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ScienceDirect

J. Differential Equations 268 (2020) 414–446

Journal of Differential Equations

www.elsevier.com/locate/jde

On a chemotaxis model with degenerate diffusion: Initial shrinking, eventual smoothness and expanding

Tianyuan Xu [a], Shanming Ji [b,*], Ming Mei [c,d], Jingxue Yin [a]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong, 510631, PR China
[b] School of Mathematics, South China University of Technology, Guangzhou, Guangdong, 510641, PR China
[c] Department of Mathematics, Champlain College Saint-Lambert, Quebec, J4P 3P2, Canada
[d] Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada

Received 16 October 2018; revised 27 April 2019; accepted 15 August 2019
Available online 27 August 2019

Abstract

We investigate the propagating profiles of a degenerate chemotaxis model describing the bacteria chemotaxis and consumption of oxygen by aerobic bacteria, in particular, the effect of the initial attractant distribution on bacterial clustering. We prove that the compact support of solutions may shrink if the signal concentration satisfies a special structure, and show the finite speed propagating property without assuming the special structure on attractant concentration, and obtain an explicit formula of the population spreading speed in terms of model parameters. The presented results suggest that bacterial cluster formation can be affected by chemotactic attractants and density-dependent dispersal.

Keywords: Chemotaxis; Degenerate diffusion; Initial shrinking; Propagating speed; Eventual smoothness; Eventual expanding

1. Introduction

We consider the following chemotaxis model with chemotactic consumption and porous media diffusion

* Corresponding author.
E-mail address: jism@scut.edu.cn (S. Ji).

https://doi.org/10.1016/j.jde.2019.08.013
0022-0396/

$$\begin{cases} u_t = \nabla(\phi(u)\nabla u) - \chi \nabla \cdot (u \nabla v), \\ v_t = \Delta v - \alpha u v, \qquad\qquad x \in \Omega, \ t > 0, \end{cases} \tag{1}$$

where u represents the number per unit volume of aerobic bacteria cells, v denotes the oxygen concentration, χ is the chemotactic coefficient, α denotes the fractional rate of oxygen consumption per unit concentration of bacteria cell. The diffusion of species is considered to be degenerate in the form of $\nabla(\phi(u)\nabla u)$ with $\phi(u) = Du^{m-1}$ and $m > 1$, which is dependent of the population density due to the population pressure, $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. This model can also describe other chemotaxis progress with nutrient consumption.

The chemotaxis model with porous medium diffusion type is motivated from a biological point of view [34]. It is worthy of mentioning that the porous medium type diffusion can represent "population pressure" in cell invasion models [28], which initially arises from the ecology literature [12,13,26,51]. In fact, experimental investigation has shown that the diffusion coefficient depends on the bacterial density [41]. In the bacterial experiments done by Ohgiwari et al. [27], they recognized that cells located inside the bacterial colonies move actively, but cells became sluggish at the outermost front with apparently low cell density. This phenomenon indicates that bacteria become active as the cell density u increases. Thus, a natural choice of the bacterial diffusion coefficient is $\phi(u) = u^{m-1}(m > 1)$, and this porous medium type bacterial diffusivity is based on the degenerate diffusion model proposed by Kawasaki et al. [16]. Recently, Leyva et al. [20] incorporated a chemotactic term into the original model by Kawasaki et al., and explored the effects of chemotaxis on bacterial aggregation patterns.

Incorporating the porous medium type diffusivity in bacterial chemotaxis models mentioned above, the system (1) appears as part of the following chemotaxis-fluid model proposed by Tuval et al. [39] to describe the motion of oxygen-driven swimming bacteria in an incompressible fluid

$$\begin{cases} u_t + w \cdot \nabla u = \Delta u^m - \nabla \cdot (u\chi(v)\nabla v), \\ v_t + w \cdot \nabla v = \Delta v - uf(v), \\ w_t + \kappa(w \cdot \nabla w) = \Delta w - \nabla P + u\nabla\phi, \\ \nabla \cdot w = 0, \quad x \in \Omega, \quad t > 0, \end{cases} \tag{2}$$

where w is the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P, κ the strength of nonlinear fluid convection, $\nabla\phi$ a gravitational force, $f(v)$ the rate of oxygen consumption, u, v $\chi(v)$ the quantities denoted as before. It was Francesco who first extended the classical fluid-chemotaxis coupled model to one with a porous medium-type diffusion of swimming bacteria [10]. Obviously, if the flow of fluid is ignored (i.e., $w = 0$) or the fluid is stationary, then (2) yields the chemotaxis model with porous media diffusion (1).

Chemotaxis is the biased migration in the direction of a chemical stimulus concentration gradient [14,17]. Bacteria can sense a large range of chemical signals, such as the concentrations of nutrients, toxins, oxygen, minerals, etc. A mathematical model for the process of aerobic motile bacteria toward oxygen which they consume was first proposed in [29]. For the classical chemotaxis model with consumption chemoattractant (i.e. $m = 1$ in (1)), the diffusion of bacteria cells are assumed to be random. When the initial data is sufficiently small, Tao [35] proved that this model admits a classical solution globally in time. Moreover, for large initial data, Tao and Winkler [36] showed that the problem admits a global weak solution, and a more interesting fact is that, the weak solution will become smooth after some time. Recently, chemotaxis models featuring a density-dependent diffusion term have drawn great attention from many authors [3, 11,23,33,42–44,48,54]. For the chemotaxis model (1) with consumption of chemoattractant and porous medium type diffusion, it was shown that the weak solution is globally solvable in two dimension for $m > 1$ [37]. In three dimensional space, the authors made great efforts to prove the global existence of weak solutions for this model for any $m > 1$. Winkler and Tao [38,47] proved that this problem admits a global weak solution for the case $m \in (1, \frac{8}{7}]$.

The qualitative analysis of the chemotaxis model with nonlinear diffusion has attracted a lot of attention and led to a variety of challenging problems [1,2,14]. In many biological cases, the diffusion coefficient $\phi(u)$ is not constant, which can be regarded as a consequence of the interaction between cells [6,22,24,32,53]. The interaction between diffusion and chemotaxis contributes substantial influences on the behavior of solutions for these models. In [7], Burger, Di Francesco, and Dolak considered the Keller-Segel model of chemotaxis with volume filling effect, which is degenerate when bacteria densities approaching either 0 or 1, and they investigated the qualitative behavior of solutions, such as finite speed of propagation and asymptotic behavior of solutions. Kim and Yao [18] studied the qualitative properties of the Patlak-Keller-Segel model with porous medium type diffusion term by using maximum principle type arguments, and they proved the finite propagation property of the compactly supported solutions generated by this type of degeneracy of diffusivity. In [9], Fischer proved finite speed of support propagation for the parabolic-elliptic chemotaxis Keller-Segel system with porous medium type diffusive term and gave sufficient criteria for support shrinking, based on the integral estimates and the Stampacchia's lemma. Moreover, mechanism of nonlinear diffusion is capable of suppressing or generating the occurrence of blow up in chemotaxis systems, such as chemotaxis with the flux limitation [3,4], the volume-filling effect [45,49].

The main feature of our model (1) lies in that the porous medium diffusive term and the chemotactic term are in competition. The dispersal term induces forward motion, whereas the chemotactic attraction may account for cohesive swarm and induce backward motion of the invasion boundary [9]. We explore the effect of density dependent diffusion and chemotactic attraction, which can account for cohesive, finite swarms with realistic density profiles.

To understand how changes in the initial conditions of chemotaxis can so dramatically alter the aggregation behavior of bacteria, we study the effects of attractant concentration on bacteria distribution. We will give (see Theorem 2.1) a mathematical understanding of the collective behavior of bacteria chemotactic toward oxygen. We find that under certain initial conditions, the boundary of $\operatorname{supp} u(\cdot, t)$ moves backward in response to the gradient of attraction at early stage. This indicates that the size of the swarm is defined by a balance of chemotactic attraction and cell dispersal: the greater the attraction the smaller its size for a given total number of organisms. This is observed biologically: bacteria exhaust the local oxygen and then react to the attractant gradient they have created, producing a flux towards the region with more oxygen. Early in bioconvection, this process generated accumulations of cells, resulting in smaller size of cell collective region. This experiment was conducted on *Bacillus subtilis* [8].

One of the intrinsic characteristics of porous medium diffusions is the population moves with a finite speed of propagation, which seems more reasonable than infinite speed in biological applications. To put it concisely, for any non-zero initial data u_0, the solution of linear diffusion equation $u(x,t) > 0$ for $t > 0$ and any $x \in \mathbb{R}^N$, thus a linear diffusion process predicts an infinite propagation [40]. However, the spatial support of the solution to the degenerate diffusion equation remains bounded for all time $t > 0$ [9].

Bacteria are known to exhibit very diverse morphological aggregation patterns depending on a variety of environmental conditions [5,25,27,41]. These experimental observations showed the bacterial enveloping front propagates outward gradually over time and the velocity of front propagation is finite. In order to explain these phenomena, a variety of mathematical models have been proposed [16,20,30,31]. The density-dependent degenerate diffusion model may capture more pattern features found experimentally and provides a better match to experimental cell density profiles. The difference between these diffusion types is that the porous medium type diffusion leads to distinct boundaries, and the population density decreases to zero at a finite point in space, rather than tends to zero asymptotically. It is therefore not surprising that the behavioral property of living organisms in these two models is different. The porous medium type models allow the cells aggregate rather than spread out. The non-physical diffusion is eliminated in this model.

Although the underlying dynamics of the chemotaxis model with degenerate mobility can be complicated, explicit description of bacteria invasion process can be given. The challenge in the mathematical analysis consists of the chemotactic term as well as the degeneracy of the diffusion term which generates compactly supported solutions. We prove several propagating properties of solutions, including the initial shrinking, finite propagation property, eventual smoothness and eventual expanding. The spreading speed is the rate at which the species with uniformly positive initial distribution over a large interval and zero distribution outside an interval expands its spatial range [19]. Theorem 2.3 below provides an explicit formula for the spreading speed in terms of model parameters. To the best of our knowledge, it is the first work that presents a precise description of the propagating speed for this model. These results provide important insight into the spatial patterns and rates of invading bacteria species in space.

Besides the porous-medium-type diffusivity, we note that the dynamics of solution's expanding and shrinking properties is also related to the nonlinearity of the chemotactic sensitivity. In [51], we considered the chemotaxis model involving the same nonlinearity in the diffusivity and the chemotactic sensitivity, which is

$$u_t = \Delta u^m - \nabla \cdot (u^n \nabla v) + u^\delta (1-u) \tag{3}$$

with $n = m > 1$, and we proved that its support is always expanding with finite speed. Roughly speaking, for the general diffusion versus chemotaxis equation (3) with $m > 1$, $n \geq 1$ and coupled or given $v(x,t)$ with suitable regularity, its support exhibits initial shrinking only for the case $n = 1$ and $v(x,t)$ satisfies some special structure. This is suggested by the exact propagating speed in Theorem 2.3 for the case $n = 1$, where the degenerate diffusion Δu^m and the chemotaxis $\chi \nabla \cdot (u \nabla v)$ maintain some precise balance such that the sign of $2mK_0^{m-1}/(m-1) - \chi\mu$ determines the expanding or shrinking of the initial support (K_0, μ are given therein). If $n > 1$, the chemotaxis $\chi \nabla \cdot (u^n \nabla v)$ is "degenerated" as $u^{n-1}(x,t)$ approaches zero near the boundary of the support, and in this case, μ is interpreted as zero and then $2mK_0^{m-1}/(m-1) - \chi\mu$ is always positive. We would like to point out that the above observation of diffusion versus chemotaxis is only concerned with the expanding or shrinking of its support, while the dynamics of the solution away from the boundary of the support is left untouched, which may exhibit different behavior.

The outline of this paper is as follows. In Section 2, we state our main results and some notations. We leave the comparison principle of the corresponding degenerate chemotaxis equation and its Hölder continuity into Section 3 as preliminaries. Section 4 is devoted to the study of the propagating properties of bacteria cells and the large time behavior of the weak solution.

2. Main results and notations

We consider the following chemotaxis system (4) with degenerate diffusion

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u^m}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $m > 1$, $\chi > 0$, $\Omega \subset \mathbb{R}^N$ is as mentioned before, u_0, v_0 are nonnegative functions, n is the unit outer normal vector.

Since degenerate diffusion equations may not have classical solutions in general, we need to formulate the following definition of generalized solutions for the initial boundary value problem (4).

