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Propagation Speed of Degenerate Diffusion Equations with Time Delay

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Abstract
We are concerned with a class of degenerate diffusion equations with time delay describing population dynamics with age structure. In our recent study [*Nonlinearity*, 33 (2020), 4013–4029], we established the existence and uniqueness of critical traveling wave for the time-delayed degenerate diffusion equations, and obtained the reducing mechanism of time delay on critical wave speed. In this paper, we now are able to show the asymptotic spreading speed and its coincidence with the critical wave speed $c^*(m, r)$ of sharp wave, and prove that the initial perturbation or the boundary of the compact support of the solution propagates at the critical wave speed $c^*(m, r)$ for the time-delayed degenerate diffusion equations. Remarkably, different from the existing studies related to spreading speeds, the time delay and the degenerate diffusion lead to some essential difficulties in the analysis of the spreading speed, because the time delay makes the critical speed of traveling waves slow down in a more complicated fashion such that the critical speed cannot be determined by the characteristic equation, and the degenerate diffusion causes the loss of regularity for the solutions. By a phase transform technique we construct upper and lower solutions with semi-compact supports and then we determine the asymptotic spreading speed. Furthermore, we propose a brand-new sharp-profile-based difference scheme to handle large variation of degenerate diffusion $(u^m)_{xx}$ near the sharp edge and carry out some numerical simulations which perfectly confirm our theoretical results.

Keywords Degenerate diffusion · Time delay · Spreading speed · Sharp waves

Mathematics Subject Classification 35K65 · 35K57 · 35C07 · 35B40 · 92D25

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1 Introduction

We consider the following degenerate diffusion equation with time delay

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - d(u) + b(u(t-r,x)), & x \in \mathbb{R}^n,\ t > 0, \\ u(s,x) = u_0(s,x), & x \in \mathbb{R}^n,\ t \in [-r,0], \end{cases} \tag{1.1}$$

where the spatial dimension $n \geq 1$, $u(t,x)$ denotes the total mature population of the species at location x and time $t > 0$, $r \geq 0$ is the maturation time, $b(u(t-r,x))$ is the birth function, $d(u)$ is the death rate function. The equation (1.1) describes the population dynamics of single species with age-structure and density-dependent diffusion ($m > 1$).

The positive density dependence on emigration has been predicted by many theoretical models and confirmed empirically in various species with age structure [12, 25, 27]. Time delays arise from the passage through sequential demographic phases in the species' life cycle [29]. Hence, time delay and nonlinear dispersal are both inextricably mechanism in population dynamics. For example, butterflies, the most popular age-structured species for dispersal studies, have been reported increased emigration at high population densities [27]. Positive density-dependent dispersal is particularly beneficial for females, because it allows laying the egg-load in less crowded patches to help their offspring avoiding severe intra-specific competition in the larval period.

In the linear diffusion case without time delay, (i.e., $r = 0$ and $m = 1$), the equation (1.1) is reduced to the Fisher and Kolmogorov-Petrovsky-Piskunov (KPP) equation [13, 17]. It is well known that there exists a critical (minimal admissible) wave speed $c^* = 2\sqrt{b'(0) - d'(0)} > 0$ (under certain conditions on the functions $b(\cdot)$ and $d(\cdot)$) for all the traveling waves connecting the two constant equilibria 0 and $\kappa > 0$, and the level set $\Gamma_\varepsilon(t) := \{x \in \mathbb{R}^n; u(t,x) = \varepsilon\}$ with $\varepsilon \in (0,\kappa)$ asymptotically propagates at the same speed c^* [4, 33]. It was Thieme and Zhao [30] who first established the theory of asymptotic spreading speed for a large class of nonlinear integral equations, which covers many time-delayed reaction and diffusion equations with linear diffusion (i.e., $r > 0$ and $m = 1$). Liang and Zhao [21] further developed the theory of spreading speeds to both discrete and continuous time monotone semiflows and investigated the application to a time-delayed evolution equation. In a series of works (e.g. [19, 38, 39]), Zhao and his collaborators investigated the traveling waves and spreading speeds of population dynamics model with nonlocal dispersal. Studies of the coincidence of the spreading speed with the critical wave speed for various evolution systems with linear diffusion systems can also be found in [9, 10, 18, 20–22, 24, 30, 40].

When the degenerate diffusion is included, the system can be used to describe biological population dynamics with density-dependent dispersal; see for instance [15, 25]. An interesting peculiarity of degenerate diffusion is the appearance of sharp type waves at the asymptotic speed [8, 31, 32]. For the case without time delay ($r = 0$), traveling wave solutions have been found by several authors [1, 2, 11, 14, 28]. Medvedev et al. [23] proved that the slowest traveling wave in the family yields the asymptotic speed of the propagation of disturbances in a class of degenerate Fisher-KPP equations. In recent works [5–7], more general cases of doubly nonlinear diffusion are considered, which includes both porous medium and p-Laplacian models.

An increasing attention has been paid to degenerate diffusion equations with time delay in order to study the effects of degenerate diffusion and time delay on the evolutionary behavior of biological systems with age structure (see [16, 34, 35, 37]). The existence of smooth monotone fronts for equations (1.1) with small time delay was proved by Huang et al. [16] based on a perturbation approach. In our recent works [34, 35], we proved that the time-delayed degenerate diffusion equation (1.1) admits a unique sharp type (semi-compactly supported) traveling wave $\phi(x + c^*t)$ for the one dimensional case, which corresponds to the minimal admissible (critical) wave speed $c^* = c^*(m, r)$. Moreover, the time delay slows down the minimal wave speed, i.e., $c^*(m, r) < c^*(m, 0)$ for $r > 0$. However, the asymptotic speeds of spreading for solutions with compactly supported initial data and the coincidence with the critical wave speed of sharp traveling wave still remain open. In this paper, we shall answer those questions on the spreading properties.

Time delay and degenerate diffusion lead to essential difficulties in the analysis of the spreading speed of (1.1). In the absence of time delay, the maximum principle and phase plane analysis proposed by Aronson and Weinberger [3, 4] yield conclusions about the asymptotic propagation speed of the linear diffusion and similar equations. This method was extended to cover degenerate diffusion equations of general Fisher-KPP sources without time delay by Medvedev et al. in [23], where all the trajectories in the phase plane are determined and correspond to special upper and lower solutions. However, time delay changes the situation dramatically. It was shown in [35] that the time delay reduces the critical wave speed $c^*(m, r)$ and the speed is not characterized by the classical phase plane analysis method. In order to construct upper and lower solutions with compact (or semi-compact) supports and with propagating speed approaching $c^*(m, r)$, we employ a new phase transform technique developed in [35] and utilize the monotone dependence in the phase space, see the phase comparison principle Lemma 2.3 in this paper. Especially, we need to treat the lower solutions technically in two ways: first, we show that the support of $u(t, \cdot)$ expands to including any given compact subset for large time; and secondly, the value of $u(t, x)$ within given compact subset grows up as time increases.

The purpose of this paper is to study the propagation speed for the equation (1.1). The main issue of the paper is to show that the initial perturbation or the boundary of the compact support of the solution propagates at the critical wave speed $c^*(m, r)$ for the time-delayed degenerate diffusion equation (1.1). Different from the existing studies related to spreading speeds, the time delay and the degenerate diffusion lead to some essential difficulties in the analysis of the spreading speed, because the time delay makes the critical speed of traveling waves slow down in a more complicated fashion, and the degenerate diffusion causes the loss of regularity for the solutions. This main result will be proved in Sect. 2. Sect. 3 is devoted to numerical computations. Since the variation of the degenerate diffusion $(u^m)_{xx}$ may be large near the sharp edge, the traditional numerical schemes are failed in the case with moving sharp-edge. Here, we propose a brand new numerical algorithm, called the sharp-profile-based difference scheme, and use this scheme to carry out some numerical simulations in different cases, which are accurate and stable in the sense of numerical performance, and also perfectly illustrate our theoretical results.

In the end of this section, we are going to state our main theorems on the propagation speed for the time-delayed degenerate-diffusion equations (1.1). A function $u(t, x)$ is said to be compactly supported for $t \in [t_1, t_2]$, if $\operatorname{supp} u(t, \cdot)$ is compact for $t \in [t_1, t_2]$. For the sake of convenience, we define the half space divided by the hyperplane through a point x_0 that has normal vector ν as

$$\Pi(x_0, \nu) := \{x \in \mathbb{R}^n; (x - x_0) \cdot \nu \geq 0\}. \tag{1.2}$$

A function $u(t, x)$ is said to be semi-compactly supported for $t \in [t_1, t_2]$, if

$$\operatorname{supp} u(t, \cdot) \subset \Pi(x(t), \nu(t)) = \{x \in \mathbb{R}^n; (x - x(t)) \cdot \nu(t) \geq 0\}$$

for $t \in [t_1, t_2]$ with some points $x(t)$ and the unit vectors $\nu(t)$. For any variable s, the positive value of s, denoted by s_+, is defined as $s_+ = \max\{s, 0\}$.

Throughout this paper, we assume that the functions $d(s)$ and $b(s)$ satisfy the following conditions:

$$\begin{aligned}&\text{There exist } u_1{=}0, u_2 > 0 \text{ such that } d, b \in C^2([0, +\infty)), d(0){=}b(0) = 0, d(u_2) = b(u_2),\\ &(b(s) - d(s)) \cdot (s - u_2) < 0, \forall s \in (0, +\infty)\backslash\{u_2\}, b'(0) > d'(0) \geq 0, d'(s) \geq 0, b'(s) \geq 0.\end{aligned} \tag{1.3}$$

Here, $u_1 = 0$ and $\kappa := u_2 > 0$ are two constant equilibria of (1.1), and the functions $b(u)$, $d(u)$ are both non-decreasing.

We first recall the properties of the sharp type traveling wave obtained in our previous study [35].

