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Large time behaviors of solutions to the unipolar hydrodynamic model of semiconductors with physical boundary effect

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ABSTRACT

In this paper, we study the asymptotic behaviors in time of solutions to the unipolar hydrodynamic model of semiconductors in the form of Euler–Poisson equations on the half line with the boundary effect, where the boundary conditions are proposed physically as the inflow/outflow/impermeable boundary or the insulating boundary. Different from the Cauchy problem, the boundary effect causes some essential difficulties in determining the asymptotic profiles for the solutions and occurs the boundary layers. First of all, by heuristically analyzing, we reasonably propose some additional boundary conditions at far field to the corresponding steady-state equations such that the steady-state systems are well-posed. Thus, we can determine the corresponding steady-states as the expected asymptotic profiles for the solutions of original IBVPs. Secondly, there are some L^2-boundary-layers between the solutions of original IBVPs and their steady-states. After investigating the exact form of gaps, we technically construct some correction functions to delete these gaps. Finally, by using the energy estimates, we further prove that the original solutions of the inflow/outflow/impermeable problem (insulating problem) time-exponentially (time-exponentially/algebraically) converge to their asymptotic profiles. Finally, we carry out some numerical simulations, which show that, the graphs for the asymptotic profiles in different boundary cases are significantly distinct.

1. Introduction

In this paper, we consider the one-dimensional unipolar hydrodynamic model of semiconductors on the half line $\mathbb{R}_+ = [0, \infty)$, represented by the following Euler–Poisson equations

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n) \right)_x = nE - J, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+. \\ E_x = n - D(x), \end{cases} \tag{1.1}$$

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Here, the unknown functions $n(x,t) > 0$, $J(x,t)$ and $E(x,t)$ represent the electron density, the current density and the electric field, respectively. The given function $D(x) > 0$ is the doping profile standing for the density of impurities in semiconductor devices. The smooth pressure function of the electron density $p(n)$ has the properties that

$$p'(s) > 0 \text{ and } s^2 p'(s) \text{ is strictly increasing for } s > 0. \tag{1.2}$$

In gas dynamics, $C := \sqrt{p'(n)}$ is the sound speed. Hence, the flow of (1.1) is said to be subsonic/sonic/supersonic if the electronic velocity $u := \frac{J}{n}$ satisfies

$$\sqrt{p'(n)} \gtreqless |u| \Longleftrightarrow n^2 p'(n) \gtreqless J^2,$$

where we call the electron density n the corresponding subsonic/sonic/supersonic density. As shown in [1], the condition (1.2) guarantees $(1.1)_1$–$(1.1)_2$ to be hyperbolic and fully subsonic under consideration.

This paper is concerned with the global existence and large time asymptotic behaviors of the solutions to the initial boundary value problem (IBVP) for the system (1.1). The initial data of (1.1) are prescribed as

$$(n, J)(x, 0) = (n_0, J_0)(x) \to (n_+, J_+) \quad \text{as } x \to \infty, \tag{1.3}$$

where $n_+ > 0$ and J_+ are state constants for the quantities at $x = \infty$. We mainly consider two types of conditions for the current density J on the boundary $x = 0$. The first one is the inflow/outflow/impermeable boundary:

$$J(0,t) = J_*, \quad E(0,t) = E_*. \tag{1.4}$$

In physics, the constant $J_* > 0$ means inflow, $J_* < 0$ means outflow, and $J_* = 0$ means that the boundary is impermeable. The second type is the insulating boundary:

$$J_x(0,t) = 0, \quad \lim_{x\to\infty} E(x,t) = f(t). \tag{1.5}$$

Here $f(t)$ is a continuous function with either the exponential or algebraic decay:

$$|f(t) - E^*| \le \begin{cases} O(e^{-\theta_1 t}) & \text{for } \theta_1 > 0, \\ O((1+t)^{-\theta_2}) & \text{for } \theta_2 > 1, \end{cases} \quad \text{as} \quad t \to \infty, \tag{1.6}$$

where E^* is a constant. In addition, we of course assume that the initial data (1.3) satisfy the boundary conditions (1.4) and (1.5) as the compatibility conditions.

The hydrodynamic model of semiconductors is usually used in the description of the charged particles such as electrons and holes in semiconductor devices [2–4], and positively and negatively charged ions in plasma [5]. Since its first introduction by Bløtekjær [2], it has been one of hot spots in mathematical physics because of its capability of modeling hot electron effects which are not accounted for in the classical drift–diffusion model. Recently, there have been many studies on the unipolar hydrodynamic model of semiconductors. For its steady-state system, Degond and Markowich [6,7] discussed the existence and uniqueness of subsonic solutions on bounded domain in one-dimensional case and in three-dimensional case for irrational flow (see also [8]), respectively. Markowich [9] investigated the existence and uniqueness of subsonic solutions in two-dimensional case. The steady subsonic flows with different boundaries were also studied in [10–12]. For the case of the transonic/supersonic flow, we refer to [13–19]. There are also many achievements on the asymptotic behavior of the solutions to the one-dimensional unipolar hydrodynamic model of semiconductors, see [1,11,20–25]. Among them, for the Cauchy problem to (1.1), Luo, Natalini and Xin [21] investigated the global existence of smooth solutions and obtained their convergence to the stationary solutions of the drift–diffusions equations under the condition $J(+\infty, 0) = J(-\infty, 0) = E(-\infty, 0) = 0$. This stiff condition physically stands for the switch-off case. For the switch-on case