Definition 2.1. Let $T \in (0, \infty)$. A pair of (u, v) is said to be a weak solution to the problem (4) in $Q_T = \Omega \times (0, T)$ if

(1) $u \in L^\infty(Q_T)$, $\nabla u^m \in L^2((0,T); L^2(\Omega))$, and $u^{m-1} u_t \in L^2((0,T); L^2(\Omega))$;

(2) $v \in L^\infty(Q_T) \cap L^2((0,T); W^{2,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$;

(3) the identities

$$\int_0^T \int_\Omega u \psi_t dxdt + \int_\Omega u_0(x) \psi(x,0) dx = \int_0^T \int_\Omega \nabla u^m \cdot \nabla \psi dxdt - \int_0^T \int_\Omega \chi u \nabla v \cdot \nabla \psi dxdt,$$

$$\int_0^T \int_\Omega v_t \varphi dxdt + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi dxdt = \int_0^T \int_\Omega wz \varphi dxdt,$$

hold for all $\psi, \varphi \in L^2((0,T); W^{1,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ with $\psi(x,T) = 0$ for $x \in \Omega$;

(4) v takes the value v_0 in the sense of trace at $t = 0$.

If (u, v) is a weak solution of (4) in Q_T for any $T \in (0, \infty)$, then we call it a global weak solution.

A pair of (u, v) is said to be a globally bounded weak solution to the problem (4) if there exists a positive constant C such that

$$\sup_{t \in \mathbb{R}^+} \left\{ \|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} \right\} \le C.$$

Throughout this paper we assume that the initial data satisfies

$$u_0 \in C(\overline{\Omega}),\ \nabla u_0^m \in L^2(\Omega),\ \frac{\partial v_0}{\partial n} = 0 \text{ on } \partial\Omega,\ v_0 \in C^{2,\alpha_0}(\overline{\Omega}) \text{ for some } \alpha_0 \in (0,1). \tag{5}$$

We are aiming at the propagating properties of the cell invasions. Let us first focus on the waiting time and the initial shrinking of the compact support caused by chemotaxis. Our approach is based on the comparison principle and the technique of self-similar weak lower and upper solutions with compact support.

Theorem 2.1 *(Initial shrinking caused by chemotaxis). Let (u, v) be a globally bounded weak solution of (4) with (i) $N = 1$; or (ii) $\sup_{t\in(0,\infty)} \|u(\cdot,t)\|_{C^{1/(2m)}(\overline{\Omega})} \le C$ for some constant $C > 0$. Further we assume that*

$$\mathit{supp}\, u_0 \subset \overline{B}_{R_0}(x_0) \subset \Omega, \quad u_0 \le K_0(R_0^2 - |x-x_0|^2)^{d_0},\ x \in B_{R_0}(x_0), \tag{6}$$

$$\nabla v_0 \cdot (x - x_0) \le -\mu|x-x_0|^2,\ x \in B_{R_0}(x_0), \tag{7}$$

for some $x_0 \in \Omega$ and positive constants $d_0 \ge 1/(m-1)$, and $R_0, K_0, \mu > 0$, such that $\chi\mu > \frac{4m}{m-1}K_0^{m-1}\max\{1, R_0^{2((m-1)d_0-1)}\}$. Then there exist a family of shrinking open sets $\{A(t)\}_{t\in(0,t_0)}$ with $t_0 > 0$ such that $A(0) = B_{R_0}(x_0)$ and

$$\mathit{supp}\, u(\cdot,t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0, t_0),$$

and $\partial A(t)$ has a finite negative derivative with respect to t.

Remark 2.1. The existence of globally bounded weak solutions of (4) is proved in [15]. We will prove in Lemma 3.5 that (i) implies (ii). The finite propagating speed (i.e. the derivative of $\partial A(t)$ with respect to t) is interpreted as in the sense of Theorem 2.3.

We show the finite speed propagating property without the special structure (7) on signal concentration.

Theorem 2.2 *(Finite speed propagating). Let the assumptions in Theorem 2.1 be valid except for (7). Then there exist a family of open sets $\{A(t)\}_{t\in(0,t_0)}$ with $t_0 > 0$ such that $A(0) = B_{R_0}(x_0)$ and*

$$\mathit{supp}\, u(\cdot,t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0, t_0),$$

and $\partial A(t)$ has a finite derivative with respect to t.

Remark 2.2. Without the structure (7) on signal concentration, we do not know the shrinking or expanding of the cells. However, Theorem 2.2 shows the propagating speed is finite.

If the cell density and the signal concentration have special structure, we will present the exact propagating speed as follows.

Theorem 2.3 *(Exact propagating speed). Let (u, v) be a globally bounded weak solution of (4) with (i) $N = 1$; or (ii) $\sup_{t\in(0,\infty)} \|u(\cdot,t)\|_{C^{1/(2m)}(\overline{\Omega})} \le C$ for some constant $C > 0$. Further we assume that the initial values satisfy*

$$\begin{cases} u_0 = K_0\big[(R_0^2 - |x-x_0|^2)_+\big]^d, \ x \in \Omega, \\ \nabla v_0 \cdot (x-x_0) = -\mu|x-x_0|^2, \ x \in B_{R_0}^{\delta}(x_0), \end{cases} \tag{8}$$

for some $x_0 \in \Omega$ and positive constants $d = 1/(m-1)$, $R_0, K_0, \mu, \delta > 0$ such that $\overline{B}_{R_0}(x_0) \subset \Omega$ and $B_{R_0}^{\delta}(x_0) := \{x \in B_{R_0}(x_0); dist(x, \partial B_{R_0}(x_0)) < \delta\}$. Here, $(R_0^2 - |x-x_0|^2)_+ = \max\{0, R_0^2 - |x-x_0|^2\}$. Then

$$supp\, u(x,t) = \{(\theta, \rho(\theta,t)); \theta \in S^{N-1}\},$$

where (θ, ρ) is the spherical coordinate centered at x_0, $\rho(\theta, 0) = R_0$ for all $\theta \in S^{N-1}$, and the propagating speed

$$\frac{\partial \rho(\theta,t)}{\partial t}\Big|_{t=0} = R_0\Big(\frac{2m}{m-1}K_0^{m-1} - \chi\mu\Big), \quad \forall \theta \in S^{N-1}.$$

With the signal being consumed as time grows, we show that the cells will eventually expand.

Theorem 2.4 *(Eventual expanding). Let (u, v) be a globally bounded weak solution of (4) with (i) $N = 1$; or (ii) $\sup_{t\in(0,\infty)} \|u(\cdot,t)\|_{C^{1/(2m)}(\overline{\Omega})} \le C$ for some constant $C > 0$. Further we assume the initial data $u_0 \ge 0$, $u_0 \not\equiv 0$ and Ω is convex. Then there exist $\hat{T} > \hat{t} > 0$ and $t_0 \in (\hat{t}, \hat{T})$, $\varepsilon_0 > 0$, and a family of expanding open sets $\{A(t)\}_{t\in(\hat{t},\hat{T})}$, such that*

$$A(t) \subset supp\, u(x,t), \quad t \in (\hat{t}, \hat{T}),$$

and $A(t) = \Omega$, $u(x,t) \ge \varepsilon_0$ for all $x \in \Omega$ and $t \in [t_0, \hat{T}]$.

Theorem 2.4 implies that the cells will eventually expand to the whole domain. After that we can show the eventual smoothness and large time behavior.

Theorem 2.5 *(Eventual smoothness). Let the assumptions in Theorem 2.4 be valid. Then $u(x,t) \ge \varepsilon_0$ for all $x \in \Omega$ and $t \ge t_0$ with $t_0 > 0$ and $\varepsilon_0 > 0$ in Theorem 2.4, $u \in C^{2,1}(\overline{\Omega} \times [t_0, \infty))$ and there exist $C > 0$ and $c > 0$ such that*

$$\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le Ce^{-ct}, \quad t > 0,$$

where $\overline{u} = \int_\Omega u_0 dx/|\Omega|$.

The main difficulty lies in the balance between the degenerate diffusion (expanding) and the possible aggregating effect (shrinking) caused by the chemotaxis. According to the exact propagating speed Theorem 2.3, it is clear that the profile near the boundary of its support competes with the gradient of the signal concentration. We first prove the comparison principle by the approximate Hohmgren's approach, and then construct several kinds of lower and upper solutions. The self similar weak lower and upper solutions with shrinking or expanding support are comparable with the Barenblatt solution to the porous medium equation

$$B(x,t) = (1+t)^{-k}\Big[\Big(1 - \frac{k(m-1)}{2mN}\frac{|x|^2}{(1+t)^{2k/N}}\Big)_+\Big]^{\frac{1}{m-1}} \tag{9}$$

with $k = 1/(m-1+2/N)$ for $m > 1$. After showing the eventual expanding property, we formulate the eventual smoothness and large time behavior.

3. Preliminaries: comparison principle and Hölder continuity

3.1. Comparison principle of degenerate diffusion equations

We present the following comparison principle of degenerate diffusion equation in general form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta A(u) - \nabla \cdot (B(u)\Phi(x,t)), \quad x \in \Omega,\ t > 0, \\ (\nabla A(u) - B(u)\Phi) \cdot n = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{10}$$

where $A(s)$ is strictly increasing and locally Lipchitz continuous for $s \in \mathbb{R}$, $B(s)$ is locally Lipchitz continuous for $s \in \mathbb{R}$, and $\Phi : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$ is bounded. Here the degenerate set $\{s \in \mathbb{R}; A'(s) = 0\}$ has no interior point and the equation (10) is weakly degenerate. The typical case is $A(u) = u^m$ with $m > 1$, $B(u) = \chi u$ and the solution u is non-negative (otherwise, one may write $A(u) = |u|^{m-1}u$).

Lemma 3.1 *(Comparison principle). Let $T > 0$ and the function space $E = \{u \in L^\infty(Q_T); \nabla A(u) \in L^2(Q_T)\}$, $u_1, u_2 \in E$, $\Phi \in L^\infty(Q_T)$, and u_1, u_2 satisfy the following differential inequality*

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta A(u_1) + \nabla \cdot (B(u_1)\Phi(x,t)), & \\ \qquad \geq \dfrac{\partial u_2}{\partial t} - \Delta A(u_2) + \nabla \cdot (B(u_2)\Phi(x,t)), & x \in \Omega, t \in (0,T), \\ (\nabla A(u_1) - B(u_1)\Phi) \cdot n \geq (\nabla A(u_2) - B(u_2)\Phi) \cdot n, & x \in \partial\Omega, t \in (0,T), \\ u_1(x,0) \geq u_2(x,0), & x \in \Omega, \end{cases}$$

in the sense that the following inequality

$$\begin{aligned} &\iint\limits_{Q_T} u_1 \varphi_t dxdt + \int\limits_{\Omega} u_{10}(x)\varphi(x,0)dx - \iint\limits_{Q_T} \nabla A(u_1) \cdot \nabla\varphi dxdt \\ &+ \iint\limits_{Q_T} B(u_1)\Phi(x,t) \cdot \nabla\varphi dxdt + \iint\limits_{\partial\Omega \times (0,T)} (\nabla A(u_1) - B(u_1)\Phi) \cdot ndxdt, \\ &\leq \iint\limits_{Q_T} u_2 \varphi_t dxdt + \int\limits_{\Omega} u_{20}(x)\varphi(x,0)dx - \iint\limits_{Q_T} \nabla A(u_2) \cdot \nabla\varphi dxd \\ &+ \iint\limits_{Q_T} B(u_2)\Phi(x,t) \cdot \nabla\varphi dxdt + \iint\limits_{\partial\Omega \times (0,T)} (\nabla A(u_2) - B(u_2)\Phi) \cdot ndxdt, \end{aligned}$$

hold for some fixed $u_{10}, u_{20} \in L^2(\Omega)$ such that $u_{10} \geq u_{20}$ on Ω and all test functions $0 \leq \varphi \in L^2((0,T); W^{1,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ with $\varphi(x,T) = 0$ on Ω. Then $u_1(x,t) \geq u_2(x,t)$ almost everywhere in Q_T.