Theorem 1.1 ([35]) *For any $m > 1$ and $r \geq 0$, the time-delayed degenerate diffusion equation* (1.1) *admits a unique (up to shift) sharp type (semi-compactly supported) traveling wave $u^*(t, x) = \phi^*(x \cdot \nu + c^*t)$ with a unique speed $c^* = c^*(m, r) > 0$ and unit vector ν such that:*

(i) $\operatorname{supp} u^*(t, \cdot) = \Pi(-c^*t\nu, \nu)$, *$\phi^*(\xi)$ is monotone increasing and $\phi^*(+\infty) = \kappa$;*
(ii) *$c^*(m, r)$ is the minimal admissible (or critical) traveling wave speed;*
(iii) *$c^*(m, r) < c^*(m, 0)$, i.e., the time delay slows down the critical traveling wave speed.*

If the initial data $u_0(s, x)$ are compactly supported (or semi-compactly supported) in a direction $\nu \in \mathbb{S}^{n-1}$, say $\operatorname{supp} u_0(s, \cdot) \subset \Pi(x_0, \nu)$ for example, it is expected that the solution $u(t, x)$ is compactly supported (or semi-compactly supported) in this direction for all the time since the diffusion of the equation (1.1) is degenerate. The solution $u(t, x)$ together with its support $\operatorname{supp} u(t, \cdot)$ expands toward opposite ν direction. In what follows, we always assume that $c^*(m, r) > 0$ is the critical (minimal admissible) wave speed of traveling waves and also is the wave speed of the unique sharp type (semi-compactly supported) traveling wave of (1.1) shown by Theorem 1.1. Now we present the following large time propagation speed of the solution with compact (or semi-compact) support.

Theorem 1.2 *Let $u(t, x)$ be the solution of* (1.1) *with bounded and continuous initial data $u_0(s, x)$ for $s \in [-r, 0]$ satisfying*

$$\operatorname{supp} u_0(s, \cdot) \subset \Pi(x_0, \nu), \qquad u_0(s, x) \geq \phi_0((x - x_0) \cdot \nu), \tag{1.4}$$

where $\phi_0(\eta)$ with $\eta = (x - x_0) \cdot \nu$ is a non-negative, continuous and non-trivial function. Then for any $0 < c_1 < c^*(m, r) < c_2$, and any $0 < \kappa_1 < \kappa < \kappa_2$, there exist a time $T = T(c_1, c_2, \kappa_1, \kappa_2) > 0$ and two functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ with semi-compact supports, such that

$$\phi_1((x - x_0) \cdot \nu + c_1 t) \leq u(t, x) \leq \phi_2((x - x_0) \cdot \nu + c_2 t), \quad \forall (x - x_0) \cdot \nu \leq 0,\ t \geq T, \tag{1.5}$$

and

$$\operatorname{supp} \phi_1(\cdot) = \operatorname{supp} \phi_2(\cdot) = \Pi(x_0, \nu), \quad \lim_{\eta \to +\infty} \phi_1(\eta) \geq \kappa_1, \quad \lim_{\eta \to +\infty} \phi_2(\eta) \leq \kappa_2.$$

Therefore,

$$\Pi(x_0 - c_1 t, \nu) \cap \Pi(x_0, -\nu) \subset \operatorname{supp} u(t, \cdot) \subset \Pi(x_0 - c_2 t, \nu), \quad t \geq T. \tag{1.6}$$

Corollary 1.1 *Under the condition of Theorem 1.2, there holds*

$$\lim_{t \to +\infty} \inf_{x \in \mathbb{R}^n,\ u(t,x) > 0} \frac{(x - x_0) \cdot \nu}{t} = -c^*(m, r),$$

and

$$\lim_{t\to+\infty} \sup_{x\in\Pi(x_0,-\nu),\ u(t,x)=0} \frac{(x-x_0)\cdot\nu}{t} = -c^*(m,r).$$

Additionally, for any $x \in \mathbb{R}^n$, it holds

$$\lim_{t\to+\infty} u(t, x - ct\nu) = \begin{cases} \kappa, & \text{if } c < c^*(m,r), \\ 0, & \text{if } c > c^*(m,r). \end{cases} \tag{1.7}$$

Theorem 1.2 implies that, the solution with non-trivial and non-negative initial data that are compactly supported in one direction (or compactly supported for the one dimensional case) propagates at the same speed as the sharp type traveling wave, which is the minimal wave speed of all the traveling waves. Particularly for the one dimensional case, any compactly supported initial perturbation remains compact and the boundary propagates at asymptotic speed $c^*(m,r)$.

2 Proof of The Main Results

Before proving our main results, we present the definition of weak solutions since the equation (1.1) is degenerate.

Definition 2.1 A function $u \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R}^n)$ with $n \ge 1$ is called a weak solution of (1.1) if $0 \le u \in L^\infty((0,+\infty)\times\mathbb{R}^n)$, $\nabla u^m \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R}^n)$, and for any $T > 0$ and $\psi \in C_0^\infty((-r,T)\times\mathbb{R}^n)$

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^n} u(t,x)\frac{\partial\psi}{\partial t}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^n} \nabla u^m\cdot\nabla\psi \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^n} d(u(t,x))\psi\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}^n} u_0(0,x)\psi(0,x)\mathrm{d}x + \int_r^{\max\{T,r\}}\int_{\mathbb{R}^n} b(u(t-r,x))\psi(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^{\min\{T,r\}}\int_{\mathbb{R}^n} b(u_0(t-r,x))\psi(x,t)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The weak upper and lower solutions are defined in a similar way (for non-negative test functions) such that the "$=$" is replaced by "$\ge$" and "$\le$" respectively.

For the time-delayed Cauchy problem (1.1), we would like to explain the solvability and comparison principle here. For $0 < t \le r$, the birth function $b(u(t-r,x))$ is already known according to the initial data $u_0(t-r,x)$. In other words, the last two integrals in the Definition 2.1 are already known and no time-delayed terms exist. Therefore, the Cauchy problem (1.1) for $0 < t \le r$ is equivalent to the following problem without time delay

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - d(u) + f(t,x), & x \in \mathbb{R}^n,\ 0 < t \le r, \\ u(0,x) = u_0(0,x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $f(t,x) = b(u_0(t-r,x))$. The existence of solutions for the Cauchy problem (2.1) can be formulated by the compactness method. In fact, we consider the following auxiliary problem on bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - d(u) + f(t,x), & x \in \Omega, 0 < t \leq r, \\ u(t,x) = u_0(0,x), & x \in \partial\Omega, 0 < t \leq r, \\ u(0,x) = u_0(0,x), & x \in \Omega. \end{cases} \tag{2.2}$$

The solvability (locally for $t \in (0,r]$) and comparison principle of (2.2) are proved in Theorem 5.14 in [31] for this type of Dirichlet problem with non-zero boundary data (the proof can be modified to cover the case with death rate function $d(u)$ since it is C^2 smooth). Utilizing the compactness analysis, we can deduce the existence of the solution for the Cauchy problem (2.1). The comparison principle for (2.1) is proved in a similar way as Proposition 9.1 in [31], noticing that the inhomogeneous term $f(t,x)$ is cancelled for comparison. The above observation is valid for $t \in (r, 2r]$, $t \in (2r, 3r]$, and so on, since once the problem (1.1) is solved for $t \in (0,r]$, the time-delayed term $b(u(t-r,x))$ is known for $t \in (r, 2r]$ and the Cauchy problem (1.1) is reduced to the same problem as (2.1). Furthermore, observing that $\underline{u} := 0$ and $\overline{u} := \max\{\kappa, \|u_0\|_{L^\infty([-r,0]\times\mathbb{R}^n)}\}$ are weak lower and upper solutions of (1.1) respectively, we see that the local solution $u(t,x)$ of (1.1) is uniformly bounded. The regularity of $u(t,x)$ can be deduced by the standard energy method for parabolic equations, thus the local solution $u(t,x)$ can not blow up in finite time and then exists globally.

We are interested in the propagation speed of the solutions. We assume that $u_0(s,x)$ for $s \in [-r,0]$ is non-trivial, non-negative, bounded and continuous, therefore the Cauchy problem (1.1) can be solved step by step such that $u(t,x)$ is non-negative, bounded and continuous on $x \in \mathbb{R}^n$ and $t \in [-r, +\infty)$. Moreover, the comparison principle holds for the Cauchy problem (1.1) and the initial boundary value problem on bounded domain since the time-delayed source $b(u(t-r,x))$ is monotone increasing with respect to $u(t-r,x)$. We shall prove that for large time scale the average speed of propagation is consistent with the sharp type traveling wave speed.

The sharp type traveling wave is the typical solution that is semi-compactly supported and propagates at a positive and finite speed $c^*(m,r)$. In order to show the large time propagation speed of general solutions with compact (or semi-compact) supports, we need to construct upper and lower solutions with compact (or semi-compact) supports and propagating at speed near $c^*(m,r)$. The case of $r = 0$ (no time delay) is proved by the phase-plane analysis, where all the trajectories are determined and correspond to special upper and lower solutions, see [23] and the references therein. Here we say that the trajectories are determined for non-delayed case in order to emphasize that the trajectories passing through any non-stationary point are uniquely determined; while this is not true for time-delayed case since the reaction term is a non-local effect and the equation cannot be regarded as a classical dynamic system. For the time-delayed case ($r > 0$), we need to employ a new phase transform approach developed in [35].

Consider the following "traveling wave" type special function defined for any $c > 0$ and $k \geq \kappa$

$$u_c^k(t,x) := \phi_c^k(x \cdot \nu + ct) = \phi_c^k(\xi), \quad \text{with } \xi := x \cdot \nu + ct, \tag{2.3}$$

such that

$$\phi_c^k(\xi) = 0, \quad \forall \xi \leq 0, \qquad \phi_c^k(\xi) \in (0,k), \quad \forall \xi \in (0, \xi_c^k), \tag{2.4}$$

for some $\xi_c^k \in (0, +\infty]$. Note that the sharp type traveling wave $\phi^*(\xi)$ in Theorem 1.1 is a traveling wave type special function $\phi_{c^*}^{\kappa}(\xi)$ corresponding to the critical traveling wave speed $c^* = c^*(m,r)$. As proved in [35], c^* is the unique speed and ϕ^* is the unique function such that ϕ^* satisfies (2.3), (2.4), and the time-delayed degenerate diffusion equation $(1.1)_1$.

