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UNIPOLAR EULER-POISSON EQUATIONS WITH TIME-DEPENDENT DAMPING: BLOW-UP AND GLOBAL EXISTENCE*

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Abstract. This paper is concerned with the Cauchy problem for one-dimensional unipolar Euler-Poisson equations with time-dependent damping, where the time-asymptotically degenerate damping in the form of $-\frac{\mu}{(1+t)^\lambda}\rho u$ for $\lambda>0$ with $\mu>0$ plays a crucial role for the structure of solutions. The main issue of the paper is to investigate the critical case with $\lambda=1$. We first prove that, for all cases with $\lambda>0$ and $\mu>0$ (including the critical case of $\lambda=1$), once the initial data is steep at a point, then the solutions are locally bounded but their derivatives will blow up in finite time, by means of the method of Riemann invariants and the technical convex analysis. Secondly, for the critical case of $\lambda=1$ with $\mu>7/3$, we prove that there exists a unique global solution, once the initial perturbation around the constant steady-state is sufficiently small. In particular, we derive the algebraic convergence rates of the solution to the constant steady-state, which are piecewise, related to the parameter μ for $7/3<\mu\le 3$, $3<\mu\le 4$ and $\mu>4$. The adopted method of proof in this critical case is the technical time-weighted energy method and the time-weight depends on the parameter μ. Finally, we carry out some numerical simulations in two cases for blow-up and global existence, respectively, which numerically confirm our theoretical results.

Keywords. Unipolar Euler-Poisson; time-asymptotically degenerate damping; blow-up; global existence; decay rates; critical.

AMS subject classifications. 35B40; 35L50; 35L60; 35L65.

1. Introduction

Modeling equations. In this paper, we consider the one-dimensional unipolar hydrodynamic model of semiconductor, which is represented by the Euler-Poisson system with time-dependent damping

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p(\rho))_x=\rho E-\dfrac{\mu}{(1+t)^\lambda}\rho u, \quad (x,t)\in\mathbb{R}\times\mathbb{R}_+, \\ E_x=\rho-D(x). \end{cases} \tag{1.1}$$

Here, the unknown functions $\rho(x,t)>0$, $u(x,t)$ and $E(x,t)$ denote the electronic density, the electronic velocity, and the electric field, respectively. The given function $D(x)>0$ denotes the doping profile which is the density of impurities in the semiconductor device, and $p=p(\rho)$ is the pressure-density function. The term $-\frac{\mu}{(1+t)^\lambda}\rho u$ with parameters $\lambda>0$ and $\mu>0$ is the so-called time-asymptotically degenerate damping effect. The hydrodynamic models of semiconductors are usually used to characterize the motion of the charged fluid particles, for example, the electrons and holes in semiconductor devices

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[2,23,31]. Recently, the theoretical study and numerical computation on hydrodynamic models of semiconductors have been one of the hot spots in mathematical physics.

Background of study. When $\lambda=0$ and $\mu>0$, the damping in the system (1.1) is reduced to the regular damping. There are many results about the existence and uniqueness of the subsonic/supersonic/transonic solution for the steady-state system of (1.1) with regular damping, we can refer to [1,5,6,8,9,27,28,32,38,39] and references therein. There are also many results about the large-time behavior of solutions to the system (1.1) with regular damping, see [13,16–18,24,29,34,35,41] for details. Among them, Li-Markowich-Mei [24] showed that the solution to the initial boundary value problem of (1.1) with regular damping exists globally and tends exponentially to the corresponding steady-state solution. Luo-Natalini-Xin [29] obtained the global existence of smooth solutions to the Cauchy problem of (1.1) with regular damping, and showed that the solutions converge to the stationary solutions of the drift-diffusion equations when the state constants on the current density and the electric field are zero (switch-off case). They required such a stiff condition owing to a technical difficulty in reformulating the perturbed system in L^2 sense. Later, Huang-Mei-Wang-Yu [18] remarkably showed that the solutions to the Cauchy problem exist globally and converge to stationary solutions when the state constants on the current density and the electric field are nonzero (switch-on case). By technically constructing correction functions, they [18] released the switch-off requirement to the switch-on case. Regarding the study on the bipolar hydrodynamic models of semiconductors with regular damping, one can refer to [7,10,19,20,33] and references therein.

When $\lambda>0$ and $\mu>0$, the damping effect is time-asymptotically degenerate. While, when $\lambda<0$ and $\mu>0$, the damping effect is time-asymptotically enhancing. In these cases, Li-Li-Mei-Zhang [26] investigated the Cauchy problem to the one-dimensional bipolar Euler-Poisson system with time-dependent damping for $-1<\lambda<1$ and $\mu=1$, and showed that the solutions time-algebraically converge to the corresponding diffusion waves when the initial perturbation is small enough. Later, Luan-Mei-Rubino-Zhu [30] proved that the solutions of the bipolar Euler-Poisson system with time-dependent damping for $\lambda=1$ and $\mu>2$ (the critical case) time-algebraically converge to constant steady-states, when the initial perturbation is small enough. But the convergence rates are not sufficient, compared with Euler equations with critical time-dependent damping [12]. Here, they [26, 30] restricted the doping profile to $D(x)=0$. For the unipolar case of Euler-Poisson system with time-dependent damping, with the doping profile $D(x)\neq 0$, Sun-Mei-Zhang [42] considered the non-critical case $\lambda\in(-1,0)\cup(0,1)$ with $\mu=1$, and showed that the solutions converge to steady-states in the sub-exponential form for $-1<\lambda<0$, and to the constant steady-states in the sub-exponential form for $0<\lambda<1$ if the doping profile is a constant.

To better understand how the time-dependent damping influences the structure of solutions for the Euler-Poisson equations, let us recall the simpler case of Euler equations with time-dependent damping:

$$\begin{cases} v_t-u_x=0, \\ u_t+p(v)_x=-\dfrac{\mu}{(1+t)^\lambda}u, \quad (x,t)\in\mathbb{R}\times\mathbb{R}_+, \\ (v,u)(x,0)=(v_0,u_0)(x)\to(v_\pm,u_\pm), \quad \text{as} \quad x\to\pm\infty, \end{cases} \tag{1.2}$$

where $v(x,t)>0$ and $u(x,t)$ represent the specific volume of the flow and fluid velocity, respectively. For states constants $v_+=v_-$, Pan [36,37] firstly showed that the critical exponents for λ and μ are $\lambda=1$ and $\mu=2$. Precisely, when $0\le\lambda<1$, $\mu>0$ or $\lambda=1$,

$\mu > 2$, the solutions of (1.2) exist globally in time, while, when $\lambda = 1$, $0 \le \mu \le 2$ or $\lambda > 1$, $\mu \ge 0$, the solutions of (1.2) will blow up in finite time. Three-dimensional cases were significantly studied by Hou-Witt-Yin [14,15] and Ji-Mei [21,22], recently. In [11], Geng-Huang-Wu concerned with the asymptotic behavior of L^∞ weak-entropy solutions to the compressible Euler equations with a vacuum and time-dependent damping. For $v_+ \neq v_-$, the parameters of λ and μ still play the crucial role for the structure of solutions. In fact, when $0 < \lambda < 1$ and $\mu > 0$, Cui-Yin-Zhang-Zhu [4] and Li-Li-Mei-Zhang [25] both independently obtained the convergence rates of the original solutions for (1.2) to the diffusion waves if the initial perturbation around the diffusion waves and the wave strength $|v_+ - v_-| + |u_+ - u_-|$ both are sufficiently small. The convergence rates in [4] are better than in [25]. Clearly, when $0 < \lambda < 1$ and $\mu > 0$, the time-asymptotically degenerate damping makes (1.2) behave like a degenerate parabolic system with diffusion phenomena. In the critical case $\lambda = 1$ and $\mu > 2$, Geng-Lin-Mei [12] remarkably observed that the hyperbolicity and the damping effect both play important roles and cannot be ignored. They [12] further proved that the solutions of (1.2) converge to the solutions of linear wave equations with critical time-dependent damping. When $\lambda = 1$, $0 \le \mu \le 2$ or $\lambda > 1$, $\mu \ge 0$, Sugiyama [40] proved that the derivative blow-up occurs in finite time with the solution itself and the pressure bounded once the initial data are steep by using Riemann invariants method. For the case $\lambda > 1$, the damping effect is so weak that it can be ignored, which makes the system (1.2) almost behave like a pure hyperbolic system and the shock waves must form. In particular, Chen-Li-Li-Mei-Zhang [3] showed that if $0 < \lambda < 1$, $\mu > 0$ or $\lambda = 1$, $\mu > 2$, even if the wave strength and the initial perturbation are big, once the derivatives of the initial data are not big, the solutions of (1.2) exist globally. However, for all $\lambda > 0$ and $\mu > 0$, once the derivatives of the initial data are big enough at some point, the solutions are still bounded but the derivatives of solutions will blow up in finite time.

Technical difficulties. Talking about the mechanism of blow-up and global existence of solutions for Euler-Poisson equations with time-dependent damping, the study is quite limited, and becomes more complicated and challenging due to the strong coupling and nonlinearity of the system (1.1). Our goal of this paper is to investigate blow-up and global solutions of the system (1.1) subjected to the initial value

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x) \to (\bar{\rho}, 0) \quad \text{as } x \to \pm\infty, \tag{1.3}$$

where $\bar{\rho} > 0$ is the state constant. We consider the case that the pressure-density function $p(\rho)$ satisfies the Gamma-law:

$$p(\rho) = \rho^\gamma / \gamma, \ \gamma > 1.$$

Here are some technical issues we need to point out.

(1) For the blow-up phenomena, different from the previous study [3] on Euler equations, there is a strong coupling for the system with the electric field $E(x,t)$, which causes the boundedness of the solutions for the system to be local only, but not globally uniform. This local boundedness of the solutions enhances difficulties in the proof of blow-up in the frame of Riemann variants.

(2) For the global existence in the critical case of $\lambda = 1$ with $\mu > 7/3$, different from the previous study [30] for bipolar Euler-Poisson system, the doping profile $D(x)$ is nonzero here. But the nonzero doping profile will cause some essential difficulty in establishing the energy estimates. In order to overcome it, here we technically choose the weight functions to be related to the physical parameter μ and introduce

some small quantities $\varepsilon_i\,(i=1,2,3)$ when we carry out the energy estimates, then we can artfully derive the algebraic convergence rates related to μ, which are piecewise in three parts: $7/3<\mu\le 3$, $3<\mu\le 4$ and $\mu>4$. These rates are much better than the rates showed in [30]. On the other hand, compared with Euler equations with critical time-dependent damping [12], the calculation of decay rates for Euler-Poisson equations with critical time-dependent damping is more complicated and trickier.

Main results. In summary, we precisely state our main results as below:

(1) Let $(\rho,u,E)\in C^1(\mathbb{R}\times[0,L])$ for $L>0$ be the solutions to the Cauchy problem (1.1), (1.3). Suppose that $D(x)=\bar{\rho}$ for $x<-N_0$, where $N_0>0$ is sufficiently large. If $\lim\limits_{x\to\pm\infty}E(x,0)=0$ and the initial data satisfy (2.5)–(2.8). Then, the solutions are locally bounded. Furthermore, for all cases $\lambda>0$ and $\mu>0$, the derivatives of solutions will blow up in finite time when the derivatives of the initial Riemann invariants with absolute value are large enough.

(2) For technical reason, we have to restrict the doping profile to $D(x)=\hat{D}$ for some positive constant $\hat{D}$, in fact, we need $\hat{D}=\bar{\rho}$. And the expected steady-state is reduced to the constant steady-state $(\hat{D},0,0)$. We then prove that the unique solution $(\rho,J,E)(x,t)$ of (1.1), (1.3) in the critical case $\lambda=1$ with $\mu>7/3$ exists globally and satisfies:

- When $7/3<\mu\le 3$, then

$$(1+t)^{\frac{\mu+1}{4}}\big(\|J(t)\|+\|E(t)\|\big)+(1+t)^{\frac{\mu}{2}}\big(\|\rho(t)-\hat{D}\|_2 +\|J_x(t)\|_1+\|E_x(t)\|_2\big)\le C\Phi_0;$$

- When $3<\mu\le 4$, then

$$(1+t)\big(\|J(t)\|+\|E(t)\|\big)+(1+t)^{\frac{\mu}{2}}\big(\|\rho(t)-\hat{D}\|_2+\|J_x(t)\|_1+\|E_x(t)\|_2\big)\le C\Phi_0;$$

- When $\mu>4$, then

$$(1+t)\big(\|J(t)\|+\|E(t)\|\big)+(1+t)^2\big(\|\rho(t)-\hat{D}\|_2+\|J_x(t)\|_1+\|E_x(t)\|_2\big)\le C\Phi_0,$$

provided that the initial perturbation Φ_0 is sufficiently small and (3.4) holds, where $J=\rho u$, $\Phi_0:=\|\omega_0\|_3+\|J_0\|_2$, and ω_0 is defined in (3.9).

Remark 1.1. Note that, for Euler equations studied in [3, 12, 14, 15, 36, 37, 40], $\lambda=1$ with $\mu>2$ is the critical case for the global existence of the solutions. However, here we have to restrict $\mu>7/3$ when $\lambda=1$. It seems that $\mu>7/3$ is not optimal for the global existence, but we could not test the global existence once $\lambda=1$ and $2<\mu\le 7/3$, because the system coupling with the electric field $E(x,t)$ and with the non-zero doping profile $D(x)$ makes the modeling equations totally different from the previous studies. Indeed, some obstacles exist in the proof.

Notations. Throughout this paper, C always denotes a generic positive constant which is independent of x and t, and may be different in different lines. $L^2(\mathbb{R})$ is the space of square-integrable real-valued functions defined on $\mathbb{R}$, with the norm $\|f\|:=\|f\|_{L^2(\mathbb{R})}$. $L^\infty(\mathbb{R})$ is the space of bounded measurable functions defined on $\mathbb{R}$, with the norm $\|f\|_{L^\infty}:=\|f\|_{L^\infty(\mathbb{R})}=\operatorname*{ess\,sup}\limits_{x\in\mathbb{R}}|f|$. $H^m(\mathbb{R})\,(m\ge 0)$ is the usual Sobolev space whose norm is abbreviated as $\|f\|_m:=\sum_{k=0}^m\|\partial_x^k f\|$.

