases} c\phi'(t) = (|(\phi^m)'(t)|^{p-2}(\phi^m)'(t))' - d(\phi(t)) + b(\phi(t-cr)), & t \in \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K. \end{cases} \tag{2.1}$$

Since (2.1) has singularity or degeneracy, we employ the following definition of sharp and smooth travelling waves. Here are some notations used throughout this paper: $L^1_{\mathrm{loc}}(\mathbb{R})$ is the set of locally Lebesgue integrable functions,

$$C^{\mathrm{b}}_{\mathrm{unif}}(\mathbb{R}) := \{\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}); \phi \text{ is uniformly continuous on } \mathbb{R}\},$$

$$C(\overline{\mathbb{R}}) := \{\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}); \lim_{t\to+\infty} \phi(t) \quad \text{and} \quad \lim_{t\to-\infty} \phi(t) \text{ exist}\},$$

and

$$W^{1,p}_{\mathrm{loc}}(\mathbb{R}) := \{\phi; \phi \in W^{1,p}(\Omega) \text{ for any compact subset } \Omega \subset \mathbb{R}\}.$$

It is clear that $C(\overline{\mathbb{R}}) \subset C^{\mathrm{b}}_{\mathrm{unif}}(\mathbb{R}) \subset C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The function spaces $C^k(\overline{\mathbb{R}})$ are defined similarly for any positive integer k.

Definition 2.1. A profile function $\phi(t)$ is said to be a travelling wave solution of (2.1) if $\phi \in C^{\mathrm{b}}_{\mathrm{unif}}(\mathbb{R})$, $0 \leqslant \phi(t) \leqslant K := u_+$, $\phi(-\infty) = 0$, $\phi(+\infty) = K$, $\phi^m \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$, $\phi(t)$ satisfies (2.1) in the sense of distributions. The travelling wave $\phi(t)$ is said to be of sharp type if the support of $\phi(t)$ is semi-compact, i.e. $\mathrm{supp}\,\phi = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$, $\phi(t) > 0$ for $t > t_0$. On the contrary, the travelling wave $\phi(t)$ is said to be of smooth type if $\phi(t) > 0$ for all $t \in \mathbb{R}$.

Without loss of generality, we may always shift t_0 to 0 for the sharp type travelling wave. Therefore, a sharp type travelling wave $\phi(t)$ is a special solution such that $\phi(t) = 0$ for $t \leqslant 0$, and $\phi(t) > 0$ for $t > 0$.

Special notations should be addressed here about the definition of sharp type travelling waves in the sense of distributions. The profile function $\phi(t)$ has the following expansion near 0 (see lemma 3.1)

$$\phi(t) = \begin{cases} \left(\dfrac{c^{\frac{1}{p-1}}(m(p-1)-1)}{m(p-1)}\right)^{\frac{p-1}{m(p-1)-1}} \cdot t^{\frac{p-1}{m(p-1)-1}} + o\left(t^{\frac{p-1}{m(p-1)-1}}\right), & t \to 0^+, \\ 0, & t \leqslant 0, \end{cases} \tag{2.2}$$

such that

$$(\phi^m(t))' = \begin{cases} c^{\frac{m}{m(p-1)-1}} \left(\frac{m(p-1)-1}{m(p-1)}\right)^{\frac{1}{m(p-1)-1}} \cdot t^{\frac{1}{m(p-1)-1}} + o\left(t^{\frac{1}{m(p-1)-1}}\right), & t \to 0^+, \\ 0, \qquad t \leqslant 0, \end{cases}$$

and

$$\begin{aligned} &(|(\phi^m(t))'|^{p-2}(\phi^m(t))')' \\ &= \begin{cases} c^{\frac{m(p-1)}{m(p-1)-1}} \left(\frac{m(p-1)-1}{m(p-1)}\right)^{\frac{p-1}{m(p-1)-1}} \frac{p-1}{(m(p-1)-1)} \cdot t^{\frac{p-m(p-1)}{m(p-1)-1}} + o\left(t^{\frac{p-m(p-1)}{m(p-1)-1}}\right), \ t \to 0^+, \\ 0, \qquad t \leqslant 0. \end{cases} \end{aligned}$$

The derivatives $(\phi^m(t))'$, $(|(\phi^m(t))'|^{p-2}(\phi^m(t))')'$, and $\phi'(t)$ belong to $L^1_{\text{loc}}(\mathbb{R})$ and are defined in the sense of distributions. Moreover, the derivative $\phi''(t)$ is not in $L^1_{\text{loc}}(\mathbb{R})$ for the case $m \geqslant \frac{p}{p-1}$. In fact, for the special case $m = \frac{p}{p-1}$, $\phi''(t)$ is the summation of a multiple of Dirac measure and a function in $L^1_{\text{loc}}(\mathbb{R})$; while for $m > \frac{p}{p-1}$, $\phi''(t)$ is more singular in the sense of distributions and can not be understood as a point-wise function. The doubly nonlinear term $|(u^m)_x|^{p-2}(u^m)_x$ is well defined for $p > 1$ provided that $(u^m)_x$ is well defined, though $|(u^m)_x|^{p-2}$ may blow up for $1 < p < 2$ and $(u^m)_x$ approaches zero. Here the solution is defined in the sense of distributions, i.e. $|(u^m)_x| \in L^p_{\text{loc}}$ is sufficient.

For any given $m > 0$, $p > 1$, such that $m(p-1) > 1$, and $r \geqslant 0$, we define the critical (or minimal) wave speed $c^*(m, p, r)$ for the degenerate diffusion equation (2.1) as follows

$$c^*(m, p, r) := \inf\{c > 0; (2.1)\text{ admits increasing travelling waves with speed } c\}. \tag{2.3}$$

For the case without time delay and with degenerate diffusion (i.e. $m(p-1) > 1$ and $r = 0$), it is proved by Benguria and Depassier in [6] that

$$\begin{aligned} c^*(m, p, 0) = \sup_{g\in\mathscr{D}} \int_0^K & \frac{p}{(p-1)^{(p-1)/p}} (-g'(\phi))^{\frac{1}{p}} (g(\phi))^{\frac{p-1}{p}} \\ & \times (m\phi^{m-1}(b(\phi) - d(\phi)))^{\frac{p-1}{p}} \,\mathrm{d}\phi, \end{aligned} \tag{2.4}$$

where $\mathscr{D} = \{g \in C^1([0, K]); \int_0^K g(s)\mathrm{d}s = 1, g(s) > 0, g'(s) < 0, \forall s \in (0, K)\}$.

Before stating the main results, we present an outline of the contents of this paper. Taking the non-delayed doubly nonlinear diffusion equation for example, i.e. (1.1) with $r = 0$, it is shown by Audrito and Vázquez [2] that:

(a) There exists a critical speed $c^* = c^*(m, p, 0) > 0$ such that (1.1) with $r = 0$ admits travelling waves for and only for the speed $c \geqslant c^*$;
(b) The travelling wave $\phi(t)$ corresponding to speed $c > c^*$ is smooth (positive for all $t \in \mathbb{R}$);
(c) The travelling wave $\phi(t)$ corresponding to speed $c = c^*$ is sharp (positive for $t > t_0$ for some t_0);
(d) The critical speed c^* is the asymptotic propagation speed.

The above properties show that the sharp travelling wave and the corresponding critical speed are essential for dynamics of degenerate diffusion equations. Here for time delayed case, we focus on the sharp travelling wave and reveal the role played by the time delay. Specifically, we prove that:

(a) (2.1) admits a unique sharp type travelling wave, and the sharp travelling wave is monotonically increasing;
(b) The sharp travelling wave corresponds to the minimal (critical) wave speed $c^*(m, p, r)$;
(c) $c^*(m, p, r) < c^*(m, p, 0)$ for any time delay $r > 0$;
(d) The regularity of sharp waves weakens as $m - \frac{1}{p-1}$ increases.

As a consequence, the time delay slows down the minimal travelling wave speed for the doubly nonlinear degenerate diffusion equations.

Our main results are as follows.

Theorem 2.1 (Critical sharp travelling wave). *Assume that $d(s)$ and $b(s)$ satisfy (H_1)–(H_2), and $m > 0$, $p > 1$, $r \geqslant 0$, such that $m(p-1) > 1$. Then the critical wave speed $c^* = c^*(m, p, r)$ defined in (2.3) is positive and satisfies $c^*(m, p, r) < c^*(m, p, 0)$ for any time delay $r > 0$, such that (2.1) admits a unique (up to shift) sharp travelling wave $\phi(x + c^*t)$ with speed c^*, which is the critical travelling wave of (2.1) and is monotonically increasing. Moreover, any other travelling wave solution must be smooth and correspond to speed $c > c^*(m, p, r)$.*

Theorem 2.2 (Regularity of sharp wave). *Assume that the conditions in theorem 2.1 hold. Let $\gamma_{m,p}$ be the largest integer that is smaller than $\frac{p-1}{m(p-1)-1}$, i.e.*

$$\gamma_{m,p} := \begin{cases} \dfrac{p-1}{m(p-1)-1} - 1, & \textit{if } \dfrac{p-1}{m(p-1)-1} \textit{ is an integer}, \\ \left[\dfrac{p-1}{m(p-1)-1}\right], & \textit{if } \dfrac{p-1}{m(p-1)-1} \textit{ is not an integer}, \end{cases}$$

and denote $\alpha_{m,p} := \frac{p-1}{m(p-1)-1} - \gamma_{m,p} \in (0, 1]$. Then the optimal regularity of sharp wave $\phi(\xi)$ is $\phi \in C^{\gamma_{m,p},\alpha_{m,p}}(\overline{\mathbb{R}})$, where $C^{\gamma_{m,p},\alpha_{m,p}}(\overline{\mathbb{R}})$ is the function space defined as: if $\frac{p-1}{m(p-1)-1}$ is an integer, then $\alpha_{m,p} = 1$, and

$$C^{\gamma_{m,p},1}(\overline{\mathbb{R}}) := \left\{ \phi \in C^{\gamma_{m,p}}(\overline{\mathbb{R}}) \,\middle|\, \partial^j\phi, \text{ for } j = 0, 1, \ldots, \frac{p-1}{m(p-1)-1} - 1, \right. \\ \left. \times \textit{ are Lipschitz continuous} \right\}; \tag{2.5}$$

while if $\frac{p-1}{m(p-1)-1}$ is not an integer, then $0 < \alpha_{m,p} < 1$, and

$$C^{\gamma_{m,p},\alpha_{m,p}}(\overline{\mathbb{R}}) := \left\{ \phi \in C^{\gamma_{m,p}}(\overline{\mathbb{R}}) \middle| \, \partial^j\phi, \textit{ for } j = 0, 1, \ldots, \left[\frac{p-1}{m(p-1)-1}\right], \right. \\ \left. \times \textit{ are } C^{\alpha_{m,p}} \textit{ Hölder continuous} \right\}. \tag{2.6}$$

Remark 2.1. If $m \geqslant \frac{p}{p-1}$, then the sharp travelling wave is not C^1 smooth; while if $m \in (\frac{1}{p-1}, \frac{p}{p-1})$, then the sharp travelling wave is C^1 smooth. See figure 1.