Therefore, we only expect that ϕ_c^k is a local solution of (1.1)$_1$ near the boundary $\xi = 0$ for $c \neq c^*$. Actually, this is solved through a delayed iteration scheme as follows.

The traveling wave type function $\phi_c^k(\xi)$ defined by (2.3) and (2.4) is a local solution of (1.1)$_1$ on $\xi \in (-\infty, \xi_c^k)$ with $\xi_c^k \in (0, +\infty]$ if

$$\begin{cases} c(\phi_c^k)'(\xi) = ((\phi_c^k)^m(\xi))'' - d(\phi_c^k(\xi)) + b(\phi_c^k(\xi - cr)), & \xi \in (-\infty, \xi_c^k), \\ \phi_c^k(\xi) = 0, \quad \forall \xi \le 0, \qquad \phi_c^k(\xi) \in (0, k), & \forall \xi \in (0, \xi_c^k). \end{cases} \tag{2.5}$$

The local solvability of the degenerate second order differential equation (2.5) is proved in the following lemma.

Lemma 2.1 *For any $c > 0$ and $k \ge \kappa$, the degenerate problem (2.5) admits a unique local solution $\phi_c^k(\xi)$ on $(-\infty, \xi_c^k)$ with $\xi_c^k \in (0, +\infty]$ (we may assume that $(-\infty, \xi_c^k)$ is the maximal existence interval) such that*

$$\phi_c^k(\xi) = \left(\frac{(m-1)c}{m}\xi\right)_+^{\frac{1}{m-1}} + o(|\xi|^{\frac{1}{m-1}}), \quad \xi \to 0.$$

Moreover, (a) $\phi_c^k(\xi)$ is strictly increasing on $(0, \xi_c^k)$ and $\phi_c^k(\xi_c^k) = k$, or (b) $\phi_c^k(\xi)$ is not strictly increasing on $(0, \xi_c^k)$ and $\phi_c^k(\xi_c^k) = 0$. For the case (b), there holds $\xi_c^k < +\infty$ and there exists a $\hat{\xi}_c^k \in (0, \xi_c^k)$ such that $\phi_c^k(\xi)$ is strictly increasing on $(0, \hat{\xi}_c^k)$ and decreasing on $(\hat{\xi}_c^k, \xi_c^k)$.

Proof Note that ϕ_c^k is semi-compactly supported and the time-delayed source function $b(\phi_c^k(\xi - cr)) = 0$ if $\xi \le cr$. Therefore, (2.5) is locally reduced to the following equation

$$\begin{cases} c\phi_c'(\xi) = (\phi_c^m(\xi))'' - d(\phi_c(\xi)), & \xi \in (-\infty, cr), \\ \phi_c(\xi) = 0, \quad \forall \xi \le 0, \qquad \phi_c(\xi) > 0, & \forall \xi \in (0, cr), \end{cases} \tag{2.6}$$

whose unique solvability is proved in [35, 37] and the solution is denoted by ϕ_c. Lemma 3.1 (and its proof) in [37] shows the locally asymptotical behavior of $\phi_c(\xi)$ near zero and the strictly increasing monotonicity of $\phi_c(\xi)$ in $(0, cr)$. If $\phi_c(cr) \ge k$, then $\xi_c^k = \sup\{\xi \in (0, cr); \phi_c(\xi) < k\}$. If $\phi_c(cr) < k$, then we solve (2.5) on $(cr, 2cr)$ as

$$\begin{cases} c\phi_c'(\xi) = (\phi_c^m(\xi))'' - d(\phi_c(\xi)) + b(\phi_c(\xi - cr)), \quad \xi \in (cr, 2cr), \\ \phi_c(cr), \phi_c'(cr) \text{ are determined from left side.} \end{cases} \tag{2.7}$$

The problem (2.7) is locally solved near cr since $\phi_c(cr) > 0$ and $b(\phi_c(\xi - cr))$ is already known from (2.6). Then three cases may happen:

(i) $\phi_c(\xi)$ is not strictly increasing on whole $(cr, 2cr)$, which means there exists a $\xi_0 \in (cr, 2cr)$ such that $\phi_c'(\xi_0) \le 0$. We employ Lemma 3.5 in [35] to derive that $\phi_c(\xi)$ is always decreasing after ξ_0 until reaching zero for $\xi > \xi_0$. If $\phi_c(\xi) > 0$ for all $\xi \in (cr, 2cr)$, then we solve (2.5) further on $(2cr, 3cr)$ and the intervals after this in a similar way as (2.7) until $\phi_c(\xi_1) = 0$ for some $\xi_1 > \xi_0 > cr$. In this case, $\xi_c^k = \sup\{\xi > cr; \phi_c(\xi) > 0\}$ and $\phi_c(\xi_c^k) = 0$. The assertion $\xi_c^k < +\infty$ is proved in a similar way as the proof of Lemma 3.5 in [37].

(ii) $\phi_c(\xi)$ is strictly increasing on $(cr, 2cr)$ and $\phi_c(2cr) \ge k$, then $\xi_c^k = \sup\{\xi \in (cr, 2cr); \phi_c(\xi) < k\}$ and $\phi_c(\xi_c^k) = k$.

(iii) $\phi_c(\xi)$ is strictly increasing on $(cr, 2cr)$ and $\phi_c(2cr) < k$, then we solve (2.5) further on $(2cr, 3cr)$ and the intervals after this until (i) or (ii) happens. Otherwise, $\phi_c(\xi)$ is strictly increasing and (2.5) is solved on $(-\infty, +\infty)$ such that $\xi_c^k = +\infty$ and $\phi_c(\xi_c^k) = k$. This happens for $c = c^*$ and $k = \kappa$ since $\phi^*(\xi) = \phi^\kappa_{c^*}(\xi)$ is the unique sharp type traveling wave. □

In order to show more precise behaviors of ϕ^k_c, we employ the following phase transform approach and formulate phase comparison principle. For any $c > 0$ and $k \geq \kappa$, let $\phi^k_c(\xi)$ be the unique solution of the degenerate problem (2.5) on its maximal existence interval $(-\infty, \xi^k_c)$ with $\xi^k_c \in (0, +\infty]$ as shown in Lemma 2.1, and let $\hat{\xi}^k_c \in (0, \xi^k_c]$ be the largest number (or equivalently, $(0, \hat{\xi}^k_c)$ be the maximal interval) such that $\phi^k_c(\xi)$ is strictly increasing on $(0, \hat{\xi}^k_c)$. Define (here we do not explicitly write down the dependence of $\psi_c(\xi)$ on k for simplicity)

$$\psi_c(\xi) := ((\phi^k_c)^m(\xi))' = m(\phi^k_c)^{m-1}(\xi) \cdot (\phi^k_c)'(\xi), \quad \xi \in (0, \hat{\xi}^k_c). \tag{2.8}$$

Now we have two functions $\phi^k_c(\xi)$ and $\psi_c(\xi)$ defined for $\xi \in (0, \hat{\xi}^k_c)$, and $\phi^k_c(\xi)$ is strictly increasing on $(0, \hat{\xi}^k_c)$, then we can interpret $\psi = \psi_c(\xi)$ as a function of $\phi = \phi^k_c(\xi)$ through the intermediate variable

$$\xi = (\phi^k_c)^{-1}(\phi), \quad \phi \in (0, \phi^k_c(\hat{\xi}^k_c)).$$

That is, we rewrite

$$\tilde{\psi}_c(\phi) := \psi_c(\xi) = \psi_c((\phi^k_c)^{-1}(\phi)), \;\; \phi \in (0, \phi^k_c(\hat{\xi}^k_c)). \tag{2.9}$$

A key transform in dealing with the time delay in the degenerate diffusion equation (2.5) is to rewrite $\phi^k_c(\xi - cr)$ as a function of $\phi = \phi^k_c(\xi)$ depending on $\tilde{\psi}_c(\phi)$ in a functional way:

$$\phi_{c,cr}(\phi) := \phi^k_c(\xi - cr) = \phi^k_c((\phi^k_c)^{-1}(\phi) - cr) = \inf_{\theta \geq 0} \left\{ \int_\theta^\phi \frac{ms^{m-1}}{\tilde{\psi}_c(s)} \mathrm{d}s \leq cr \right\}, \quad \phi \in (0, \phi^k_c(\hat{\xi}^k_c)). \tag{2.10}$$

Lemma 2.2 (Phase transform) *The functional interpretation* (2.10) *is well-defined for the sharp type functions* $\phi^k_c(\xi)$ *for* $\phi \in (0, \phi^k_c(\hat{\xi}^k_c))$.

Proof We divide the proof into two cases.