The rest of this paper is organized as follows. In Section 2, we prove that for all $\lambda>0$ and $\mu>0$, the derivatives of solutions to the Cauchy problem (1.1), (1.3) will blow up in finite time. The global existence and large-time behavior of the solutions to the Cauchy problem (1.1), (1.3) in critical case $\lambda=1$ with $\mu>7/3$ are obtained in Section 3.

2. Blow up for the steep initial data

In this section, for all cases of $\lambda>0$ and $\mu>0$, we show that the derivatives of the solutions to Cauchy problem (1.1) and (1.3) will blow up in finite time if the derivatives of the initial data are sufficiently large.

We introduce the Riemann invariants to the system (1.1):

$$r(x,t):=\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}(x,t)-\bar\rho^{\frac{\gamma-1}{2}}\big)-u(x,t),\quad s(x,t):=\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}(x,t)-\bar\rho^{\frac{\gamma-1}{2}}\big)+u(x,t). \tag{2.1}$$

Then (r,s) satisfy the following system

$$\begin{cases} r_t+r_x\big(u-\rho^{\frac{\gamma-1}{2}}\big)=-E+\dfrac{\mu}{2(1+t)^\lambda}(s-r),\\ s_t+s_x\big(u+\rho^{\frac{\gamma-1}{2}}\big)=E-\dfrac{\mu}{2(1+t)^\lambda}(s-r),\\ r(x,0)=\frac{2}{\gamma-1}\big(\rho_0^{\frac{\gamma-1}{2}}(x)-\bar\rho^{\frac{\gamma-1}{2}}\big)-u_0(x)=:r_0(x),\\ s(x,0)=\frac{2}{\gamma-1}\big(\rho_0^{\frac{\gamma-1}{2}}(x)-\bar\rho^{\frac{\gamma-1}{2}}\big)+u_0(x)=:s_0(x). \end{cases} \tag{2.2}$$

Denote

$$A_1(t)=\exp\Big(\int_0^t\frac{\mu}{(1+\tau)^\lambda}d\tau\Big),\qquad A_2(t)=\exp\Big(\int_0^t\frac{\mu}{2(1+\tau)^\lambda}d\tau\Big),\qquad C^*=\sup_{x\in\mathbb{R}}D(x),$$

and

$$\theta=\frac12\Big(\mu+\sqrt{\mu^2+2^{2\gamma/(\gamma-1)}\bar\rho}\Big), \tag{2.3}$$

$$M_0=\frac{2^{2/(\gamma-1)}\bar\rho}{\theta}=\frac12\Big(\sqrt{\mu^2+2^{2\gamma/(\gamma-1)}\bar\rho}-\mu\Big). \tag{2.4}$$

We state the main result of this section as follows.

THEOREM 2.1. *Let $(\rho,u,E)\in C^1(\mathbb{R}\times[0,L])$ for $L>0$ be the solutions to the Cauchy problem (1.1) and (1.3). Suppose that $D(x)=\bar\rho$ for $x<-N_0$, where $N_0>0$ is sufficiently large. If $\lim_{x\to\pm\infty}E(x,0)=0$ and the initial data satisfy*

$$\int_{-\infty}^{+\infty}(|\rho_0(x)-\bar\rho|+|u_0(x)|+|\rho_0(x)-D(x)|)dx<+\infty, \tag{2.5}$$

$$|h_-(0)+h_+(0)-2\bar h+u_0(x_+(0))-u_0(x_-(0))|\le 2K, \tag{2.6}$$

$$|h_+(0)-h_-(0)+u_0(x_-(0))+u_0(x_+(0))|\le 2K, \tag{2.7}$$

and

$$\Big|\int_{-\infty}^x(\rho_0-D)(y)dy\Big|<KM_0, \tag{2.8}$$

for any $K\in(0,\bar{h})$*, where* $x_\pm(t)$ *are the characteristic curves satisfying*

$$\frac{d}{dt}x_\pm(t)=u(x_\pm(t),t)\pm\rho^{\frac{\gamma-1}{2}}(x_\pm(t),t),$$

and $h_\pm(t)$ *and* $\bar{h}$ *are defined by*

$$h_\pm(t)=\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}(x_\pm(t),t),\qquad \bar{h}=\frac{2}{\gamma-1}\bar{\rho}^{\frac{\gamma-1}{2}}.$$

Then, the solutions (ρ,u,E) *are uniformly bounded for* $0<t<t^*:=\frac{1}{\theta}\ln\frac{\bar{h}}{K}$*. Furthermore, for all cases* $\lambda>0$ *and* $\mu>0$*, there exists a sufficiently large positive constant* $N=N(\gamma,\mu,\lambda,C^*,\bar{\rho},K)$ *such that, if* $r_x(x,0)\geq N$ *or* $s_x(x,0)\leq -N$*, then the derivatives of the solutions will blow up before* $t=t_*:=\frac{t^*}{2}$*, that is*

$$\lim_{t\to t_*^-}\|(\rho_x,u_x)(t)\|_{L^\infty}=+\infty.$$

In order to obtain the blow-up result, we first need to derive the uniform boundedness of the solutions to (1.1), (1.3). We prepare the following two lemmas for the boundedness of the solutions.

LEMMA 2.1. *Let* $(\rho,u,E)\in C^1(\mathbb{R}\times[0,L])$ *be the solutions to the Cauchy problem* (1.1) *and* (1.3)*,* ρ *and* u *be uniformly bounded in* $\mathbb{R}\times[0,L]$ *and* $\rho\geq\delta_0>0$*. Suppose that* $D(x)=\bar{\rho}$ *for* $x<-N_0$*, where* $N_0>0$ *is sufficiently large. If* $\lim\limits_{x\to\pm\infty}E(x,0)=0$ *and* (2.5) *holds, then*

$$\lim_{x\to\pm\infty}E(x,t)=0,\qquad \lim_{x\to\pm\infty}\rho(x,t)=\bar{\rho},\qquad \lim_{x\to\pm\infty}u(x,t)=0. \tag{2.9}$$

Proof. Let $x_\pm(t)$ be the plus and minus characteristic curves which satisfy the following differential equations:

$$\frac{d}{dt}x_\pm(t)=u(x_\pm(t),t)\pm\rho^{\frac{\gamma-1}{2}}(x_\pm(t),t). \tag{2.10}$$

Differentiating $r(x_-(t),t)A_1(t)$ with respect to t to get

$$\begin{aligned}\frac{d}{dt}[r(x_-(t),t)A_1(t)]=&A_1(t)\frac{d}{dt}r(x_-(t),t)+\frac{\mu A_1(t)}{(1+t)^\lambda}r(x_-(t),t)\\ =&A_1(t)\big(r_t+r_x\big(u-\rho^{\frac{\gamma-1}{2}}\big)\big)+\frac{\mu A_1(t)}{(1+t)^\lambda}r\\ =&A_1(t)\big(\rho^{\frac{\gamma-3}{2}}\rho_t-u_t+\big(\rho^{\frac{\gamma-3}{2}}\rho_x-u_x\big)\big(u-\rho^{\frac{\gamma-1}{2}}\big)\big)\\ &+\frac{\mu A_1(t)}{(1+t)^\lambda}\left(\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}-\bar{\rho}^{\frac{\gamma-1}{2}}\big)-u\right)\\ =&A_1(t)\left(-E+\frac{\mu}{(1+t)^\lambda}u\right)+\frac{\mu A_1(t)}{(1+t)^\lambda}\left(\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}-\bar{\rho}^{\frac{\gamma-1}{2}}\big)-u\right)\\ =&-A_1(t)E+\frac{\mu A_1(t)}{(1+t)^\lambda}\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}-\bar{\rho}^{\frac{\gamma-1}{2}}\big).\end{aligned} \tag{2.11}$$

Analogous to (2.11), we obtain

$$\frac{d}{dt}[s(x_+(t),t)A_1(t)]=A_1(t)E+\frac{\mu A_1(t)}{(1+t)^\lambda}\frac{2}{\gamma-1}\big(\rho^{\frac{\gamma-1}{2}}-\bar{\rho}^{\frac{\gamma-1}{2}}\big). \tag{2.12}$$

Denote

$$h_\pm(t)=\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}(x_\pm(t),t),\qquad \bar h=\frac{2}{\gamma-1}\bar\rho^{\frac{\gamma-1}{2}}.$$

Integrating (2.11) and (2.12) over $[0,t]$, respectively, one has

$$\begin{aligned} r(x_-(t),t)=&A_1^{-1}(t)r(x_-(0),0)-A_1^{-1}(t)\int_0^t A_1(\tau)E(x_-(\tau),\tau)d\tau\\ &+A_1^{-1}(t)\int_0^t\frac{\mu A_1(\tau)}{(1+\tau)^\lambda}(h_-(\tau)-\bar h)d\tau, \end{aligned}\tag{2.13}$$

$$\begin{aligned} s(x_+(t),t)=&A_1^{-1}(t)s(x_+(0),0)+A_1^{-1}(t)\int_0^t A_1(\tau)E(x_+(\tau),\tau)d\tau\\ &+A_1^{-1}(t)\int_0^t\frac{\mu A_1(\tau)}{(1+\tau)^\lambda}(h_+(\tau)-\bar h)d\tau. \end{aligned}\tag{2.14}$$

Recalling that, at each point, we have two characteristic curves intersecting at (x,t), namely, $x_-(t)=x_+(t)$ for fixed t. Adding (2.13) to (2.14) yields

$$\begin{aligned} h_\pm(t)-\bar h=&\frac12A_1^{-1}(t)(r(x_-(0),0)+s(x_+(0),0))+\frac12A_1^{-1}(t)\int_0^tA_1(\tau)(E(x_+(\tau),\tau)\\ &-E(x_-(\tau),\tau))d\tau+\frac12A_1^{-1}(t)\int_0^t\frac{\mu A_1(\tau)}{(1+\tau)^\lambda}[(h_-(\tau)-\bar h)+(h_+(\tau)-\bar h)]d\tau\\ =&\frac12A_1^{-1}(t)[h_-(0)-\bar h+h_+(0)-\bar h+u_0(x_+(0))-u_0(x_-(0))]\\ &+\frac12A_1^{-1}(t)\int_0^tA_1(\tau)(E(x_+(\tau),\tau)-E(x_-(\tau),\tau))d\tau\\ &+\frac12A_1^{-1}(t)\int_0^t\frac{\mu A_1(\tau)}{(1+\tau)^\lambda}[(h_-(\tau)-\bar h)+(h_+(\tau)-\bar h)]d\tau, \end{aligned}\tag{2.15}$$

and subtracting (2.13) from (2.14) leads

$$\begin{aligned} u(x_\pm(t),t)=&\frac12A_1^{-1}(t)[h_+(0)-\bar h-(h_-(0)-\bar h)+u_0(x_-(0))+u_0(x_+(0))]\\ &+\frac12A_1^{-1}(t)\int_0^tA_1(\tau)(E(x_-(\tau),\tau)+E(x_+(\tau),\tau))d\tau\\ &+\frac12A_1^{-1}(t)\int_0^t\frac{\mu A_1(\tau)}{(1+\tau)^\lambda}[(h_+(\tau)-\bar h)-(h_-(\tau)-\bar h)]d\tau. \end{aligned}\tag{2.16}$$

We first prove $\lim\limits_{x\to-\infty}E(x,t)=0$, $\lim\limits_{x\to-\infty}\rho(x,t)=\bar\rho$ and $\lim\limits_{x\to-\infty}u(x,t)=0$. For any point (x,t_1), there are two characteristic curves

$$x_\pm(t)=x_\pm(\tau)+\int_\tau^t(u(x_\pm(s),s)\pm\rho^{\frac{\gamma-1}{2}}(x_\pm(s),s))ds,$$

where $0\le\tau<t\le t_1$. Since $x_+(t_1)=x_-(t_1)$, we have

$$x_+(\tau)-x_-(\tau)=\int_\tau^{t_1}((u-\rho^{\frac{\gamma-1}{2}})|_{x=x_-(s)}-(u+\rho^{\frac{\gamma-1}{2}})|_{x=x_+(s)})ds.$$

Since ρ and u are uniformly bounded, there is a constant $B>0$, such that

$$|x_+(\tau)-x_-(\tau)|\le B(t_1-\tau). \tag{2.17}$$

It follows from $(1.1)_3$ that

$$\begin{aligned} E(x_-(\tau),\tau)-E(x_+(\tau),\tau) &= \int_{x_+(\tau)}^{x_-(\tau)} (\rho(y,\tau)-D(y))dy \\ &= \int_{x_+(\tau)}^{x_-(\tau)} (\rho(y,\tau)-\bar\rho)dy, \end{aligned} \tag{2.18}$$

if $x_+(\tau)<x_-(\tau)<-N_0$. Note that

$$\begin{aligned} h_\pm(\tau)-\bar h &= \frac{2}{\gamma-1}\int_0^1 \frac{d}{dr}[r\rho+(1-r)\bar\rho]^{\frac{\gamma-1}{2}}dr \\ &= \int_0^1 [r\rho+(1-r)\bar\rho]^{\frac{\gamma-3}{2}}dr(\rho(x_\pm(\tau),\tau)-\bar\rho), \end{aligned}$$

then, there is a constant $N_1>0$, such that

$$|\rho(x_\pm(\tau),\tau)-\bar\rho|\le N_1|h_\pm(\tau)-\bar h|, \tag{2.19}$$

and

$$|\rho(y,\tau)-\bar\rho|\le N_1 f(\tau), \tag{2.20}$$

where

$$f(\tau)=\max_{x(\tau)\in[x_+(\tau),x_-(\tau)]}|h_-(\tau)-\bar h|=\max_{x(\tau)\in[x_+(\tau),x_-(\tau)]}|h_+(\tau)-\bar h|$$

and $x(\tau)$ is a characteristic curve between $[x_+(\tau),x_-(\tau)]$.