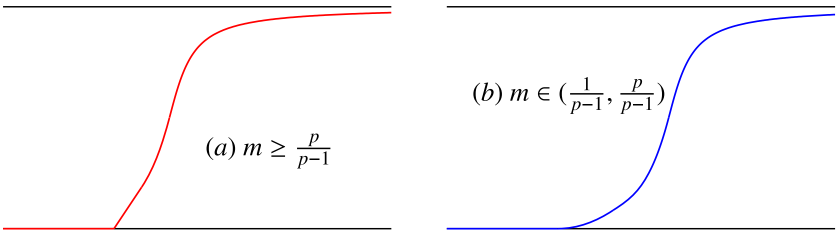

Figure 1. Travelling waves: (a) non-C^1 sharp type for $m \geqslant \frac{p}{p-1}$; (b) C^1 sharp type for $m \in (\frac{1}{p-1}, \frac{p}{p-1})$.

The idea of the proof is the following. The sharp travelling wave $\phi(t)$ is a special solution of (2.1) satisfying:

(a) The support of $\phi(t)$ is $[0, +\infty)$;
(b) $\phi(t)$ is monotone increasing and $\phi(+\infty) = K$, i.e. $\phi(t)$ grows up to K as $t \to +\infty$;
(c) The speed c is a special one, actually, $c = c^*(m, p, r)$, which is the unique critical wave speed.

We note that the parameter c in property (c) and also in the equation (2.1) is not *a priori* known. Meanwhile the property (a) says the local behaviour of $\phi(t)$ near the edge of the support, and the property (b) tells us the global behaviour.

It seems impossible for us to find the sharp travelling wave satisfying all these properties in a single step. To overcome the difficulties, we observe that the precise support of sharp wave and the time-delayed structure of differential equation (2.1) make it possible to construct local solutions with undetermined speed step by step, and then we compare those local solutions with different speeds via a generalized phase plane analysis method. In other word, we solve the local solution of (2.1) with property (a) and undetermined speed c, then we adjust the speed c such that the local solution satisfies the global property (b), hence we find the sharp travelling wave with critical speed $c^*(m, p, r)$.

3. Proof of the main results

For any given $m > 0$, $p > 1$, and $r > 0$, such that $m(p-1) > 1$, we solve (2.1) locally for any $c > 0$ and then we single out a special solution that is a sharp travelling wave with critical wave speed. First, noticing that the sharp wave solution $\phi(t) = 0$ for $t \leqslant 0$ and then $\phi(t - cr) = 0$ for $t \in [0, cr)$, (2.1) is locally reduced to

$$\begin{cases} c\phi'(t) = (|(\phi^m(t))'|^{p-2}(\phi^m(t))')' - d(\phi(t)), \\ \phi(0) = 0, \quad (\phi^m)'(0) = 0, \quad t \in (0, cr). \end{cases} \tag{3.1}$$

The problem (3.1) is a second-order equation with initial value $\phi^m(0) = 0$ and $(\phi^m)'(0) = 0$. However, singularity arises and $\phi^m(t) \equiv 0$ is not the unique solution. To see this more clearly, we set $\zeta(t) = \phi^m(t)$ and rewrite (3.1) as

$$|\zeta'(t)|^{p-2}\zeta'(t) = c\zeta^{\frac{1}{m}}(t) + \int_0^t d\left(\zeta^{\frac{1}{m}}(s)\right) \mathrm{d}s,$$

which is

$$\zeta'(t) = \left(c\zeta^{\frac{1}{m}}(t) + \int_0^t d\left(\zeta^{\frac{1}{m}}(s)\right) \mathrm{d}s\right)^{\frac{1}{p-1}} \approx c^{\frac{1}{p-1}}\zeta^{\frac{1}{m(p-1)}}(t).$$

The following prototypical problem

$$\begin{cases} \zeta'(t) = c^{\frac{1}{p-1}}\zeta^{\frac{1}{m(p-1)}}(t), \quad t > 0, \\ \zeta(0) = 0, \end{cases} \tag{3.2}$$

has infinitely many solutions since $m(p-1) > 1$ and $\zeta^{\frac{1}{m(p-1)}}$ is not Lipschitz continuous with respect to ζ. Actually, for any $\tau \geqslant 0$,

$$\zeta_\tau(t) := \begin{cases} \left(\dfrac{c^{\frac{1}{p-1}}(m(p-1)-1)}{m(p-1)}\right)^{\frac{p-1}{m(p-1)-1}} \cdot (t-\tau)^{\frac{m(p-1)}{m(p-1)-1}}, \quad t > \tau, \\ 0, \quad t \leqslant \tau, \end{cases}$$

is a solution to (3.2), and $\zeta_0(t)$ is the maximal one and also the unique one such that $\zeta_0(t) > 0$ for $t > 0$. The problem (3.1) behaves similarly as (3.2), such that its solutions are not unique and we choose the maximal one satisfying $\phi(t) > 0$ for $t \in (0, cr)$ as shown in lemma 3.1. Here, $(\phi^m)'(0^+) = 0$ is a necessary and sufficient condition such that the zero extension of $\phi(t)$ to the left satisfies (2.1) locally near 0 in the sense of distributions. In fact, if $(\phi^m)'(0^+) > 0$, then the generalized derivative $(|(\phi^m(t))'|^{p-2}(\phi^m(t))')'$ of the zero extension is not a locally integrable function. Specifically, $(|(\phi^m(t))'|^{p-2}(\phi^m(t))')'$ is a locally integrable function added by a multiple of Dirac measure and the differential equation (2.1) cannot be valid in the sense of distributions.

The proof follows from the similar outline as in [46], the difference lies in the asymptotic behaviour of $\psi(t) := |(\phi^m(t))'|^{p-2}(\phi^m(t))'$ in the singular phase plane of (ϕ, ψ) for the sharp wave solution $\phi(t)$. Here we mainly sketch the proofs that have differences for the sake of simplicity.

Lemma 3.1. *For any $c > 0$, the degenerate ODE (3.1) admits a unique maximal solution $\phi_c^1(t)$ on $(0, cr)$ such that $\phi_c^1(t) > 0$ and $(\phi_c^1)'(t) > 0$ on $(0, cr)$ and*

$$\phi(t) = \left(\frac{c^{\frac{1}{p-1}}(m(p-1)-1)}{m(p-1)}\right)^{\frac{p-1}{m(p-1)-1}} \cdot t^{\frac{p-1}{m(p-1)-1}} + o\left(t^{\frac{p-1}{m(p-1)-1}}\right), \quad as \quad t \to 0^+. \tag{3.3}$$

Proof. A positive solution $\phi(t) > 0$ on $(0, cr)$ with locally $\phi'(t) > 0$ near 0 to the degenerate ODE (3.1) satisfies the following singular differential system on $(0, cr)$

$$\begin{cases} \phi'(t) = \dfrac{\psi^{\frac{1}{p-1}}(t)}{m\phi^{m-1}(t)}, \\ \psi'(t) = c\dfrac{\psi^{\frac{1}{p-1}}(t)}{m\phi^{m-1}(t)} + d(\phi(t)), \end{cases} \tag{3.4}$$

with $\psi(t) := |(\phi^m(t))'|^{p-2}(\phi^m(t))'$. We seek for a solution to (3.4) such that $\phi(t) > 0$ and $\psi(t) > 0$ for $t \in (0, cr)$ with $\psi(0) = 0$ and $\phi(0) = 0$. If this kind of solution exists, then it is a solution to

(3.1). The system (3.4) has singularity at some points where $\phi(t)=0$. For functions $\phi(t)>0$ on $(0,cr)$ such that $\frac{1}{m\phi^{m-1}(t)}$ is integrable, we make change of variables such that

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{m\phi^{m-1}(t)},\quad s=0\,\text{for}\,t=0;$$

while for functions $\phi(t)>0$ on $(0,cr)$ such that $\frac{1}{m\phi^{m-1}(t)}$ is not integrable, we make change of variables as

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{m\phi^{m-1}(t)},\quad s=0\,\text{for}\,t=cr,\ \ s=-\infty\,\text{for}\,t=0.$$

The system (3.4) is autonomous and the change of variable does not affect the shape of trajectories. Then system (3.4) is converted to

$$\begin{cases}\dfrac{\mathrm{d}\phi}{\mathrm{d}s}=\psi^{\frac{1}{p-1}}(t(s)),\\ \dfrac{\mathrm{d}\psi}{\mathrm{d}s}=c\psi^{\frac{1}{p-1}}(t(s))+m\phi^{m-1}(t(s))\cdot d(\phi(t(s))).\end{cases}\tag{3.5}$$

The behaviour of trajectories for (3.5) is not clear for ψ near zero. Therefore, we solve (3.5) with the condition $(\phi_\varepsilon(0),\psi_\varepsilon(0))=(0,\varepsilon)$, i.e. we consider the trajectory $(\phi_\varepsilon(t(s)),\psi_\varepsilon(t(s)))$ passing through $(0,\varepsilon)$ for $t\to 0^+$ (for $s\to 0^+$ or for $s\to-\infty$ depending on the integrability of $\frac{1}{m\phi^{m-1}(t)}$) in the phase plane (3.5).