Case I. If for some $\phi = \phi^k_c(\xi)$ with $\xi \in (0, \hat{\xi}^k_c)$ and $\phi \in (0, \phi^k_c(\hat{\xi}^k_c))$ there holds

$$\int_0^\phi \frac{ms^{m-1}}{\tilde{\psi}_c(s)} \mathrm{d}s > cr,$$

then we rewrite the above integral through the method of substitution of $s = \phi^k_c(t)$ for $s \in (0, \phi)$ and $t \in (0, \xi)$ to find

$$cr < \int_0^\phi \frac{ms^{m-1}}{\tilde{\psi}_c(s)} \mathrm{d}s = \int_0^\xi \frac{m(\phi^k_c)^{m-1}(t)}{m(\phi^k_c)^{m-1}(t) \cdot (\phi^k_c)'(t)} (\phi^k_c)'(t)\, \mathrm{d}t = \xi.$$

Therefore, $\xi - cr > 0$ and $\phi_{c,cr}(\phi) = \phi^k_c(\xi - cr)$ is the unique value such that

$$\int_{\phi_{c,cr}(\phi)}^\phi \frac{ms^{m-1}}{\tilde{\psi}_c(s)} \mathrm{d}s = cr.$$

Case II. If for some $\phi = \phi^k_c(\xi)$ with $\xi \in (0, \hat{\xi}^k_c)$ and $\phi \in (0, \phi^k_c(\hat{\xi}^k_c))$ there holds

$$\int_0^\phi \frac{ms^{m-1}}{\tilde{\psi}_c(s)} \mathrm{d}s \leq cr,$$

then $\xi - cr \le 0$ and $\phi_c^k(\xi - cr) = 0$ since ϕ_c^k is sharp type such that $\phi_c^k(t) \equiv 0$ for all $t \le 0$. □

Lemma 2.3 (Monotone dependence) *For any $c > 0$ and $k \ge \kappa$, let $\phi_c^k(\xi)$ be the unique solution of the degenerate problem (2.5) on its maximal existence interval $(-\infty, \xi_c^k)$ with $\xi_c^k \in (0, +\infty]$ and let $\tilde{\psi}_c(\phi)$ and $\phi_{c,cr}(\phi)$ be the phase transform functions defined by (2.9) and (2.10). Then for $c_1 > c_2 > 0$, there holds*

$$\tilde{\psi}_{c_1}(\phi) > \tilde{\psi}_{c_2}(\phi), \quad \forall \phi \in (0, \min\{\phi_{c_1}^k(\hat{\xi}_{c_1}^k), \phi_{c_2}^k(\hat{\xi}_{c_2}^k)\}), \tag{2.11}$$

and

$$\phi_{c_1}^k(\xi) > \phi_{c_2}^k(\xi), \quad \forall \xi \in (0, \min\{\hat{\xi}_{c_1}^k, \hat{\xi}_{c_2}^k\}). \tag{2.12}$$

Proof The monotone dependence of $\phi_c^k(\xi)$ with respect to c is proved in Lemma 3.6 in [35]. The monotone dependence $\tilde{\psi}_{c_1}(\phi) > \tilde{\psi}_{c_2}(\phi)$ means that

$$(\phi_{c_1}^k)'(\xi_1) > (\phi_{c_2}^k)'(\xi_2), \quad \text{at where } \phi_{c_1}^k(\xi_1) = \phi = \phi_{c_2}^k(\xi_2),$$

or equivalently,

$$(\phi_{c_1}^k)'((\phi_{c_1}^k)^{-1}(\phi)) > (\phi_{c_2}^k)'((\phi_{c_2}^k)^{-1}(\phi)). \tag{2.13}$$

In contrast to the comparison between two functions $\phi_{c_1}^k(\xi)$ and $\phi_{c_2}^k(\xi)$ at the same point ξ, (2.13) shows the comparison of their derivatives at where they take the same value, hence we would call it the phase comparison principle. The prototype of (2.13) (and (2.11)) is already formulated in the proof of Lemma 3.6 in [35]. Here we omit the details. □

The above monotone dependence is used to construct special upper and lower solutions at speed near the critical wave speed $c^*(m, r)$.

Lemma 2.4 *For any $c > c^*(m, r)$, there exists a number $k > \kappa$, such that*

$$\overline{u}(t, x) := \overline{\phi}_c^k(\xi) := \begin{cases} \phi_c^k(\xi), & \xi < \xi_c^k, \\ k, & \xi \ge \xi_c^k, \end{cases} \qquad \xi = x \cdot \nu + ct,$$

is an upper solution of (1.1) with the initial data $\overline{u}_0(s, x) := \overline{\phi}_c^k(x \cdot \nu + cs)$ for $s \in [-r, 0]$, where $\phi_c^k(\xi)$ is the unique solution of the degenerate problem (2.5) on its maximal existence interval $(-\infty, \xi_c^k)$ with $\xi_c^k \in (0, +\infty]$. Similarly, for any $c \in (0, c^*(m, r))$,

$$\underline{u}(t, x) := \underline{\phi}_c^\kappa(\xi) := \begin{cases} \phi_c^\kappa(\xi), & \xi < \xi_c^\kappa, \\ 0, & \xi \ge \xi_c^\kappa, \end{cases} \qquad \xi = x \cdot \nu + ct,$$

is a lower solution of (1.1) with the initial data $\underline{u}_0(s, x) := \underline{\phi}_c^\kappa(x \cdot \nu + cs)$ for $s \in [-r, 0]$, $\xi_c^\kappa < +\infty$, and $\sup_{\xi \in \mathbb{R}} \phi_c^\kappa(\xi) < \kappa$. Moreover, for any $c \in (0, c^*(m, r))$,

$$\hat{\underline{u}}(t, x) := \hat{\underline{\phi}}_c^\kappa(\xi) := \begin{cases} \phi_c^\kappa(\xi), & \xi < \hat{\xi}_c^\kappa, \\ \phi_c^\kappa(\hat{\xi}_c^\kappa), & \xi \ge \hat{\xi}_c^\kappa, \end{cases} \qquad \xi = x \cdot \nu + ct,$$

also is a lower solution of (1.1) with initial data $\hat{\underline{u}}_0(s, x) := \hat{\underline{\phi}}_c^\kappa(x \cdot \nu + cs)$ for $s \in [-r, 0]$. Additionally, it holds

$$\lim_{c \to (c^*(m,r))^-} \sup_{\xi \in \mathbb{R}} \phi_c^\kappa(\xi) = \kappa.$$

Proof For the critical wave speed $c = c^*$, the unique solution of the degenerate problem (2.5) is the sharp traveling wave $\phi^*(\xi) = \phi^\kappa_{c^*}(\xi)$, and the phase transform function $\tilde{\psi}_{c^*}(\phi)$ satisfies $\tilde{\psi}_{c^*}(\phi) > 0$ for $\phi \in (0, \kappa)$ and $\tilde{\psi}_{c^*}(0) = \tilde{\psi}_{c^*}(\kappa) = 0$.

For any $c > c^*$, according to the phase comparison principle (2.11) in Lemma 2.3, we see that for any $k > \kappa$, $\tilde{\psi}_c(\kappa) > \tilde{\psi}_{c^*}(\kappa) = 0$, that is, $(\phi^k_c)'((\phi^k_c)^{-1}(\kappa)) > 0$. We choose $k > \kappa$ such that case (a) in Lemma 2.1 occurs, i.e., $\phi^k_c(\xi)$ is strictly increasing on $(0, \xi^k_c)$ and $\phi^k_c(\xi^k_c) = k > \kappa$. Therefore, $\overline{\phi}^k_c$ is an upper solution of the second order differential equation $(1.1)_1$.

For any $0 < c < c^*$, similar to the above analysis, according to the phase comparison principle (2.11), we have $\tilde{\psi}_c(\hat{\phi}) = 0 < \tilde{\psi}_{c^*}(\hat{\phi})$ for some $\hat{\phi} \in (0, \kappa)$ since $\tilde{\psi}_{c^*}(\kappa) = 0$. Therefore, $\phi^\kappa_c(\xi)$ is increasing up to $\hat{\phi} < \kappa$ and then decreases to zero, which means Case (b) in Lemma 2.1 occurs. In this case, we have $\xi^\kappa_c < +\infty$ and $\phi^\kappa_c(\xi^\kappa_c) = 0$. Then it follows that $\underline{\phi}^\kappa_c$ is a lower solution of the second order differential equation $(1.1)_1$. Furthermore, $\underline{\hat{u}}(t, x) = \underline{\hat{\phi}}^\kappa_c(\xi)$ is a lower solution of (1.1) since $\phi^{\kappa\prime}_c(\hat{\xi}^\kappa_c) = 0$ at the cut-off edge. The limit of $\lim_{c\to(c^*(m,r))^-} \sup_{\xi\in\mathbb{R}} \phi^\kappa_c(\xi) = \kappa$ follows from the continuous dependence (see Lemma 3.4 in [35] for example), the monotone dependence Lemma 2.3, and the fact that $\sup_{\xi\in\mathbb{R}} \phi^\kappa_{c^*(m,r)}(\xi) = \kappa$. □

Now we investigate the large-time evolution of the solution with semi-compact support.

Lemma 2.5 *Let $u(t, x)$ be the solution of (1.1) with the initial data $u_0(s, x)$ semi-compactly supported and bounded*

$$\operatorname{supp} u_0(s, \cdot) \subset \Pi(x_0, \nu), \quad u_0 \in L^\infty([-r, 0] \times \mathbb{R}^n). \tag{2.14}$$

Then

$$\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^n} u(t, x) \le \kappa.$$

Proof Consider the following differential problem

$$\begin{cases} U'(t) = -d(U) + b(U(t-r)), & t > 0, \\ U(s) = U_0(s) \equiv \|u_0\|_{L^\infty([-r,0]\times\mathbb{R}^n)}, & s \in [-r, 0]. \end{cases} \tag{2.15}$$

The large-time asymptotic analysis of the time-delayed ordinary differential equation (2.15) shows that $\lim_{t\to+\infty} U(t) = \kappa$. Based on the comparison principle, and taking $U(t)$ as an upper solution of (1.1), we have

$$\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^n} u(t, x) \le \limsup_{t\to+\infty} U(t) = \kappa.$$

The proof is completed. □

Lemma 2.6 *Let $u(t, x)$ be the solution of (1.1) with initial data $u_0(s, x)$ satisfying*

$$u_0(s, x) \ge \phi_0((x - x_0) \cdot \nu), \tag{2.16}$$

where $\phi_0(\eta)$ with $\eta = (x - x_0) \cdot \nu$ is a non-negative, continuous and non-trivial function. Then for any compact subset $K \subset \mathbb{R}^n$,

$$\liminf_{t\to+\infty} \inf_{x\in K} u(t, x) \ge \kappa.$$

Proof The proof is divided into two steps. The first one is to show that the support of $u(t,\cdot)$ expands to including any given compact subset for large time, and the second one is to show that the value of $u(t,x)$ within given compact subset grows up as time increases.