$J(+\infty,0)\neq J(-\infty,0)$, Huang, Mei, Wang and Yu [20] successfully obtained global existence of the smooth solutions and their convergence to the corresponding stationary solutions by technically constructing a new kind of correction functions to remove the gaps. Regarding the study on the multidimensional case, relaxation limits, shock schemes and entropy solutions, we refer to [26–38]. There are also many papers about the studies on the bipolar hydrodynamic model of semiconductors, see [39–47] and the references therein.

Note that, there are few results about the unipolar hydrodynamic model of semiconductors with boundary effect as far as we know. Motivated by [44,48], we investigate the large time asymptotic behaviors of the solutions to (1.1) with the above two types of boundary conditions on $x=0$. For the Cauchy problem to (1.1), the solutions were shown in [20] to time-exponentially converge to their stationary solutions, the so-called stationary waves. Inspired by this, we naturally expect the asymptotic profiles $(\tilde n,\tilde J,\tilde E)(x)$ to (1.1) satisfying the steady-state equations

$$\begin{cases}\tilde J_x=0,\\ \left(\dfrac{\tilde J^2}{\tilde n}+p(\tilde n)\right)_x=\tilde n\tilde E-\tilde J, \quad x\in\mathbb{R}_+,\\ \tilde E_x=\tilde n-D(x),\end{cases} \tag{1.7}$$

with the corresponding boundary conditions, either the inflow/outflow/impermeable boundary:

$$\tilde J(0)=J_*,\quad \tilde E(0)=E_*, \tag{1.8}$$

or the insulating boundary:

$$\tilde E_x(0)=n_0(0)-D(0),\quad \lim_{x\to\infty}\tilde E(x)=E^*. \tag{1.9}$$

Here, for the insulating problem, we derive $n_t(0,t)=0$ from $(1.1)_1$ by $J_x(0,t)=0$, i.e. $n(0,t)=n_0(0)$, and further have $E_x(0,t)=n_0(0)-D(0)$ from the third equation of (1.1).

However, different from the Cauchy problem to (1.1), the boundary effect causes some essential difficulties in determining the asymptotic profiles for the solutions and occurs the boundary layers. In fact, the system (1.7) with the inflow/outflow/impermeable boundary condition (1.8) or the insulating boundary (1.9) may not be well-posed, when we look for the bounded solutions in C^2.

Indeed, for the system (1.7) subjected to (1.8), from the first equation of (1.7), we have $\tilde J\equiv J_*$. Dividing the second equation of (1.7) by $\tilde n$, and substituting $\tilde n=\tilde E_x+D(x)$ from the third equation to the second one, we have the following equation for the electric field $\tilde E$:

$$\left[\frac{p'(\tilde E_x+D(x))}{\tilde E_x+D(x)}-\frac{J_*^2}{(\tilde E_x+D(x))^3}\right](\tilde E_{xx}+D'(x))=\tilde E-\frac{J_*}{\tilde E_x+D(x)}. \tag{1.10}$$

This could be a uniform second-order elliptic equation under the consideration of subsonic flow, namely, $\tilde E_x$ is small enough and the doping profile is subsonic which satisfies

$$D^2p'(D)>J_*^2.$$

But such an equation with the boundary condition (1.8) is under-determined, because we only have one boundary condition $\tilde E(0)=E_*$. Now, we have to reasonably add one more condition for $\tilde E$ at far field $x=\infty$. Heuristically, we look for the solutions such that

$$\lim_{x\to\infty}\tilde n_x(x)=0,$$

once we focus on the solution space in Sobolev space $H^2(\mathbb{R}_+)$. Then, from the original initial data (1.3), we can expect

$$\lim_{x\to\infty}\tilde n(x)=n_+.$$

So, from the second equation of (1.7), by taking the limit as $x \to \infty$, we formally obtain the other boundary condition

$$\lim_{x\to\infty} \tilde{E}(x) = \frac{J_*}{n_+}. \tag{1.11}$$

With these two boundary conditions, the second-order elliptic equation (1.10) is well-posed, therefore the asymptotic profile can be uniquely determined by the system of (1.7), (1.8) and (1.11). For details, we refer to the next section.