Locally $\psi_\varepsilon(t)>0$ and then $\phi_\varepsilon'(t)>0$, we can make change of variables such that we take ϕ_ε as an independent variable and regard ψ_ε as a function of ϕ_ε, denoted by $\tilde{\psi}_\varepsilon(\phi)$. Then we have

$$\begin{cases}\dfrac{\mathrm{d}\tilde{\psi}_\varepsilon}{\mathrm{d}\phi}=c+\dfrac{m\phi^{m-1}d(\phi)}{\tilde{\psi}_\varepsilon^{\frac{1}{p-1}}},\quad \phi>0,\\ \tilde{\psi}_\varepsilon(0)=\varepsilon.\end{cases}\tag{3.6}$$

The initial value problem (3.6) has a unique solution since $\tilde{\psi}_\varepsilon(\phi)\geqslant\varepsilon$ and the right-hand side function is Lipschitz continuous with respect to both ϕ and $\tilde{\psi}_\varepsilon$. The monotone dependence and the continuous dependence of $\tilde{\psi}_\varepsilon(\phi)$ with respect to ε follow from the classical ODE theory. Moreover, $\tilde{\psi}_\varepsilon(\phi)\geqslant c\phi$, and then

$$\begin{aligned}\frac{\mathrm{d}\tilde{\psi}_\varepsilon}{\mathrm{d}\phi}&=c+\frac{m\phi^{m-1}d(\phi)}{\tilde{\psi}_\varepsilon^{\frac{1}{p-1}}}\leqslant c+\frac{m\phi^{m-1}d(\phi)}{(c\phi)^{\frac{1}{p-1}}}\\ &\leqslant c+\frac{mM}{c^{\frac{1}{p-1}}}\phi^{m-\frac{1}{p-1}},\quad \phi\in[0,u_+],\end{aligned}$$

for some constant $M>0$ such that $d(\phi)\leqslant M\phi$ for $\phi\in[0,u_+]$. Therefore,

$$c\phi\leqslant\tilde{\psi}_\varepsilon(\phi)\leqslant\varepsilon+c\phi+\frac{mM}{c^{\frac{1}{p-1}}\left(m-\frac{1}{p-1}\right)}\phi^{m-\frac{1}{p-1}+1}.$$

The limiting function $\tilde{\psi}(\phi)$ of $\tilde{\psi}_\varepsilon(\phi)$ as $\varepsilon \to 0^+$ exists and satisfies

$$c\phi \leqslant \tilde{\psi}(\phi) \leqslant c\phi + \frac{mM}{c^{\frac{1}{p-1}}\left(m - \frac{1}{p-1}\right)}\phi^{m-\frac{1}{p-1}+1}, \quad \phi \in [0, u_+]. \tag{3.7}$$

Since $(0, 0)$ is the unique stationary point in $\{(\phi, \psi);\ \phi < u_+\}$ at the phase plane of (3.5), no trajectories intersect with each other at points except $(0, 0)$. Noticing that $\tilde{\psi}(\phi)$ is the limiting function of $\tilde{\psi}_\varepsilon(\phi)$ corresponding to trajectories passing through $(0, \varepsilon)$, we see that $\tilde{\psi}(\phi)$ is the maximal (if not unique) solution such that its trajectory passes through $(0, 0)$.

Now we show that $\tilde{\psi}(\phi)$ is the unique one such that its trajectory passes through $(0, 0)$ into the first quadrant. Let $\tilde{\psi}_i(\phi)$, $i = 1, 2$, be two solutions with $\tilde{\psi}_1(\phi_0) > \tilde{\psi}_2(\phi_0) > 0$ for some $\phi_0 > 0$. Then

$$\begin{aligned}\frac{\mathrm{d}(\tilde{\psi}_1 - \tilde{\psi}_2)}{\mathrm{d}\phi} &= \frac{m\phi^{m-1}d(\phi)}{\tilde{\psi}_1^{\frac{1}{p-1}}} - \frac{m\phi^{m-1}d(\phi)}{\tilde{\psi}_2^{\frac{1}{p-1}}} \\ &= -\frac{m\phi^{m-1}d(\phi)\left(\tilde{\psi}_1^{\frac{1}{p-1}} - \tilde{\psi}_2^{\frac{1}{p-1}}\right)}{\tilde{\psi}_1^{\frac{1}{p-1}}\tilde{\psi}_2^{\frac{1}{p-1}}}.\end{aligned}$$

It follows that $\tilde{\psi}_1 - \tilde{\psi}_2$ decreases to $\tilde{\psi}_1(\phi_0) - \tilde{\psi}_2(\phi_0) > 0$ and contradicts to $\tilde{\psi}_1(0) = \tilde{\psi}_2(0) = 0$.

We recover $(\phi(t), \psi(t))$ from $\tilde{\psi}(\phi)$ according to the relation

$$\psi(t) = |(\phi^m(t))'|^{p-2}(\phi^m(t))' = \tilde{\psi}(\phi(t)).$$

Asymptotic analysis in (3.7) shows that (note that $m(p-1) > 1$)

$$\tilde{\psi}(\phi) = c\phi + o(\phi), \quad \text{as} \quad \phi \to 0^+,$$

then

$$|(\phi^m(t))'|^{p-2}(\phi^m(t))' = \tilde{\psi}(\phi(t)) = c\phi(t) + o(\phi(t)), \quad \text{as} \quad t \to 0^+. \tag{3.8}$$

Furthermore, let $\phi^m(t) > 0$ on $(0, cr)$ with $\phi^m(0) = 0$ be the maximal solution (there are infinitely many solutions without the condition $\phi^m(t) > 0$ on $(0, cr)$) to the following singular first order differential equation

$$\begin{aligned}(\phi^m(t))' &= \left(\tilde{\psi}(\phi(t))\right)^{\frac{1}{p-1}} \\ &= c^{\frac{1}{p-1}}(\phi^m(t))^{\frac{1}{m(p-1)}} + o\left((\phi^m(t))^{\frac{1}{m(p-1)}}\right), \quad \text{as} \quad t \to 0^+.\end{aligned}$$

Therefore,

$$\phi^m(t) = \left(\frac{c^{\frac{1}{p-1}}(m(p-1)-1)}{m(p-1)}\right)^{\frac{m(p-1)}{m(p-1)-1}} \cdot t^{\frac{m(p-1)}{m(p-1)-1}} + o\left(t^{\frac{m(p-1)}{m(p-1)-1}}\right), \quad \text{as} \quad t \to 0^+.$$

The proof of (3.3) is completed. □

Next, let $\phi_c^2(t)$ be the solution of the following initial value second order ODE problem

$$\begin{cases} c\phi'(t) = (|(\phi^m(t))'|^{p-2}(\phi^m(t))')' - d(\phi(t)) + b(\phi_c^1(t-cr)), \quad t \in (cr, 2cr), \\ \phi(cr) = \phi_c^1(cr), \quad \phi'(cr) = (\phi_c^1)'(cr). \end{cases} \tag{3.9}$$

The problem (3.9) is locally solvable and has no singularity near $t = cr$ since $\phi_c^1(cr) > 0$. The above steps can be continued unless $\phi_c^k(t)$ blows up or decays to zero in finite time for some $k \in \mathbb{N}^+$. Let $\phi_c(t)$ be the connecting function of those functions on each step, i.e.

$$\phi_c(t) = \begin{cases} \phi_c^1(t), & t \in [0, cr), \\ \phi_c^2(t), & t \in [cr, 2cr), \\ \vdots \\ \phi_c^k(t), & t \in [(k-1)cr, kcr), \\ \vdots \end{cases} \tag{3.10}$$

for some finite steps such that $\phi_c(t)$ blows up or decays to zero, or for infinite steps such that $\phi_c(t)$ is defined on $(0, +\infty)$ and zero extended to $(-\infty, 0)$ for convenience.

We assert that once $\phi_c(t)$ is decreasing then it is always decreasing after that.

Lemma 3.2. *If there exists a $t_0 > 0$ such that $\phi_c'(t_0) < 0$, then $\phi_c(t)$ decreases to 0 after t_0.*

Proof. According to lemma 3.1, $\phi_c(t)$ is strictly increasing on $(0, cr)$. Let $t_1 \in [cr, t_0)$ be the maximal one such that $\phi_c(t)$ is strictly monotone increasing on $(0, t_1)$, and let $t_2 > t_1$ be the maximal one such that $\phi_c(t)$ is strictly monotone decreasing on (t_1, t_2).

Here we would like to explain more details about the existence of a right neighbourhood $(t_1, t_1 + \delta)$ with some $\delta > 0$ such that $\phi_c(t)$ is strictly monotone decreasing. Generally, a function with zero derivative at t_1 can oscillate on the right of t_1 (see $(t - t_1)^3 \sin(1/(t - t_1))$ for example) and no right neighbourhood with monotonicity exists. We argue by contradiction and assume that no monotone right neighbourhood of $\phi_c(t)$ at t_1 exists. Thanks to the construction of $\phi_c(t)$ in (3.1) and (3.9) such that it satisfies the differential equation (2.1) on its existence interval, we find that there must hold $\phi_c'(t_1) = 0$, $(\phi_c^m)'(t_1) = 0$, and $(|(\phi_c^m)'(t)|^{p-2}(\phi_c^m)'(t))'|_{t=t_1} = 0$ (the existence is guaranteed by the equation and the equality to zero follows from the oscillations), then $d(\phi_c(t_1)) = b(\phi_c(t_1 - cr))$ according to (2.1). Note that $\phi_c(t)$ is strictly increasing near $t_1 - cr$ since $t_1 - cr \in (0, t_1)$ and $\phi_c'(t_1) = 0$, we show that $b(\phi_c(t - cr)) - d(\phi_c(t))$ is strictly increasing and then positive at a right neighbourhood of t_1, denoted by $(t_1, t_1 + \delta)$. Therefore, within $(t_1, t_1 + \delta)$, $\phi_c(t)$ satisfies the following one-dimensional quasi-linear elliptic problem

$$-(|(\phi^m)'(t)|^{p-2}(\phi^m)'(t))' + c\phi'(t) = b(\phi(t-cr)) - d(\phi(t)) > 0, \quad t \in (t_1, t_1 + \delta). \tag{3.11}$$

The positivity of the right-hand side of (3.11) implies that no inner minimum point exists in $(t_1, t_1 + \delta)$ and thus no oscillations exist.