Step I. Define a Barenblatt type function

$$g(t,x) = \varepsilon(\tau+t)^{-\sigma}\Big[\Big(\gamma^2 - \frac{|x-x_*|^2}{(\tau+t)^\beta}\Big)_+\Big]^d, \quad x \in \mathbb{R}^n,\ t \geq 0,$$

where $d = 1/(m-1)$, $\beta, \sigma, \varepsilon, \gamma$, and τ are positive constants, $x_* \in \mathbb{R}^n$. Then by appropriately selecting $\beta, \varepsilon, \tau, \sigma, \gamma$, and x_*, the function $g(t,x)$ is a weak lower solution of (1.1) for all the time $t > 0$. The detailed calculations can be found in the proof of Lemma 4.4 in [36]. We note that Lemma 4.4 in [36] was proved for a smooth bounded domain Ω, where the weak lower solution means the differential inequality in the sense of distributions together with the comparison of initial data and the comparison of boundary value on $\partial\Omega$. Here for the Cauchy problem (1.1), the weak lower solution (as defined in Definition 2.1) is equivalent to the differential inequality in the sense of distributions and the comparison of initial data (no boundary value is needed). The differential inequality is calculated in the same way and the comparison of initial data is valid by suitably choosing x_* and the parameters. Although the value of $g(t,x)$ is decaying, its support is expanding at a rate as $(\tau+t)^{\frac{\beta}{2}}$ for some $\beta > 0$. Therefore, for any given compact subsets $K_1 \subset K_2 \subset \mathbb{R}^n$, there exists a time $t_1 > 0$ such that $K_2 \subset \operatorname{supp} u(t,\cdot)$ for any $t \geq t_1$ and $\inf_{t\in[t_1,t_2],x\in K_1} u(t_1,x) > 0$ for any $t_2 > t_1$.

We point out here that the above proof of expanding support relies on the construction of lower solutions of Barenblatt type whose initial data is $\varepsilon\tau^{-\sigma}[(\gamma^2 - \frac{|x-x_*|^2}{\tau^\beta})_+]^d$ (which could be small enough) within $B_{\gamma\tau^{\beta/2}}(x_*)$ such that the radius $\gamma\tau^{\beta/2}$ is chosen to be small and the center x_* lies in the positive set of $u_0(0,x)$. Note that the initial data condition (2.16) is valid for the translation perpendicular to the vector ν, i.e., the lower solution can be chosen to be the same under the translations (perpendicular to the vector ν) of the center x_*. Therefore, the above proof in Step I is valid for any given subset K that is compact in the ν direction: $K \subset \Pi(x_1,\nu) \cap \Pi(x_2,-\nu)$ for some $x_1, x_2 \in \mathbb{R}^n$.

Step II. We assert that for any $\hat{x} \in K$ and any $\hat{\kappa} < \kappa$, there exist a time $\hat{t}$ and an open neighbourhood $B(\hat{x})$ of $\hat{x}$ such that $u(t,x) \geq \hat{\kappa}$ for all $t \geq \hat{t}$ and $x \in B(\hat{x})$. Then the assertion $\liminf_{t\to+\infty}\inf_{x\in K} u(t,x) \geq \kappa$ follows from the finite covering theorem. For any given $\hat{\kappa} < \kappa$, we define

$$\hat{d}(s) := d(s) + \lambda_0 s$$

with $\lambda_0 > 0$ sufficiently small such that $b(s) > \hat{d}(s)$ for all $s \in (0,\hat{\kappa}]$ due to $b(s) > d(s)$ for all $s \in (0,\kappa)$. That is, the minimal positive equilibrium for $b(s)$ and $\hat{d}(s)$ is located in $(\hat{\kappa},\kappa)$. Consider the following separated variable function

$$U(t,x) := (\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}} \cdot g(t) \cdot \chi_Z(x), \tag{2.17}$$

with $\mu_0 > 0$ and function $g(t) > 0$ to be determined. Here $\chi_Z(x)$ is the characteristic function of the zone $Z := \{x \in \mathbb{R}^n; \mu_0(x-\hat{x})\cdot\nu \in (-\frac{\pi}{2},\frac{\pi}{2})\}$. In other words, $\overline{Z} = \Pi(\hat{x}+R_0\cdot\nu,-\nu) \cap \Pi(\hat{x}-R_0\cdot\nu,\nu)$ with $R_0 := \frac{\pi}{2\mu_0}$.

We have

$$\Delta U^m(t,x) = -\mu_0^2(\cos(\mu_0(x-\hat{x})\cdot\nu))_+ \cdot g^m(t) = -\mu_0^2 U^m(t,x), \quad \forall x \in Z, \tag{2.18}$$

and the generalized derivative (which is not Lebesgue integrable) satisfies

$$\Delta U^m(t,x) \geq -\mu_0^2(\cos(\mu_0(x-\hat{x})\cdot\nu))_+ \cdot g^m(t)\cdot\chi_Z(x) \geq -\mu_0^2 U^m(t,x), \tag{2.19}$$

in the sense of distributions. In fact, for the scalar case such that $x \in \mathbb{R}$ and $\nu = 1, \hat{x} = 0$, for example, $U^m(t,x) = (\cos(\mu_0 x))_+ \cdot g^m(t)\cdot\chi_Z(x)$ with $Z = (-\frac{\pi}{2\mu_0}, \frac{\pi}{2\mu_0})$. We can compute that the generalized derivatives

$$\begin{aligned}\frac{\partial U^m}{\partial x} &= -\,\mu_0\sin(\mu_0 x)\cdot g^m(t)\cdot\chi_Z(x),\\ \frac{\partial^2 U^m}{\partial x^2} &= -\,\mu_0^2\cos(\mu_0 x)\cdot g^m(t)\cdot\chi_Z(x) + \mu_0\cdot g^m(t)\cdot\delta_{\pm\frac{\pi}{2\mu_0}},\end{aligned}$$

where $\delta_{\pm\frac{\pi}{2\mu_0}}$ is the Dirac measure at two points $\pm\frac{\pi}{2\mu_0}$. Note that

$$\begin{aligned}\frac{\partial U^m}{\partial x}\Big|_{x=(\frac{\pi}{2\mu_0})^+} = 0 \text{ and } \frac{\partial U^m}{\partial x}\Big|_{x=(\frac{\pi}{2\mu_0})^-}\\ = -\mu_0\sin(\mu_0 x)\cdot g^m(t)\big|_{x=(\frac{\pi}{2\mu_0})^-} = -\mu_0\cdot g^m(t) < 0,\end{aligned}$$

and also

$$\begin{aligned}\frac{\partial U^m}{\partial x}\Big|_{x=(-\frac{\pi}{2\mu_0})^-} = 0 \text{ and } \frac{\partial U^m}{\partial x}\Big|_{x=(-\frac{\pi}{2\mu_0})^+}\\ = -\mu_0\sin(\mu_0 x)\cdot g^m(t)\big|_{x=(-\frac{\pi}{2\mu_0})^+} = \mu_0\cdot g^m(t) > 0.\end{aligned}$$

It follows that the singular term in the second order generalised derivative of U^m (in one-dimensional case) is a positive measure supported at two points and thus the differential inequality (2.19) holds in the sense of distributions. For multi-dimensional case, for any $0 \leq \psi \in C_0^\infty((-r,T)\times\mathbb{R}^n)$, let $\Omega(t) := \{x \in \mathbb{R}^n; \psi(t,x) > 0\}$. Without loss of generality, we assume that $\nu = (1,0,\cdots,0)$ such that $Z = (\hat{x}_1 - R_0, \hat{x}_1 + R_0)\times\mathbb{R}^{n-1}$, and then $\Omega(t)\cap Z = \{x \in \mathbb{R}^n; x_1 \in (\hat{x}_1 - R_0, \hat{x}_1 + R_0), \omega(x_1, x_2, \cdots, x_n) < 0\}$ for some smooth function ω for simplicity. Further we denote $x' := (x_2,\cdots,x_n)$, ∇' and Δ' are derivatives with respect to x' in the following, and compute that

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla U^m\cdot\nabla\psi\mathrm{d}x &= \int_{\Omega(t)\cap Z}\nabla U^m\cdot\nabla\psi\mathrm{d}x\\ &= \int_{(\hat{x}_1-R_0,\hat{x}_1+R_0)}\int_{[\omega(x_1,x')<0]}\nabla U^m\cdot\nabla\psi\mathrm{d}x'\mathrm{d}x_1\\ &= \int_{\hat{x}_1-R_0}^{\hat{x}_1+R_0}\int_{[\omega(x_1,x')<0]}\partial_{x_1}U^m\cdot\partial_{x_1}\psi\mathrm{d}x'\mathrm{d}x_1\\ &\quad+\int_{\hat{x}_1-R_0}^{\hat{x}_1+R_0}\int_{[\omega(x_1,x')<0]}\nabla' U^m\cdot\nabla'\psi\mathrm{d}x'\mathrm{d}x_1.\end{aligned} \tag{2.20}$$

Note that ψ and $\nabla\psi$ vanish at the boundary $\partial\Omega(t)$, integration by parts implies that

$$\int_{\hat{x}_1-R_0}^{\hat{x}_1+R_0}\int_{[\omega(x_1,x')<0]}\nabla' U^m\cdot\nabla'\psi\mathrm{d}x'\mathrm{d}x_1 = -\int_{\Omega(t)\cap Z}\Delta' U^m\cdot\psi\mathrm{d}x. \tag{2.21}$$

The above inequality (2.21), valid for general function U, is trivial if one has observed that here the special function U is independent of x'. More importantly,

$$\begin{aligned}
&\int_{\hat{x}_1-R_0}^{\hat{x}_1+R_0}\int_{[\omega(x_1,x')<0]}\partial_{x_1}U^m\cdot\partial_{x_1}\psi\,\mathrm{d}x'\mathrm{d}x_1\\
&=\int_{\Omega(t)\cap Z}\partial_{x_1}(\partial_{x_1}U^m\cdot\psi)\mathrm{d}x'\mathrm{d}x_1-\int_{\Omega(t)\cap Z}\partial_{x_1}^2U^m\cdot\psi\,\mathrm{d}x'\mathrm{d}x_1\\
&=\int_{[\omega(\hat{x}_1+R_0,x')<0]}\partial_{x_1}U^m\cdot\psi\,\mathrm{d}x'-\int_{[\omega(\hat{x}_1-R_0,x')<0]}\partial_{x_1}U^m\cdot\psi\,\mathrm{d}x'\\
&\quad-\int_{\Omega(t)\cap Z}\partial_{x_1}^2U^m\cdot\psi\,\mathrm{d}x'\mathrm{d}x_1\\
&=: I_1+I_2+I_3.
\end{aligned}\tag{2.22}$$