Proof. The following inequality

$$\iint_{Q_T} (u_1 - u_2)\varphi_t dxdt \leq \iint_{Q_T} \nabla(A(u_1) - A(u_2)) \cdot \nabla\varphi dxdt - \iint_{Q_T} (B(u_1) - B(u_2))\Phi(x,t) \cdot \nabla\varphi dxdt,$$

holds for all $0 \leq \varphi \in L^2((0,T); W^{1,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ with $\varphi(x,T) = 0$. If we further assume that $\frac{\partial\varphi}{\partial n} = 0$ for $x \in \partial\Omega$ and $t \in (0,T)$, then we have

$$\iint_{Q_T} (u_1 - u_2)\varphi_t dxdt \leq \iint_{Q_T} \Big(-(A(u_1) - A(u_2))\Delta\varphi - (B(u_1) - B(u_2))\Phi(x,t) \cdot \nabla\varphi\Big) dxdt. \quad (11)$$

Let

$$a(x,t) = \int_0^1 A'(su_1 + (1-s)u_2)ds = \begin{cases} \dfrac{A(u_1) - A(u_2)}{u_1 - u_2}, & u_1(x,t) \neq u_2(x,t), \\ A'(u_1), & u_1(x,t) = u_2(x,t), \end{cases}$$

$$b(x,t) = \int_0^1 B'(su_1 + (1-s)u_2)ds \cdot \Phi(x,t) = \begin{cases} \dfrac{(B(u_1) - B(u_2))\Phi(x,t)}{u_1 - u_2}, & u_1(x,t) \neq u_2(x,t), \\ B'(u_1)\Phi(x,t), & u_1(x,t) = u_2(x,t), \end{cases}$$

and

$$c_\delta^\eta(x,t) = \begin{cases} (\eta + a(x,t))^{-\frac{1}{2}} b(x,t), & |u_1(x,t) - u_2(x,t)| \geq \delta, \\ 0, & |u_1(x,t) - u_2(x,t)| < \delta, \end{cases}$$

for any $\eta > 0$ and $\delta > 0$. Further, for any fixed $\gamma > 0$, we denote

$$F_\gamma = \{(x,t) \in Q_T; |u_1(x,t) - u_2(x,t)| \geq \gamma\},$$

and

$$G_\gamma = \{(x,t) \in Q_T; |u_1(x,t) - u_2(x,t)| < \gamma\}.$$

Now, (11) reads

$$\iint\limits_{Q_T} (u_1 - u_2)\Big(- \varphi_t - a(x,t)\Delta\varphi - b(x,t)\cdot\nabla\varphi\Big) dxdt \geq 0, \tag{12}$$

for all $0 \leq \varphi \in L^2((0,T); W^{1,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ with $\varphi(x,T) = 0$ for $x \in \Omega$ and $\frac{\partial\varphi}{\partial n} = 0$ for $x \in \partial\Omega$ and $t \in (0,T)$. Since $\Phi(x,t), u_1, u_2$ are bounded and $A(s)$, $B(s)$ are locally Lipchitz continuous, there exists a constant $C > 0$ such that $|a|$, $|b|$ and $|u_1|$, $|u_2| \leq C$. Henceforth, a generic positive constant (possibly changing from line to line) is denoted by C. According to the strictly increasing property of $A(s)$ and the boundedness of u_1, u_2, there exists a constant $L(\gamma) > 0$ such that

$$a(x,t) \geq L(\gamma), \qquad \text{for all } (x,t) \in F_\gamma,$$

and therefore

$$|c_\delta^\eta| \leq L(\delta)^{-\frac{1}{2}}|b| \leq L(\delta)^{-\frac{1}{2}}C =: K(\delta).$$

We employ the standard duality proof method or the approximate Hohmgren's approach to complete this proof (see Theorem 6.5 in [40], Chapter 1.3 and 3.2 in [50], see also the comparison principle Lemma 3.4 in [52] on unbounded domain and Lemma 4.1 in [51]). For any smooth function $0 \leq \psi(x,t) \in C_0^2(Q_T)$, consider the following approximated dual problem

$$\begin{cases} -\varphi_t - (\eta + a_\varepsilon(x,t))\Delta\varphi - c_{\delta,\varepsilon}^\eta(x,t)(\eta + a_\varepsilon(x,t))^{\frac{1}{2}} \cdot \nabla\varphi = \psi, & (x,t) \in Q_T, \\ \dfrac{\partial\varphi}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \varphi(x,T) = 0, \quad x \in \Omega, \end{cases} \tag{13}$$

where $\eta > 0$, $\delta > 0$, $\varepsilon > 0$, a_ε is a smooth approximation of a in $L^4(Q_T)$, $a_\varepsilon \geq a$, and $c_{\delta,\varepsilon}^\eta(x,t)$ is a smooth approximation of $c_\delta^\eta(x,t)$ in $L^4(Q_T)$. Here we note that (13) is a standard parabolic problem as the initial data is imposed at the end time $t = T$. Therefore, it has a smooth solution $\varphi \geq 0$. Maximum principle shows the boundedness of φ such that $0 \leq \varphi \leq C(\psi)$. Then we get from (12) and (13) the estimate

$$\begin{aligned} \iint\limits_{Q_T} (u_1 - u_2)\psi dxdt \geq & - \iint\limits_{Q_T} |u_1 - u_2||a - a_\varepsilon||\Delta\varphi| dxdt \\ & - \eta \iint\limits_{Q_T} |u_1 - u_2||\Delta\varphi| dxdt \\ & - \iint\limits_{Q_T} |u_1 - u_2||c_{\delta,\varepsilon}^\eta(\eta + a_\varepsilon)^{\frac{1}{2}} - b||\nabla\varphi| dxdt \\ =: & - I_1 - I_2 - I_3. \end{aligned} \tag{14}$$

Next, we need the a priori estimate on $(\eta + a_\varepsilon)|\Delta\varphi|^2$. We multiply the equation (13) by $-\Delta\varphi$. Integrating over Q_T yields

$$\begin{aligned}
&\iint_{Q_T} \varphi_t \Delta\varphi dxdt + \iint_{Q_T} (\eta + a_\varepsilon)(\Delta\varphi)^2 dxdt \\
&\le \iint_{Q_T} |c^\eta_{\delta,\varepsilon}|(\eta + a_\varepsilon)^{\frac{1}{2}} |\nabla\varphi||\Delta\varphi| dxdt + \iint_{Q_T} \psi \Delta\varphi dxdt \\
&\le \frac{1}{4} \iint_{Q_T} (\eta + a_\varepsilon)(\Delta\varphi)^2 dxdt + \iint_{Q_T} |c^\eta_{\delta,\varepsilon}|^2 |\nabla\varphi|^2 dxdt + \iint_{Q_T} |\Delta\psi||\varphi| dxdt \\
&\le \frac{1}{4} \iint_{Q_T} (\eta + a_\varepsilon)(\Delta\varphi)^2 dxdt + (K(\delta))^2 \iint_{Q_T} |\nabla\varphi|^2 dxdt + C(\psi).
\end{aligned}$$

Using $\varphi(x, T) = 0$ and $\frac{\partial\varphi}{\partial n} = 0$ on $\partial\Omega$, we have

$$\begin{aligned}
\iint_{Q_T} \varphi_t \Delta\varphi dxdt &= -\iint_{Q_T} \nabla\varphi \cdot \nabla\varphi_t dxdt = -\frac{1}{2} \iint_{Q_T} \frac{\partial}{\partial t} |\nabla\varphi|^2 dxdt \\
&= \frac{1}{2} \int_\Omega |\nabla\varphi(x, 0)|^2 dx \ge 0,
\end{aligned}$$

and

$$\begin{aligned}
\iint_{Q_T} |\nabla\varphi|^2 dxdt &= \iint_{Q_T} \nabla\varphi \cdot \nabla\varphi dxdt = -\iint_{Q_T} \varphi \Delta\varphi dxdt \\
&\le \frac{1}{4(K(\delta))^2} \iint_{Q_T} (\eta + a_\varepsilon)(\Delta\varphi)^2 dxdt + \eta^{-1} (K(\delta))^2 C(\psi).
\end{aligned}$$

Therefore,

$$(K(\delta))^2 \iint_{Q_T} |\nabla\varphi|^2 dxdt + \iint_{Q_T} (\eta + a_\varepsilon)(\Delta\varphi)^2 dxdt \le C(\psi)(K(\delta))^4 \eta^{-1}, \tag{15}$$

and

$$\|\Delta\varphi\|_{L^2(Q_T)} \le (C(\psi)(K(\delta))^4 \eta^{-2})^{\frac{1}{2}} \le C(\psi)(K(\delta))^2 \eta^{-1}.$$

It follows that

$$\begin{aligned}
I_1 &= \iint_{Q_T} |u_1 - u_2||a - a_\varepsilon||\Delta\varphi| dxdt \\
&\le C \|\Delta\varphi\|_{L^2(Q_T)} \|a - a_\varepsilon\|_{L^2(Q_T)} \le C(\psi)(K(\delta))^2 \eta^{-1} \|a - a_\varepsilon\|_{L^2(Q_T)},
\end{aligned}$$

which converges to zero if we let $\varepsilon \to 0$. We can estimate I_2 as follows

$$
\begin{aligned}
I_2 &= \eta \iint_{Q_T} |u_1 - u_2||\Delta\varphi| dxdt \\
&\le \eta \iint_{G_\gamma} |u_1 - u_2||\Delta\varphi| dxdt + \eta \iint_{F_\gamma} |u_1 - u_2||\Delta\varphi| dxdt \\
&\le \gamma \iint_{G_\gamma} \eta|\Delta\varphi| dxdt + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} \iint_{F_\gamma} a^{\frac{1}{2}} |\Delta\varphi| dxdt \\
&\le \gamma \iint_{G_\gamma} \eta|\Delta\varphi| dxdt + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} \iint_{F_\gamma} a_\varepsilon^{\frac{1}{2}} |\Delta\varphi| dxdt \\
&\le C\gamma\eta^{\frac{1}{2}} \Big(\iint_{Q_T} \eta|\Delta\varphi|^2 dxdt \Big)^{\frac{1}{2}} + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} \Big(\iint_{Q_T} a_\varepsilon |\Delta\varphi|^2 dxdt \Big)^{\frac{1}{2}} \\
&\le C\gamma\eta^{\frac{1}{2}} C(\psi)(K(\delta))^2 \eta^{-\frac{1}{2}} + \frac{C\eta}{L(\gamma)^{\frac{1}{2}}} C(\psi)(K(\delta))^2 \eta^{-\frac{1}{2}} \\
&= \gamma C(\psi)(K(\delta))^2 + \eta^{\frac{1}{2}} C(\psi)(K(\delta))^2 / L(\gamma)^{\frac{1}{2}}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
I_3 &= \iint_{Q_T} |u_1 - u_2||c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}} - b||\nabla\varphi| dxdt \\
&\le \iint_{G_\delta} |u_1 - u_2||c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}}||\nabla\varphi| dxdt + \iint_{G_\delta} |u_1 - u_2||b||\nabla\varphi| dxdt \\
&\quad + \iint_{F_\delta} |u_1 - u_2||c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}} - b||\nabla\varphi| dxdt \\
&\le \delta \|c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}}\|_{L^2(G_\delta)} \|\nabla\varphi\|_{L^2(Q_T)} + C\delta \iint_{G_\delta} |\nabla\varphi| dxdt \\
&\quad + C\|c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}} - b\|_{L^2(F_\delta)} \|\nabla\varphi\|_{L^2(Q_T)}.
\end{aligned}
$$

We note that

$$
c^\eta_{\delta,\varepsilon}(\eta + a_\varepsilon)^{\frac{1}{2}} \to c^\eta_\delta(\eta + a)^{\frac{1}{2}} = \begin{cases} 0, & (x,t) \in G_\delta, \\ b(x,t), & (x,t) \in F_\delta, \end{cases}
$$

almost everywhere and also in $L^2(Q_T)$. It follows that

$$
\limsup_{\varepsilon\to 0} I_3 \le C\delta \iint_{G_\delta} |\nabla\varphi| dxdt.
$$

We leave the uniform L^1 estimate of $\|\nabla\varphi\|_{L^1(Q_T)} \le C(\psi)$ to the next lemma (Lemma 3.2), and we combine the above estimates to find

$$\limsup_{\varepsilon\to 0}(I_1+I_2+I_3) \le \gamma C(\psi)(K(\delta))^2 + \eta^{\frac{1}{2}} C(\psi)(K(\delta))^2/L(\gamma)^{\frac{1}{2}} + C(\psi)\delta.$$