Now that we have proved the existence of a monotone right neighbourhood of $\phi_c(t)$ at t_1, we see that t_2 exists. If $t_2 = +\infty$, which means $\phi_c(t)$ is decreasing on $(t_1, +\infty)$, we assert that $\phi_c(t)$ decreases to zero. Otherwise, the monotonicity implies that $\phi_c(t)$ decreases to some point $\kappa \in (0, K)$. The monotonicity of $\phi_c(t)$ also shows the existence of

$\lim_{t\to+\infty} d(\phi(t)) = d(\kappa)$ and $\lim_{t\to+\infty} b(\phi(t-cr)) = b(\kappa)$, $\lim_{t\to+\infty} \phi'(t) = 0$, and further according to (2.1) $\lim_{t\to+\infty} (|(\phi^m)'(t)|^{p-2}(\phi^m)'(t))'$ exists, which must be zero if exists. Then using (2.1) again, $d(\kappa) = b(\kappa)$, which contradicts to the fact that no equilibrium exists in $(0, K)$.

If t_2 is finite and $\phi_c(t_2)$ does not reach 0, then $\phi_c'(t_1) = 0 = \phi_c'(t_2)$ and further we find that

$$(|(\phi^m)'(t)|^{p-2}(\phi^m)'(t))'\big|_{t=t_1} \leqslant 0,$$

since $(\phi^m)'(t) \geqslant 0$ for $t < t_1$ and $(\phi^m)'(t) \leqslant 0$ for $t_1 < t < t_2$. Therefore, the equation (2.1) implies that

$$d(\phi(t_1)) \leqslant b(\phi(t_1 - cr)). \tag{3.12}$$

Similarly, analysis at t_2 shows that

$$(|(\phi^m)'(t)|^{p-2}(\phi^m)'(t))'\big|_{t=t_2} \geqslant 0,$$

and

$$d(\phi(t_2)) \geqslant b(\phi(t_2 - cr)). \tag{3.13}$$

Since $d(s) \leqslant b(s)$ for $s \in (0, K)$, (3.13) implies that

$$\phi(t_2 - cr) \leqslant \phi(t_2) < \phi(t_1).$$

Noticing that $\phi_c(t)$ is strictly increasing on $(0, t_1)$, and strictly decreasing on (t_1, t_2), there exists a $t_* \in (0, t_1)$ such that $\phi_c(t_*) = \phi_c(t_2)$. Therefore $t_2 - cr \leqslant t_*$. In other words, $t_2 - cr$ and $t_1 - cr$ both reside in $(0, t_*)$ and then $\phi(t_2 - cr) > \phi(t_1 - cr)$ according to the monotonicity of ϕ_c on $(0, t_*)$. Now we have

$$b(\phi(t_2 - cr)) > b(\phi(t_1 - cr)), \quad d(\phi(t_2)) < d(\phi(t_1)),$$

which contradicts to (3.12) and (3.13). □

Lemma 3.3. *For any given $r > 0$, there exist two numbers $\overline{c} > \underline{c} > 0$ such that if $0 < c \leqslant \underline{c}$, then $\phi_c(t)$ decays to zero; if $c \geqslant \overline{c}$, then $\phi_c(t)$ grows up to $+\infty$ as t tends to $+\infty$.*

Proof. The above assertions for the special case of $m > 1$ and $p = 2$ are proved in [47]. Here we consider the following dynamical system

$$\begin{cases} \phi'(t) = \dfrac{\psi^{\frac{1}{p-1}}(t)}{m\phi^{m-1}(t)}, \\ \psi'(t) = c\dfrac{\psi^{\frac{1}{p-1}}(t)}{m\phi^{m-1}(t)} + d(\phi(t)) - b(\phi(t - cr)), \end{cases} \tag{3.14}$$

with $\psi(t) := |(\phi^m(t))'|^{p-2}(\phi^m(t))'$. For the case without time delay, i.e. $r = 0$, the phase plane corresponding to (3.14) can be analysed by investigating the behaviour of the following first order equation (especially in the first quadrant)

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = c - \frac{m\phi^{m-1}(b(\phi) - d(\phi))}{\psi^{\frac{1}{p-1}}}.$$

While for the time-delayed case, the pair $(\phi(t), \psi(t))$ for any solution $\phi(t)$ still draws trajectories in the phase plane of (ϕ, ψ). But the analysis of this phase plane is more subtle. The trajectories may intersect with each other at non-stationary points due to the non-local property of time delay. We would call it a generalized phase plane.

According to lemma 3.1, $\phi_c(t)$ is strictly increasing on $(0, cr)$. There exists a $t_0 \geqslant cr$ such that $\phi_c(t)$ is strictly monotone increasing on $(0, t_0)$, and we may take $(0, t_0)$ as the maximal one. That is,

$$t_0 := \sup \{s \geqslant cr; \phi_c(t) \text{ is strictly increasing on } (0, s)\}.$$

For the local solution $\phi_c(t)$ on its strictly monotone increasing interval $(0, t_0)$, we take ϕ_c as an independent variable and regard ψ_c as a function of ϕ_c, denoted by

$$\tilde{\psi}_c(\phi_c) = \psi_c(t^{-1}(\phi_c)), \text{ such that } t^{-1}(\phi_c) \text{ is the inverse function of } \phi_c(t). \tag{3.15}$$

We may drop the subscripts in ϕ_c, ψ_c, $\tilde{\psi}_c$, and simply write ϕ, ψ, $\tilde{\psi}$, for a given $c > 0$. Define

$$\phi_{cr} := \inf_{\theta \in [0,\phi]} \left\{ \int_\theta^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \, \mathrm{d}s \leqslant cr \right\}. \tag{3.16}$$

We note that $\int_\theta^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \mathrm{d}s$ is decreasing with respect to $\theta \in (0, \phi]$. If $\int_0^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \mathrm{d}s \leqslant cr$, then $\phi_{cr} = 0$; while if $\int_0^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \mathrm{d}s > cr$, then $\phi_{cr} > 0$ and ϕ_{cr} is the minimal one such that $\int_\theta^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \mathrm{d}s \leqslant cr$ and the unique one satisfying $\int_{\phi_{cr}}^\phi \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)} \mathrm{d}s = cr$. The definition of ϕ_{cr} recovers $\phi(t - cr)$ from $\phi(t)$ and $\psi(t)$. Then the function $\tilde{\psi}(\phi)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = c - \dfrac{m\phi^{m-1} \cdot (b(\phi_{cr}) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)}, \\ \tilde{\psi}(0) = 0, \quad \tilde{\psi}(\phi) > 0 \text{ for } \phi \in (0, \phi^*), \end{cases} \tag{3.17}$$

where $\phi^* = \phi_c(t_0)$ and $(0, t_0)$ is the maximum interval such that $\phi_c(t)$ is strictly monotone increasing.

(a) We show that $\phi_c(t)$ decays to zero if c is sufficiently small. Noticing that $\phi_{cr} < \phi$ and $b(s)$ is increasing, we have

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = c - \frac{m\phi^{m-1} \cdot (b(\phi_{cr}) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)} > c - \frac{m\phi^{m-1} \cdot (b(\phi) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)} =: F_1(\phi, \psi), \tag{3.18}$$

and

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = c - \frac{m\phi^{m-1} \cdot (b(\phi_{cr}) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)} < c + \frac{m\phi^{m-1} \cdot d(\phi)}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)} =: F_2(\phi, \psi). \tag{3.19}$$

The analysis of trajectories corresponding to $F_1(\phi, \psi)$ and $F_2(\phi, \psi)$ is trivial. We see that

$$\tilde{\psi}(\phi) \leqslant C_1 \phi, \quad \phi \in (0, \phi^*).$$

According to the definition of ϕ_{cr}, we have

$$\begin{aligned} cr &\geqslant \int_{\phi_{cr}}^{\phi} \frac{ms^{m-1}}{\tilde{\psi}^{\frac{1}{p-1}}(s)}\,\mathrm{d}s \\ &\geqslant \int_{\phi_{cr}}^{\phi} \frac{ms^{m-1}}{C_1^{\frac{1}{p-1}} s^{\frac{1}{p-1}}}\,\mathrm{d}s \\ &= \frac{m}{C_1^{\frac{1}{p-1}}\left(m-\frac{1}{p-1}\right)}\left(\phi^{m-\frac{1}{p-1}} - \phi_{cr}^{m-\frac{1}{p-1}}\right). \end{aligned} \tag{3.20}$$

It follows that ϕ_{cr} is close to ϕ if cr is small. Specifically, as discussed in the proof of lemma 4.1 in [47], we fix $\eta = \min\{\phi^*, u_+/2\}$ and choose $\varepsilon > 0$ such that

$$\int_0^{\varepsilon} s^{m-1} d(s)\mathrm{d}s < \frac{1}{4}\int_{\varepsilon}^{\eta} s^{m-1}(b(s)-d(s))\mathrm{d}s,$$

and let $\delta := \inf_{s\in(\varepsilon,\eta)} (b(s)-d(s)) > 0$. Assume that cr is sufficiently small such that $\phi \geqslant \varepsilon$ implies $\phi_{cr} \geqslant \varepsilon/2$ based on (3.20). Further, we take cr smaller if necessary such that

$$b(\phi) - b(\phi_{cr}) \leqslant \frac{\delta}{2} \leqslant \frac{b(\phi)-d(\phi)}{2}$$

for $\phi \in [\varepsilon, \eta]$ and $\phi_{cr} \geqslant \varepsilon/2$ since ϕ_{cr} is close to ϕ. Therefore, for any $\phi \in (\varepsilon, \eta)$,

$$b(\phi_{cr}) - d(\phi) = (b(\phi)-d(\phi)) - (b(\phi)-b(\phi_{cr})) \geqslant \frac{b(\phi)-d(\phi)}{2}.$$