For simplicity, we denote

$$w(x)=\frac{1}{2}|h_-(0)+h_+(0)-2\bar h+u_0(x_+(0))-u_0(x_-(0))|.$$

Choosing $x<-N_0-Bt_1$, we deduce, from (2.15), (2.18), (2.17) and (2.20), that

$$\begin{aligned} |h_\pm(t)-\bar h| &\le A_1^{-1}(t)w(x)+\frac{N_1}{2}\int_0^t f(\tau)|x_-(\tau)-x_+(\tau)|d\tau \\ &\quad +\frac{\mu}{2}\int_0^t(|h_+(\tau)-\bar h|+|h_-(\tau)-\bar h|)d\tau \\ &\le w(x)+N_1Bt_1\int_0^t f(\tau)d\tau+\mu\int_0^t f(\tau)d\tau, \end{aligned}$$

or

$$f(t)\le w(x)+(N_1Bt_1+\mu)\int_0^t f(\tau)d\tau,$$

or

$$f(t)\le w(x)\mathrm{e}^{(N_1Bt_1+\mu)t}, \tag{2.21}$$

for $t\le t_1$. Hence, from (2.18), (2.20) and (2.21), for any $x_1<x_2<-N_0-Bt_1$,

$$|E(x_2,t_1)-E(x_1,t_1)|\le\int_{x_1}^{x_2}|\rho(y,t_1)-\bar{\rho}|dy\le N_1\mathrm{e}^{(N_1Bt_1+\mu)t_1}\int_{x_1}^{x_2}w(x)dx. \tag{2.22}$$

By (2.5), the value in right-hand side can be arbitrary small as long as $|x_2|$ is sufficiently large, which implies that $\lim_{x\to-\infty}E(x,t_1)=0$.

Now, we prove $\lim_{x\to-\infty}\rho(x,t)=\bar{\rho}$ and $\lim_{x\to-\infty}u(x,t)=0$. For any $\varepsilon_1>0\,(\varepsilon_1<\bar{h})$ and any fixed $t_1>0$, let $\delta_1=\varepsilon_1/(2\mathrm{e}^{M_1t_1})$, where $M_1=2+2\mu$. There exists a positive constant N_2 such that

$$|u_0(x)|\le\delta_1,\qquad |h_\pm(0)-\bar{h}|\le\delta_1,\qquad |E(x,t)|\le\delta_1/M_1, \tag{2.23}$$

for all $x<-N_2$. We claim that

$$|u(x,t)|<2\delta_1\mathrm{e}^{M_1t},\qquad |h_\pm(t)-\bar{h}|<2\delta_1\mathrm{e}^{M_1t}, \tag{2.24}$$

for all $x<-N_2$. Suppose that (2.24) is not satisfied, then, there exists a point (x_3,t_2) with $x_3\le-N_2-Bt_2-1$ and $t_2<t_1$, such that either

$$|u(x_3,t_2)|=2\delta_1\mathrm{e}^{M_1t_2},\quad\text{or}\quad |h_\pm(t_2)-\bar{h}|=2\delta_1\mathrm{e}^{M_1t_2}, \tag{2.25}$$

but (2.24) is true for $t<t_2$. There are two characteristic curves intersecting at (x_3,t_2), which are denoted by

$$x_\pm(t;x_3)=x_\pm(0;x_3)+\int_0^t(u(x,s)\pm\rho^{\frac{\gamma-1}{2}}(x,s))|_{x=x_\pm(s;x_3)}ds,$$

satisfying $x_+(0;x_3)<x_-(0;x_3)$. In fact, we have $x_-(0;x_3)<-N_2$. Since ρ and u are uniformly bounded, there is a constant $B>0$ such that

$$\begin{aligned}x_-(0;x_3)\le&x_-(t_2;x_3)+\int_0^{t_2}(|u(x_-(s),s)|+|\rho^{\frac{\gamma-1}{2}}(x_-(s),s)|)ds\\ \le&-N_2-Bt_2-1+Bt_2\\ =&-N_2-1.\end{aligned}$$

It follows from (2.15), (2.23) and (2.24) that

$$\begin{aligned}|h_\pm(t_2)-\bar{h}|<&2\delta_1+\int_0^{t_2}\frac{\delta_1}{M_1}d\tau+2\mu\delta_1\int_0^{t_2}\mathrm{e}^{M_1\tau}d\tau\\ \le&2\delta_1+\frac{\delta_1(1+2\mu)}{M_1}(\mathrm{e}^{M_1t_2}-1)\\ \le&2\delta_1+\delta_1(\mathrm{e}^{M_1t_2}-1)\\ \le&2\delta_1\mathrm{e}^{M_1t_2}.\end{aligned}$$

Similarly, we have

$$|u(x_3,t_2)|<2\delta_1\mathrm{e}^{M_1t_2},$$

which is a contradiction to (2.25). Thus, $\lim_{x\to-\infty}\rho(x,t)=\bar{\rho}$ and $\lim_{x\to-\infty}u(x,t)=0$.

Next, we prove $\lim_{x\to+\infty} E(x,t)=0$, $\lim_{x\to+\infty}\rho(x,t)=\bar\rho$ and $\lim_{x\to+\infty}u(x,t)=0$. Integrating $(1.1)_1$ over $(0,t)\times(-\infty,x)$, and integrating $(1.1)_3$ over $(-\infty,x)$, we get

$$E(x,t)=\int_{-\infty}^{x}(\rho_0-D)(y)dy-\int_0^t(\rho u)(x,\tau)d\tau. \tag{2.26}$$

For any $\varepsilon_2>0\,(\varepsilon_2<\bar h)$ and any fixed $t_3>0$, let

$$\delta_2=\varepsilon_2/(2\mathrm{e}^{M_2t_3}), \tag{2.27}$$

where $M_2=\max\{1+2\mu,2((\gamma-1)(\bar h+\varepsilon_2)/2)^{2/(\gamma-1)}\}+1$. Thus, there is a constant $N_3>0$ such that

$$|E(x,0)|\le\delta_2,\qquad |h_\pm(0)-\bar h|\le\delta_2,\qquad |u_0(x)|\le\delta_2,$$

for all $x>N_3$. We claim that

$$|E(x,t)|<\delta_2\mathrm{e}^{M_2t},\qquad |h_\pm(t)-\bar h|<2\delta_2\mathrm{e}^{M_2t},\qquad |u(x,t)|<2\delta_2\mathrm{e}^{M_2t}, \tag{2.28}$$

for all $x>N_3$. Suppose that (2.28) is not satisfied, then, there is a point (x_4,t_4) with $x_4\ge N_3+Bt_4+1$ and $t_4<t_3$, such that either

$$|E(x_4,t_4)|=\delta_2\mathrm{e}^{M_2t_4},\quad \text{or}\quad |h_\pm(t_4)-\bar h|=2\delta_2\mathrm{e}^{M_2t_4},\quad \text{or}\quad |u(x_4,t_4)|=2\delta_2\mathrm{e}^{M_2t_4}, \tag{2.29}$$

but (2.28) is true for $t<t_4$.

Note that, at (x_4,t_4), there are two characteristic curves denoted by

$$x_\pm(t;x_4)=x_\pm(0;x_4)+\int_0^t(u(x,\tau)\pm\rho^{\frac{\gamma-1}{2}}(x,\tau))|_{x=x_\pm(\tau;x_4)}d\tau,$$

satisfying $x_+(0;x_4)<x_-(0;x_4)$. We need to show that $x_+(0;x_4)>N_3$. In fact, since ρ and u are uniformly bounded, there exists a positive constant B such that

$$\begin{aligned}x_+(0;x_4)\ge&x_+(t_4;x_4)-\int_0^{t_4}(|u(x_+(\tau),\tau)|+|\rho^{\frac{\gamma-1}{2}}(x_+(\tau),\tau)|)d\tau\\ \ge&N_3+Bt_4+1-Bt_4\\ =&N_3+1.\end{aligned}$$

By $(2.28)_2$, we have

$$\left[\frac{\gamma-1}{2}(\bar h-\varepsilon_2)\right]^{\frac{2}{\gamma-1}}<\rho(x,t)<\left[\frac{\gamma-1}{2}(\bar h+\varepsilon_2)\right]^{\frac{2}{\gamma-1}}, \tag{2.30}$$

for all $x>N_3$ and $t<t_4$, From (2.26) and (2.30), we obtain, along two characteristic curves $x_\pm(t;x_4)$,

$$\begin{aligned}&|E(x_\pm(t),t)|\le|E(x_\pm(0),0)|+\int_0^t|\rho u|\big|_{x=x_\pm(\tau)}d\tau\\ &<\delta_2+2\delta_2\left[\frac{\gamma-1}{2}(\bar h+\varepsilon_2)\right]^{\frac{2}{\gamma-1}}\int_0^t\mathrm{e}^{M_2\tau}d\tau\le\delta_2\mathrm{e}^{M_2t},\end{aligned} \tag{2.31}$$

for all $t \le t_4$. By (2.15) and (2.31), we find

$$\begin{aligned}|h_\pm(t_4)-\bar{h}| &< 2\delta_2+\delta_2\int_0^{t_4}\mathrm{e}^{M_2\tau}d\tau+2\mu\delta_2\int_0^{t_4}\mathrm{e}^{M_2\tau}d\tau\\ &=2\delta_2+\frac{\delta_2(1+2\mu)}{M_2}(\mathrm{e}^{M_2t_4}-1)\le 2\delta_2\mathrm{e}^{M_2t_4}.\end{aligned}$$

Similarly, we have

$$|u(x_4,t_4)|<2\delta_2\mathrm{e}^{M_2t_4},$$

which is a contradiction to (2.29). Hence, $\lim_{x\to+\infty}E(x,t)=0$, $\lim_{x\to+\infty}\rho(x,t)=\bar{\rho}$ and $\lim_{x\to+\infty}u(x,t)=0$. □

Now we are able to prove the uniform boundedness of the solutions to the Cauchy problem (1.1), (1.3).

Lemma 2.2. *Suppose that conditions in Lemma 2.1 are satisfied. For any $K\in(0,\bar{h})$, if (2.6)–(2.8) hold, then,*

$$|E(x,t)|<M_0K\mathrm{e}^{\theta t},\qquad |h_\pm(t)-\bar{h}|<K\mathrm{e}^{\theta t},\qquad |u(x,t)|<K\mathrm{e}^{\theta t}, \tag{2.32}$$

for

$$0<t<t^*=\frac{1}{\theta}\ln\frac{\bar{h}}{K}. \tag{2.33}$$

Furthermore, ρ satisfies

$$\left(\bar{\rho}^{\frac{\gamma-1}{2}}-\frac{(\gamma-1)}{2}K\mathrm{e}^{\theta t}\right)^{\frac{2}{\gamma-1}}<\rho(x,t)<2^{\frac{2}{\gamma-1}}\bar{\rho},\quad 0<t<t^*. \tag{2.34}$$

Proof. If (2.32) is not satisfied, then there is a point (x_5,t_5) with $t_5\in(0,t^*)$ such that one of (2.32) is an equality at (x_5,t_5) and (2.32) holds for $0<t<t_5$. We denote the two characteristic curves by $x_\pm(t;x_5)$ which intersect at (x_5,t_5).

Note that, by (2.33), we have

$$K\mathrm{e}^{\theta t}<\bar{h}\quad\text{for}\quad t\le t_5.$$

From $|h_\pm(t)-\bar{h}|<K\mathrm{e}^{\theta t}<\bar{h}$ for $t<t_5$, we get

$$\bar{h}-K\mathrm{e}^{\theta t}<h_\pm(t)<2\bar{h},\quad t<t_5,$$

or

$$((\gamma-1)(\bar{h}-K\mathrm{e}^{\theta t})/2)^{2/(\gamma-1)}<\rho(x_\pm(t),t)<2^{2/(\gamma-1)}\bar{\rho},\quad t<t_5.$$

Using (2.26), (2.8) and (2.4), we deduce

$$\begin{aligned}|E(x_5,t_5)| &<KM_0+2^{2/(\gamma-1)}\bar{\rho}K\int_0^{t_5}\mathrm{e}^{\theta\tau}d\tau\\ &=KM_0+\frac{2^{2/(\gamma-1)}\bar{\rho}}{\theta}K(\mathrm{e}^{\theta t_5}-1)\end{aligned}$$

$$
\begin{aligned}
&= KM_0 + KM_0(e^{\theta t_5} - 1) \\
&= KM_0 e^{\theta t_5}.
\end{aligned}
\tag{2.35}
$$

From (2.15), (2.6), and (2.4), we obtain

$$
\begin{aligned}
|h_\pm(t_5) - \bar{h}| &< K + KM_0 \int_0^{t_5} e^{\theta\tau} d\tau + \mu K \int_0^{t_5} e^{\theta\tau} d\tau \\
&= K + \frac{K(M_0+\mu)}{\theta}(e^{\theta t_5} - 1) \\
&= K e^{\theta t_5}.
\end{aligned}
\tag{2.36}
$$

Similarly, it follows from (2.16), (2.7), and (2.4) that

$$
|u(x_5,t_5)| < K e^{\theta t_5}. \tag{2.37}
$$

This is a contradiction with our assumption that one of (2.32) is an equality at (x_5,t_5). Hence, (2.32) holds for $t<t^*$. ☐

Lemma 2.2 enables us to prove Theorem 2.1.