Now we conclude according to (14) that

$$\iint_{Q_T} (u_1-u_2)\psi\, dxdt \ge -\Big\{\gamma C(\psi)(K(\delta))^2 + \eta^{\frac{1}{2}} C(\psi)(K(\delta))^2/L(\gamma)^{\frac{1}{2}} + C(\psi)\delta\Big\},$$

for any given $\delta>0$, $\eta>0$, $\gamma>0$ and $\psi\ge 0$, which yields that

$$\iint_{Q_T} (u_1-u_2)\psi\, dxdt \ge 0,$$

by taking $\eta\to 0$, then $\gamma\to 0$, and at last $\delta\to 0$. Since $0\le\psi\in C_0^2(Q_T)$ is arbitrary selected, we see that $u_1\ge u_2$ almost everywhere on Q_T. □

Lemma 3.2. *Let φ be the solution of the approximated dual problem (13) in the proof of Lemma 3.1. Then there holds*

$$\sup_{t\in(0,T)} \int_\Omega |\nabla\varphi(x,t)|dx \le \iint_{Q_T} |\nabla\psi| dxdt.$$

Proof. Since φ is smooth enough, $\varphi(x,T)=0$ on Ω and $\frac{\partial\varphi}{\partial n}=0$ on $\partial\Omega$, we take the gradient of (13) and then multiply it by $|\nabla\varphi|^{\beta-1}\nabla\varphi$ with $\beta\in(0,1)$, integrate over $Q_{t,T}=\Omega\times(t,T)$, to find

$$\begin{aligned}
&\frac{1}{\beta+1}\int_\Omega |\nabla\varphi(x,t)|^{\beta+1}dx + \beta\iint_{Q_{t,T}} (\eta+a_\varepsilon)|\Delta\varphi|^2|\nabla\varphi|^{\beta-1}dxdt\\
&= -\beta\iint_{Q_{t,T}} c^\eta_{\delta,\varepsilon}(\eta+a_\varepsilon)^{\frac{1}{2}}\cdot\nabla\varphi|\nabla\varphi|^{\beta-1}\Delta\varphi dxdt + \iint_{Q_{t,T}} \nabla\psi\cdot|\nabla\varphi|^{\beta-1}\nabla\varphi dxdt\\
&\le \beta\iint_{Q_{t,T}} (\eta+a_\varepsilon)|\Delta\varphi|^2|\nabla\varphi|^{\beta-1}dxdt + \beta\iint_{Q_{t,T}} |c^\eta_{\delta,\varepsilon}|^2|\nabla\varphi|^{\beta+1}dxdt\\
&\quad + \iint_{Q_{t,T}} |\nabla\psi||\nabla\varphi|^\beta dxdt. \qquad (16)
\end{aligned}$$

According to (15), we see that

$$\iint_{Q_T} |c^\eta_{\delta,\varepsilon}|^2|\nabla\varphi|^{\beta+1}dxdt \le \iint_{Q_T} (K(\delta))^2(1+|\nabla\varphi|^2)dxdt \le C(\psi)(K(\delta))^4\eta^{-1},$$

and

$$\limsup_{\beta\to 0} \iint_{Q_T} |\nabla\psi||\nabla\varphi|^{\beta} dxdt \le \iint_{Q_T} |\nabla\psi| dxdt,$$

by the dominated convergence theorem. Now we let β tend to zero, then (16) implies that

$$\int_{\Omega} |\nabla\varphi(x,t)| dx \le \iint_{Q_T} |\nabla\psi| dxdt,$$

for all $t \in (0,T)$. The proof is completed. □

The comparison principle together with specially constructed weak lower and upper solutions are used to show the propagating properties. Hence we define the following weak lower and upper solutions of the first equation in (4).

Definition 3.1 *(Weak lower and upper solutions).* A function $g(x,t)$ is said to be a weak lower (or upper) solution of the first equation in (4) on Q_T corresponding to the initial value u_0 and a given function v such that $\nabla v \in L^\infty(Q_T)$, if $0 \le g \in L^\infty(Q_T)$, $\nabla g^m \in L^2(Q_T)$, and it satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial g}{\partial t} \le (\ge) \Delta g^m - \nabla\cdot(g\nabla v), & x\in\Omega, t\in(0,T), \\ \dfrac{\partial g^m}{\partial n} - g\nabla v\cdot n \le (\ge) 0, & x\in\partial\Omega, t\in(0,T), \\ g(x,0)\ge 0, \quad g(x,0)\le(\ge) u_0(x), & x\in\Omega, \end{cases}$$

where the first two inequality are satisfied in the following sense

$$\begin{aligned} &\iint_{Q_T} g\varphi_t dxdt + \int_{\Omega} g(x,0)\varphi(x,0)dx \\ &\ge (\le) \iint_{Q_T} \nabla g^m\cdot\nabla\varphi dxdt - \iint_{Q_T} g\nabla v\cdot\nabla\varphi dxdt, \end{aligned}$$

holds for all test functions $0\le\varphi\in L^2((0,T);W^{1,2}(\Omega))\cap W^{1,2}((0,T);L^2(\Omega))$ with $\varphi(x,T)=0$ on Ω.

Lemma 3.3 *(Comparison principle). Let (u,v) be a globally bounded weak solution of (4). If $g(x,t)$ is a weak lower (or upper) solution of the first equation in (4) on Q_T, then*

$$u(x,t) \ge (\le) g(x,t), \quad \forall (x,t)\in Q_T.$$

Proof. This is a simple corollary of comparison principle Lemma 3.1. □

3.2. Regularity of Hölder continuity

In order to show the propagation properties of the degenerate chemotaxis system (4), we need to know the existence, global boundedness, regularity and large time behavior of its solutions.

We recall the existence and the global boundedness of solutions to the degenerate chemotaxis model (4).

Lemma 3.4 *([15]). Assume that $u_0 \in L^\infty(\Omega)$, $\nabla u_0^m \in L^2(\Omega)$, $v_0 \in W^{2,\infty}(\Omega)$, $u_0, v_0 \geq 0$ and $m > 1$, the spacial dimension $N = 3$. Then the problem (4) admits a nonnegative global bounded weak solution (u, v) with*

$$\sup_{t\in(0,\infty)} (\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)}) \leq C,$$

$$\sup_{t\in(0,\infty)} \int_\Omega |\nabla u^m|^2 dx + \sup_{t\in(0,\infty)} \|u^{\frac{m+1}{2}}\|_{W_2^{1,1}(\Omega\times(t,t+1))} \leq C,$$

$$\sup_{t\in(0,\infty)} \|v\|_{W_p^{2,1}(\Omega\times(t,t+1))} \leq C(p), \quad \forall p > 1.$$

Furthermore,

$$\lim_{t\to\infty} \|v\|_{L^\infty(\Omega)} = 0, \quad \lim_{t\to\infty} \|u - \bar{u}\|_{L^p(\Omega)} = 0, \ \forall p > 1,$$

where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u_0 dx > 0$.

Remark 3.1. The same global boundedness and asymptotic behavior results hold for the lower spatial dimensional case $N = 1, 2$.

Remark 3.2. We note that the boundedness of $\|u\|_{L^\infty(Q_T)}$ and $\|v\|_{L^\infty(Q_T)}$ is insufficient for the boundedness of $\|\Delta v\|_{L^\infty(Q_T)}$ according to the strong theory of the second equation in (4). Hence the $W_p^{2,1}$ estimate for $p = \infty$ is not obtained in the above Lemma 3.4.

Remark 3.3. One of the basic features for the degenerate diffusion equations, such as the porous medium equation, is the property of finite speed of propagation. Therefore, the first component u may not have positive minimum for some time $t > 0$. For the large time behavior, it is proved in Lemma 3.4 that $u(x,t)$ converges to $\bar{u}$ in $L^p(\Omega)$ for $p < \infty$, while the $L^\infty(\Omega)$ and some other more regular convergence are not deduced.

In a special case that $v_0 \equiv 0$, we see that $v(x,t) \equiv 0$ and u satisfies the porous medium equation. The Barenblatt solution (9) of the porous medium equation shows that the best regularity of the first equation in (4) is no better than Hölder continuous $C^{\frac{1}{m-1}}(\overline{Q}_T)$ (for $m > 2$) even for the one spatial dimensional case $N = 1$. In what follows, we will show the Hölder continuous of u with respect to space, and the boundedness of $\|\Delta v\|_{L^\infty(Q_T)}$. Actually, we will prove that $\Delta v \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ for some $\alpha \in (0, 1)$.

Lemma 3.5. *Let $N = 1$ and (u, v) be the globally bounded weak solution of* (4). *Then there exists a constant $C > 0$ such that*

$$\sup_{t\in(0,\infty)} \left\{ \|u^m(\cdot,t)\|_{C^{1/2}(\overline{\Omega})} + \|u(\cdot,t)\|_{C^{1/(2m)}(\overline{\Omega})} \right\} \le C.$$

Proof. According to Lemma 3.4, $\|\nabla u^m(\cdot,t)\|_{L^2(\Omega)}$ is uniformly bounded. The Sobolev embedding theorem for one dimensional case implies the uniform boundedness of $\|u^m(\cdot,t)\|_{C^{1/2}(\overline{\Omega})}$.

We assert that for $m > 1$,

$$|a-b|^m \le C(M)|a^m - b^m|, \quad \forall a, b \in [0, M].$$

This is a simple result of calculus. Actually, we can choose $C(M) = 1$. Therefore,

$$\Big(\frac{|u(x_1,t)-u(x_2,t)|}{|x_1-x_2|^{1/(2m)}}\Big)^m \le C(\sup_{t\in(0,\infty)} \|u\|_{L^\infty(\Omega)}) \frac{|u^m(x_1,t)-u^m(x_2,t)|}{|x_1-x_2|^{1/2}}, \quad x_1 \ne x_2.$$

That is, the uniform $C^{1/2}$ regularity of $u^m(\cdot,t)$ implies the uniform $C^{1/(2m)}$ regularity of $u(\cdot,t)$. □

The following continuity of $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ and boundedness of $\|\Delta v(\cdot,t)\|_{L^\infty(\Omega)}$ will be used to formulate varies types of upper and lower solutions in the next section.

Lemma 3.6. *Let (u, v) be the globally bounded weak solution of* (4) *such that*

$$\sup_{t\in(0,\infty)} \|u(\cdot,t)\|_{C^{1/(2m)}(\overline{\Omega})} \le C,$$

and $v_0 \in C^{2,\alpha_0}(\overline{\Omega})$ for some $\alpha_0 \in (0, 1)$, $\frac{\partial v_0}{\partial n} = 0$ on $\partial\Omega$. Then $\nabla v(\cdot,t)$ is continuous in the $\|\cdot\|_{L^\infty(\Omega)}$ norm with respect to time and there exist $\alpha \in (0, 1)$ and a constant $C(T,\delta) > 0$ such that

$$\|\Delta v(x,t)\|_{C^\alpha(\overline{\Omega}_\delta\times[0,T])} \le C(T,\delta),$$

where $\Omega_\delta = \{x \in \Omega; dist(x, \partial\Omega) > \delta\}$.

Proof. Since $\|v\|_{W_p^{2,1}(\Omega\times(t,t+1))}$ is uniformly bounded for $p > 1$ in Lemma 3.4, we see that $\sup_{t\in(0,\infty)} \|v(\cdot,t)\|_{C^\beta(\overline{\Omega})} \le C$ for some $\beta \in (0, 1)$. Therefore,

$$\sup_{t\in(0,\infty)} \|(uv)(\cdot,t)\|_{C^\alpha(\overline{\Omega})} \le C$$

for some $\alpha \in (0, 1)$. Indeed, we can choose $\alpha = \min\{1/(2m), \beta\}$. The Schauder theory via Campanato space theory in [21] implies the interior Hölder continuity of Δv with respect to space and time, and the Hölder continuity of v_t with respect to space (the Hölder continuity of v_t with respect to time is insufficient). □

For large time behavior, we present the following regularity.