Integrating (3.17) (multiplied by $\tilde{\psi}^{\frac{1}{p-1}}(\phi)$) over $(0, \eta)$ shows that

$$\begin{aligned} c\int_0^{\eta} \tilde{\psi}^{\frac{1}{p-1}}(\phi)\mathrm{d}\phi &= \frac{p-1}{p}\tilde{\psi}^{\frac{p}{p-1}}(\phi)\Big|_0^{\eta} + \int_0^{\eta} m\phi^{m-1}(b(\phi_{cr})-d(\phi))\mathrm{d}\phi \\ &\geqslant \int_0^{\varepsilon} m\phi^{m-1}(0-d(\phi))\mathrm{d}\phi + \int_{\varepsilon}^{\eta} m\phi^{m-1}(b(\phi_{cr})-d(\phi))\mathrm{d}\phi \\ &\geqslant \left(-\frac{1}{4}+\frac{1}{2}\right)\int_{\varepsilon}^{\eta} m\phi^{m-1}(b(\phi)-d(\phi))\mathrm{d}\phi. \end{aligned}$$

Meanwhile,

$$c\int_0^{\eta} \tilde{\psi}^{\frac{1}{p-1}}(\phi)\mathrm{d}\phi \leqslant c\int_0^{\eta} C_1^{\frac{1}{p-1}}\phi^{\frac{1}{p-1}}\,\mathrm{d}\phi.$$

The above two inequalities cannot be valid together if c is sufficiently small and $\phi^* \geqslant u_+/2$ (such that $\eta = u_+/2$ in this case). That is, $\phi_c(t)$ cannot increase to $u_+/2$ if c is sufficiently small. Noticing that $(0, 0)$ is the unique stationary point in $\{(\phi, \psi);\ \phi \leqslant u_+/2\}$ on the phase plane, $\phi_c(t)$ cannot increase to some positive equilibrium and once $\phi_c(t)$ decreases, it must decrease to zero according to lemma 3.2.

(b) Next, we show that $\phi_c(t)$ grows up to $+\infty$ if c is sufficiently large. According to the relation (3.18), we compare the phase plane of (ϕ, ψ) with that corresponds to $F_1(\phi, \psi)$ in (3.18), where the latter one is a dynamical system without time delay. This is proved in lemma 3.3 of [47].

The local solution $\phi_c(t)$ may grow beyond the positive equilibrium $K = u_+ > 0$ or decay to zero in finite interval. The sharp travelling wave is the special one (the uniqueness will be proved) such that $\phi_c(t)$ exists globally and is monotone increasing on $(0, +\infty)$, together with the speed c being identical to the critical wave speed $c^*(m, p, r)$. The existence and other properties of the sharp wave for the case of $m > 1$ and $p = 2$ are proved in [46, 47]. Specifically, the existence of sharp wave in the above settings is proved in [47] by a continuation argument for general non-monotone birth function $b(s)$; and the uniqueness is prove in [46] via monotone dependence of $\phi_c(t)$ with respect to c for monotone birth function.

Lemma 3.4. *The solution $\phi_c(t)$ is locally continuously dependent on c and is strictly monotonically increasing with respect to c on their joint existence interval. To be more precisely, let $\phi_{c_1}(t)$ and $\phi_{c_2}(t)$ be the solutions corresponding to $c_1 > c_2 > 0$, then $\phi_{c_1}(t) > \phi_{c_2}(t)$ on $(0, T_{c_1}) \cap (0, T_{c_2})$, where $(0, T_{c_i})$ is the maximal existence interval of $\phi_{c_i}(t)$.*

Proof. The continuous dependence of $\phi_c(t)$ on c is proved in two steps. According to lemma 3.1, the locally asymptotic behaviour of $\phi_c(t)$ near zero shows that $\psi_c(t) := |(\phi_c^m(t))'|^{p-2}(\phi_c^m(t))'$, denoted by $\tilde{\psi}_c(\phi_c)$ as a function of ϕ_c, satisfies the estimates (3.7) and the following ODE

$$\frac{\mathrm{d}\tilde{\psi}_c^{\frac{p}{p-1}}}{\mathrm{d}\phi} = c\frac{p}{p-1}\tilde{\psi}_c^{\frac{1}{p-1}} + \frac{p}{p-1}m\phi^{m-1}d(\phi), \quad \tilde{\psi}_c(0) = 0. \tag{3.21}$$

The estimates (3.7) imply that $\tilde{\psi}_c(\phi)$ is the maximal solution of (3.21), which is continuously dependent on c. Therefore, for $t \in (0, t_1)$ with some $0 < t_1 \leqslant cr/2$, $\phi_c(t)$ is continuous with respect to c; while for $t \in (t_1, T)$ with $T > cr$, the continuous dependence is a trivial problem since $\phi_c(t) \geqslant \phi_c(t_1) > 0$ for $t_1 < t < T$ and (2.1) is a regular delayed ODE since the dynamic system (3.14) has no singularity as $\phi(t)$ has positive infimum.

The monotone dependence is proved via the phase plane analysis of system (3.14). We follow the same line as the proof of lemma 3.6 in [46]. Here in the proof of lemma 3.3, we convert the system (3.14) into the problem (3.17) with ϕ_{cr} defined by (3.16). Let $c_1 > c_2 > 0$ and $\phi_{c_1}(t)$, $\phi_{c_2}(t)$ be the local solution constructed in (3.10). The function $\tilde{\psi}_c(\phi_c)$ defined in (3.15) corresponding to c_i is denoted by $\tilde{\psi}_i(\phi)$ for $i = 1, 2$. Then we have

$$\frac{\mathrm{d}\tilde{\psi}_i}{\mathrm{d}\phi} = c_i - \frac{m\phi^{m-1}\cdot(b(\phi^i_{c_ir}) - d(\phi))}{\tilde{\psi}_i^{\frac{1}{p-1}}(\phi)}, \quad \phi \in (0, \phi_i^*), i = 1, 2, \tag{3.22}$$

where ϕ_i^* is the supremum of $\phi_{c_i}(t)$ on its strictly increasing interval and $\phi^i_{c_ir}$ depends on the function $\phi_{c_i}(t)$ such that we add a superscript on each ϕ_{c_ir}. For $\phi \in (0, \phi_{c_1}(c_1r))$, or equivalently, $t \in (0, c_1r)$ for the function $\phi_{c_1}(t)$, we have (note that $\phi^1_{c_1r} = 0$ and $\phi^2_{c_2r} \geqslant 0$)

$$\frac{\mathrm{d}\tilde{\psi}_1}{\mathrm{d}\phi} = c_1 + \frac{m\phi^{m-1}\cdot d(\phi)}{\tilde{\psi}_1^{\frac{1}{p-1}}(\phi)} > c_2 + \frac{m\phi^{m-1}\cdot d(\phi)}{\tilde{\psi}_1^{\frac{1}{p-1}}(\phi)},$$

and

$$\frac{\mathrm{d}\tilde{\psi}_2}{\mathrm{d}\phi} = c_2 - \frac{m\phi^{m-1}\cdot(b(\phi^2_{c_2r}) - d(\phi))}{\tilde{\psi}_2^{\frac{1}{p-1}}(\phi)} \leqslant c_2 + \frac{m\phi^{m-1}\cdot d(\phi)}{\tilde{\psi}_2^{\frac{1}{p-1}}(\phi)}.$$

The dynamic behaviour of the following ODE

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = c + \frac{m\phi^{m-1}\cdot d(\phi)}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)}, \quad \tilde{\psi}(0) = 0,\ \ \tilde{\psi}(\phi) > 0, \phi > 0,$$

show that (see (3.7) in the proof of lemma 3.1)

$$c_i\phi \leqslant \tilde{\psi}_i(\phi) \leqslant c_i\phi + \frac{mM}{c_i^{\frac{1}{p-1}}\left(m-\frac{1}{p-1}\right)}\phi^{m-\frac{1}{p-1}+1}, \quad \phi \in [0, \phi_i^*],\ i = 1, 2.$$

It follows that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ locally near $\phi = 0$.

We further show that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for all $\phi \in (0, \min\{\phi_1^*, \phi_2^*\})$. According to the definition of ϕ_{cr} in (3.16) (or equivalently, the property of $\phi_c(t - cr)$), for $t > c_1 r$, we have (note that $\phi^1_{c_1 r} > 0$ and $\phi^2_{c_2 r} > 0$)

$$\int_{\phi^1_{c_1 r}}^{\phi} \frac{ms^{m-1}}{\tilde{\psi}_1^{\frac{1}{p-1}}(s)}\,\mathrm{d}s = c_1 r, \qquad \int_{\phi^2_{c_2 r}}^{\phi} \frac{ms^{m-1}}{\tilde{\psi}_2^{\frac{1}{p-1}}(s)}\,\mathrm{d}s = c_2 r. \tag{3.23}$$