Proof. **(Proof of Theorem 2.1.)** Differentiating $(2.2)_1$ and $(2.2)_2$ with respect to x to give

$$
\begin{cases}
r_{xt} + r_{xx}(u - \rho^{\frac{\gamma-1}{2}}) + r_x\left(u_x - \dfrac{\gamma-1}{2}\rho^{\frac{\gamma-3}{2}}\rho_x\right) = -E_x + \dfrac{\mu}{2(1+t)^\lambda}(s_x - r_x), \\
s_{xt} + s_{xx}(u + \rho^{\frac{\gamma-1}{2}}) + s_x\left(u_x + \dfrac{\gamma-1}{2}\rho^{\frac{\gamma-3}{2}}\rho_x\right) = E_x - \dfrac{\mu}{2(1+t)^\lambda}(s_x - r_x).
\end{cases}
\tag{2.38}
$$

Denote $z=r_x$ and $w=s_x$. Since

$$
u_x = \frac{1}{2}(s_x - r_x), \qquad \rho^{\frac{\gamma-3}{2}}\rho_x = \frac{1}{2}(s_x + r_x),
$$

then, (2.38) can be rewritten as

$$
\begin{cases}
\dfrac{d}{dt}z(x_-(t),t) = \dfrac{\gamma-3}{4}wz + \dfrac{\gamma+1}{4}z^2 + \dfrac{\mu}{2(1+t)^\lambda}(w-z) - E_x, \\
\dfrac{d}{dt}w(x_+(t),t) = -\dfrac{\gamma-3}{4}wz - \dfrac{\gamma+1}{4}w^2 - \dfrac{\mu}{2(1+t)^\lambda}(w-z) + E_x.
\end{cases}
\tag{2.39}
$$

From the Equation $(1.1)_1$, we get

$$
\begin{cases}
\dfrac{d}{dt}\rho(x_-(t),t) = -(\rho w)(x_-(t),t), \\
\dfrac{d}{dt}\rho(x_+(t),t) = (\rho z)(x_+(t),t).
\end{cases}
\tag{2.40}
$$

Multiplying $(2.39)_1$ and $(2.39)_2$ by $A_2(t)\rho^{\frac{\gamma-3}{4}}(x_-(t),t)$ and $A_2(t)\rho^{\frac{\gamma-3}{4}}(x_+(t),t)$, respectively, then by (2.40), one has

$$
\begin{cases}
\dfrac{d}{dt}\left(A_2(t)(\rho^{\frac{\gamma-3}{4}}z)(x_-(t),t)\right) \\
= \dfrac{\gamma+1}{4}A_2(t)\rho^{\frac{\gamma-3}{4}}z^2 + \dfrac{\mu A_2(t)}{2(1+t)^\lambda}\rho^{\frac{\gamma-3}{4}}w - A_2(t)\rho^{\frac{\gamma-3}{4}}E_x, \\
\dfrac{d}{dt}\left(A_2(t)(\rho^{\frac{\gamma-3}{4}}w)(x_+(t),t)\right) \\
= -\dfrac{\gamma+1}{4}A_2(t)\rho^{\frac{\gamma-3}{4}}w^2 + \dfrac{\mu A_2(t)}{2(1+t)^\lambda}\rho^{\frac{\gamma-3}{4}}z + A_2(t)\rho^{\frac{\gamma-3}{4}}E_x.
\end{cases}
\tag{2.41}
$$

Let

$$H_1(t) := A_2(t)\big(\rho^{\frac{\gamma-3}{4}} z\big)(x_-(t),t), \qquad H_2(t) := A_2(t)\big(\rho^{\frac{\gamma-3}{4}} w\big)(x_+(t),t).$$

It follows from (2.40) and (2.41) that

$$\begin{cases} \dfrac{d}{dt}H_1(t) = \dfrac{\gamma+1}{4}A_2^{-1}(t)\rho^{\frac{3-\gamma}{4}}(H_1(t))^2 - \dfrac{\mu A_2(t)}{2(1+t)^\lambda}\dfrac{d}{dt}\theta(\rho) \\ \qquad\qquad -A_2(t)\rho^{\frac{\gamma-3}{4}}(\rho - D(x_-(t))), \\ \dfrac{d}{dt}H_2(t) = -\dfrac{\gamma+1}{4}A_2^{-1}(t)\rho^{\frac{3-\gamma}{4}}(H_2(t))^2 + \dfrac{\mu A_2(t)}{2(1+t)^\lambda}\dfrac{d}{dt}\theta(\rho) \\ \qquad\qquad +A_2(t)\rho^{\frac{\gamma-3}{4}}(\rho - D(x_+(t))), \end{cases} \tag{2.42}$$

where

$$\theta(\rho) = \begin{cases} \dfrac{4}{\gamma-3}\rho^{\frac{\gamma-3}{4}}, & \gamma \neq 3, \\ \ln\rho, & \gamma = 3. \end{cases}$$

Integrating $(2.42)_1$ over $[0,t]$ gives

$$\begin{aligned} H_1(t) =& H_1(0) + \frac{\gamma+1}{4}\int_0^t A_2^{-1}(\tau)\rho^{\frac{3-\gamma}{4}}(H_1(\tau))^2 d\tau - \int_0^t \frac{\mu A_2(\tau)}{2(1+\tau)^\lambda}\frac{d}{d\tau}\theta(\rho)d\tau \\ & - \int_0^t A_2(\tau)\rho^{\frac{\gamma-3}{4}}(\rho - D)d\tau \\ =& H_1(0) + \frac{\mu}{2}\theta(\rho_0) - \frac{\mu A_2(t)}{2(1+t)^\lambda}\theta(\rho) + \int_0^t \frac{\mu A_2(\tau)}{2(1+\tau)^\lambda}\left(\frac{\mu}{2(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right)\theta(\rho)d\tau \\ & - \int_0^t A_2(\tau)\rho^{\frac{\gamma-3}{4}}(\rho - D)d\tau + \frac{\gamma+1}{4}\int_0^t A_2^{-1}(\tau)\rho^{\frac{3-\gamma}{4}}(H_1(\tau))^2 d\tau. \end{aligned}$$

For $\lambda > 0$, $\mu > 0$ and $t \le t_* = \frac{t^*}{2}$, there exist two positive constants C_1 and C_2 which depend only on $\lambda, \mu, \gamma, \bar{\rho}$, K, and C^* such that

$$H_1(t) > H_1(0) - C_1 + C_2\int_0^t (H_1(\tau))^2 d\tau. \tag{2.43}$$

Next we prove that, if $r_x(x_-(0),0) \ge N$ is sufficiently large, (2.43) indicates that $H_1(t)$ will blow up before $t = t_*$. In fact, we consider the integral equation

$$q(t) = \frac{1}{C_2 t_*} + C_2\int_0^t q^2(\tau)d\tau, \tag{2.44}$$

which indicates that

$$q'(t) = C_2 q^2(t), \quad q(0) = \frac{1}{C_2 t_*}.$$

That is

$$C_2 t_* - \frac{1}{q(t)} = C_2 t,$$

which will blow up for $t \le t_*$. We choose $H_1(0) = C_1 + \frac{1}{C_2 t_*} + 1$, subtracting (2.44) from (2.43) to get

$$H_1(t) - q(t) > 1 + C_2 \int_0^t (H_1(\tau) - q(\tau))(H_1(\tau) + q(\tau)) d\tau,$$

thus, $H_1(t) > q(t)$. Hence, $H_1(t)$ will blow up before $t = t_*$. Similarly, if $-s_x(x,0) \ge N$ is sufficiently large, one can get that $-H_2(t)$ will blow up before $t = t_*$. The proof of Theorem 2.1 is completed. □

3. Global existence of the solutions in critical case

In this section, we study the global existence and large-time behavior of solutions to the Cauchy problem (1.1) and (1.3) in the critical case of $\lambda = 1$ with $\mu > 7/3$.

3.1. Reformulation of the problem and main results. Let $J = \rho u$ be the current density, then the system (1.1) with $\lambda = 1$ becomes

$$\begin{cases} \rho_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{\rho} + p(\rho) \right)_x = \rho E - \dfrac{\mu}{1+t} J, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ E_x = \rho - D(x). \end{cases} \tag{3.1}$$

For the technical reason (for example, see [42]), we have to restrict $D(x) \equiv \text{constant} = \hat{D} > 0$ in this section. Then, the asymptotic profile of (3.1) with $D(x) = \hat{D}$ is constant steady-state $(\hat{D}, 0, 0)$.

Precisely, we consider the system

$$\begin{cases} \rho_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{\rho} + p(\rho) \right)_x = \rho E - \dfrac{\mu}{1+t} J, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ E_x = \rho - \hat{D}, \end{cases} \tag{3.2}$$

subjected to the initial value

$$(\rho, J)(x,0) = (\rho_0, J_0)(x) \to (\bar{\rho}, 0) \quad \text{as } x \to \pm\infty. \tag{3.3}$$

In what follows, we are going to prove the global existence and large-time behavior of solutions to the Cauchy problem (3.2)–(3.3) nearby the constant steady-constant $(\hat{D}, 0, 0)$. By Lemma 2.1, if

$$\bar{\rho} = \hat{D}, \qquad \lim_{x \to \pm\infty} E(x,0) = 0, \qquad \int_{-\infty}^{+\infty} (|\rho_0(x) - \hat{D}| + |u_0(x)|) < +\infty, \tag{3.4}$$

then,

$$\lim_{x \to \pm\infty} \rho(x,t) = \hat{D}, \qquad \lim_{x \to \pm\infty} J(x,t) = 0, \qquad \lim_{x \to \pm\infty} E(x,t) = 0.$$

Now, we are able to reformulate the problem (3.2)–(3.3). Let us set

$$\varphi(x,t) = \rho(x,t) - \hat{D}, \qquad \psi(x,t) = J(x,t) - 0, \qquad \omega(x,t) = E(x,t) - 0. \tag{3.5}$$

Then, we arrive at a new system

$$\begin{cases} \varphi_t+\psi_x=0, \\ \psi_t+\left(\dfrac{\psi^2}{\hat{D}+\varphi}+p(\hat{D}+\varphi)\right)_x=(\hat{D}+\varphi)\omega-\dfrac{\mu}{1+t}\psi, \\ \omega_x=\varphi. \end{cases} \tag{3.6}$$

It is easy to verify that

$$\psi=-\omega_t, \qquad \varphi=\omega_x. \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\omega_{tt}+\frac{\mu}{1+t}\omega_t+\hat{D}\omega-(p'(\hat{D})\omega_x)_x=-\omega\omega_x+(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_x+\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_x. \tag{3.8}$$

We integrate $(3.2)_3$ to get

$$E(x,0)=\int_{-\infty}^{x}(\rho_0(y)-\hat{D})dy=:\omega_0(x). \tag{3.9}$$

Therefore, by (3.5), (3.7) and (3.9), we obtain the initial value for the Equation (3.8) as

$$\omega(x,0)=\omega_0(x), \qquad \omega_t(x,0)=-J_0(x). \tag{3.10}$$

Next, we state the second result of this paper as follows.

THEOREM 3.1. *Assume that (3.4) holds. Let $(\omega_0,J_0)(x)\in H^3(\mathbb{R})\times H^2(\mathbb{R})$, and $\Phi_0:=\|\omega_0\|_3+\|J_0\|_2$ is sufficiently small. Then, the Cauchy problem (3.2)–(3.3) in the critical case $\lambda=1$ with $\mu>7/3$ admits a unique global solution $(\rho,J,E)(x,t)$, which satisfies:*

- *When $7/3<\mu\le 3$, then*

$$(1+t)^{\frac{\mu+1}{4}}\left(\|J(t)\|+\|E(t)\|\right)+(1+t)^{\frac{\mu}{2}}\left(\|\rho(t)-\hat{D}\|_2+\|J_x(t)\|_1+\|E_x(t)\|_2\right)\le C\Phi_0; \tag{3.11}$$

- *When $3<\mu\le 4$, then*

$$(1+t)\left(\|J(t)\|+\|E(t)\|\right)+(1+t)^{\frac{\mu}{2}}\left(\|\rho(t)-\hat{D}\|_2+\|J_x(t)\|_1+\|E_x(t)\|_2\right)\le C\Phi_0; \tag{3.12}$$

- *When $\mu>4$, then*

$$(1+t)\left(\|J(t)\|+\|E(t)\|\right)+(1+t)^2\left(\|\rho(t)-\hat{D}\|_2+\|J_x(t)\|_1+\|E_x(t)\|_2\right)\le C\Phi_0. \tag{3.13}$$

Using Sobolev inequality

$$\|f\|_{L^\infty}\le C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}, \tag{3.14}$$

we can further derive following estimates.

COROLLARY 3.1. *Under the conditions of Theorem 3.1, it holds that*

$$\|\rho(t)-\hat{D}\|_{L^\infty}\le\begin{cases} C\Phi_0(1+t)^{-\frac{\mu}{2}}, & \text{for } 7/3<\mu\le 3, \\ C\Phi_0(1+t)^{-\frac{\mu}{2}}, & \text{for } 3<\mu\le 4, \\ C\Phi_0(1+t)^{-2}, & \text{for } \mu>4, \end{cases} \tag{3.15}$$

and

$$\|J(t)\|_{L^\infty} \le \begin{cases} C\Phi_0(1+t)^{-\frac{3\mu+1}{8}}, & \text{for } 7/3<\mu\le 3,\\ C\Phi_0(1+t)^{-\frac{\mu+2}{4}}, & \text{for } 3<\mu\le 4,\\ C\Phi_0(1+t)^{-\frac{3}{2}}, & \text{for } \mu>4. \end{cases} \tag{3.16}$$

Let $T\in(0,+\infty]$, we define the solution space as

$$X(T):=\{\omega(x,t);\ \partial_t^j\omega\in C(0,T;H^{3-j}(\mathbb{R})),\ j=0,1,\ 0\le t\le T\}$$

with the norm

$$N_1(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{\frac{\mu+1}{2}}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{\mu}\|\partial_x^i\omega(t)\|^2 + \sum_{i=1}^{2}(1+t)^{\mu}\|\partial_x^i\omega_t(t)\|^2\Bigg) \tag{3.17}$$

for $7/3<\mu\le 3$; and

$$N_2(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{2}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{\mu}\|\partial_x^i\omega(t)\|^2+\sum_{i=1}^{2}(1+t)^{\mu}\|\partial_x^i\omega_t(t)\|^2\Bigg) \tag{3.18}$$

for $3<\mu\le 4$; and

$$N_3(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{2}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{4}\|\partial_x^i\omega(t)\|^2+\sum_{i=1}^{2}(1+t)^{4}\|\partial_x^i\omega_t(t)\|^2\Bigg) \tag{3.19}$$

for $\mu>4$.