Lemma 3.7. *Let* (u, v) *be the globally bounded weak solution of* (4). *Then*

$$\lim_{t\to\infty} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} = 0.$$

Proof. Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary condition. Then the solution v can be expressed as follows

$$v(x,t) = e^{t\Delta}v_0(x) - \int_0^t e^{(t-s)\Delta}(uv)(x,s)ds, \quad t \geq t_0 \geq 0.$$

According to the $L^p - L^q$ estimates for the Neumann heat semigroup (see for example [46]),

$$\begin{aligned}
\|\nabla v(x,t)\|_{L^\infty(\Omega)} &\leq \|\nabla e^{t\Delta}v_0(x)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}(uv)(x,s)\|_{L^\infty(\Omega)}ds \\
&\leq C\big(1+t^{-\frac{1}{2}}\big)e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + \int_0^t C\big(1+(t-s)^{-\frac{1}{2}}\big)e^{-\lambda_1(t-s)}\|(uv)(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\leq C\big(1+t^{-\frac{1}{2}}\big)e^{-\lambda_1 t}\|v_0\|_{L^\infty(\Omega)} + C\int_0^{t-1} e^{-\lambda_1(t-s)}ds \\
&\quad + C\int_{t-1}^t \big(1+(t-s)^{-\frac{1}{2}}\big)ds \sup_{\tau\in(t-1,t)} \|v(\cdot,\tau)\|_{L^\infty(\Omega)},
\end{aligned}$$

which tends to zero since $\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ are uniformly bounded and $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ tends to zeros as $t \to \infty$ from Lemma 3.4. □

Lemma 3.8. *Let the conditions in Lemma 3.6 be valid. Then*

$$\lim_{t\to\infty} \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)} = 0.$$

Proof. We rewrite $v = v_1 + w$ such that

$$\begin{cases} v_{1t} = \Delta v_1, & x\in\Omega,\ t>0, \\ v_1(x,0) = v_0(x), & x\in\Omega, \\ \frac{\partial v_1}{\partial n} = 0, & x\in\partial\Omega,\ t>0, \end{cases}$$

and

$$\begin{cases} w_t = \Delta w - uv, \quad x \in \Omega, \ t > 0, \\ w(x,0) = 0, \quad x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega, \ t > 0. \end{cases}$$

The Neumann heat semigroup theory shows $\lim_{t\to\infty} \|\Delta v_1(\cdot,t)\|_{L^\infty(\Omega)} = 0$. We note that

$$\|(uv)(\cdot,t)\|_{C^\alpha(\overline{\Omega})} \le \|u(\cdot,t)\|_{L^\infty(\Omega)} \|v(\cdot,t)\|_{C^\alpha(\overline{\Omega})} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \|u(\cdot,t)\|_{C^\alpha(\overline{\Omega})} \to 0,$$

as t tends to infinity since $\lim_{t\to\infty} \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} = 0$ according to Lemma 3.7 and $\|u(\cdot,t)\|_{C^\alpha(\overline{\Omega})}$ are uniformly bounded in Lemma 3.6 for some $\alpha \in (0,1)$. The Schauder theory in [21] shows the Hölder continuity

$$\|\Delta w(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \sup_{s\in[t/2,t]} \|(uv)(\cdot,s)\|_{C^\alpha(\overline{\Omega})} + C_2(t) \sup_{s\in(0,\infty)} \|(uv)(\cdot,s)\|_{C^\alpha(\overline{\Omega})},$$

where $C_1 > 0$ is a constant and $C_2(t)$ decays to zeros as t tends to infinity. □

4. Propagation properties: shrinking versus expanding

This section is devoted to the study of the propagating properties of bacteria cells and the large time behavior of the weak solution (u, v) to the problem (4). In contrast with the heat equation, it is known that the porous medium equation has the property of finite speed of propagation. Therefore, the first component u may not have positive minimum for some time $t > 0$. We use the comparison principle together with weak lower solutions.

Our interest lies in the propagating properties of the cell invasions. Let us first focus on the waiting time and initial shrinking of the compact support. Our approach is the combination of the comparison principle Lemma 3.3 and weak lower and upper solutions with compact support.

4.1. Initial shrinking caused by the chemotaxis

The Barenblatt solution (9) of the classical porous medium equation indicates the slow diffusion with finite speed of expanding support; while the chemotaxis may cause backward diffusion, i.e. the aggregation, which in competition with the slow diffusion results in a initial shrinking of the support provided specified structures of the signal concentration.

We consider a typical situation in which the cells are concentrated in a compact support and the signal concentration has the aggregation effect. Specifically speaking, assume that

$$\begin{cases} \operatorname{supp} u_0 \subset \overline{B}_{R_0}(x_0) \subset \Omega, \quad u_0 \le K_0(R_0^2 - |x-x_0|^2)^{d_0}, \ x \in B_{R_0}(x_0), \\ \nabla v_0 \cdot (x - x_0) \le -\mu |x-x_0|^2, \ x \in B_{R_0}(x_0), \end{cases} \tag{17}$$

for some $x_0 \in \Omega$ and positive constants $d_0 \ge 1/(m-1)$, and $R_0, K_0, \mu > 0$.

We construct self similar upper and lower solution with compact support to show the propagating property. We note that for the degenerate porous medium type equation and the self similar function of the form $g = [(1-|x|^2)_+]^d$ with $md > 1$, we can check that ∇g^m is continuous and $\Delta g^m \in L^q(\Omega)$ for some $q > 1$. This shows that the differential inequality for an upper (or lower) solution only need to be valid almost everywhere, without the possible Radon measures on the boundary of its support, which is completely different from the uniform parabolic cases.

Lemma 4.1. *Let the conditions in Lemma 3.6 be valid with the initial values satisfying (17) and* $\chi\mu > \frac{4m}{m-1}K_0^{m-1}\max\{1, R_0^{2((m-1)d_0-1)}\}$*. Define a function*

$$g(x,t) = \varepsilon(\tau+t)^\sigma\Big[\Big(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^\beta}\Big)_+\Big]^d, \quad x\in\Omega,\ t\geq 0,$$

where $d = 1/(m-1)$*,* $\beta,\sigma\in\mathbb{R}$*,* $\varepsilon>0$*,* $\eta>0$*,* $\tau>0$*. Then by appropriately selecting* $\beta<0$*,* $\sigma>0$*,* ε*,* η *and* τ*, the support of* $g(x,t)$ *is contained in* Ω *and shrinks for* $t\in(0,t_0)$ *with some* $t_0>0$ *and the function* $g(x,t)$ *is an upper solution of the first equation in (4) on* $\Omega\times(0,t_0)$ *corresponding to* $v(x,t)$ *and the initial date* u_0*. Therefore,* $u(x,t)\leq g(x,t)$ *and there exist a family of shrinking open sets* $\{A(t)\}_{t\in(0,t_0)}$ *such that*

$$supp\, u(\cdot,t)\subset\overline{A}(t)\subset\Omega, \quad t\in(0,t_0),$$

and $\partial A(t)$ *has a finite derivative with respect to* t*.*

Proof. For simplicity, we let

$$h(x,t) = \Big(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^\beta}\Big)_+, \quad x\in\Omega,\ t\geq 0,$$

and

$$A(t) = \Big\{x\in\Omega;\ \frac{|x-x_0|^2}{(\tau+t)^\beta} < \eta^2\Big\}, \quad t\geq 0.$$

Without loss of generality, we may assume that $x_0=0$ and write $B_R = B_R(0)$. Straightforward computation shows that

$$\begin{aligned}
g_t &= \sigma\varepsilon(\tau+t)^{\sigma-1}h^d + \varepsilon(\tau+t)^\sigma dh^{d-1}\frac{\beta|x|^2}{(\tau+t)^{\beta+1}},\\
\nabla g &= -\varepsilon(\tau+t)^\sigma dh^{d-1}\frac{2x}{(\tau+t)^\beta},\\
\nabla g^m &= -\varepsilon^m(\tau+t)^{m\sigma}mdh^{md-1}\frac{2x}{(\tau+t)^\beta},\\
\Delta g^m &= \varepsilon^m(\tau+t)^{m\sigma}md(md-1)h^{md-2}\frac{4|x|^2}{(\tau+t)^{2\beta}} - \varepsilon^m(\tau+t)^{m\sigma}mdh^{md-1}\frac{2N}{(\tau+t)^\beta},
\end{aligned}$$

for all $x\in A(t)$ and $t>0$. According to the initial condition (17) and the regularity result Lemma 3.6, we see that at the initial time

$$\begin{aligned}
\nabla g(x,0)\cdot\nabla v(x,0) &= \nabla g(x,0)\cdot\nabla v_0(x)\\
&= -\varepsilon\tau^\sigma dh^{d-1}\frac{2x}{\tau^\beta}\cdot\nabla v_0(x) \geq \varepsilon\tau^{\sigma-\beta}dh^{d-1}2\mu|x|^2,
\end{aligned}$$

and there exists a $\hat{t}>0$ by the continuity such that

$$\nabla v(x,t)\cdot x \le -\frac{\mu}{2}|x|^2, \quad x \in B_{R_0}\backslash B_{R_0/2},\ t\in[0,t_0],$$
$$\nabla v(x,t)\cdot x \le \frac{\mu}{2}R_0^2, \quad x \in B_{R_0/2},\ t\in[0,\hat{t}].$$

Therefore,

$$\nabla g(x,t)\cdot\nabla v(x,t) = -\varepsilon(\tau+t)^\sigma d h^{d-1}\frac{2x}{(\tau+t)^\beta}\cdot\nabla v(x,t)$$
$$\ge \varepsilon(\tau+t)^{\sigma-\beta} d h^{d-1}\mu|x|^2, \quad x\in B_{R_0}\backslash B_{R_0/2},\ t\in[0,\hat{t}]. \tag{18}$$

Let $\tau > 0$ be determined and

$$\eta^2 = \frac{R_0^2}{\tau^\beta}, \quad t_0 = \min\{\tau,\hat{t}\}. \tag{19}$$

According to the definition of $g(x,t)$, we see that $A(0) = B_{R_0}(0)$, $\operatorname{supp} u_0 \subset \overline{A}(0) \subset \Omega$, and $A(t) \subset B_{R_0}(0) \subset \Omega$ for $t \in [0,t_0]$. Therefore, $\frac{\partial g}{\partial n} = 0$ and $\frac{\partial g^m}{\partial n} = 0$ on $\partial\Omega$ for all $t\in(0,t_0)$, and

$$g(x,0) = \varepsilon\tau^\sigma\left[\left(\eta^2 - \frac{|x|^2}{\tau^\beta}\right)_+\right]^d = \varepsilon\tau^\sigma\left(\frac{R_0^2}{\tau^\beta} - \frac{|x|^2}{\tau^\beta}\right)^d\cdot 1_{B_{R_0}(0)}$$
$$= \varepsilon\tau^{\sigma-d\beta}(R_0^2-|x|^2)^d\cdot 1_{B_{R_0}(0)} \ge K_0(R_0^2-|x|^2)^{d_0}\cdot 1_{B_{R_0}(0)} \ge u_0(x), \quad x\in\Omega,$$

provided that

$$\varepsilon\tau^{\sigma-d\beta} \ge K_0\max\{1, R_0^{2(d_0-d)}\}. \tag{20}$$

In order to find a weak upper solution g, we only need to check the following differential inequality on $A(t)$

$$\frac{\partial g}{\partial t} \ge \Delta g^m - \chi\nabla\cdot(g\nabla v) = \Delta g^m - \chi\nabla g\cdot\nabla v - \chi g\Delta v, \quad x\in A(t),\ t\in(0,t_0). \tag{21}$$

We denote $C_1 = \|\nabla v\|_{L^\infty(\Omega\times[0,1])}$ and $C_2 = \|\Delta v\|_{L^\infty(\Omega\times[0,1])}$ for convenience, since they are bounded according to Lemma 3.6. A sufficient condition of inequality (21) is

$$\sigma\varepsilon(\tau+t)^{\sigma-1}h^d + \varepsilon(\tau+t)^\sigma d h^{d-1}\frac{\beta|x|^2}{(\tau+t)^{\beta+1}}$$
$$+\varepsilon^m(\tau+t)^{m\sigma} m d h^{md-1}\frac{2N}{(\tau+t)^\beta} + \chi\nabla g\cdot\nabla v$$
$$\ge \varepsilon^m(\tau+t)^{m\sigma} m d(md-1)h^{md-2}\frac{4|x|^2}{(\tau+t)^{2\beta}} + C_2\chi\varepsilon(\tau+t)^\sigma h^d, \tag{22}$$

for all $x\in A(t)$, $t\in(0,t_0)$. As we have chosen $d = 1/(m-1)$, we rewrite (22) into

$$\sigma\varepsilon(\tau+t)^{\sigma-1}h+\frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}}$$
$$+2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}}+h^{1-d}\chi\nabla g\cdot\nabla v$$
$$\geq\frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}}+C_2\chi\varepsilon(\tau+t)^{\sigma}h, \tag{23}$$

for all $x\in A(t)$, $t\in(0,t_0)$. For simplicity, we denote (23) by $LHS\geq RHS$.