In the interval $(0, \phi_0)$ where $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$, the relation $c_1 > c_2$ in (3.23) implies

$$\phi^1_{c_1 r} < \phi^2_{c_2 r}.$$

This comparison of delayed values is essential for the comparison of functions $\phi_c(t)$ with different speed parameters c. We choose ϕ_0 to be the maximal one such that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_0)$ and $\tilde{\psi}_1(\phi_0) = \tilde{\psi}_2(\phi_0) =: \tilde{\psi}_0$. Now we can conclude from (3.22) that at $\phi = \phi_0$

$$\begin{aligned}\frac{\mathrm{d}\tilde{\psi}_1}{\mathrm{d}\phi} - \frac{\mathrm{d}\tilde{\psi}_2}{\mathrm{d}\phi} &= (c_1 - c_2) - \left(\frac{m\phi^{m-1}\cdot(b(\phi^1_{c_1 r}) - d(\phi))}{\tilde{\psi}_1^{\frac{1}{p-1}}(\phi)} - \frac{m\phi^{m-1}\cdot(b(\phi^2_{c_2 r}) - d(\phi))}{\tilde{\psi}_2^{\frac{1}{p-1}}(\phi)}\right)\\ &= (c_1 - c_2) - \frac{m\phi^{m-1}\cdot(b(\phi^1_{c_1 r}) - b(\phi^2_{c_2 r}))}{\tilde{\psi}_0^{\frac{1}{p-1}}}\\ &> c_1 - c_2 > 0,\end{aligned}$$

where we have used $\phi^1_{c_1 r} < \phi^2_{c_2 r}$. This contradicts to the situation that $\tilde{\psi}_1(\phi) - \tilde{\psi}_2(\phi) > 0$ in $(0, \phi_0)$ and $\tilde{\psi}_1(\phi_0) - \tilde{\psi}_2(\phi_0) = 0$. The proof is completed. □

Lemma 3.5. *There exists a unique number $c^* = c^*(m, p, r) > 0$ such that $\phi_{c^*}(t)$ is strictly increasing on $(0, +\infty)$ with $\phi_{c^*}(+\infty) = K$, and the function $\phi_{c^*}(t)$ is the unique travelling wave solution of sharp type. The speed of any smooth travelling wave is greater than $c^*(m, p, r)$, and no travelling waves $\phi(x + ct)$ exist when $c < c^*$. Namely, c^* is the minimal admissible travelling wave speed.*

Proof. This is proved in a similar way as lemmas 3.7 and 3.9 in [46]. Lemma 3.4 shows the strictly monotone dependence of $\phi_c(t)$ on c and lemma 3.3 shows that $\phi_c(t)$ decays to zero for small c and grows to $+\infty$ for large c. Define

$$c^* = c^*(m, p, r) := \inf\{c > 0;\ \phi_c(t) \text{ grows up to } K \text{ in finite time}\}. \tag{3.24}$$

Then c^* is the unique speed of the sharp travelling wave. We assert that $\phi_{c^*}(t)$ is bounded. Otherwise, assume that there exists a $t_1 > cr$ such that $\phi_{c^*}(t_1) > K$. Note that c^* is the infimum of speed c such that $\phi_c(t)$ grows up to K in finite time. In other words, for $c < c^*$, $\phi_c(t)$ cannot grow up to K in finite time and then $\sup\limits_{t\in(0,t^*)} \phi_c(t) \leqslant K$, where $(0, t^*)$ is the existence interval of $\phi_c(t)$. Now we conclude that for $c < c^*$, $\phi_c(t_1) < K$; and for $c \geqslant c^*$, $\phi_c(t_1) \geqslant \phi_{c^*}(t_1) > K$

according to the monotone dependence of $\phi_c(t)$ on c as proved in lemma 3.4. This contradicts to the continuous dependence of $\phi_c(t)$ on c in lemma 3.4 since for $c < c^*$ sufficiently close to c^*, there always exists a gap $(K, \phi_{c^*}(t_1))$ between $\phi_c(t_1)$ and $\phi_{c^*}(t_1)$.

Next, we prove that the speed of any smooth travelling wave is greater than $c^*(m, p, r)$. Here we need to note the following asymptotic behaviour near zero of the phase function $\tilde{\psi}(\phi)$, defined as (3.15) for any travelling wave solution $\phi(t)$ with $\psi(t) := |(\phi^m(t))'|^{p-2}(\phi^m(t))'$:

(a) If $\phi(t)$ is a sharp travelling wave with speed c, then $\tilde{\psi}(\phi) = c\phi + o(\phi)$, as $\phi \to 0^+$, according to lemma 3.1;

(b) If $\phi(t)$ is a smooth travelling wave with speed c, then

$$\tilde{\psi}(\phi) \sim \left(\frac{m(b'(0)e^{-\lambda cr} - d'(0))}{c}\right)^{p-1} \phi^{m(p-1)}, \quad \text{as} \quad \phi \to 0^+, \tag{3.25}$$

where $\lambda > 0$ is the unique root of the equation $c\lambda + d'(0) = b'(0)\mathrm{e}^{-\lambda cr}$.

The asymptotic expansion (3.25) follows from the characteristic equation of (2.1) corresponding to smooth travelling wave $\phi(t) \sim a\mathrm{e}^{\lambda t}$ with $a > 0$ and $\lambda > 0$ as $t \to -\infty$, such that

$$ca\lambda \mathrm{e}^{\lambda t} \sim a^{m(p-1)}(m\lambda)^p(p-1)\mathrm{e}^{m(p-1)\lambda t} - d'(0)a\mathrm{e}^{\lambda t} + b'(0)a\mathrm{e}^{\lambda(t-cr)}, \quad t \to -\infty.$$

The characteristic equation is $c\lambda = -d'(0) + b'(0)\mathrm{e}^{-\lambda cr}$. And the definition of $\psi(t) = |(\phi^m(t))'|^{p-2}(\phi^m(t))' \sim a^{m(p-1)}(m\lambda)^{p-1}\mathrm{e}^{m(p-1)\lambda t} \sim (m\lambda)^{p-1}\phi^{m(p-1)}(t)$ as $t \to -\infty$ with $\lambda = \frac{b'(0)\mathrm{e}^{-\lambda cr} - d'(0)}{c}$. Here we have *a priori* assumed the existence of smooth travelling wave and try to estimate its speed.

Suppose that $\hat{\phi}(t)$ is a smooth travelling wave with speed $c > 0$. The phase function corresponds to $\hat{\phi}(t)$ is denoted by $\hat{\psi}(\phi)$. Let $\phi_c(t)$ be the local solution of sharp type with the same speed and $\tilde{\psi}(\phi)$ be its phase function. Locally near zero,

$$\hat{\psi}(\phi) \sim \left(\frac{m(b'(0)\mathrm{e}^{-\lambda cr} - d'(0))}{c}\right)^{p-1} \phi^{m(p-1)} < c\phi \sim \tilde{\psi}(\phi), \quad \text{as} \quad \phi \to 0^+,$$

since $m(p-1) > 1$. It follows that the trajectory $\tilde{\psi}(\phi)$ corresponding to sharp travelling wave (if it exists) locates above the trajectory $\hat{\psi}(\phi)$ corresponding to smooth travelling wave (if it exists), though they correspond to the same speed. Therefore, the monotone dependence (see the proof of lemma 3.4) shows that $\tilde{\psi}(\phi) > \hat{\psi}(\phi)$ for $\phi \in (0, K]$. Note that corresponding to the critical speed c^*, $\tilde{\psi}(\phi)$ decays to zero at K such that $\tilde{\psi}(K) = 0$ since $\phi_{c^*}(+\infty) = K$ and $\phi'_{c^*}(+\infty) = 0$ (which means $\tilde{\psi}(\phi) = 0$ at $\phi = K$). The monotone dependence of $\phi_c(t)$ on c implies that corresponding to any speed $c \leqslant c^*$, there exists a $\kappa \in (0, K]$ such that $\tilde{\psi}(\phi)$ decays to zero at $\phi = \kappa$. Thus, the relation $\tilde{\psi}(\phi) > \hat{\psi}(\phi)$ for $\phi \in (0, K]$ shows that $\hat{\psi}(\phi)$ decays to zero before $\phi = K$ for any speed $c \leqslant c^*$. In other words, the smooth solution for speed $c \leqslant c^*$ (if exists) starts to decrease before reaching K, and hence it never reaches K similar to the proof of lemma 3.2. It follows that the speed of smooth solution must be greater than $c^*(m, p, r)$. □

We also need to describe the asymptotic behaviour of the phase function $\tilde{\psi}(\phi)$ near the positive equilibrium K, which is important to the variational characterization of critical wave speed.

Lemma 3.6. *The phase function $\tilde{\psi}(\phi)$ defined in (3.15) for the unique sharp travelling wave solution $\phi(t)$ satisfies the following asymptotic expansion:*

(*a*) *If $p = 2$, then*

$$\tilde{\psi}(\phi) = \kappa_2(K - \phi) + o(|K - \phi|), \quad as \ \ \phi \to K^-,$$

where $\lambda > 0$ is the unique positive root of

$$mK^{m-1}\lambda^2 + c\lambda + b'(K)e^{\lambda cr} - d'(K) = 0,$$

and $\kappa_2 := mK^{m-1}\lambda$;

(*b*) *If $p > 2$, then*

$$\tilde{\psi}(\phi) = \kappa_p(K - \phi)^{p-1} + o(|K - \phi|^{p-1}), \quad as \ \ \phi \to K^-,$$

where $\lambda > 0$ is the unique positive root of

$$c\lambda + b'(K)e^{\lambda cr} - d'(K) = 0,$$

and $\kappa_p := (mK^{m-1}\lambda)^{p-1}$ for $p > 2$;

(*c*) *If $1 < p < 2$, then*

$$\tilde{\psi}(\phi) = \kappa_p(K - \phi)^{\frac{2(p-1)}{p}} + o\left(|K - \phi|^{\frac{2(p-1)}{p}}\right), \quad as \ \ \phi \to K^-,$$

where $\lambda > 0$ is the unique positive root of

$$m^{p-1}K^{(m-1)(p-1)}\frac{2(p-1)}{p}\left(\frac{p}{2-p}\lambda\right)^p + b'(K) - d'(K) = 0.$$

and $\kappa_p := m^{p-1}K^{(m-1)(p-1)}\left(\frac{p}{2-p}\lambda\right)^{p-1}$ for $p \in (1, 2)$.