We point out that left side limit $\partial_{x_1}U^m<0$ at $x_1=(\hat{x}_1+R_0)^-$ and $\psi(x)\geq 0$, thus the first integral I_1 in the end of (2.22) is non-positive. The same is true for I_2 since the right side limit $\partial_{x_1}U^m>0$ at $x_1=(\hat{x}_1-R_0)^+$ and it starts with a minus sign. Therefore, according to (2.21) and (2.22),

$$\begin{aligned}
<\Delta U^m,\psi>:=&-\int_{\mathbb{R}^n}\nabla U^m\cdot\nabla\psi\,\mathrm{d}x\\
&=\int_{\Omega(t)\cap Z}\Delta' U^m\cdot\psi\,\mathrm{d}x-I_1-I_2+\int_{\Omega(t)\cap Z}\partial_{x_1}^2U^m\cdot\psi\,\mathrm{d}x'\mathrm{d}x_1\\
&\geq\int_{\Omega(t)\cap Z}[\Delta U^m]_{\mathrm{reg}}\cdot\psi\,\mathrm{d}x,
\end{aligned}\tag{2.23}$$

where $[\Delta U^m]_{\mathrm{reg}}$ is the regular part as calculated in (2.18) of the generalized derivative ΔU^m. Hence the differential inequality (2.19) is valid in the sense of distributions.

Let us choose $g(t)\in(0,\kappa)$ and $\mu_0=\sqrt{\lambda_0/\kappa^{m-1}}$, then

$$\Delta U^m(t,x)\geq-\mu_0^2U^m(t,x)\geq-\mu_0^2\kappa^{m-1}U(t,x)=-\lambda_0U(t,x).$$

In order to construct $U(t,x)$ as a lower solution of (1.1) for $t>t_1$ with some $T>0$, it suffices to set

$$\begin{cases}\dfrac{\partial U}{\partial t}\leq-\hat{d}(U)+b(U(t-r,x)), & x\in\mathbb{R}^n,\ t>T,\\ U(s,x)\leq u(s,x), & x\in\mathbb{R}^n,\ s\in[T-r,T].\end{cases}\tag{2.24}$$

Therefore, we have

$$\begin{aligned}
\frac{\partial U}{\partial t}&\leq-\hat{d}(U)+b(U(t-r,x))\\
&=-\lambda_0U-d(U)+b(U(t-r,x))\\
&\leq\Delta U^m(t,x)-d(U)+b(U(t-r,x)),
\end{aligned}$$

which is the differential inequality in the sense of distributions for lower solutions. Similar to the proof of (2.23), the above differential inequality in the sense of distributions is equivalent to the integral inequality in the definition of lower solutions (the only singular term is $<\Delta U^m,\psi>$ and is realized as the integral of $-\nabla U^m\cdot\nabla\psi$). As to the comparison of the initial data, we have

$$U(s,x)\leq g(s)\cdot\chi_Z(x).$$

According to Step I, by setting $K_1=Z$ and $T=t_1+r$, $t_2=T$, we further have

$$u(s,x)\geq\inf_{t\in[t_1,t_2],x\in K_1}u(t,x):=\varepsilon_0>0,\quad\forall x\in Z,s\in[T-r,T].$$

It follows that a sufficient condition for (2.24) is

$$\begin{cases} (\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}}\cdot g'(t) \\ \qquad \le -\hat{d}((\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}}\cdot g(t)) + b((\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}}\cdot g(t-r)), \quad x\in Z,\ t>T, \\ g(s)=\varepsilon_0, \quad s\in[T-r,T], \qquad g(t)\in(0,\kappa), \quad t>T, \end{cases} \tag{2.25}$$

or alternatively,

$$\begin{cases} g'(t) \le \inf\limits_{\lambda\in(0,1)} \dfrac{b(\lambda g(t-r)) - \hat{d}(\lambda g(t))}{\lambda}, \qquad t>T, \\ g(s)=\varepsilon_0, \quad s\in[T-r,T], \qquad g(t)\in(0,\kappa), \quad t>T. \end{cases} \tag{2.26}$$

Note that

$$\lim_{\lambda\to 0^+} \frac{b(\lambda s)-\hat{d}(\lambda s)}{\lambda} = b'(0)s - \hat{d}'(0)s = b'(0)s - (d'(0)+\lambda_0)s,$$

which is strictly increasing for all $s>0$ since $b'(0)>d'(0)\ge 0$ (λ_0 is sufficiently small), and $b(s)>\hat{d}(s)$ for all $s\in(0,\hat{\kappa}]$. There exists a constant $\delta_0>0$ such that

$$\inf_{\lambda\in(0,1)} \frac{b(\lambda s)-\hat{d}(\lambda s)}{\lambda} \ge \delta_0 > 0, \quad \forall s\in[\varepsilon,\hat{\kappa}].$$

We now solve the following time-delayed ordinary differential equation step by step:

$$\begin{cases} g'(t) = \inf\limits_{\lambda\in(0,1)} \dfrac{b(\lambda g(t-r)) - \hat{d}(\lambda g(t))}{\lambda}, \quad t>T, \\ g(s)=\varepsilon_0, \quad t\in[T-r,T]. \end{cases} \tag{2.27}$$

Firstly, for $t\in[T,T+r]$, we have

$$g'(T) = \inf_{\lambda\in(0,1)} \frac{b(\lambda\varepsilon_0)-\hat{d}(\lambda\varepsilon_0)}{\lambda} \ge \delta_0 > 0,$$

which means $g(t)$ is strictly increasing until $t\ge T+r$ or

$$\inf_{\lambda\in(0,1)} \frac{b(\lambda\varepsilon_0)-\hat{d}(\lambda g(t))}{\lambda} = 0. \tag{2.28}$$

Since there exist two constants $C_2\ge C_2>0$ such that $C_1 g(t)\le \hat{d}(\lambda g(t))/\lambda \le C_2 g(t)$, the asymptotic analysis of linear differential inequality shows that (2.28) cannot happen in finite time. Therefore, $g(t)$ is strictly increasing on $[T,T+r]$ and

$$\inf_{\lambda\in(0,1)} \frac{b(\lambda\varepsilon_0)-\hat{d}(\lambda g(t))}{\lambda} > 0, \quad \forall t\in[T,T+r]. \tag{2.29}$$

Secondly, for $t\in[T+r,T+2r]$, we have

$$g'(T+r) = \inf_{\lambda\in(0,1)} \frac{b(\lambda g(T))-\hat{d}(\lambda g(T+r))}{\lambda} = \inf_{\lambda\in(0,1)} \frac{b(\lambda\varepsilon_0)-\hat{d}(\lambda g(T+r))}{\lambda} > 0,$$

due to (2.29). It follows that $g(t)$ is strictly increasing until $t\ge T+2r$ or

$$\inf_{\lambda\in(0,1)} \frac{b(\lambda g(t-r))-\hat{d}(\lambda g(t))}{\lambda} = 0, \tag{2.30}$$

where $g(t-r)$ is already known as $t-r \in [T, T+r]$. An asymptotic analysis shows that (2.30) cannot happen in finite time, especially, in $[T+r, T+2r]$. Otherwise, let $t^* \in (T+r, T+2r]$ be the minimal time such that (2.30) is valid. Then $g'(t) = 0$ and $b(\lambda g(t-r))/\lambda$ is strictly increasing. Hence there exists a $\hat{t}^* \in (T+r, t^*)$ such that

$$\inf_{\lambda\in(0,1)} \frac{b(\lambda g(t-r)) - \hat{d}(\lambda g(t))}{\lambda} < 0, \quad t \in (\hat{t}^*, t^*), \qquad \text{and} \inf_{\lambda\in(0,1)} \frac{b(\lambda g(\hat{t}^*-r)) - \hat{d}(\lambda g(\hat{t}^*))}{\lambda} = 0,$$

which contradicts to the minimality of t^*. Repeating the above arguments, we see that $g(t)$ is increasing and the minimal positive equilibrium of $\inf_{\lambda\in(0,1)} (b(\lambda s) - \hat{d}(\lambda s))/\lambda$ is greater than $\hat{\kappa}$. There exists a time $\hat{t} > T$ such that $g(t) > \hat{\kappa}$ for all $t \geq \hat{t}$. Furthermore, by the comparison principle,

$$u(t,x) \geq U(t,x) = (\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}} \cdot g(t) > (\cos(\mu_0(x-\hat{x})\cdot\nu))_+^{\frac{1}{m}} \cdot \hat{\kappa}, \quad t \geq \hat{t}.$$

That is,

$$u(t,\hat{x}) \geq U(t,\hat{x}) = g(t) > \hat{\kappa}, \quad t \geq \hat{t}.$$

Based on the uniformly continuity of $(\cos(\mu_0(x-\hat{x})\cdot\nu))_+$ near $\hat{x}$ with respect to t, we can find a neighborhood $B(\hat{x})$ of $\hat{x}$, independent of time, such that $u(t,x) \geq \hat{\kappa}$, for all $x \in B(\hat{x})$ and $t \geq \hat{t}$. The proof is complete. □

Next, in order to get the large time speed of propagation, we are going to prove it by combining the large time evolution of the solution proved in Lemma 2.5, Lemma 2.6, and the special upper and lower solutions in Lemma 2.4.