Now, we give sufficient conditions of (23) to be valid on $B_{R_0/2}$ and $B_{R_0}\backslash B_{R_0/2}$ respectively (Note that $A(t)\subset B_{R_0}(0)$ for $t\in(0,t_0)$ as $\beta<0$). For $x\in(B_{R_0}\backslash B_{R_0/2})\cap A(t)$ and $t\in(t,t_0)$, we have according to the estimate (18) that

$$\begin{aligned}
LHS-RHS\geq{}&\sigma\varepsilon(\tau+t)^{\sigma-1}h+\frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}}\\
&+2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}}+\chi\varepsilon(\tau+t)^{\sigma-\beta}d\mu|x|^2\\
&-\frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}}-C_2\chi\varepsilon(\tau+t)^{\sigma}h\\
\geq{}&\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta+1}-C_2\chi(\tau+t)\Big)\varepsilon(\tau+t)^{\sigma-1}h\\
&+\Big(d\chi\mu+\frac{\beta}{m-1}(\tau+t)^{-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2\\
\geq{}&\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1}-2C_2\chi\tau\Big)\varepsilon(\tau+t)^{\sigma-1}h\\
&+\Big(d\chi\mu+\frac{\beta}{m-1}\tau^{-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2.
\end{aligned}\tag{24}$$

For $x\in(B_{R_0/2})\cap A(t)$ and $t\in(t,t_0)$, we also have

$$\begin{aligned}
LHS-RHS\geq{}&\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1}-2C_2\chi\tau\Big)\varepsilon(\tau+t)^{\sigma-1}h\\
&+\Big(\frac{\beta}{m-1}\tau^{-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2\\
&-d\chi\mu\varepsilon(\tau+t)^{\sigma-\beta}R_0^2.
\end{aligned}\tag{25}$$

Let $\beta\in[-2\ln(4/3)/\ln 2,0)$, i.e. $2^{\beta/2}\in[3/4,1)$. For $t\in(0,t_0)$, we see that

$$A(t)=B_{\eta(\tau+t)^{\beta/2}}(0)=B_{R_0(1+t/\tau)^{\beta/2}}(0)\supset B_{2^{\beta/2}R_0}(0)\supset B_{3R_0/4}(0).$$

Further if $x\in(B_{R_0/2})\cap A(t)=B_{R_0/2}$ and $t\in(0,t_0)\subset(0,\tau)$,

$$h(x,t)=\Big(\eta^2-\frac{|x|^2}{(\tau+t)^{\beta}}\Big)_+=\frac{R_0^2}{\tau^{\beta}}-\frac{|x|^2}{(\tau+t)^{\beta}}\geq\frac{R_0^2}{\tau^{\beta}}-\frac{(R_0/2)^2}{(2\tau)^{\beta}}\geq\frac{5}{9}\frac{R_0^2}{\tau^{\beta}}.$$

Then (25) reads

$$
\begin{aligned}
LHS - RHS \geq & \left(\sigma + 2N\frac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1} - 2C_2\chi\tau\right)\varepsilon(\tau+t)^{\sigma-1}\frac{5}{9}\frac{R_0^2}{\tau^\beta} \\
& + \left(\frac{\beta}{m-1}\tau^{-1} - \frac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta} - 4d\chi\mu\right)\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0^2}{4} \\
& \geq \left[\left(\sigma + 2N\frac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1} - 2C_2\chi\tau\right)\frac{5}{9\tau^\beta}\right. \\
& \left. + \left(\frac{\beta}{m-1}\tau^{-1} - \frac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta} - 4d\chi\mu\right)\frac{(\tau+t)^{1-\beta}}{4}\right]\varepsilon(\tau+t)^{\sigma-1}R_0^2.
\end{aligned} \tag{26}
$$

Let $\varepsilon > 0$, $\beta \in [-2\ln(4/3)/\ln 2, 0)$, $\sigma > 0$, $\tau > 0$, $\eta > 0$ and $t_0 > 0$ be chosen such that (19), (20) are valid and

$$
\begin{cases}
\sigma + 2N\dfrac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1} - 2C_2\chi\tau \geq 0, \\
d\chi\mu + \dfrac{\beta}{m-1}\tau^{-1} - \dfrac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta} \geq 0, \\
\left(\sigma + 2N\dfrac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1} - 2C_2\chi\tau\right)\dfrac{5}{9\tau^\beta} \\
\qquad + \left(\dfrac{\beta}{m-1}\tau^{-1} - \dfrac{4m}{(m-1)^2}\varepsilon^{m-1}(2\tau)^{(m-1)\sigma-\beta} - 4d\chi\mu\right)\dfrac{(2\tau)^{1-\beta}}{4} \geq 0.
\end{cases} \tag{27}
$$

We can fix $\tau = 1$, η and t_0 to be determined by (19), $\varepsilon = K_0 \max\{1, R_0^{2(d_0-d)}\}$ as (20) is valid, and $\beta < 0$ with $|\beta|$ being sufficiently small such that the second inequality in (27) is true since $\chi\mu > \frac{4m}{m-1}K_0^{m-1}\max\{1, R_0^{2((m-1)d_0-1)}\}$, and at last we choose $\sigma > 0$ to be sufficiently large such that the first and the third inequalities are satisfied. Now, (27) is valid for those parameters. Then according to the inequalities (24), (25), (26), we find that

$$
LHS - RHS \geq 0, \qquad x \in B_{R_0} \cap A(t) = A(t),\ t \in (0, t_0),
$$

which yields (21), (23), and then $g(x,t)$ is an upper solution.

The comparison principle Lemma 3.3 implies that $u(x,t) \leq g(x,t)$ for all $x \in \Omega$ and $t \in (0, t_0)$. Thus,

$$
\operatorname{supp} u(\cdot, t) \subset \overline{A}(t) = \{x \in \Omega;\ |x - x_0|^2 < \eta^2(\tau+t)^\beta\}, \quad t \in (0, t_0),
$$

and

$$
\partial A(t) = \{x \in \Omega;\ |x - x_0| = \eta(\tau+t)^{\frac{\beta}{2}}\}, \quad t \in (0, t_0),
$$

which has finite derivative with respect to t. The family of sets $\{A(t)\}_{t\in(0,t_0)}$ is shrinking with respect to t since $\beta < 0$. □

Remark 4.1. We compare the self similar weak upper solution $g(x,t)$ in the proof of Lemma 4.1 to the Barenblatt solution of porous medium equation

$$B(x,t) = (1+t)^{-k}\Big[\Big(1 - \frac{k(m-1)}{2mN}\frac{|x|^2}{(1+t)^{2k/N}}\Big)_+\Big]^{\frac{1}{m-1}},$$

with $k = 1/(m-1+2/N)$. The Barenblatt solution $B(x,t)$ is decaying at the rate $(1+t)^{-1/(m-1+2/N)}$ in $L^\infty(\mathbb{R}^N)$ and the support is expanding at the rate $(1+t)^{k/N}$. Here, the upper solution is increasing at the rate $(\tau+t)^\sigma$ and its support is shrinking at the rate $(\tau+t)^{\beta/2}$. The increasing of $g(x,t)$ makes it possible to be an upper solution, which is crucial in the proof.

4.2. Finite speed propagating and the exact propagating speed

We have proved that the compact support may shrink if the signal concentration satisfies a special structure such as (17). Now, we will show the finite speed propagating property without assuming the special structure on signal concentration. Assume that

$$\operatorname{supp} u_0 \subset \overline{B}_{R_0}(x_0) \subset \Omega, \quad u_0 \le K_0(R_0^2 - |x-x_0|^2)^{d_0}, \quad x \in B_{R_0}(x_0), \tag{28}$$

for some $x_0 \in \Omega$ and positive constants $d_0 \ge 1/(m-1)$ and $R_0, K_0 > 0$.

Lemma 4.2. *Let the conditions in Lemma 3.6 be valid with the initial values satisfying* (28)*. Define a function*

$$g(x,t) = \varepsilon(\tau+t)^\sigma\Big[\Big(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^\beta}\Big)_+\Big]^d, \quad x \in \Omega,\ t \ge 0,$$

where $d = 1/(m-1)$, $\beta, \sigma \in \mathbb{R}$, $\varepsilon > 0$, $\eta > 0$, $\tau > 0$. Then by appropriately selecting $\beta > 0$, $\sigma > 0$ ε, η and τ, the support of $g(x,t)$ is contained in Ω for $t \in (0,t_0)$ with some $t_0 > 0$ and the function $g(x,t)$ is an upper solution of the first equation in (4) *on $\Omega \times (0,t_0)$ corresponding to $v(x,t)$ and the initial data u_0. Therefore, $u(x,t) \le g(x,t)$ and there exist a family of open sets $\{A(t)\}_{t\in(0,t_0)}$ such that*

$$\operatorname{supp} u(\cdot,t) \subset \overline{A}(t) \subset \Omega, \quad t \in (0,t_0),$$

and $\partial A(t)$ has a finite derivative with respect to t.

Proof. This proof is similar to the proof of Lemma 4.1, except there is no structure condition (18) and we need minor modifications. We still define $h(x,t)$ and $A(t)$ as in the proof of Lemma 4.1 and we assume $x_0 = 0$ for simplicity. Let

$$\eta^2 = \frac{R_0^2}{\tau^\beta}, \quad \varepsilon\tau^{\sigma-d\beta} \ge K_0 \max\{1, R_0^{2(d_0-d)}\}, \tag{29}$$

and C_1, C_2 be defined as in the proof of Lemma 4.1. We need to check the differential inequality (21) (i.e. (22)). A sufficient condition of (22) is

$$\sigma\varepsilon(\tau+t)^{\sigma-1}h+\frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}}$$
$$+2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}}-C_1\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{2|x|}{m-1}$$
$$\geq\frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}}+C_2\chi\varepsilon(\tau+t)^{\sigma}h, \tag{30}$$

for all $x\in A(t)$, $t\in(0,t_0)$. For simplicity, we denote (30) by $LHS\geq RHS$.

According to (28), $\overline{B}_{R_0}(0)\subset\Omega$, there exists a $R>R_0$ such that $\overline{B}_{R_0}(0)\subset B_R(0)\subset\subset\Omega$. Let $\hat{t}>0$ depending on β and τ such that

$$\left(1+\frac{\hat{t}}{\tau}\right)^{\beta}\leq\frac{R^2}{R_0^2}. \tag{31}$$

Let $t_0=\min\{\tau,\hat{t}\}$. We see that for $t\in(0,t_0)$,

$$\operatorname{supp} g(x,t)=\overline{A}(t)=\overline{B}_{\eta(\tau+t)^{\beta/2}}\cap\Omega=\overline{B}_{R_0(1+t/\tau)^{\beta/2}}\cap\Omega\subset\overline{B}_R\cap\Omega=\overline{B}_R\subset\Omega.$$

Then $\frac{\partial g}{\partial n}=0$ and $\frac{\partial g^m}{\partial n}=0$ on $\partial\Omega$ for all $t\in(0,t_0)$. For $x\in A(t)\backslash B_{R_0/2}$ and $t\in(0,t_0)$, we have

$$LHS-RHS\geq\sigma\varepsilon(\tau+t)^{\sigma-1}h+\frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}}$$
$$+2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}}-C_1\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{4|x|^2}{(m-1)R_0}$$
$$-\frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}}-C_2\chi\varepsilon(\tau+t)^{\sigma}h$$
$$\geq\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta+1}-C_2\chi(\tau+t)\Big)\varepsilon(\tau+t)^{\sigma-1}h$$
$$+\Big(\frac{\beta}{m-1}(\tau+t)^{-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta}-\frac{4C_1\chi}{(m-1)R_0}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2$$
$$\geq\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta+1}\min\{1,2^{(m-1)\sigma-\beta+1}\}-2C_2\chi\tau\Big)\varepsilon(\tau+t)^{\sigma-1}h$$
$$+\Big(\frac{\beta}{m-1}(2\tau)^{-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta}\max\{1,2^{(m-1)\sigma-\beta}\}$$
$$-\frac{4C_1\chi}{(m-1)R_0}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2. \tag{32}$$