THEOREM 3.2. *Under the conditions of Theorem 3.1. If $N_i(T)\ll 1\,(i=1,2,3)$, then, the Cauchy problem (3.8), (3.10) in the critical case $\lambda=1$ with $\mu>7/3$ admits a unique global solution $\omega(x,t)$, which satisfies:*

- *When $7/3<\mu\le 3$, then*

$$\sum_{i=0}^{1}(1+t)^{\frac{\mu+1}{4}}\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega_t(t)\|\le C\Phi_0; \tag{3.20}$$

- *When $3<\mu\le 4$, then*

$$\sum_{i=0}^{1}(1+t)\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega_t(t)\|\le C\Phi_0; \tag{3.21}$$

- *When $\mu>4$, then*

$$\sum_{i=0}^{1}(1+t)\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^{2}\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^{2}\|\partial_x^i\omega_t(t)\|\le C\Phi_0. \tag{3.22}$$

Proof. **(Proof of Theorem 3.1.)** Once Theorem 3.2 is proved, we can immediately obtain Theorem 3.1. ∎

3.2. Proof of Theorem 3.2. We will prove Theorem 3.2 by employing a standard extension argument based on the local existence and the *a priori* estimates. The local existence of the solution to the Cauchy problem (3.8), (3.10) can be obtained by the iteration method, so we omit its detail. The key step is to establish the *a priori* estimates (3.20)–(3.22) and the continuity arguments. The rest of this subsection is to establish the a priori estimates (3.20)–(3.22).

From (3.17)–(3.19) and Sobolev inequality (3.14), if $N_i(T)\ll 1\,(i=1,2,3)$, we have, for $\mu>7/3$,

$$0<\hat{D}/2\le\hat{D}+\omega_x\le 2\hat{D}. \tag{3.23}$$

We define the norm as follows:

- for $7/3<\mu\le 3$, it is

$$N_{1,\varepsilon_1}(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{\mu-\varepsilon_1}\|\partial_x^i\omega(t)\|^2 \\ +\sum_{i=1}^{2}(1+t)^{\mu-\varepsilon_1}\|\partial_x^i\omega_t(t)\|^2\Bigg), \tag{3.24}$$

  where $0<\varepsilon_1<(3\mu-7)/8$;

- for $3<\mu\le 4$, it is

$$N_{2,\varepsilon_2}(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{2}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{\mu-\varepsilon_2}\|\partial_x^i\omega(t)\|^2 \\ +\sum_{i=1}^{2}(1+t)^{\mu-\varepsilon_2}\|\partial_x^i\omega_t(t)\|^2\Bigg), \tag{3.25}$$

  where $0<\varepsilon_2<1/2$;

- for $\mu>4$, it is

$$N_{3,\varepsilon_3}(T)^2:=\sup_{0\le t\le T}\Bigg(\sum_{i=0}^{1}(1+t)^{2}\|\partial_t^i\omega(t)\|^2+\sum_{i=1}^{3}(1+t)^{4-\varepsilon_3}\|\partial_x^i\omega(t)\|^2 \\ +\sum_{i=1}^{2}(1+t)^{4-\varepsilon_3}\|\partial_x^i\omega_t(t)\|^2\Bigg), \tag{3.26}$$

  where $0<\varepsilon_3<1$.

LEMMA 3.1. *Under the conditions of Theorem 3.2, when $7/3<\mu\le 3$, it holds that*

$$\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\int_{\mathbb{R}}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dx \\ +\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dxd\tau \\ \le C(\|\omega_0\|_1^2+\|J_0\|^2)+CN_{1,\varepsilon_1}(T)^3, \tag{3.27}$$

where $0<\varepsilon_1<(3\mu-7)/8$.

Proof. Multiplying (3.8) by $(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega$ and integrating the resulting equation with respect to x over $\mathbb{R}$ and using integration by parts give

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\bigg[\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\omega_t\\
&\quad+\left(\frac{(\mu+1)^2}{16}+\frac{(\mu+1)\varepsilon_1}{8}\right)(1+t)^{\frac{\mu-3}{2}-\varepsilon_1}\omega^2\bigg]dx\\
&\quad+\frac{\mu-1+\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_t^2dx+\frac{\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}p'(\hat{D})\omega_x^2dx\\
&\quad+\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\hat{D}-\frac{1}{32}(\mu+1)(\mu-3-2\varepsilon_1)(\mu+1+2\varepsilon_1)(1+t)^{\frac{\mu-5}{2}-\varepsilon_1}\right]\omega^2dx\\
&=-\int_{\mathbb{R}}\omega\omega_x\left[(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dx\\
&\quad+\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_x\left[(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dx\\
&\quad+\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_x\left[(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dx.
\end{aligned}\tag{3.28}$$

Integrating (3.28) over $[0,t]$, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\bigg[\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)+\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\omega_t\\
&\quad+\left(\frac{(\mu+1)^2}{16}+\frac{(\mu+1)\varepsilon_1}{8}\right)(1+t)^{\frac{\mu-3}{2}-\varepsilon_1}\omega^2\bigg]dx\\
&\quad+\frac{\mu-1+\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_t^2dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}p'(\hat{D})\omega_x^2dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\bigg[\frac{\varepsilon_1}{2}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\hat{D}\\
&\quad-\frac{1}{32}(\mu+1)(\mu-3-2\varepsilon_1)(\mu+1+2\varepsilon_1)(1+\tau)^{\frac{\mu-5}{2}-\varepsilon_1}\bigg]\omega^2dxd\tau\\
&\le C(\|\omega_0\|_1^2+\|J_0\|^2)-\int_0^t\int_{\mathbb{R}}\omega\omega_x\left[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_x\left[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_x\left[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dxd\tau\\
&=C(\|\omega_0\|_1^2+\|J_0\|^2)+I_1+I_2+I_3.
\end{aligned}\tag{3.29}$$

Next, we focus on estimating I_1, I_2 and I_3 as below. From Hölder inequality, the Sobolev inequality (3.14), and (3.24), we can estimate I_1 as

$$\begin{aligned}
I_1=&-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}((1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_x\omega^2)dxd\tau-\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_x\omega^2dxd\tau\\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_{xt}\omega^2dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\omega_0\|_1^2+(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega\|_{L^\infty}\|\omega\|\|\omega_x\|+\int_0^t(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega\|_{L^\infty}\|\omega\|\|\omega_x\|d\tau\\
&\quad+\int_0^t(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega\|_{L^\infty}\|\omega\|\|\omega_{xt}\|d\tau\\
&\le C\|\omega_0\|_1^2+C(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega\|^{\frac32}\|\omega_x\|^{\frac32}+C\int_0^t(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega\|^{\frac32}\|\omega_x\|^{\frac32}d\tau\\
&\quad+C\int_0^t(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega\|^{\frac32}\|\omega_x\|^{\frac12}\|\omega_{xt}\|d\tau\\
&=C\|\omega_0\|_1^2+C(1+t)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac32}(1+t)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac32}(1+t)^{-\frac12(\frac{\mu+1}{2}-\varepsilon_1)}\\
&\quad+C\int_0^t(1+\tau)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac32}(1+\tau)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac32}(1+\tau)^{-\frac{\mu+5-2\varepsilon_1}{4}}d\tau\\
&\quad+C\int_0^t(1+\tau)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac32}(1+\tau)^{\frac14(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac12}\\
&\quad\cdot(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xt}\|(1+\tau)^{-\frac{\mu-\varepsilon_1}{2}}d\tau\\
&\le C\|\omega_0\|_1^2+CN_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.30}
$$

It follows from Taylor's formula, Hölder inequality, (3.14), and (3.24) that I_2 can be estimated as

$$
\begin{aligned}
I_2&=\int_0^t\int_{\mathbb{R}}(p(\hat D+\omega_x)-p(\hat D)-p'(\hat D)\omega_x)_x\left[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\right]dxd\tau\\
&=-\int_0^t\int_{\mathbb{R}}(p(\hat D+\omega_x)-p(\hat D)-p'(\hat D)\omega_x)\Big[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_{xt}\\
&\qquad+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_x\Big]dxd\tau\\
&\le C\int_0^t\int_{\mathbb{R}}\omega_x^2\big[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}|\omega_{xt}|+(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}|\omega_x|\big]dxd\tau\\
&\le C\int_0^t(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega_x\|_{L^\infty}\|\omega_x\|\|\omega_{xt}\|d\tau+C\int_0^t(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega_x\|_{L^\infty}\|\omega_x\|^2d\tau\\
&\le C\int_0^t(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega_x\|^{\frac32}\|\omega_{xx}\|^{\frac12}\|\omega_{xt}\|d\tau+C\int_0^t(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega_x\|^{\frac52}\|\omega_{xx}\|^{\frac12}d\tau\\
&=C\int_0^t(1+\tau)^{\frac34(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac32}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xx}\|^{\frac12}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xt}\|(1+\tau)^{-\frac{5\mu-1-4\varepsilon_1}{8}}d\tau\\
&\quad+C\int_0^t(1+\tau)^{\frac54(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac52}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xx}\|^{\frac12}(1+\tau)^{-\frac{3\mu+9-4\varepsilon_1}{8}}d\tau\\
&\le CN_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.31}
$$

Now we are going to deal with I_3, by Hölder inequality, (3.14), (3.23), and (3.24), one can obtain

$$
\begin{aligned}
I_3&=-\int_0^t\int_{\mathbb{R}}\frac{\omega_t^2}{\hat D+\omega_x}\left[(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_{xt}+\frac{\mu+1}{4}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_x\right]dxd\tau\\
&\le C\int_0^t(1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega_t\|_{L^\infty}\|\omega_t\|\|\omega_{xt}\|d\tau+C\int_0^t(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega_x\|_{L^\infty}\|\omega_t\|^2d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^t (1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega_t\|^{\frac{3}{2}}\|\omega_{xt}\|^{\frac{3}{2}}d\tau + C\int_0^t (1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\|\omega_x\|^{\frac{1}{2}}\|\omega_{xx}\|^{\frac{1}{2}}\|\omega_t\|^2 d\tau\\
&= C\int_0^t (1+\tau)^{\frac{3}{4}(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_t\|^{\frac{3}{2}}(1+\tau)^{\frac{3(\mu-\varepsilon_1)}{4}}\|\omega_{xt}\|^{\frac{3}{2}}(1+\tau)^{-\frac{5\mu-1-4\varepsilon_1}{8}}d\tau\\
&\quad + C\int_0^t (1+\tau)^{\frac{1}{4}(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega_x\|^{\frac{1}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xx}\|^{\frac{1}{2}}\\
&\quad \cdot (1+\tau)^{\frac{\mu+1}{2}-\varepsilon_1}\|\omega_t\|^2(1+\tau)^{-\frac{3\mu+9-4\varepsilon_1}{8}}d\tau\\
&\le C N_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.32}
$$

Inserting (3.30)–(3.32) into (3.29) to have

$$
\begin{aligned}
&\int_{\mathbb{R}}\bigg[\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}(\omega_t^2 + p'(\hat{D})\omega_x^2 + \hat{D}\omega^2) + \frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\omega_t\\
&\qquad + \left(\frac{(\mu+1)^2}{16} + \frac{(\mu+1)\varepsilon_1}{8}\right)(1+t)^{\frac{\mu-3}{2}-\varepsilon_1}\omega^2\bigg]dx\\
&\quad + \frac{\mu-1+\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\omega_t^2 dxd\tau + \frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}p'(\hat{D})\omega_x^2 dxd\tau\\
&\quad + \int_0^t\int_{\mathbb{R}}\bigg[\frac{\varepsilon_1}{2}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}\hat{D}\\
&\quad - \frac{1}{32}(\mu+1)(\mu-3-2\varepsilon_1)(\mu+1+2\varepsilon_1)(1+\tau)^{\frac{\mu-5}{2}-\varepsilon_1}\bigg]\omega^2 dxd\tau\\
&\le C(\|\omega_0\|_1^2 + \|J_0\|^2) + CN_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.33}
$$

Noting that

$$
\left|\frac{\mu+1}{4}(1+t)^{\frac{\mu-1}{2}-\varepsilon_1}\omega\omega_t\right| \le \frac{1}{4}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\omega_t^2 + \frac{(\mu+1)^2}{16}(1+t)^{\frac{\mu-3}{2}-\varepsilon_1}\omega^2. \tag{3.34}
$$

Since $7/3<\mu\le 3$ and $0<\varepsilon_1<(3\mu-7)/8$, then, the desired estimate (3.27) follows from (3.33) and (3.34). ∎

LEMMA 3.2. *Under the conditions of Theorem 3.2, when* $7/3<\mu\le 3$, *it holds that*

$$
\begin{aligned}
&\frac{1}{2}(1+t)^{\mu-\varepsilon_1}\int_{\mathbb{R}}(\omega_{xt}^2 + p'(\hat{D})\omega_{xx}^2 + \hat{D}\omega_x^2)dx\\
&\quad + \varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(\omega_{xt}^2 + p'(\hat{D})\omega_{xx}^2 + \hat{D}\omega_x^2)dxd\tau\\
&\le C(\|\omega_0\|_2^2 + \|J_0\|_1^2) + CN_{1,\varepsilon_1}(T)^3 + C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-3-\varepsilon_1}\omega_x^2 dxd\tau,
\end{aligned}
\tag{3.35}
$$

where $0<\varepsilon_1<(3\mu-7)/8$.