Proof. For $p \geqslant 2$, we utilize the ansatz of expansion $\tilde{\psi}(\phi) \sim \kappa(K - \phi)^{p-1}$ and $\phi(t) - K \sim -\mu e^{-\lambda t}$ as $t \to +\infty$ and $\phi \to K^-$ for some positive constants κ, μ and λ. Further, $d(\phi) - d(K) \sim d'(K)(\phi - K)$, and $b(\phi_{cr}) - b(K) \sim b'(K)(\phi_{cr} - K)$, as $t \to +\infty$, where $\phi_{cr} = \phi(\cdot - cr)$. Noticing that $b(K) = d(K)$, we see that

$$\begin{aligned} b(\phi_{cr}) - d(\phi) &\sim b'(K)(\phi_{cr} - K) - d'(K)(\phi - K) \\ &\sim (b'(K) - d'(K))(\phi - K) + b'(K)(\phi_{cr} - \phi) \\ &\sim (b'(K) - d'(K))(\phi - K) + b'(K)(e^{\lambda cr} - 1)(\phi - K) \\ &\sim (b'(K)e^{\lambda cr} - d'(K))(\phi - K), \end{aligned}$$

as $t \to +\infty$ since $\frac{\phi_{cr}-\phi}{\phi-K} \sim \mathrm{e}^{\lambda cr} - 1$. According to (3.14) and (3.17), near K, $\tilde{\psi}$ behaves similar as

$$\begin{cases} \dfrac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} \sim c - \dfrac{mK^{m-1} \cdot (b'(K)\mathrm{e}^{\lambda cr} - d'(K))(\phi - K)}{\tilde{\psi}^{\frac{1}{p-1}}(\phi)}, \\ \tilde{\psi}(K) = 0, \quad \tilde{\psi}(\phi) > 0 \text{ for } \phi \in (0, K). \end{cases} \tag{3.26}$$

For the special case $p = 2$, $\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} \sim -\kappa$, and the singular ODE (3.26) admits a solution satisfying the expansion $\tilde{\psi}(\phi) \sim \kappa(K - \phi)$ provided that $\kappa > 0$ is the unique positive root of the following equation

$$-\kappa = c + \frac{mK^{m-1} \cdot (b'(K)\mathrm{e}^{\lambda cr} - d'(K))}{\kappa}. \tag{3.27}$$

Additionally, a necessary condition for the characteristic value $\lambda > 0$ of the travelling wave of (2.1) satisfying $\phi(t) - K \sim -\mu \mathrm{e}^{-\lambda t}$ as $t \to +\infty$ is

$$mK^{m-1}\lambda^2 + c\lambda + b'(K)\mathrm{e}^{\lambda cr} - d'(K) = 0. \tag{3.28}$$

Moreover, according to the asymptotic expansions $\tilde{\psi}(\phi) \sim \kappa(K - \phi)$ and $\phi(t) - K \sim -\mu \mathrm{e}^{-\lambda t}$, noticing that $\tilde{\psi}(\phi) = \psi(t) = (\phi^m(t))' \sim mK^{m-1}\phi'(t)$ for $p = 2$, there must hold

$$\kappa = mK^{m-1}\lambda. \tag{3.29}$$

Since $d'(K) > b'(K) \geqslant 0$, the characteristic equation (3.28) admits a unique positive root $\lambda > 0$, and then (3.27) is equivalent to (3.29).

For the case $p > 2$, $\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = o(1)$ as $\phi \to K^-$, and in this situation (3.27) reads as

$$0 = c + \frac{mK^{m-1} \cdot (b'(K)\mathrm{e}^{\lambda cr} - d'(K))}{\kappa^{\frac{1}{p-1}}}. \tag{3.30}$$

The characteristic equation (3.28) now is

$$c\lambda + b'(K)\mathrm{e}^{\lambda cr} - d'(K) = 0. \tag{3.31}$$

Similar to (3.29) the relation between the expansions $\tilde{\psi}(\phi) \sim \kappa(K - \phi)^{p-1}$ and $\phi(t) - K \sim -\mu \mathrm{e}^{-\lambda t}$ implies

$$\tilde{\psi}(\phi) = \psi(t) = |(\phi^m(t))'|^{p-2}(\phi^m(t))' \sim (mK^{m-1}\phi'(t))^{p-1} \sim \kappa(K - \phi)^{p-1}.$$

That is,

$$\kappa = (mK^{m-1}\lambda)^{p-1}. \tag{3.32}$$

The characteristic equation (3.31) has a unique positive root $\lambda > 0$ and (3.30) is equivalent to (3.32) in this case.

The case of $1 < p < 2$ is quite different, we utilize the ansatz $\tilde{\psi}(\phi) \sim \kappa(K - \phi)^{\frac{2(p-1)}{p}}$ and $\phi(t) - K \sim -\mu(1 + \lambda t)^{-\frac{p}{2-p}}$ as $t \to +\infty$ and $\phi \to K^-$ for some positive constants κ, μ and λ. It should be addressed that the sharp travelling wave approached the positive equilibrium K algebraically instead of exponentially. Note that $\frac{2(p-1)}{p} \in (0, 1)$, $\frac{2(p-1)}{p} - 1 = 1 - \frac{2}{p}$, and

$\frac{p}{2-p} \in (1, +\infty)$ for $p \in (1, 2)$. The algebraically decay behaves differently from the exponential decay, such that

$$\begin{aligned} b(\phi_{cr}) - d(\phi) &\sim b'(K)(\phi_{cr} - K) - d'(K)(\phi - K) \\ &\sim (b'(K) - d'(K))(\phi - K) + b'(K)(\phi_{cr} - \phi) \\ &\sim (b'(K) - d'(K))(\phi - K), \end{aligned}$$

as $t \to +\infty$ since $\frac{\phi_{cr}-\phi}{\phi-K} = o(1)$. The singular ODE (3.26) now shows that

$$\frac{2(p-1)}{p} \cdot \kappa = -\frac{mK^{m-1} \cdot (b'(K) - d'(K))}{\kappa^{\frac{1}{p-1}}}. \tag{3.33}$$

For the degenerate case such that $\phi(t) - K \sim -\mu(1 + \lambda t)^{-\frac{p}{2-p}}$ (assume $\mu = 1$ by rescaling), we have

$$\begin{aligned} (|(\phi^m(t))'|^{p-2}(\phi^m(t))')' &= (m^{p-1}\phi^{(m-1)(p-1)}|\phi'|^{p-2}\phi')' \\ &= m^{p-1}\phi^{(m-1)(p-1)}(p-1)|\phi'|^{p-2}\phi'' \\ &\quad + m^{p-1}(m-1)(p-1)\phi^{(m-1)(p-1)-1}|\phi'|^p \\ &\sim m^{p-1}\phi^{(m-1)(p-1)}(p-1)|\phi'|^{p-2}\phi'', \end{aligned}$$

since $|\phi'|^p = o(|\phi'|^{p-2}\phi'')$ according to $\frac{p}{2-p} > 1$. Moreover, $\phi' = o(|K - \phi|) = o(b(\phi_{cr}) - d(\phi))$, and then the characteristic equation of (2.1) is

$$\begin{aligned} -m^{p-1}\phi^{(m-1)(p-1)}(p-1)|\phi'|^{p-2}\phi'' &\sim b(\phi_{cr}) - d(\phi) \\ &\sim (b'(K) - d'(K))(\phi - K), \end{aligned}$$

which means

$$m^{p-1}K^{(m-1)(p-1)}(p-1)\left(\frac{p}{2-p}\lambda\right)^{p-2}\frac{p}{2-p}\frac{2}{2-p}\lambda^2 + b'(K) - d'(K) = 0.$$

That is,

$$m^{p-1}K^{(m-1)(p-1)}\frac{2(p-1)}{p}\left(\frac{p}{2-p}\lambda\right)^p + b'(K) - d'(K) = 0. \tag{3.34}$$

Moreover, according to the asymptotic expansions $\tilde{\psi}(\phi) \sim \kappa(K - \phi)^{\frac{2(p-1)}{p}}$ and $\phi(t) - K \sim -(1 + \lambda t)^{-\frac{p}{2-p}}$, we have

$$\begin{aligned} \tilde{\psi}(\phi) =& |(\phi^m(t))'|^{p-2}(\phi^m(t))' \sim (mK^{m-1}\phi'(t))^{p-1} \\ \sim& m^{p-1}K^{(m-1)(p-1)}\left(\frac{p}{2-p}(1 + \lambda t)^{-\frac{2}{2-p}}\lambda\right)^{p-1} \\ \sim& \kappa\left((1 + \lambda t)^{-\frac{p}{2-p}}\right)^{\frac{2(p-1)}{p}}. \end{aligned}$$

Therefore,

$$\kappa = m^{p-1}K^{(m-1)(p-1)}\left(\frac{p}{2-p}\lambda\right)^{p-1}. \tag{3.35}$$

The characteristic equation (3.34) has a unique positive root λ, and then (3.35) is identical to (3.33). □

The critical wave speed $c^*(m, p, 0)$ for non-delayed case is characterized via a variational approach by Benguria and Depassier [4–6]. For time-delayed case, we utilize the variational characterization method to show the dependence of the critical wave speed $c^*(m, p, r)$ with respect to the time delay r.

Lemma 3.7. *The minimal travelling wave speed $c^*(m, p, r)$ for the time delay $r > 0$ is strictly smaller than that without time delay, i.e. $c^*(m, p, r) < c^*(m, p, 0)$.*

Proof. Let $\phi(t)$ be the unique sharp type travelling wave corresponding to the speed $c = c^*(m, p, r)$ according to lemma 3.5. This kind of special solution is the unique one that is strictly increasing on $(0, +\infty)$, $\phi(+\infty) = K$ and $\phi'(+\infty) = 0$, according to the monotone dependence lemmas 3.4 and 3.5.