Proof of Theorem 1.2. First of all, from Lemma 2.5 and Lemma 2.6, we have the large time evolution such that

$$\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^n} u(t,x) \leq \kappa,$$

and for any compact subset $K \subset \mathbb{R}^n$

$$\liminf_{t\to+\infty} \inf_{x\in K} u(t,x) \geq \kappa.$$

Note that the initial condition (1.4) is translation invariant in the direction perpendicular to ν, similar to the proof of Lemma 2.6, for any finite numbers $s_1 < s_2$, there holds

$$\liminf_{t\to+\infty} \inf_{(x-x_0)\cdot\nu\in[s_1,s_2]} u(t,x) \geq \kappa.$$

Without loss of generality, we may assume that $\phi_0(\eta)$ is symmetric (after shifting if necessary) with respect to $\eta = 0$. Otherwise, we can choose another function with symmetry and smaller than $\phi_0(\eta)$. Then the lower solution $u^*(t,x)$ with the initial data given by $u_0^*(s,x) = \phi_0((x-x_0)\cdot\nu)$ is also symmetric with respect to $\eta = 0$. The propagation properties of $u^*(t,x)$ at one side of $\eta < 0$ is equivalent to the initial boundary value problem with homogeneous Neumann condition in the half space according to the refection principle.

For any $c > c^*(m,r)$, let $\hat{c} \in (c^*(m,r), c)$ and $\overline{u}(t,x) = \overline{\phi}_{\hat{c}}^k(\xi)$ with $\xi = x\cdot\nu + \hat{c}t$ be the upper solution of (1.1) corresponding to $\hat{c} > c^*(m,r)$ as proved in Lemma 2.4. Note that $\lim_{\xi\to+\infty} \overline{\phi}_{\hat{c}}^k(\xi) = k > \kappa$, we can change the initial time to some $T > 0$ such that

$$\sup_{x\in\mathbb{R}^n} u(t,x) < \frac{k+\kappa}{2} < k, \quad \forall t > T,$$

and then shift $\overline{u}(t,x) = \overline{\phi}^k_{\hat{c}}(\xi)$ such that the comparison of the initial data is valid. The comparison principle shows that

$$\begin{aligned}\lim_{t\to+\infty} u(t, x - ct\nu) \le \lim_{t\to+\infty} \overline{u}(t, x - ct\nu) &= \lim_{t\to+\infty} \overline{\phi}^k_{\hat{c}}((x - ct\nu)\cdot\nu + \hat{c}t)\\ &= \lim_{t\to+\infty} \overline{\phi}^k_{\hat{c}}(x\cdot\nu - (c-\hat{c})t) = 0,\end{aligned}$$

since $c > \hat{c}$ and $\overline{\phi}^k_{\hat{c}}(\xi) = 0$ for $\xi \le \xi_0$, where ξ_0 is given after the shifting.

Similarly, for any $c < c^*(m, r)$, let $\hat{c} \in (c, c^*(m, r))$ and $\underline{\hat{u}}(t, x) = \underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi)$ with $\xi = x\cdot\nu + \hat{c}t$ be the lower solution of (1.1) corresponding to $\hat{c} < c^*(m, r)$ constructed in Lemma 2.4. Since $\sup_{\xi\in\mathbb{R}} \underline{\phi}^\kappa_{\hat{c}}(\xi) < \kappa$, we change the initial time to some $T > 0$ such that

$$\inf_{x\cdot\nu\in[-(\hat{\xi}^\kappa_{\hat{c}}+\hat{c}r+1),0]} u(t,x) > \sup_{\xi\in\mathbb{R}} \underline{\phi}^\kappa_{\hat{c}}(\xi), \qquad t \ge T,$$

Note that $\underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi) = 0$ for $\xi \le 0$ and $\underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi) = \phi^\kappa_{\hat{c}}(\hat{\xi}^\kappa_{\hat{c}})$ for $\xi \ge \hat{\xi}^\kappa_{\hat{c}}$, we shift $\underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi)$ such that $\underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi) = 0$ for $\xi \le -(\hat{\xi}^\kappa_{\hat{c}} + 1)$ and $\underline{\hat{\phi}}^{\kappa\prime}_{\hat{c}}(\xi) = 0$ for $\xi \ge -1$. Therefore, $\underline{\hat{u}}(t, x) = \underline{\hat{\phi}}^\kappa_{\hat{c}}(\xi)$ is a lower solution of the corresponding homogeneous Neumann problem (1.1) on the half space. According to the comparison principle, we have

$$\begin{aligned}\lim_{t\to+\infty} u(t, x - ct\nu) \ge \lim_{t\to+\infty} \underline{\hat{u}}(t, x - ct\nu) &= \lim_{t\to+\infty} \underline{\hat{\phi}}^k_{\hat{c}}((x - ct\nu)\cdot\nu + \hat{c}t)\\ &= \lim_{t\to+\infty} \underline{\hat{\phi}}^k_{\hat{c}}(x\cdot\nu + (\hat{c}-c)t) = \phi^\kappa_{\hat{c}}(\hat{\xi}^\kappa_{\hat{c}}),\end{aligned}$$

since $\hat{c} > c$. According to Lemma 2.4,

$$\lim_{\hat{c}\to(c^*(m,r))^-} \sup_{\xi\in\mathbb{R}} \phi^\kappa_{\hat{c}}(\xi) = \lim_{\hat{c}\to(c^*(m,r))^-} \phi^\kappa_{\hat{c}}(\hat{\xi}^\kappa_{\hat{c}}) = \kappa,$$

and $\hat{c} \in (c, c^*(m, r))$ is arbitrary, we see that $\lim_{t\to+\infty} u(t, x - ct\nu) \ge \kappa$. Combining this with the fact that $\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^n} u(t, x) \le \kappa$, we have $\lim_{t\to+\infty} u(t, x - ct\nu) = \kappa$. The proof is completed. □

3 Numerical Simulations

This section is devoted to the numerical simulations for the propagation properties of the degenerate diffusion equation (1.1) with time delay. The most artful part is the numerical calculation of Δu^m near the sharp edge (i.e., the boundary of the support). For simplicity, we only consider the 1-dimensional case. We note that the sharp traveling wave $\phi^*(x + c^*t)$ is a typical solution that propagates to the left direction with fixed speed $c^* > 0$ and $c^*(r) < c^*(0)$ for time delay $r > 0$ according to Theorem 1.1.

We point out that the classical second order difference scheme

$$(u^m)_{xx}\big|_{x_k} \approx \frac{u^m(t, x_{k+1}) + u^m(t, x_{k-1}) - 2u^m(t, x_k)}{(\Delta x)^2} \tag{3.1}$$

does not work well near the boundary, since the solution of the degenerate equation has sharp moving edge. In fact, Lemma 2.1 shows that the sharp traveling wave

$$\phi^*(\xi) = \left(\frac{(m-1)c^*}{m}\xi\right)_+^{\frac{1}{m-1}} + o(|\xi|^{\frac{1}{m-1}}), \quad \xi \to 0. \tag{3.2}$$

Therefore, if we take $\phi^*(x)$ as the initial data, the solution $u(t,x)$ propagates to the left direction with the same profile. Remarkably, $(u^m)_{xx}$ is not continuous near the boundary for $m \geq 2$, hence the second order difference scheme based on the values at nearby discrete points loses accuracy.

Sharp-profile-based difference scheme. In order to handle the large variation of $(u^m)_{xx}$ near the boundary, we propose the following sharp-profile-based difference scheme based on the expansion of the profile in (3.2). We take the case of $m = 2$ for example. Other cases can be converted to a similar equation of the so-called pressure function $v(t,x) := \frac{m}{m-1}u^{m-1}(t,x)$.

(i) Besides the partition points $x_{-N} < x_{-N+1} < \cdots < x_0 < \cdots < x_{N-1} < x_N$, we additionally introduce an edge point $\hat{x} \in (x_{k-1}, x_k]$ for some $-N < k \leq N$ depending on time t such that $u(t,x_j) = 0$ for $j < k$ and $u(t,x_j) > 0$ for $j > k$;

(ii) The second order derivatives $(u^2)_{xx}\big|_{x_j}$ away from the boundary (i.e., for $j > k$) are calculated by the classical second order difference scheme (3.1) and $(u^2)_{xx}\big|_{x_j} = 0$ for $j < k$ since locally $u(t,x) \equiv 0$ near x_j;

(iii) For $(u^2)_{xx}$ near the boundary, i.e., for $(u^2)_{xx}\big|_{x_k}$, we use the profile ansatz according to (3.2)

$$u(t,x) = c_1(x-\hat{x})_+ + c_2(x-\hat{x})_+^2 + o((x-\hat{x})_+^2), \quad x \to \hat{x},$$

and the values $u(t,x_k)$, $u(t,x_{k+1})$, $u(t,x_{k+2})$ to fit the coefficients c_1 and c_2, then

$$u^2(t,x) = c_1^2(x-\hat{x})_+^2 + 2c_1c_2(x-\hat{x})_+^3 + o((x-\hat{x})_+^3), \quad x \to \hat{x},$$

such that we calculate $(u^2)_{xx}\big|_{x_k} = 2c_1^2 + 12c_1c_2(x_k - \hat{x})_+$.

(iv) We compute the values $u(t+\Delta t, x_j)$ according to the equation (1.1), and special attention should be paid to the values $u(t+\Delta t, x_j)$ near the edge. We use the ansatz

$$u(t+\Delta t, x) = c_1'(x-\hat{x}')_+ + c_2'(x-\hat{x}')_+^2 + o((x-\hat{x}')_+^2), \quad x \to \hat{x}',$$

where $\hat{x}'$ is the new edge point, and the new values $u(t+\Delta t, x_k)$, $u(t+\Delta t, x_{k+1})$, $u(t+\Delta t, x_{k+2})$ to fit the new coefficients c_1', c_2', and the new edge $\hat{x}'$. If the distance $x_k - \hat{x}'$ is larger than or equal to Δx, we modify

$$u(t+\Delta t, x_{k-1}) = c_1'(x_{k-1} - \hat{x}')_+ + c_2'(x_{k-1} - \hat{x}')_+^2,$$

which means the solution propagates across x_{k-1} and the values are calculated based on the edge profile instead of the classical difference scheme based on nearby values.