Proof. Differentiating (3.8) with respect to x yields

$$
\begin{aligned}
&\omega_{xtt} + \frac{\mu}{1+t}\omega_{xt} + \hat{D}\omega_x - p'(\hat{D})\omega_{xxx}\\
&= -\omega_x^2 - \omega\omega_{xx} + (p(\hat{D}+\omega_x) - p'(\hat{D})\omega_x)_{xx} + \left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xx}.
\end{aligned}
\tag{3.36}
$$

Multiplying (3.36) by $(1+t)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x$ and integrating it over $\mathbb{R}$ and using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\omega_{xt}\right.\\
&\quad\left.+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_x^2\right]dx\\
&\quad+\frac{\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\mu-1-\varepsilon_1}\omega_{xt}^2dx+\frac{\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xx}^2dx\\
&\quad+\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+t)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+t)^{\mu-3-\varepsilon_1}\right]\omega_x^2dx\\
=&-\int_{\mathbb{R}}(\omega_x^2+\omega\omega_{xx})\left[(1+t)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\right]dx\\
&\quad+\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_{xx}\left[(1+t)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\right]dx\\
&\quad+\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xx}\left[(1+t)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\right]dx. \qquad (3.37)
\end{aligned}$$

Integrating (3.37) over $[0,t]$, one has

$$\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\omega_{xt}\right.\\
&\quad\left.+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_x^2\right]dx\\
&\quad+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xt}^2dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xx}^2dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+\tau)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+\tau)^{\mu-3-\varepsilon_1}\right]\omega_x^2dxd\tau\\
\le&C(\|\omega_0\|_2^2+\|J_0\|_1^2)-\int_0^t\int_{\mathbb{R}}(\omega_x^2+\omega\omega_{xx})\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_x\right]dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_{xx}\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_x\right]dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xx}\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_x\right]dxd\tau\\
=&C(\|\omega_0\|_2^2+\|J_0\|_1^2)+I_4+I_5+I_6. \qquad (3.38)
\end{aligned}$$

Furthermore, I_4, I_5 and I_6 can be estimated as follows. By Hölder inequality, (3.14), and (3.24), we can estimate I_4 as

$$\begin{aligned}
I_4\le&C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_x\|^{\frac{3}{2}}\|\omega_{xx}\|^{\frac{1}{2}}\|\omega_{xt}\|d\tau+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\omega_{xx}\|\|\omega_{xt}\|d\tau\\
&+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_x\|^{\frac{5}{2}}\|\omega_{xx}\|^{\frac{1}{2}}d\tau+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega\|^{\frac{1}{2}}\|\omega_x\|^{\frac{3}{2}}\|\omega_{xx}\|d\tau\\
=&C\int_0^t(1+\tau)^{\frac{3(\mu-\varepsilon_1)}{4}}\|\omega_x\|^{\frac{3}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xx}\|^{\frac{1}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xt}\|(1+\tau)^{-\frac{\mu-\varepsilon_1}{2}}d\tau
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t(1+\tau)^{\frac{1}{4}(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac{1}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac{1}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xx}\| \\
&\cdot(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xt}\|(1+\tau)^{-\frac{3\mu+1-4\varepsilon_1}{8}}d\tau \\
&+C\int_0^t(1+\tau)^{\frac{5(\mu-\varepsilon_1)}{4}}\|\omega_x\|^{\frac{5}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xx}\|^{\frac{1}{2}}(1+\tau)^{-\frac{\mu+2-\varepsilon_1}{2}}d\tau \\
&+C\int_0^t(1+\tau)^{\frac{1}{4}(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac{1}{2}}(1+\tau)^{\frac{3(\mu-\varepsilon_1)}{4}}\|\omega_x\|^{\frac{3}{2}} \\
&\cdot(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xx}\|(1+\tau)^{-\frac{3\mu+9-4\varepsilon_1}{8}}d\tau\le CN_{1,\varepsilon_1}(T)^3.
\end{aligned}\tag{3.39}
$$

From Taylor's formula, Hölder inequality, (3.14), and (3.24), I_5 can be estimated as

$$
\begin{aligned}
I_5=&-\int_0^t\int_{\mathbb{R}}(p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xx}\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dxd\tau \\
=&-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}((1+\tau)^{\mu-\varepsilon_1}(p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xx}^2)dxd\tau \\
&-\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xx}^2dxd\tau \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}p''(\hat{D}+\omega_x)\omega_{xt}\omega_{xx}^2dxd\tau \\
\le& C\|\omega_0\|_2^2+C(1+t)^{\mu-\varepsilon_1}\|\omega_x\|^{\frac{1}{2}}\|\omega_{xx}\|^{\frac{5}{2}}+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_x\|^{\frac{1}{2}}\|\omega_{xx}\|^{\frac{5}{2}}d\tau \\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xt}\|^{\frac{1}{2}}\|\omega_{xxt}\|^{\frac{1}{2}}\|\omega_{xx}\|^2d\tau \\
=& C\|\omega_0\|_2^2+C(1+t)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac{1}{2}}(1+t)^{\frac{5(\mu-\varepsilon_1)}{4}}\|\omega_{xx}\|^{\frac{5}{2}}(1+t)^{-\frac{\mu-\varepsilon_1}{2}} \\
&+C\int_0^t(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac{1}{2}}(1+\tau)^{\frac{5(\mu-\varepsilon_1)}{4}}\|\omega_{xx}\|^{\frac{5}{2}}(1+\tau)^{-\frac{\mu+2-\varepsilon_1}{2}}d\tau \\
&+C\int_0^t(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xt}\|^{\frac{1}{2}}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_{xxt}\|^{\frac{1}{2}}(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xx}\|^2(1+\tau)^{-\frac{\mu-\varepsilon_1}{2}}d\tau \\
\le& C\|\omega_0\|_2^2+CN_{1,\varepsilon_1}(T)^3.
\end{aligned}\tag{3.40}
$$

Before we estimate I_6, we first show the estimate $\|\omega_{tt}\|^2$. Multiplying (3.8) by ω_{tt} and integrating it over $\mathbb{R}$, we obtain

$$
\int_{\mathbb{R}}\omega_{tt}^2dx=\int_{\mathbb{R}}\left[-\frac{\mu}{1+t}\omega_t-\hat{D}\omega-\omega\omega_x+p(\hat{D}+\omega_x)_x+\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_x\right]\omega_{tt}dx,
$$

it then follows from Hölder inequality and Sobolev inequality (3.14) that

$$
\begin{aligned}
\|\omega_{tt}\|^2\le& C\big((1+t)^{-2}\|\omega_t\|^2+\|\omega\|^2+\|\omega\|\|\omega_x\|^3+\|\omega_{xx}\|^2 \\
&+\|\omega_t\|\|\omega_{xt}\|^3+\|\omega_t\|^2\|\omega_{xt}\|^2\|\omega_{xx}\|^2\big).
\end{aligned}\tag{3.41}
$$

Now, we deal with I_6. By a direct calculation from Hölder inequality, (3.14), (3.23), (3.24), and (3.41) that

$$
I_6=-\int_0^t\int_{\mathbb{R}}\left(\frac{2\omega_t\omega_{xt}}{\hat{D}+\omega_x}-\frac{\omega_t^2\omega_{xx}}{(\hat{D}+\omega_x)^2}\right)\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dxd\tau
$$

$$
\begin{aligned}
=&\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}\left((1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xx}^2}{(\hat{D}+\omega_x)^2}\right)dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_{xt}^3}{\hat{D}+\omega_x}dxd\tau\\
&-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{xt}^2\omega_{xx}}{(\hat{D}+\omega_x)^2}dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\frac{\omega_t^2\omega_{xx}^2}{(\hat{D}+\omega_x)^2}dxd\tau\\
&-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{tt}\omega_{xx}^2}{(\hat{D}+\omega_x)^2}dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xx}^2\omega_{xt}}{(\hat{D}+\omega_x)^3}dxd\tau\\
&-\mu\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\frac{\omega_t\omega_{xt}\omega_{xx}}{\hat{D}+\omega_x}dxd\tau\\
\le& C(\|\omega_0\|_2^2+\|J_0\|_1^2)+C(1+t)^{\mu-\varepsilon_1}\|\omega_t\|\|\omega_{xt}\|\|\omega_{xx}\|^2\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xt}\|^{\frac{5}{2}}\|\omega_{xxt}\|^{\frac{1}{2}}d\tau\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_t\|^{\frac{1}{2}}\|\omega_{xt}\|^{\frac{5}{2}}\|\omega_{xx}\|^{\frac{1}{2}}\|\omega_{xxx}\|^{\frac{1}{2}}d\tau\\
&+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_t\|\|\omega_{xt}\|\|\omega_{xx}\|^2d\tau\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_t\|\|\omega_{xt}\|^{\frac{3}{2}}\|\omega_{xx}\|^2\|\omega_{xxt}\|^{\frac{1}{2}}d\tau\\
&+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_t\|^{\frac{1}{2}}\|\omega_{xt}\|^{\frac{3}{2}}\|\omega_{xx}\|d\tau\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_t\|^{\frac{1}{2}}\|\omega_{xt}\|^{\frac{1}{2}}\|\omega_{xx}\|^{\frac{3}{2}}\|\omega_{xxx}\|^{\frac{1}{2}}\|\omega_{tt}\|d\tau\\
\le& C(\|\omega_0\|_2^2+\|J_0\|_1^2)+CN_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.42}
$$

Substituting (3.39), (3.40), and (3.42) into (3.38) to get

$$
\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\omega_{xt}\right.\\
&\quad\left.+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_x^2\right]dx\\
&\quad+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xt}^2dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xx}^2dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+\tau)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+\tau)^{\mu-3-\varepsilon_1}\right]\omega_x^2dxd\tau\\
\le& C(\|\omega_0\|_2^2+\|J_0\|_1^2)+CN_{1,\varepsilon_1}(T)^3.
\end{aligned}
\tag{3.43}
$$

Recalling that

$$
\left|\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_x\omega_{xt}\right|\le\frac{1}{4}(1+t)^{\mu-\varepsilon_1}\omega_{xt}^2+\frac{\mu^2}{4}(1+t)^{\mu-2-\varepsilon_1}\omega_x^2,\tag{3.44}
$$

combining (3.27) and (3.43)–(3.44), we get the desired estimate (3.35). □

LEMMA 3.3. *Under the conditions of Theorem 3.2, when* $7/3<\mu\le 3$*, it holds that*

$$
\frac{1}{2}(1+t)^{\mu-\varepsilon_1}\int_{\mathbb{R}}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dx
$$

$$+\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dxd\tau$$
$$\le C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{1,\varepsilon_1}(T)^3+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-3-\varepsilon_1}(\omega_x^2+\omega_{xx}^2)dxd\tau, \tag{3.45}$$

where $0<\varepsilon_1<(3\mu-7)/8$.

Proof. Differentiating (3.36) in x to obtain

$$\begin{aligned}&\omega_{xxtt}+\frac{\mu}{1+t}\omega_{xxt}+\hat{D}\omega_{xx}-p'(\hat{D})\omega_{xxxx}\\&=-3\omega_x\omega_{xx}-\omega\omega_{xxx}+(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_{xxx}+\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xxx}.\end{aligned} \tag{3.46}$$

Multiplying (3.46) by $(1+t)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}$ and integrating it over $\mathbb{R}$ and using integration by parts, one can get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\omega_{xxt}\right.\\&\left.\quad+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_{xx}^2\right]dx\\&\quad+\frac{\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\mu-1-\varepsilon_1}\omega_{xxt}^2dx+\frac{\varepsilon_1}{2}\int_{\mathbb{R}}(1+t)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xxx}^2dx\\&\quad+\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+t)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+t)^{\mu-3-\varepsilon_1}\right]\omega_{xx}^2dx\\&=-\int_{\mathbb{R}}(3\omega_x\omega_{xx}+\omega\omega_{xxx})\left[(1+t)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dx\\&\quad+\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_{xxx}\left[(1+t)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dx\\&\quad+\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xxx}\left[(1+t)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dx.\end{aligned} \tag{3.47}$$

Integrating (3.47) over $[0,t]$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\omega_{xxt}\right.\\&\left.\quad+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_{xx}^2\right]dx\\&\quad+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xxt}^2dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xxx}^2dxd\tau\\&\quad+\int_0^t\int_{\mathbb{R}}\left[\frac{\varepsilon_1}{2}(1+\tau)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+\tau)^{\mu-3-\varepsilon_1}\right]\omega_{xx}^2dxd\tau\\&\le C(\|\omega_0\|_3^2+\|J_0\|_2^2)\\&\quad-\int_0^t\int_{\mathbb{R}}(3\omega_x\omega_{xx}+\omega\omega_{xxx})\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dxd\tau\\&\quad+\int_0^t\int_{\mathbb{R}}(p(\hat{D}+\omega_x)-p'(\hat{D})\omega_x)_{xxx}\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dxd\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}\left(\frac{\omega_t^2}{\hat{D}+\omega_x}\right)_{xxx}\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xx}\right]dxd\tau\\
=&C(\|\omega_0\|_3^2+\|J_0\|_2^2)+I_7+I_8+I_9. \qquad (3.48)
\end{aligned}
$$

Next, we estimate I_7, I_8 and I_9 as below. Similar to (3.39), it is easy to compute that

$$
\begin{aligned}
I_7\le& C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\omega_{xx}\|\|\omega_{xxx}\|d\tau+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_x\|^{\frac12}\|\omega_{xx}\|^{\frac52}d\tau\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\omega_{xxx}\|\|\omega_{xxt}\|d\tau\\
&+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_x\|^{\frac12}\|\omega_{xx}\|^{\frac32}\|\omega_{xxt}\|d\tau\\
=&C\int_0^t(1+\tau)^{\frac14(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac12}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac12}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xx}\|\\
&\cdot(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xxx}\|(1+\tau)^{-\frac{3\mu+9-4\varepsilon_1}{8}}d\tau\\
&+C\int_0^t(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac12}(1+\tau)^{\frac{5(\mu-\varepsilon_1)}{4}}\|\omega_{xx}\|^{\frac52}(1+\tau)^{-\frac{\mu+2-\varepsilon_1}{2}}d\tau\\
&+C\int_0^t(1+\tau)^{\frac14(\frac{\mu+1}{2}-\varepsilon_1)}\|\omega\|^{\frac12}(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac12}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xxx}\|\\
&\cdot(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xxt}\|(1+\tau)^{-\frac{3\mu+1-4\varepsilon_1}{8}}d\tau\\
&+C\int_0^t(1+\tau)^{\frac{\mu-\varepsilon_1}{4}}\|\omega_x\|^{\frac12}(1+\tau)^{\frac{3(\mu-\varepsilon_1)}{4}}\|\omega_{xx}\|^{\frac32}(1+\tau)^{\frac{\mu-\varepsilon_1}{2}}\|\omega_{xxt}\|(1+\tau)^{-\frac{\mu-\varepsilon_1}{2}}d\tau\\
\le& CN_{1,\varepsilon_1}(T)^3. \qquad (3.49)
\end{aligned}
$$