We note that $B_{R_0/2}\subset B_{R_0}\subset A(t)$ for $t\in(0,t_0)$ since $\beta>0$. For $x\in(B_{R_0/2})\cap A(t)$ and $t\in(t,t_0)$, we find that

$$h(x,t)=\left(\eta^2-\frac{|x|^2}{(\tau+t)^{\beta}}\right)_+=\frac{R_0^2}{\tau^{\beta}}-\frac{|x|^2}{(\tau+t)^{\beta}}\geq\frac{R_0^2}{\tau^{\beta}}-\frac{(R_0/2)^2}{\tau^{\beta}}\geq\frac{3}{4}\frac{R_0^2}{\tau^{\beta}},$$

then we also have

$$
\begin{aligned}
LHS - RHS \geq\, & \sigma\varepsilon(\tau+t)^{\sigma-1}h + \frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}} \\
& + 2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}} - C_1\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1} \\
& - \frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}} - C_2\chi\varepsilon(\tau+t)^{\sigma}h \\
\geq & \Big(\sigma + 2N\frac{m}{m-1}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta+1} - C_2\chi(\tau+t)\Big)\varepsilon(\tau+t)^{\sigma-1}h \\
& + \Big(\frac{\beta}{m-1}(\tau+t)^{-1} - \frac{4m}{(m-1)^2}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2 \\
& - C_1\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1} \\
\geq & \Big(\sigma - 2C_2\chi\tau\Big)\varepsilon(\tau+t)^{\sigma-1}\frac{3}{4}\frac{R_0^2}{\tau^{\beta}} - C_1\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1} \\
& + \Big(\frac{\beta}{m-1}(2\tau)^{-1} - \frac{4m}{(m-1)^2}\varepsilon^{m-1}\tau^{(m-1)\sigma-\beta}\max\{1, 2^{(m-1)\sigma-\beta}\}\Big)\varepsilon(\tau+t)^{\sigma-\beta}|x|^2, \qquad (33)
\end{aligned}
$$

provided that $\sigma \geq 2C_2\chi\tau$.

Let $\tau = 1$, $\eta = R_0$, $\varepsilon = K_0\max\{1, R_0^{2(d_0-d)}\}$, and $\beta = (m-1)\sigma$ with $\sigma > 0$ being sufficiently large such that

$$
\begin{cases}
\dfrac{\beta}{2(m-1)} - \dfrac{4m}{(m-1)^2}\varepsilon^{m-1} - \dfrac{4C_1\chi}{(m-1)R_0} \geq 0, \\
(\sigma - 2C_2\chi)\dfrac{3R_0}{4\tau^{\beta}}\min\{1, 2^{\beta-1}\} - \dfrac{C_1\chi}{m-1} \geq 0.
\end{cases}
$$

Then (33) tells us $LHS \geq RHS$ for all $x \in A(t)$ and $t \in (0, t_0)$. It follows that $g(x,t)$ is an upper solution. The comparison principle Lemma 3.3 completes the proof. □

Lemma 4.2 implies the finite speed propagating property of the degenerate diffusion equation. We will present the exact propagating speed for a special structure initial data.

Lemma 4.3 *(Exact propagating speed). Let the conditions in Lemma 3.6 be valid with the initial values satisfying*

$$
\begin{cases}
u_0 = K_0\big[(R_0^2 - |x-x_0|^2)_+\big]^d, \ x \in \Omega, \\
\nabla v_0 \cdot (x - x_0) = -\mu|x-x_0|^2, \ x \in B_{R_0}^{\delta}(x_0),
\end{cases} \qquad (34)
$$

for some $x_0 \in \Omega$ and positive constants $d = 1/(m-1)$, $R_0, K_0, \mu, \delta > 0$ such that $\overline{B}_{R_0}(x_0) \subset \Omega$ and $B_{R_0}^{\delta}(x_0) := \{x \in B_{R_0}(x_0); dist(x, \partial B_{R_0}(x_0)) < \delta\}$. Then

$$
supp\, u(x,t) = \{(\theta, \rho(\theta,t)); \theta \in S^{N-1}\},
$$

where (θ, ρ) is the spherical coordinate centered at x_0, $\rho(\theta, 0) = R_0$ for all $\theta \in S^{N-1}$, and the propagating speed

$$\frac{\partial \rho(\theta, t)}{\partial t}\Big|_{t=0} = R_0\Big(\frac{2m}{m-1}K_0^{m-1} - \chi\mu\Big), \quad \forall \theta \in S^{N-1}.$$

Proof. Define

$$g_\pm(x,t) = \varepsilon(\tau+t)^{\sigma_\pm}\Big[\Big(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta_\pm}}\Big)_+\Big]^d, \quad x \in \Omega,\ t \geq 0,$$

with $\varepsilon = K_0$, $\tau = 1$, $\eta = R_0$, $\sigma_\pm \in \mathbb{R}$, $\beta_\pm \in \mathbb{R}$ are to be determined. We have

$$g_\pm(x,0) = K_0\big[(R_0^2 - |x-x_0|^2)_+\big]^d = u_0, \quad x \in \Omega,$$

and $\frac{\partial g_\pm}{\partial n} = 0$, $\frac{\partial g_\pm^m}{\partial n} = 0$ on $\partial\Omega$ at least for a small time interval since $\overline{B}_{R_0} \subset \Omega$. Here we only aim to find the exact propagating speed and we only need to construct upper and lower solutions on a small time interval. We note that

$$\begin{aligned}\nabla g_\pm(x,0)\cdot\nabla v_0 &= -\varepsilon(\tau+t)^{\sigma_\pm-\beta} dh^{d-1} 2(x-x_0)\cdot\nabla v_0 \\ &= 2\mu\varepsilon(\tau+t)^{\sigma_\pm-\beta} dh^{d-1}|x-x_0|^2,\end{aligned}$$

for $x \in B_{R_0}^\delta(x_0)$. Let

$$\beta = \frac{4m}{m-1}K_0^{m-1} - 2\chi\mu,$$

and $\beta_\pm$ approach β from above and below. Take $\sigma_+ > 0$ sufficiently large and $\sigma_- < 0$ with $|\sigma_-|$ being sufficiently large, we can check as in the proof of Lemma 4.1 and next Lemma 4.4 that $g_\pm(x,t)$ are upper and lower solutions for a small time interval $(0, T_\pm)$, where $T_\pm > 0$ depend on $|\beta_\pm - \beta|$. Here we omit the details. Then the comparison principle Lemma 3.3 implies that there exists $\{A_{\beta_\pm}(t)\}_{t\in(0,T_\pm)}$ such that

$$A_{\beta_\pm}(t) = B_{R_0(1+t)^{\beta_\pm/2}}(x_0), \quad t \in (0, T_\pm),$$

and

$$A_{\beta_-}(t) \subset \operatorname{supp} u(x,t) \subset \overline{A}_{\beta_+}(t), \quad t \in (0, T_\pm).$$

Therefore,

$$\frac{\partial \rho(\theta,t)}{\partial t}\Big|_{t=0} \in [R_0\beta_-/2, R_0\beta_+/2].$$

Since $\beta_\pm$ approach β, we have $\frac{\partial\rho(\theta,0)}{\partial t} = R_0\beta/2$. □

4.3. Eventual smoothness and expanding

The large time behavior in Lemma 3.4 and Lemma 3.7 shows that $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ tends to zero as time grow. This indicates that the chemotaxis effect decays and the support will expand to the whole domain. Now we construct a self similar weak lower solution with expanding support.

Lemma 4.4. *Let the conditions in Lemma 3.6 be valid with the initial data $u_0 \geq 0$, $u_0 \not\equiv 0$ and Ω be convex. Define a function*

$$g(x,t) = \varepsilon(\tau+t)^{\sigma}\left[\left(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta}}\right)_+\right]^d, \quad x \in \Omega,\ t > -\tau,$$

where $d = 1/(m-1)$, $\beta > 0$, $\sigma < 0$, ε, $\eta > 0$, $\tau \in \mathbb{R}$ and $x_0 \in \Omega$. Then by appropriately selecting β, ε, τ, σ, η and x_0, the function $g(x,t)$ is a weak lower solution of the first equation in (4) on $\Omega \times (\hat{t}, \hat{T})$ corresponding to $v(x,t)$ and u_0 for some $\hat{T} > \hat{t} > 0$. Therefore, $u(x,t) \geq g(x,t)$ and there exist $t_0 \in (\hat{t}, \hat{T})$, $\varepsilon_0 > 0$, and a family of expanding open sets $\{A(t)\}_{t\in(\hat{t},\hat{T})}$, such that

$$A(t) \subset \operatorname{supp} u(x,t), \quad t \in (\hat{t}, \hat{T}),$$

and $A(t) = \Omega$, $u(x,t) \geq \varepsilon_0$ for all $x \in \Omega$ and $t \in [t_0, \hat{T}]$.

Proof. Since $u_0 \geq 0$, $u_0 \not\equiv 0$ and $u_0 \in C(\overline{\Omega})$, the first equation in (4) shows that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x) > 0, \quad t > 0.$$

For any $t > 0$, there exists a $x_0(t) \in \Omega$ such that $u(x_0(t),t) \geq \overline{u} := \frac{1}{|\Omega|}\int_\Omega u_0(x) > 0$. According to the uniform Hölder continuity of $u(\cdot,t)$, we find that there exists a $R_0 > 0$ independent of t such that

$$u(x,t) \geq \frac{\overline{u}}{2} =: \varepsilon_1, \quad \forall x \in B_{R_0}(x_0(t)). \tag{35}$$

We denote $C_1(t) = \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ and $C_2(t) = \|\Delta v(\cdot,t)\|_{L^\infty(\Omega)}$ for convenience. According to Lemma 3.7 and Lemma 3.8, $C_1(t)$ and $C_2(t)$ tend to zero. For fixed $\delta > 0$ to be determined, let $\hat{t} > 0$ depend on δ such that

$$C_1(t) \leq \delta, \quad C_2(t) \leq \delta, \qquad \forall t \geq \hat{t}. \tag{36}$$

Note that $u(x,\hat{t}) \geq \varepsilon_1$ on $B_{R_0}(x_0(\hat{t}))$. Without loss of generality, we may assume that $B_{R_0} = B_{R_0}(x_0(\hat{t})) \subset \Omega$ and $x_0 = x_0(\hat{t}) = 0$.

Similar to the proof of Lemma 4.1, we let

$$h(x,t) = \left(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta}}\right)_+, \quad x \in \Omega,\ t \geq 0,$$

and

$$A(t) = \left\{x \in \Omega; \frac{|x - x_0|^2}{(\tau + t)^\beta} < \eta^2\right\}, \quad t \geq 0.$$

According to the definition of g, we see that $\frac{\partial g}{\partial n} \leq 0$ and $\frac{\partial g^m}{\partial n} \leq 0$ on $\partial\Omega$ since Ω is convex, and for $\tau = 1 - \hat{t}$ we have

$$g(x, \hat{t}) = \varepsilon[(\eta^2 - |x|^2)_+]^d \leq \varepsilon_1 1_{B_{R_0}(x_0)} \leq u_0(x), \quad x \in \Omega,$$

provided that

$$\eta \leq R_0, \quad \varepsilon\eta^{2d} \leq \varepsilon_1. \tag{37}$$

In order to find a weak lower solution g, we only need to check the following differential inequality on $A(t)$

$$\frac{\partial g}{\partial t} \leq \Delta g^m - \chi\nabla \cdot (g\nabla v) = \Delta g^m - \chi\nabla g \cdot \nabla v - \chi g\Delta v, \quad x \in A(t),\ t \in (\hat{t}, \hat{T}), \tag{38}$$

for some $\hat{T} > \hat{t}$ to be determined.