Testing the sharp-profile-based difference scheme with known profiles. The above sharp-profile-based scheme works perfectly for the degenerate diffusion equations with and without time delay. For the non-delayed case, it is known that the following degenerate diffusion equation with Fisher-KPP source

$$u_t = (u^2)_{xx} + u - u^2, \tag{3.3}$$

admits an explicit sharp traveling wave solution $u(t,x) = (1 - e^{\frac{-x-t}{2}})_+$, see [26]. Therefore, the minimal wave speed and the propagation speed is $c^*(2,0) = 1$ for $m = 2$ and $r = 0$ since the sharp traveling wave is unique (up to translation) according to [35]. Note that $c^*(1,0) = 2$ for the non-degenerate case $m = 1$ and $r = 0$.

We first take the non-smooth initial value

$$u_0(x) := (1 - e^{-\frac{x}{2}})_+ \tag{3.4}$$

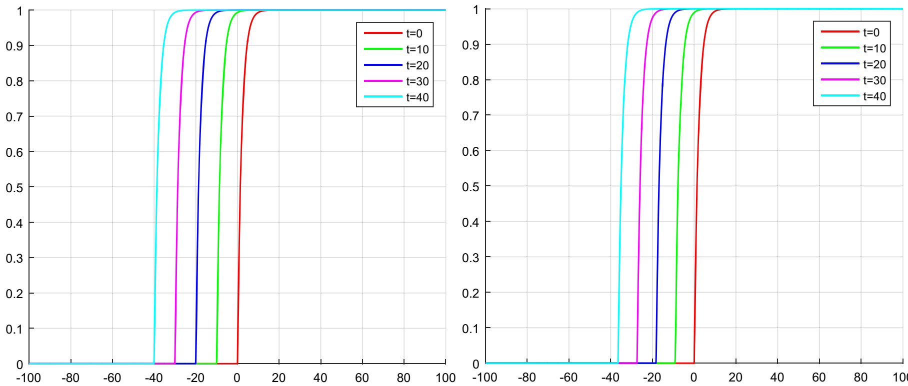

Fig. 1 The sharp-profile-based scheme for degenerate diffusion equations: (left) non-delayed case; (right) delayed case with the time delay $r = 0.1$

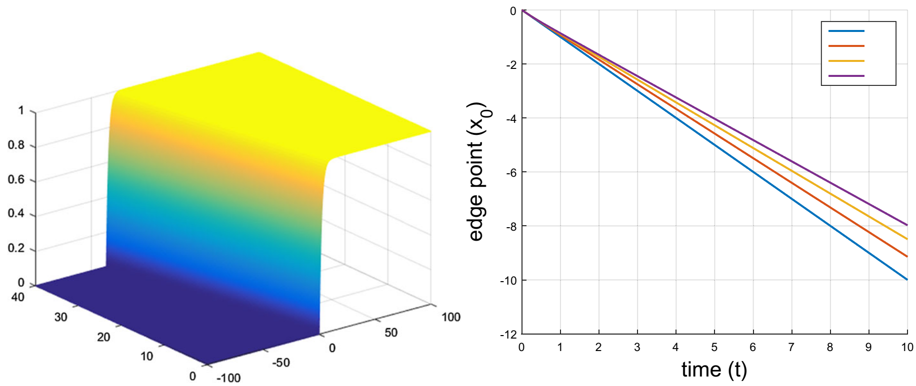

Fig. 2 (left) The 3-dimensional image of solution for $r = 0.1$; (right) The edge point as a function of time for $r = 0$, $r = 0.1$, $r = 0.2$, and $r = 0.3$, respectively

for the non-delayed degenerate diffusion equation (3.3), and the expected explicit solution is $u(t,x) = (1 - e^{\frac{-x-t}{2}})_+$, which has a sharp edge propagating to the left with speed $c^*(2,0) = 1$. The numerical simulation using the sharp-profile-based difference scheme shows that the solution propagates with the same profile and the sharp edge is preserved. See the illustration Fig. 1.

Time-delayed degenerate diffusion equations. For the time-delayed case, we apply the above sharp-profile-based scheme to

$$u_t(t,x) = (u^2(t,x))_{xx} + u(t-r,x) - u^2(t,x), \tag{3.5}$$

with the given initial data

$$u_0(s,x) := (1 - e^{-\frac{x}{2}})_+, \quad \text{uniformly in } s \in [-r, 0]. \tag{3.6}$$

Numerical simulation shows that the solution propagates to the left with a smaller speed $c^*(2,r) < c^*(2,0)$, which coincides with the theoretical result in Theorem 1.2. See the illustration Fig. 1 and the 3-dimensional image in Fig. 2.

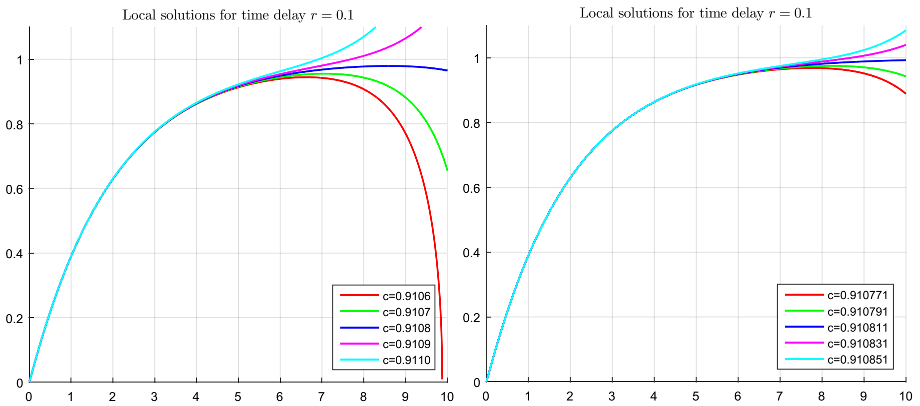

Fig. 3 The local solutions for searching of critical wave speed for $r = 0.1$

Propagation speed influenced by time delay. We take the time delay $r = 0, 0.1, 0.2, 0.3$, respectively. Numerical simulations show that the corresponding propagation speed is monotonically decreasing with respect to the time delay. Illustrated figure is presented in Fig. 2. Here, we show that the asymptotic propagation speed after the evolution on $[0, T]$ with time $T = 10$ is

$$c^*(2,0) \approx 1.0000, \quad c^*(2,0.1) \approx 0.9115, \quad c^*(2,0.2) \approx 0.8439, \quad c^*(2,0.3) \approx 0.7891, \tag{3.7}$$

for the time delay $r = 0,\ 0.1,\ 0.2$, and 0.3, respectively.

Coincidence with the critical wave speed for sharp traveling waves. The critical wave speed for the non-delayed case is know explicitly as $c^*(2,0) = 1$, while the speed of the time-delayed case is characterized by a variational inequality in [35] such that $c^*(m,r) < c^*(m,0)$ without information for the (numerical) calculation. According to the proof in Sect. 3, the sharp wave is determined as the local solution $\phi_c^k(\xi)$ of a singular ODE (2.5) such that $\phi_c^k(\xi)$ admits global extension and is monotone increasing. This primary idea of proof provides a numerical method to calculate the critical wave speed, although it may not be efficient:

(i) for large $c > c^*$, the local solution $\phi_c^k(\xi)$ grows up to beyond the positive equilibrium 1;
(ii) for small $c < c^*$, the local solution $\phi_c^k(\xi)$ eventually declines down to 0;
(iii) the local solution $\phi_c^k(\xi)$ is monotonically increasing with respect to c.

Based on the above theoretical observations, we carry out the following simulation of the critical wave speed for the time delay $r = 0.1$ in Fig. 3. This kind of simulation shows that

$$c^*(2,0) \approx 1.0000, \quad c^*(2,0.1) \approx 0.9108, \quad c^*(2,0.2) \approx 0.8430, \quad c^*(2,0.3) \approx 0.7880, \tag{3.8}$$

for $r = 0,\ 0.1, 0.2$, and 0.3, respectively. We see that the numerical propagation speeds shown in (3.7) coincide with these calculated critical wave speeds (3.8), and the numerical errors are somewhat like 10^{-4}, which are pretty good.

Numerical stability. We take the initial value

$$u_0(s,x) := (1 - e^{-\frac{x}{2}})_+ + 0.2\sin(\pi(x-2)/20) \cdot \chi_{[2,42]}(x), \quad \text{uniformlly in } s \in [-r, 0], \tag{3.9}$$

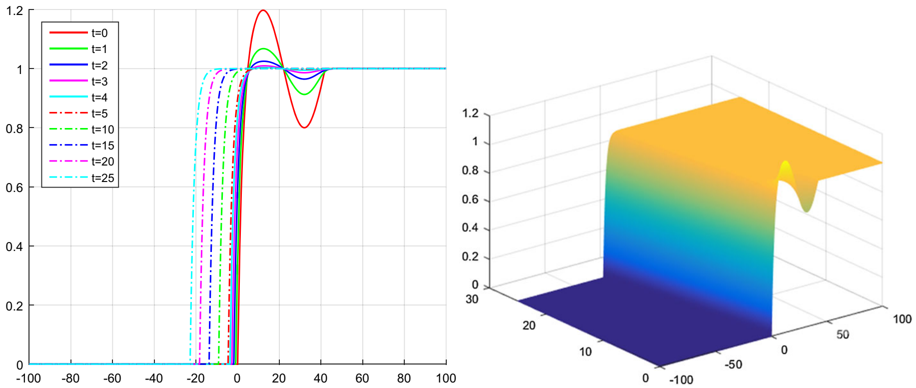

Fig. 4 The evolution of perturbation: (left) the 2-dimensional image; (right) the 3-dimensional image

where the perturbation is chosen within the support of the profile. According to the numerical simulation in Fig. 4, we see that the perturbation decays to zero, and the solution still propagates to the left with sharp edge.

The datasets generated during the current study are available from the corresponding author on reasonable request.

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