Analogous to (3.40), I_8 can be estimated as

$$
\begin{aligned}
I_8=&-\int_0^t\int_{\mathbb{R}}\left((p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xxx}+p''(\hat{D}+\omega_x)\omega_{xx}^2\right)\\
&\cdot\left[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xxx}\right]dxd\tau\\
=&-\frac12\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}((1+\tau)^{\mu-\varepsilon_1}(p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xxx}^2)dxd\tau\\
&-\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(p'(\hat{D}+\omega_x)-p'(\hat{D}))\omega_{xxx}^2dxd\tau\\
&+\frac12\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}p''(\hat{D}+\omega_x)\omega_{xt}\omega_{xxx}^2dxd\tau\\
&-\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}((1+\tau)^{\mu-\varepsilon_1}p''(\hat{D}+\omega_x)\omega_{xx}^2\omega_{xxx})dxd\tau\\
&+\frac{\mu-2\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}p''(\hat{D}+\omega_x)\omega_{xx}^2\omega_{xxx}dxd\tau\\
&+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}p'''(\hat{D}+\omega_x)\omega_{xx}^2\omega_{xt}\omega_{xxx}dxd\tau\\
&+2\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}p''(\hat{D}+\omega_x)\omega_{xx}\omega_{xxt}\omega_{xxx}dxd\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\omega_0\|_3^2+C(1+t)^{\mu-\varepsilon_1}\|\omega_x\|^{\frac12}\|\omega_{xx}\|^{\frac12}\|\omega_{xxx}\|^2\\
&\quad+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_x\|^{\frac12}\|\omega_{xx}\|^{\frac12}\|\omega_{xxx}\|^2d\tau\\
&\quad+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xt}\|^{\frac12}\|\omega_{xxt}\|^{\frac12}\|\omega_{xxx}\|^2d\tau+C(1+t)^{\mu-\varepsilon_1}\|\omega_{xx}\|^{\frac32}\|\omega_{xxx}\|^{\frac32}\\
&\quad+C\int_0^t(1+\tau)^{\mu-1-\varepsilon_1}\|\omega_{xx}\|^{\frac32}\|\omega_{xxx}\|^{\frac32}d\tau+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xx}\|\|\omega_{xt}\|\|\omega_{xxx}\|^2d\tau\\
&\quad+C\int_0^t(1+\tau)^{\mu-\varepsilon_1}\|\omega_{xx}\|^{\frac12}\|\omega_{xxx}\|^{\frac32}\|\omega_{xxt}\|d\tau\\
&\le C\|\omega_0\|_3^2+CN_{1,\varepsilon_1}(T)^3. \qquad (3.50)
\end{aligned}
$$

Before we deal with I_9, we first derive the L^2 estimate of ω_{xtt}. We multiply (3.36) by ω_{xtt} and integrate it over $\mathbb{R}$ to get

$$
\int_{\mathbb{R}}\omega_{xtt}^2dx=\int_{\mathbb{R}}\Big[-\frac{\mu}{1+t}\omega_{xt}-\hat{D}\omega_x-\omega_x^2-\omega\omega_{xx}+p(\hat{D}+\omega_x)_{xx}+\Big(\frac{\omega_t^2}{\hat{D}+\omega_x}\Big)_{xx}\Big]\omega_{xtt}dx,
$$

then,

$$
\begin{aligned}
\|\omega_{xtt}\|^2\le C\big(&(1+t)^{-2}\|\omega_{xt}\|^2+\|\omega_x\|^2+\|\omega_x\|^3\|\omega_{xx}\|+\|\omega\|\|\omega_x\|\|\omega_{xx}\|^2+\|\omega_{xx}\|^3\|\omega_{xxx}\|\\
&+\|\omega_{xxx}\|^2+\|\omega_{xt}\|^3\|\omega_{xxt}\|+\|\omega_t\|\|\omega_{xt}\|\|\omega_{xxt}\|^2+\|\omega_t\|\|\omega_{xt}\|^2\|\omega_{xx}\|^2\|\omega_{xxt}\|\\
&+\|\omega_t\|^2\|\omega_{xt}\|^2\|\omega_{xxx}\|^2+\|\omega_t\|^2\|\omega_{xt}\|^2\|\omega_{xx}\|^3\|\omega_{xxx}\|\big). \qquad (3.51)
\end{aligned}
$$

Now, we estimate I_9. By a direct calculation and using Hölder inequality, (3.14), (3.23), (3.24), and (3.51), we obtain

$$
\begin{aligned}
I_9=&-\int_0^t\int_{\mathbb{R}}\Big(\frac{2\omega_{xt}^2}{\hat{D}+\omega_x}+\frac{2\omega_t\omega_{xxt}}{\hat{D}+\omega_x}-\frac{4\omega_t\omega_{xt}\omega_{xx}}{(\hat{D}+\omega_x)^2}-\frac{\omega_t^2\omega_{xxx}}{(\hat{D}+\omega_x)^2}+\frac{2\omega_t^2\omega_{xx}^2}{(\hat{D}+\omega_x)^3}\Big)\\
&\qquad\cdot\Big[(1+\tau)^{\mu-\varepsilon_1}\omega_{xxxt}+\frac{\mu}{2}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xxx}\Big]dxd\tau\\
=&-\frac{\mu}{2}\int_0^t\int_{\mathbb{R}}\Big(\frac{2\omega_{xt}^2}{\hat{D}+\omega_x}+\frac{2\omega_t\omega_{xxt}}{\hat{D}+\omega_x}-\frac{4\omega_t\omega_{xt}\omega_{xx}}{(\hat{D}+\omega_x)^2}+\frac{2\omega_t^2\omega_{xx}^2}{(\hat{D}+\omega_x)^3}\Big)(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xxx}dxd\tau\\
&+5\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_{xt}\omega_{xxt}^2}{\hat{D}+\omega_x}dxd\tau-5\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{xx}\omega_{xxt}^2}{(\hat{D}+\omega_x)^2}dxd\tau\\
&-6\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_{xt}^2\omega_{xx}\omega_{xxt}}{(\hat{D}+\omega_x)^2}dxd\tau-4\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{xt}\omega_{xxx}\omega_{xxt}}{(\hat{D}+\omega_x)^2}dxd\tau\\
&-6\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xx}^3\omega_{xxt}}{(\hat{D}+\omega_x)^4}dxd\tau+\frac12\int_0^t\int_{\mathbb{R}}\frac{d}{d\tau}\Big((1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xxx}^2}{(\hat{D}+\omega_x)^2}\Big)dxd\tau\\
&+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\frac{\omega_t^2\omega_{xxx}^2}{(\hat{D}+\omega_x)^2}dxd\tau-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{tt}\omega_{xxx}^2}{(\hat{D}+\omega_x)^2}dxd\tau\\
&+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xt}\omega_{xxx}^2}{(\hat{D}+\omega_x)^3}dxd\tau+12\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t\omega_{xt}\omega_{xx}^2\omega_{xxt}}{(\hat{D}+\omega_x)^3}dxd\tau\\
&+4\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-\varepsilon_1}\frac{\omega_t^2\omega_{xx}\omega_{xxx}\omega_{xxt}}{(\hat{D}+\omega_x)^3}dxd\tau
\end{aligned}
$$

$$\leq C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{1,\varepsilon_1}(T)^3. \tag{3.52}$$

Putting (3.49), (3.50) and (3.52) into (3.48) gives

$$\begin{aligned}
&\int_{\mathbb{R}}\Big[\frac{1}{2}(1+t)^{\mu-\varepsilon_1}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\omega_{xxt}\\
&\quad+\frac{\mu}{4}(1+\varepsilon_1)(1+t)^{\mu-2-\varepsilon_1}\omega_{xx}^2\Big]dx\\
&\quad+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}\omega_{xxt}^2dxd\tau+\frac{\varepsilon_1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}p'(\hat{D})\omega_{xxx}^2dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\Big[\frac{\varepsilon_1}{2}(1+\tau)^{\mu-1-\varepsilon_1}\hat{D}-\frac{\mu}{4}(\mu-2-\varepsilon_1)(1+\varepsilon_1)(1+\tau)^{\mu-3-\varepsilon_1}\Big]\omega_{xx}^2dxd\tau\\
&\leq C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{1,\varepsilon_1}(T)^3.
\end{aligned} \tag{3.53}$$

Noting that

$$\Big|\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_1}\omega_{xx}\omega_{xxt}\Big|\leq\frac{1}{4}(1+t)^{\mu-\varepsilon_1}\omega_{xxt}^2+\frac{\mu^2}{4}(1+t)^{\mu-2-\varepsilon_1}\omega_{xx}^2. \tag{3.54}$$

Therefore, the desired estimate (3.45) can be derived from (3.35), (3.53) and (3.54). □

PROPOSITION 3.1. *Under the conditions of Theorem 3.2, when* $7/3<\mu\leq 3$, *it holds that*

$$\sum_{i=0}^{1}(1+t)^{\frac{\mu+1}{4}}\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega_t(t)\|\leq C\Phi_0. \tag{3.55}$$

Proof. It follows from Lemmas 3.1–3.3 that

$$\begin{aligned}
&\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\int_{\mathbb{R}}(\hat{D}\omega^2+p'(\hat{D})\omega_x^2+\omega_t^2)dx\\
&\quad+\frac{1}{2}(1+t)^{\mu-\varepsilon_1}\int_{\mathbb{R}}(\hat{D}\omega_x^2+(p'(\hat{D})+\hat{D})\omega_{xx}^2+\omega_{xt}^2+p'(\hat{D})\omega_{xxx}^2+\omega_{xxt}^2)dx\\
&\quad+\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dxd\tau\\
&\quad+\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(\hat{D}\omega_x^2+(p'(\hat{D})+\hat{D})\omega_{xx}^2+\omega_{xt}^2+p'(\hat{D})\omega_{xxx}^2+\omega_{xxt}^2)dxd\tau\\
&\leq C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{1,\varepsilon_1}(T)^3+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-3-\varepsilon_1}(\omega_x^2+\omega_{xx}^2)dxd\tau.
\end{aligned}$$

Using Gronwall's inequality, we have

$$\begin{aligned}
&\frac{1}{2}(1+t)^{\frac{\mu+1}{2}-\varepsilon_1}\int_{\mathbb{R}}(\hat{D}\omega^2+p'(\hat{D})\omega_x^2+\omega_t^2)dx\\
&\quad+\frac{1}{2}(1+t)^{\mu-\varepsilon_1}\int_{\mathbb{R}}(\hat{D}\omega_x^2+(p'(\hat{D})+\hat{D})\omega_{xx}^2+\omega_{xt}^2+p'(\hat{D})\omega_{xxx}^2+\omega_{xxt}^2)dx\\
&\quad+\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\frac{\mu-1}{2}-\varepsilon_1}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dxd\tau\\
&\quad+\varepsilon_1\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_1}(\hat{D}\omega_x^2+(p'(\hat{D})+\hat{D})\omega_{xx}^2+\omega_{xt}^2+p'(\hat{D})\omega_{xxx}^2+\omega_{xxt}^2)dxd\tau\\
&\leq C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{1,\varepsilon_1}(T)^3.
\end{aligned}$$

Then, by taking $\varepsilon_1\to 0$, and due to $N_1(T)\ll 1$, we can get (3.55). ∎

LEMMA 3.4. *Under the conditions of Theorem 3.2, when $3<\mu\le 4$, it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^2\int_{\mathbb{R}}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dx+(\mu-3)\int_0^t\int_{\mathbb{R}}(1+\tau)(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dxd\tau\\ &\le C(\|\omega_0\|_1^2+\|J_0\|^2)+CN_{2,\varepsilon_2}(T)^3,\end{aligned}\tag{3.56}$$

where $0<\varepsilon_2<1/2$.

Proof. Multiplying (3.8) by $(1+t)^2\omega_t+\frac{\mu-1}{2}(1+t)\omega$, and applying similar argument as in Lemma 3.1, we obtain the desired estimate (3.56). ∎

LEMMA 3.5. *Under the conditions of Theorem 3.2, when $3<\mu\le 4$, it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^{\mu-\varepsilon_2}\int_{\mathbb{R}}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)dx\\ &\quad+\varepsilon_2\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_2}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)dxd\tau\\ &\le C(\|\omega_0\|_2^2+\|J_0\|_1^2)+CN_{2,\varepsilon_2}(T)^3+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-3-\varepsilon_2}\omega_x^2dxd\tau,\end{aligned}\tag{3.57}$$

where $0<\varepsilon_2<1/2$.

Proof. Multiplying (3.36) by $(1+t)^{\mu-\varepsilon_2}\omega_{xt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_2}\omega_x$, similarly to the proof of Lemma 3.2, we get (3.57). ∎

LEMMA 3.6. *Under the conditions of Theorem 3.2, when $3<\mu\le 4$, it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^{\mu-\varepsilon_2}\int_{\mathbb{R}}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dx\\ &\quad+\varepsilon_2\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-1-\varepsilon_2}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dxd\tau\\ &\le C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{2,\varepsilon_2}(T)^3+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\mu-3-\varepsilon_2}(\omega_x^2+\omega_{xx}^2)dxd\tau,\end{aligned}\tag{3.58}$$

where $0<\varepsilon_2<1/2$.

Proof. Multiplying (3.46) by $(1+t)^{\mu-\varepsilon_2}\omega_{xxt}+\frac{\mu}{2}(1+t)^{\mu-1-\varepsilon_2}\omega_{xx}$, and by a similar calculation to Lemma 3.3, we have the desired estimate (3.58). ∎

Combining Lemmas 3.4–3.6 and Gronwall's inequality, we can prove the following proposition, whose proof is similar to Proposition 3.1 and we omit it here.