A sufficient condition of inequality (38) is

$$\begin{aligned}
&\sigma\varepsilon(\tau + t)^{\sigma-1}h + \frac{\varepsilon\beta}{m-1}(\tau + t)^\sigma \frac{|x|^2}{(\tau + t)^{\beta+1}} \\
&+ 2N\frac{m}{m-1}\varepsilon^m(\tau + t)^{m\sigma}\frac{h}{(\tau + t)^\beta} + C_1(t)\chi\varepsilon(\tau + t)^{\sigma-\beta}\frac{2|x|}{m-1} \\
&\leq \frac{m}{(m-1)^2}\varepsilon^m(\tau + t)^{m\sigma}\frac{4|x|^2}{(\tau + t)^{2\beta}} - C_2(t)\chi\varepsilon(\tau + t)^\sigma h,
\end{aligned} \tag{39}$$

for all $x \in A(t)$, $t \in (\hat{t}, \hat{T})$. For simplicity, we denote (39) by $LHS \leq RHS$. The estimates on the above inequality is quite similar to (30) in the proof of Lemma 4.2 except some terms are with inverse signs. Here, (32) and (33) are changed into

$$\begin{aligned}
LHS - RHS \leq\ &\sigma\varepsilon(\tau + t)^{\sigma-1}h + \frac{\varepsilon\beta}{m-1}(\tau + t)^\sigma\frac{|x|^2}{(\tau + t)^{\beta+1}} \\
&+ 2N\frac{m}{m-1}\varepsilon^m(\tau + t)^{m\sigma}\frac{h}{(\tau + t)^\beta} + C_1(t)\chi\varepsilon(\tau + t)^{\sigma-\beta}\frac{4|x|^2}{(m-1)R_0} \\
&- \frac{m}{(m-1)^2}\varepsilon^m(\tau + t)^{m\sigma}\frac{4|x|^2}{(\tau + t)^{2\beta}} + C_2(t)\chi\varepsilon(\tau + t)^\sigma h \\
\leq &\Big(\sigma + 2N\frac{m}{m-1}\varepsilon^{m-1}(\tau + t)^{(m-1)\sigma-\beta+1} + C_2(t)\chi(\tau + t)\Big)\varepsilon(\tau + t)^{\sigma-1}h \\
&+ \Big(\frac{\beta}{m-1} - \frac{4m}{(m-1)^2}\varepsilon^{m-1}(\tau + t)^{(m-1)\sigma-\beta+1} + \frac{4C_1(t)\chi(\tau + t)}{(m-1)R_0}\Big)\varepsilon(\tau + t)^{\sigma-\beta-1}|x|^2 \\
\leq &\Big(\sigma + 2N\frac{m}{m-1}\varepsilon^{m-1}\max\{1, (\tau + \hat{T})^{(m-1)\sigma-\beta+1}\} + C_2(t)\chi(\tau + \hat{T})\Big)\varepsilon(\tau + t)^{\sigma-1}h
\end{aligned}$$

$$+\Big(\frac{\beta}{m-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}\min\{1,(\tau+\hat{T})^{(m-1)\sigma-\beta+1}\}$$
$$+\frac{4C_1(t)\chi(\tau+\hat{T})}{(m-1)R_0}\Big)\varepsilon(\tau+t)^{\sigma-\beta-1}|x|^2,\quad x\in A(t)\backslash B_{R_0/2},\ t\in(\hat{t},\hat{T}),\tag{40}$$

and (note that $\sigma<0$)

$$\begin{aligned}
LHS-RHS\le{}&\sigma\varepsilon(\tau+t)^{\sigma-1}h+\frac{\varepsilon\beta}{m-1}(\tau+t)^{\sigma}\frac{|x|^2}{(\tau+t)^{\beta+1}}\\
&+2N\frac{m}{m-1}\varepsilon^m(\tau+t)^{m\sigma}\frac{h}{(\tau+t)^{\beta}}+C_1(t)\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1}\\
&-\frac{m}{(m-1)^2}\varepsilon^m(\tau+t)^{m\sigma}\frac{4|x|^2}{(\tau+t)^{2\beta}}+C_2(t)\chi\varepsilon(\tau+t)^{\sigma}h\\
\le{}&\Big(\sigma+2N\frac{m}{m-1}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta+1}+C_2(t)\chi(\tau+t)\Big)\varepsilon(\tau+t)^{\sigma-1}h\\
&+\Big(\frac{\beta}{m-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}(\tau+t)^{(m-1)\sigma-\beta+1}\Big)\varepsilon(\tau+t)^{\sigma-\beta-1}|x|^2\\
&\quad+C_1(t)\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1}\\
\le{}&\Big(2N\frac{m}{m-1}\varepsilon^{m-1}\max\{1,(\tau+\hat{T})^{(m-1)\sigma-\beta+1}\}+C_2(t)\chi(\tau+\hat{T})\Big)\varepsilon(\tau+t)^{\sigma-1}\eta^2\\
&+\Big(\frac{\beta}{m-1}-\frac{4m}{(m-1)^2}\varepsilon^{m-1}\min\{1,(\tau+\hat{T})^{(m-1)\sigma-\beta+1}\}\Big)\varepsilon(\tau+t)^{\sigma-\beta-1}|x|^2\\
&+\sigma\varepsilon(\tau+t)^{\sigma-1}\frac{3}{4}\eta^2+C_1(t)\chi\varepsilon(\tau+t)^{\sigma-\beta}\frac{R_0}{m-1},\quad x\in(B_{R_0/2})\cap A(t),\ t\in(\hat{t},\hat{T}).
\end{aligned}\tag{41}$$

Since Ω is bounded, there exists $R>R_0$ such that $\Omega\subset B_R(x_0)$. Let $\eta=R_0$, $\varepsilon>0$, $\beta\in(0,1)$, $\tau=1-\hat{t}$, $\hat{T}>\hat{t}$ and $\sigma=-\frac{1-\beta}{m-1}<0$ be chosen such that

$$\begin{cases}\varepsilon\eta^{2d}\le\varepsilon_1,\quad 2N\frac{m}{m-1}\varepsilon^{m-1}\le-\sigma/4,\quad \beta\le\frac{2m}{m-1}\varepsilon^{m-1},\\ \delta\chi(\hat{T}-\hat{t}+1)\le-\sigma/4,\quad 4\delta\chi(\hat{T}-\hat{t}+1)\le\frac{2m}{m-1}\varepsilon^{m-1}R_0,\\ \delta\chi(\hat{T}-\hat{t}+1)^{1-\beta}\frac{R_0}{m-1}\le-\sigma\eta^2/4,\quad(\hat{T}-\hat{t}+1)^{\beta/2}\ge 2R/R_0.\end{cases}\tag{42}$$

The above seven inequalities can be satisfied simultaneously in the following way. We first fix $\beta\in(0,1)$ sufficiently small such that $N\beta\le(1-\beta)/(4(m-1))$. Then we set $\varepsilon=\varepsilon(\beta)>0$ such that $\frac{2m}{m-1}\varepsilon^{m-1}=\beta$. Now we can modify β to be smaller such that $\varepsilon\eta^{2d}\le\varepsilon_1$. The first three inequalities are valid. Let $L=e^{\frac{2}{\beta}\ln\frac{2R}{R_0}}-1$ and

$$\delta=\min\{-\sigma/(4\chi(L+1)),\ \frac{2m}{m-1}\varepsilon^{m-1}R_0/(4\chi(L+1)),\ -\sigma\eta^2(m-1)/(4\chi(L+1)^{1-\beta}R_0)\}.$$

For this $\delta>0$, let $\hat{t}$ be chosen such that (36) is fulfilled and $\hat{T}=\hat{t}+L$.

For those parameters, we see that (42) is valid and (40), (41) tells us $LHS \le RHS$ for all $x \in A(t)$ and $t \in (\hat{t}, \hat{T})$, i.e. (39). It follows that $g(x,t)$ is a lower solution. The comparison principle Lemma 3.3 shows that

$$u(x,t) \ge g(x,t) = \varepsilon(\tau+t)^{\sigma}\left[\left(\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta}}\right)_+\right]^d,$$

for all $x \in \Omega$ and $t \in (\hat{t}, \hat{T})$. We note that for this lower solution, its support satisfies

$$A(\hat{t}) = B_{\eta(\tau+\hat{t})^{\beta/2}}(x_0) \cap \Omega = B_{R_0}(x_0),$$

and

$$A(\hat{T}) = B_{\eta(\tau+\hat{T})^{\beta/2}}(x_0) \cap \Omega = B_{R_0(\hat{T}-\hat{t}+1)^{\beta/2}}(x_0) \cap \Omega \supset B_{2R}(x_0) \cap \Omega = \Omega,$$

since $(\hat{T}-\hat{t}+1)^{\beta/2} \ge 2R/R_0$ in (42) and $\Omega \subset B_R(x_0)$. There exists a $t_1 \in (\hat{t}, \hat{T})$ such that

$$\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta}} \ge 0, \quad \forall x \in \Omega,\ t \in (t_1, \hat{T}),$$

which means $A(t) = \Omega$ for $t \in (t_1, \hat{T})$. And there exists a $t_0 \in (t_1, \hat{T})$ such that

$$\eta^2 - \frac{|x-x_0|^2}{(\tau+t)^{\beta}} \ge \frac{\eta^2}{2}, \quad \forall x \in \Omega,\ t \in (t_0, \hat{T}),$$

and thus

$$u(x,t) \ge g(x,t) \ge \varepsilon(\hat{T}-\hat{t}+1)^{\sigma}\left(\frac{\eta^2}{2}\right)^d =: \varepsilon_0, \quad \forall x \in \Omega,\ t \in (t_0, \hat{T}).$$

The proof is completed. □

Remark 4.2. It is interesting to compare the self similar weak lower solution $g(x,t)$ in the proof of Lemma 4.4 to the Barenblatt solution of porous medium equation

$$B(x,t) = (1+t)^{-k}\left[\left(1 - \frac{k(m-1)}{2mN}\frac{|x|^2}{(1+t)^{2k/N}}\right)_+\right]^{\frac{1}{m-1}},$$

with $k = 1/(m-1+2/N)$. The Barenblatt solution $B(x,t)$ is decaying at the rate $(1+t)^{-1/(m-1+2/N)}$ in $L^{\infty}(\mathbb{R}^N)$ and the support is expanding at the rate $(1+t)^{k/N}$. While the self similar weak lower solution $g(x,t)$ is decaying at the rate $(1+t)^{-(1-\beta)/(m-1)}$ and its support is expanding at the rate $(1+t)^{\beta/2}$. Here in the proof we have selected $\beta > 0$ sufficiently small, which means the support of g is expanding with a much slower rate and the maximum of g is decaying at a slightly faster rate.

Now that we have proved the lower bound of $u(x,t)$ on $\Omega \times (t_0, \hat{T})$, we will show the globally lower bound at large time, as well as the non-degeneracy, regularity for large time behavior.

Lemma 4.5 *(Eventual smoothness). Let the conditions in Lemma 4.4 be valid. Then* $u(x,t) \geq \varepsilon_0$ *for all* $x \in \Omega$ *and* $t \geq t_0$ *with* $t_0 > 0$ *and* $\varepsilon_0 > 0$ *being defined as in the proof of Lemma 4.4,* $u \in C^{2,1}(\overline{\Omega} \times [t_0, \infty))$ *and there exist* $C > 0$ *and* $c > 0$ *such that*

$$\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq Ce^{-ct}, \quad t > 0,$$

where $\overline{u} = \int_\Omega u_0 dx/|\Omega|$.

Proof. We point out that

$$\varepsilon_0 = \varepsilon(\hat{T} - \hat{t} + 1)^\sigma \left(\frac{\eta^2}{2}\right)^d = \varepsilon(L+1)^\sigma \left(\frac{\eta^2}{2}\right)^d$$

is independent of δ and $\hat{t}$ therein, since L only depends on β, R_0 and R (note that $\beta, \sigma, \varepsilon$ depend only on ε_1 and $\varepsilon_1 = \overline{u}/2$ is fixed). Therefore, we can take $\hat{t}$ larger to be $\hat{t} + \theta$ with any $\theta > 0$ such that (36) is also valid. Lemma 4.4 shows that $u(x,t) \geq \varepsilon_0$ for all $x \in \Omega$ and $t \in [t_0 + \theta, \hat{T} + \theta]$. Since $\varepsilon_0 > 0$ is fixed and $\theta > 0$ is arbitrary, we have $u(x,t) \geq \varepsilon_0$ for all $x \in \Omega$ and $t \geq t_0$. It follows that the first equation in (4) is non-degenerate and uniform parabolic. The Hölder regularity and exponential decay can be verified similar to the proof of Theorem 1.3 in [51]. □

Acknowledgment

The research of T. Xu is supported by the Innovation Project of Graduate School of South China Normal University Grant No. 2018LKXM005. The research of S. Ji is supported by NSFC Grant No. 11701184. The research of M. Mei is supported in part by NSERC Grant RGPIN 354724-2016, and FRQNT Grant No. 2019-CO-256440. The research of J. Yin is supported in part by NSFC Grant No. 11771156 and NSF of Guangzhou Grant No. 201804010391.

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