In the proof of lemma 3.3, we formulate the generalized phase plane (3.14) and (3.17), where ϕ_{cr} is defined by (3.16). Moreover, $\tilde{\psi}(0) = 0$, $\tilde{\psi}(K) = 0$ since $\phi'(+\infty) = 0$, $\tilde{\psi}(\phi) > 0$ for all $\phi \in (0, K)$. We rewrite (3.17) into

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi} = c - \frac{m\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}} + \frac{m\phi^{m-1}(b(\phi) - b(\phi_{cr}))}{\tilde{\psi}^{\frac{1}{p-1}}}, \quad \phi \in (0, K). \tag{3.36}$$

For any $g \in \mathscr{D} = \{g \in C^1([0, K]); \int_0^K g(s)\mathrm{d}s = 1, g(s) > 0, g'(s) < 0, \forall s \in (0, K)\}$, we multiply (3.36) by $g(s)$ and integrate it over $(0, K)$ to find

$$\begin{aligned} c &= \int_0^K g(\phi)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\phi}\,\mathrm{d}\phi + \int_0^K g(\phi)\frac{m\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}}\,\mathrm{d}\phi \\ &\quad - \int_0^K g(\phi)\frac{m\phi^{m-1}(b(\phi) - b(\phi_{cr}))}{\tilde{\psi}^{\frac{1}{p-1}}}\,\mathrm{d}\phi \\ &= \int_0^K - g'(\phi)\tilde{\psi}(\phi)\mathrm{d}\phi + \int_0^K g(\phi)\frac{m\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}}\,\mathrm{d}\phi \\ &\quad + \left[g(\phi)\tilde{\psi}(\phi)\right]\Big|_{\phi=0}^{\phi=K} - \int_0^K g(\phi)\frac{m\phi^{m-1}(b(\phi) - b(\phi_{cr}))}{\tilde{\psi}^{\frac{1}{p-1}}}\,\mathrm{d}\phi \\ &= F(g, \tilde{\psi}) - \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}\,\mathrm{d}\phi, \end{aligned} \tag{3.37}$$

according to $\int_0^K g(s)\mathrm{d}s = 1$ and $[g(\phi)\tilde{\psi}(\phi)]\Big|_{\phi=0}^{\phi=K} = 0$ as $\tilde{\psi}(0) = 0 = \tilde{\psi}(K)$, where the functional

$$F(g, \tilde{\psi}) := \int_0^K - g'(\phi)\tilde{\psi}(\phi)\mathrm{d}\phi + \int_0^K g(\phi)\frac{m\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^{\frac{1}{p-1}}}\,\mathrm{d}\phi.$$

Next, we consider the function $F(g, \tilde{\psi})$ over all set $\mathscr{D}$. Utilizing Young's inequality, we see that

$$F(g, \tilde{\psi}) \geqslant \int_0^K \frac{p}{(p-1)^{(p-1)/p}} (-g'(\phi))^{\frac{1}{p}} (g(\phi))^{\frac{p-1}{p}} (m\phi^{m-1}(b(\phi) - d(\phi)))^{\frac{p-1}{p}} \, \mathrm{d}\phi, \quad (3.38)$$

see also in [6] for non-delayed case. The equality in (3.38) is attainable if there exists a function $\hat{g} \in \mathscr{D}$ such that

$$m\phi^{m-1}(b(\phi) - d(\phi))\hat{g}(\phi) = (p-1)(-\hat{g}'(\phi))\tilde{\psi}^{\frac{p}{p-1}}(\phi), \quad \phi \in (0, K). \quad (3.39)$$

The existence of a $\hat{g} \in \mathscr{D}$ solving (3.39) relies heavily on the asymptotic behaviour of $\tilde{\psi}(\phi)$ near K and near 0 as shown in lemmas 3.6 and 3.1. According to lemma 3.6,

$$\tilde{\psi}^{\frac{p}{p-1}}(\phi) \sim \begin{cases} \kappa(K-\phi)^p, & p \geqslant 2, \\ \kappa(K-\phi)^2, & p \in (1, 2), \end{cases}$$

where $\kappa > 0$ is a positive number. Therefore, $\hat{g}(K) = 0$. Otherwise,

$$\frac{-\hat{g}'(\phi)}{\hat{g}(\phi)} = \frac{m\phi^{m-1}(b(\phi) - d(\phi))}{(p-1)\tilde{\psi}^{\frac{p}{p-1}}(\phi)} \sim \frac{mK^{m-1}(d'(K) - b'(K))}{(p-1)\kappa(K-\phi)^{\max\{p-1,1\}}}, \quad \text{as} \quad \phi \to K^-, \quad (3.40)$$

which is not integrable, a contradiction. Consider the singular ODE (3.40) near K with the condition $\hat{g}(K) = 0$, and $\hat{g}'(\phi) < 0$ for $\phi \in (0, K)$, since $\max\{p-1, 1\} \geqslant 1$ for all $p > 1$, it has infinitely many solutions such that $\hat{g}(\phi) > 0$ for $\phi \in (0, K)$. If $\hat{g}(0)$ is finite, we can normalize $\hat{g}$ such that $\int_0^K \hat{g}(\phi)\mathrm{d}\phi = 1$ and then $\hat{g} \in \mathscr{D}$. According to lemma 3.1, $\tilde{\psi}(\phi) \sim c\phi$ as $\phi \to 0^+$. Then

$$\frac{-\hat{g}'(\phi)}{\hat{g}(\phi)} = \frac{m\phi^{m-1}(b(\phi) - d(\phi))}{(p-1)\tilde{\psi}^{\frac{p}{p-1}}(\phi)} \sim \frac{m(b'(0) - d'(0))\phi^m}{(p-1)c^{\frac{p}{p-1}}\phi^{\frac{p}{p-1}}}, \quad \text{as} \quad \phi \to K^-. \quad (3.41)$$

It follows that $\hat{g}(0) < +\infty$ since $m - \frac{p}{p-1} > -1$ as $m(p-1) > 1$ such that $\phi^{m-\frac{p}{p-1}}$ is integrable near zero.

Finally, for $\hat{g} \in \mathscr{D}$, we have

$$\begin{aligned} c &= F(\hat{g}, \tilde{\psi}) - \int_0^K \hat{g}(\phi) \frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}} \, \mathrm{d}\phi \\ &< F(\hat{g}, \tilde{\psi}) \\ &\leqslant \sup_{g \in \mathscr{D}} \int_0^K \frac{p}{(p-1)^{(p-1)/p}} (-g'(\phi))^{\frac{1}{p}} (g(\phi))^{\frac{p-1}{p}} (m\phi^{m-1}(b(\phi) - d(\phi)))^{\frac{p-1}{p}} \, \mathrm{d}\phi \\ &= c^*(m, p, 0), \end{aligned}$$

where the last equality is the variational characterization of the speed for non-delayed case as proved in [6]. The proof is completed. □

Proof of theorem 2.1. The existence and uniqueness of sharp type travelling wave are proved in lemma 3.5. According to lemma 3.7, we see that time delay slows down the critical wave speed. Any other travelling waves must be positive and the regularity is trivial since (2.1) is non-degenerate at where $\phi(t) > 0$. □

Proof of theorem 2.2. This is proved according to the asymptotic behaviour near 0 in lemma 3.1. □

4. Model formulation

The models with degenerate diffusion but without time-delay were firstly introduced in [17, 31], and the models with time-delay and regular diffusion were widely studied in [10, 23, 26–28, 36, 39, 40]. However, the derivation of the models with both effects of degenerate diffusion and time-delay is not officially derived, even if such models had been proposed and studied in our previous research works [18, 45–47] based on the mathematical concerning. In this section, we develop a degenerate diffusion model with time delay that arises in the modelling of age-structured populations. Here, we give a brief derivation of the equations we treat.

The problem is as follows. Let a denote chronological age, t denote time and x denote spatial position, and let $u(a, t, x)$ denote the population density of age a at time t and at position x. Here, we concerns species whose life cycles consist of two demographically distinct phases incorporating immature and reproductive periods. By $r \geqslant 0$ we denote the maturation time that divides the two phases, so the matured population density at location x and time t is

$$u(x, t) = \int_r^{\infty} v(t, x, a)\mathrm{d}a. \tag{4.1}$$

Also, since only the mature can reproduce, the functional dependence of birth rate β is assumed to enter only through dependence on u and so that $\beta = \beta(u)$.

Assuming that the emigration in species due to intraspecific competition in a way that makes the flux of individuals proportional to the gradient of the mature population, in [8], the following age-structured population model with degenerate diffusion is derived

$$\begin{cases} \dfrac{\partial v}{\partial a} + \dfrac{\partial v}{\partial t} = g(t, x, u, \nabla u)\cdot\nabla v + h(t, x, u, \nabla u, D^2 u)v - \mu v, \qquad x \in \mathbb{R}^n,\ \ t > 0, \\ v(0, t, x) = \displaystyle\int_r^{\infty} \beta(u)v(a, t, x)\mathrm{d}a, \\ v(a, 0, x) = v_0(a, x). \end{cases} \tag{4.2}$$

Using (4.1) and integrating the partial differential equation (4.2) from r to $+\infty$, we obtain for $t > 0$ and $x \in \mathbb{R}^n$

$$\frac{\partial u}{\partial t} = \tilde{g}(t, x, u, \nabla u)\cdot\nabla u + \tilde{h}(t, x, u, \nabla u, D^2 u)u + \beta(u(t - r, x))u(t - r, x)S - \mu u. \tag{4.3}$$

In obtaining (4.3) the following biological realistic assumptions are necessary [41]: (i) $v(a, t, x) \to 0$ as $a \to +\infty$; (ii) the birth rate at time t reduces to $v(0, t, x) = \beta(u)u$; and (iii) $v(r, t, x) = v(0, t - r, x)S$, where S is the fraction of individuals that survives through the first demographic phase. The last point reflects the fact that the individuals of age r is made up of the survivors that were born at time $t - r$.

The spatial diffusion term $\tilde{g} \cdot \nabla u + \tilde{h}u$ includes operators of the form Δu^m and doubly nonlinear operator in (1.1). When $r = 0, p = 2, m > 1$, (4.3) may reduce to density dependent models of population dynamics of the form

$$\frac{\partial u}{\partial t} = \Delta u^m + \beta(u)u - \mu u, \quad x \in \mathbb{R}^n, \ t > 0, \tag{4.4}$$

which was considered by means of existing theory (Aronson [1], Gurtin and MacCamy [16]). Our model includes a large number of evolution equations in biology, for example, the degenerate delayed Fisher–KPP equations, the degenerate Nicholson's blowflies equation, and the Mackey–Glass equation.

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