PROPOSITION 3.2. *Under the conditions of Theorem 3.2, when $3<\mu\le 4$, it holds that*

$$\sum_{i=0}^{1}(1+t)\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^{\frac{\mu}{2}}\|\partial_x^i\omega_t(t)\|\le C\Phi_0.\tag{3.59}$$

LEMMA 3.7. *Under the conditions of Theorem 3.2, when $\mu>4$, it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^2\int_{\mathbb{R}}(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dx\\ &\quad+(\mu-3)\int_0^t\int_{\mathbb{R}}(1+\tau)(\omega_t^2+p'(\hat{D})\omega_x^2+\hat{D}\omega^2)dxd\tau\end{aligned}$$

$$\le C(\|\omega_0\|_1^2+\|J_0\|^2)+CN_{3,\varepsilon_3}(T)^3, \tag{3.60}$$

where $0<\varepsilon_3<1$.

Proof. Multiplying (3.8) by $(1+t)^2\omega_t+\frac{\mu-1}{2}(1+t)\omega$, similarly to the process for deriving Lemma 3.1, we can get (3.60). □

LEMMA 3.8. *Under the conditions of Theorem 3.2, when* $\mu>4$, *it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^{4-\varepsilon_3}\int_{\mathbb{R}}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)dx\\&\quad+\varepsilon_3\int_0^t\int_{\mathbb{R}}(1+\tau)^{3-\varepsilon_3}(\omega_{xt}^2+p'(\hat{D})\omega_{xx}^2+\hat{D}\omega_x^2)dxd\tau\\&\le C(\|\omega_0\|_2^2+\|J_0\|_1^2)+CN_{3,\varepsilon_3}(T)^3,\end{aligned} \tag{3.61}$$

where $0<\varepsilon_3<1$.

Proof. Multiplying (3.36) by $(1+t)^{4-\varepsilon_3}\omega_{xt}+2(1+t)^{3-\varepsilon_3}\omega_x$, and applying similar argument as in Lemma 3.2, we can obtain the desired estimate (3.61). □

LEMMA 3.9. *Under the conditions of Theorem 3.2, when* $\mu>4$, *it holds that*

$$\begin{aligned}&\frac{1}{2}(1+t)^{4-\varepsilon_3}\int_{\mathbb{R}}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dx\\&\quad+\varepsilon_3\int_0^t\int_{\mathbb{R}}(1+\tau)^{3-\varepsilon_3}(\omega_{xxt}^2+p'(\hat{D})\omega_{xxx}^2+\hat{D}\omega_{xx}^2)dxd\tau\\&\le C(\|\omega_0\|_3^2+\|J_0\|_2^2)+CN_{3,\varepsilon_3}(T)^3+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{1-\varepsilon_3}\omega_{xx}^2dxd\tau,\end{aligned} \tag{3.62}$$

where $0<\varepsilon_3<1$.

Proof. Multiplying (3.46) by $(1+t)^{4-\varepsilon_3}\omega_{xxt}+2(1+t)^{3-\varepsilon_3}\omega_{xx}$, similarly to the proof of Lemma 3.3, we have (3.62). □

Combining Lemmas 3.7–3.9 and using Gronwall's inequality, and applying similar argument as in Proposition 3.1, we can prove the following proposition. The detail of proof is also omitted.

PROPOSITION 3.3. *Under the conditions of Theorem 3.2, when* $\mu>4$, *it holds that*

$$\sum_{i=0}^{1}(1+t)\|\partial_t^i\omega(t)\|+\sum_{i=1}^{3}(1+t)^2\|\partial_x^i\omega(t)\|+\sum_{i=1}^{2}(1+t)^2\|\partial_x^i\omega_t(t)\|\le C\Phi_0. \tag{3.63}$$

Proof. **(Proof of Theorem 3.2.)** Propositions 3.1–3.3 imply Theorem 3.2. □

From the Equation (3.8), the inequalities (3.20)–(3.22), and Hölder inequality, we can prove the following lemma and we omit its proof here.

LEMMA 3.10. *Suppose that conditions in Theorem 3.2 are satisfied, then,*

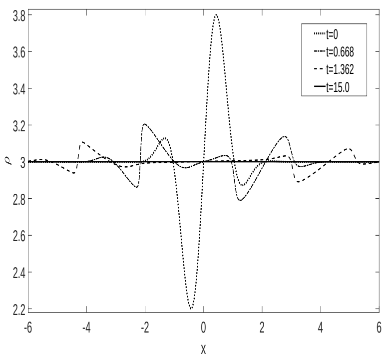
$$\begin{aligned}&(1+t)^{\frac{\mu+1}{4}}(\|\omega_{tt}(t)\|+\|\omega_{ttt}(t)\|)+(1+t)^{\frac{\mu}{2}}\|\omega_{xtt}(t)\|\le C\Phi_0,\quad \textit{for}\ \ 7/3<\mu\le 3,\\&(1+t)(\|\omega_{tt}(t)\|+\|\omega_{ttt}(t)\|)+(1+t)^{\frac{\mu}{2}}\|\omega_{xtt}(t)\|\le C\Phi_0,\quad \textit{for}\ \ 3<\mu\le 4,\\&(1+t)(\|\omega_{tt}(t)\|+\|\omega_{ttt}(t)\|)+(1+t)^2\|\omega_{xtt}(t)\|\le C\Phi_0,\quad \textit{for}\ \ \mu>4.\end{aligned}$$

4. Numerical Simulations

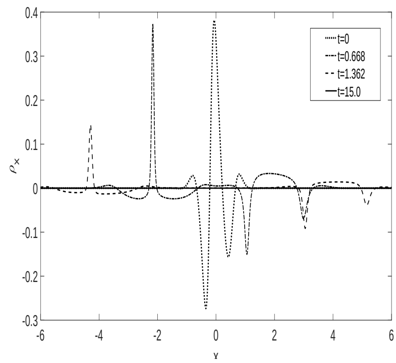
In this section, we present numerical simulations in the critical case of $\lambda=1$ to demonstrate the global solutions for some non-large initial derivatives and the arising of blow-up solution at a point for some large initial derivatives. To obtain a stable numerical solution, we use (2.1), (2.2) and (2.26) to form the following equivalent system:

$$\begin{cases} r_t+r_x(u-\rho^{\frac{\gamma-1}{2}})=-E+\dfrac{\mu}{2(1+t)^{\lambda}}(s-r), \\ s_t+s_x(u+\rho^{\frac{\gamma-1}{2}})=E-\dfrac{\mu}{2(1+t)^{\lambda}}(s-r), \\ E_t=-\dfrac{(s-r)}{2}\left(\dfrac{\gamma-1)(s+r)}{4}+\bar{\rho}^{\frac{\gamma-1}{2}}\right)^{\frac{2}{\gamma-1}} \\ r(x,0)=\dfrac{2}{\gamma-1}(\rho_0^{\frac{\gamma-1}{2}}(x)-\bar{\rho}^{\frac{\gamma-1}{2}})-u_0(x), \\ s(x,0)=\dfrac{2}{\gamma-1}(\rho_0^{\frac{\gamma-1}{2}}(x)-\bar{\rho}^{\frac{\gamma-1}{2}})+u_0(x) \\ E(x,0)=\int_{-\infty}^{x}(\rho_0-D)(y)dy. \end{cases} \tag{4.1}$$

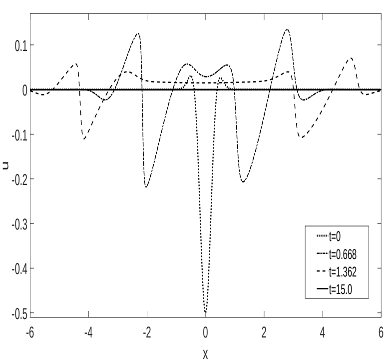
We use Lax-Friedrichs scheme to numerically study the following two examples with $\lambda=1,\mu=3,\gamma=3$ and $D(x)\equiv 3$.



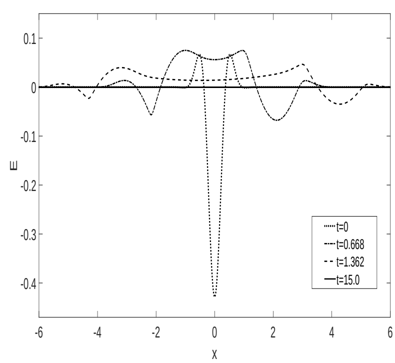
(a) x,ρ plane



(b) x,ρ_x plane

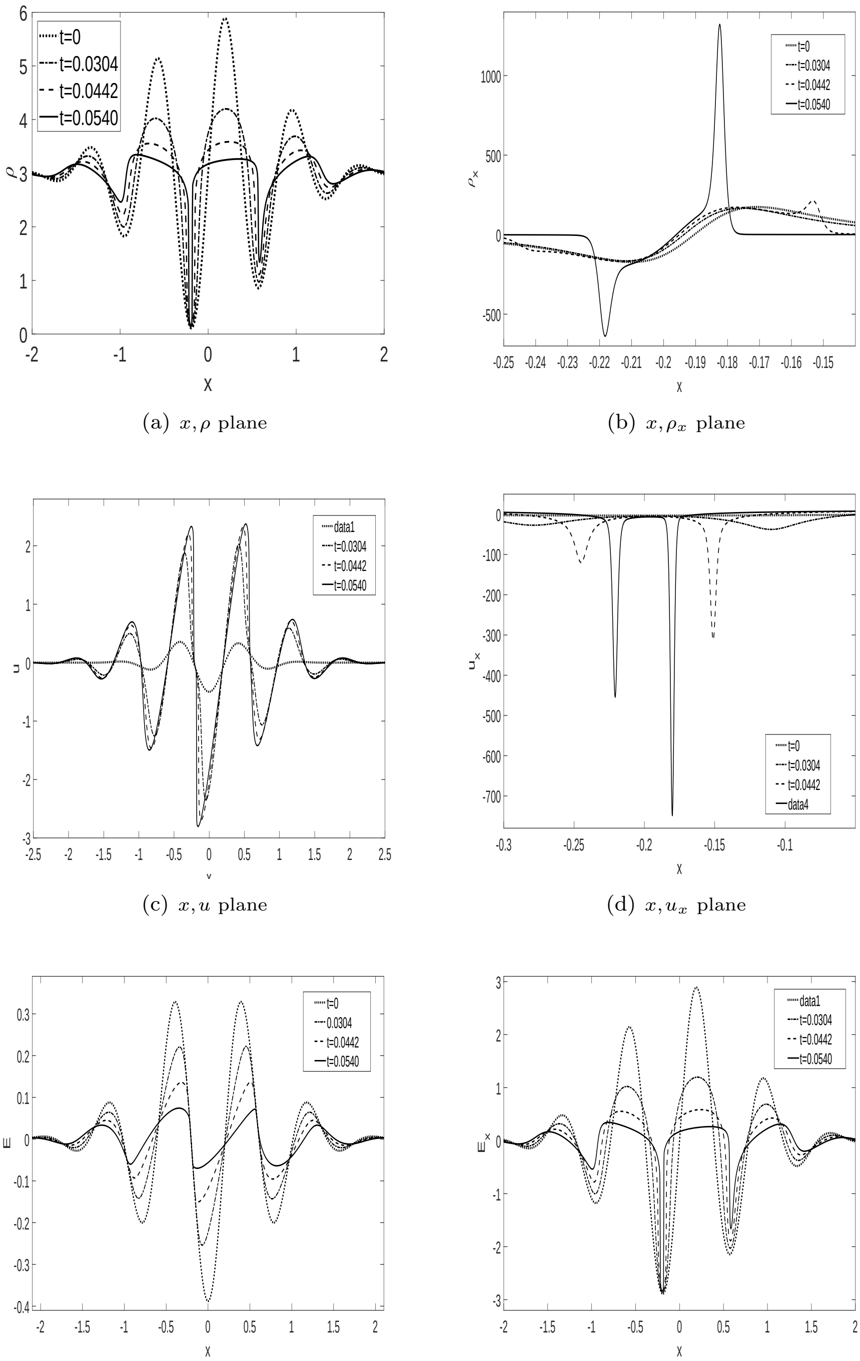


(c) x, u plane



(d) x,E plane

FIG. 4.1. *Global solutions for non-large initial derivatives*



(a) x, ρ plane

(b) x, ρ_x plane

(c) x, u plane

(d) x, u_x plane

(e) x, E plane

(f) x, E_x plane

Fig. 4.2. *Blowup solutions for large initial derivatives*

EXAMPLE 4.1 (Global solutions for non-large initial derivatives). Here, we choose the initial data

$$\rho_0(x)=3+\exp(-x^2)\sin(3x),\ u_0(x)=-0.5\exp(-2x^2)\cos(2x).$$

The computational domain is $[-20,20]$ with Neumann boundary conditions and 100001 uniform mesh points. When $t=15$, the maximum values of ρ, $|u|$ and $|E|$ are all less than 0.0004, which are small enough. The numerical results presented in Figure 4.1 show the global existence of the solutions, which time-asymptotically behave as the steady-states: $(\rho,u,E)(x,t)\to(3,0,0)$ and $(\rho_x,u_x,E_x)(x,t)\to(0,0,0)$ as $t\to\infty$.

EXAMPLE 4.2 (Blowup solutions for large initial derivatives). We choose initial data as

$$\rho_0(x)=3+3\exp(-x^2)\sin(8x),\ u_0(x)=-0.5\exp(-2(x+0.02)^2)\cos(7x).$$

The initial data are steep as showed in Figure 4.2. The computational domain is $[-5,5]$ with Neumann boundary conditions and 500001 uniform mesh points. The solutions $(\rho,u,E)(x,t)$ are bounded, but their derivatives (ρ_x,u_x,E_x) will blow up near $t\approx 0.055$.

Declarations

Ethical Approval: non-applicable for this study.

Competing interests: The authors declare that this work does not have any conflicts of interests.

Authors' contributions: the contributions made by all co-authors are equal.

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