For the system (1.7) subjected to (1.9), although there have been two boundary conditions for $\tilde{E}$ in (1.9), the system (1.7) with the insulating boundary (1.9) is also ill-posed because the value of $\tilde{J}$ has not been specified. The same as the inflow/outflow/impermeable problem, we also look for the solutions satisfying

$$\lim_{x\to\infty} \tilde{n}_x(x) = 0 \text{ and } \lim_{x\to\infty} \tilde{n}(x) = n_+.$$

By taking the limit as $x \to \infty$ to the second equation of (1.7), we get

$$\tilde{J} = n_+ E^*. \tag{1.12}$$

Thus, the corresponding equation for the electric field $\tilde{E}$ turns into

$$\left[\frac{p'(\tilde{E}_x + D(x))}{\tilde{E}_x + D(x)} - \frac{(n_+E^*)^2}{(\tilde{E}_x + D(x))^3}\right](\tilde{E}_{xx} + D'(x)) = \tilde{E} - \frac{n_+E^*}{\tilde{E}_x + D(x)}. \tag{1.13}$$

Meanwhile, if $\tilde{E}_x$ is small enough and the doping profile is subsonic which satisfies

$$D^2 p'(D) > (n_+E^*)^2,$$

Eq. (1.13) could also be uniformly second-order elliptic and well-posed with the boundary (1.9). Now, the corresponding asymptotic profiles for the solutions of original insulating IBVP can be uniquely determined by the system (1.7), (1.9) with (1.12) .

Another difficult issue for the system (1.1) is about boundary layers caused by the boundary effect, namely, there are some gaps between the solutions of original IBVP and their asymptotic profiles (stationary solutions) in L^2-sense. For example, for the inflow/outflow/impermeable problem, it will cause a boundary layer between $\tilde{J}(x) \equiv J_*$ and $J_0(x)$ at $x = \infty$ when $J_* \neq J_+$ (switch-on case). In order to delete these gaps, inspired by [43,44], we first heuristically analyze the explicit forms for the gaps, then we technically construct some correction functions to delete these gaps.

Finally, after carefully setting the perturbed system around the corresponding asymptotic profiles, we prove that the solutions of original IBVP converge to their corresponding asymptotic profiles by using energy method. More precisely, when the initial perturbation around the stationary solution $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ is small enough, we get the following results:

- The solutions of the inflow/outflow/impermeable problem globally exist and converge to the corresponding stationary solutions in the form

  $$\|(n-\tilde{n}, J-\tilde{J}, E-\tilde{E})(t)\|_\infty = Ce^{-\sigma_1 t}$$

  for some $\sigma_1 > 0$.
- The solutions of the insulating problem globally exist and converge to the corresponding stationary solutions in the forms

  $$\|(n-\tilde{n}, J-\tilde{J}, E-\tilde{E})(t)\|_\infty = \begin{cases} Ce^{-\sigma_2 t} & \text{as } |f(t)-E^*| = O(e^{-\theta_1 t}), \quad \theta_1 > 0, \\ C(1+t)^{-\theta_2} & \text{as } |f(t)-E^*| = O((1+t)^{-\theta_2}), \quad \theta_2 > 1, \end{cases}$$

  for some $\sigma_2 > 0$.

Throughout this paper, let $D^* := \sup_{x\ge 0} D(x)$, $D_* := \inf_{x\ge 0} D(x)$ and assume that

$$\lim_{x\to\infty} D(x) = n_+ \quad \text{and} \quad D - n_+ \in H^2(\mathbb{R}_+). \tag{1.14}$$

The remaining part of this paper is organized as follows. After we state the notations, in Section 2, we treat the inflow/outflow/impermeable problem. In Section 3, we consider the insulating boundary problem. Finally, in Section 4, we carry out some numerical simulations, which show that, the graphs for the asymptotic profiles in different boundary cases are significantly distinct.

Notations. Throughout this paper, $(\tilde n, \tilde J, \tilde E)(x)$ denotes the stationary solution which is the solution of the steady-state equation. $(\hat n, \hat J, \hat E)(x,t)$ denotes the correction function. $C > 0$ denotes a generic constant while $C_0, \tilde C_0, d_{11}\cdots$ represent some specific positive constants. The derivatives of f are denoted by f_x, f_{xx} or $\partial_x^k f$ $(k = 0,1,2,\ldots)$. $L^p(\mathbb{R}_+)$ $(1 \le p \le \infty)$ is the usual Lebesgue space with the norm $\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}$ for $1\le p<\infty$, and $\|f\|_\infty = \sup_{x\ge 0}|f(x)|$ for $p=\infty$. Sometimes, we write $\|f\| = \|f\|_2$ for brevity. $H^k(R_+)$ $(k\ge 0)$ is the usual Sobolev space with the norm $\|f\|_{H^k} = \left(\sum_{i=0}^k \int_0^\infty |\partial_x^i f|^2 dx\right)^{\frac{1}{2}}$. For simplicity, we also denote $\|(f,g,k)\|^2 = \|f\|^2 + \|g\|^2 + \|h\|^2$. Let $T>0$ and $\mathcal{B}$ be a Banach space, $C([0,T];\mathcal{B})$ is the space of $\mathcal{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathcal{B})$ is the space of $\mathcal{B}$-valued L^2-functions on $[0,T]$. The other spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ can be defined similarly.

2. The inflow/outflow/impermeable problem

In this section, we first prove the existence and uniqueness of solutions to the corresponding steady-state equations with well-proposed boundary conditions, which are the expected asymptotic profiles for the original solutions to (1.1) with the inflow/outflow/impermeable boundary conditions. Secondly, we look into the explicit forms for the gaps between the solutions of original IBVP and the steady-state solutions at $x=\infty$. To delete the gaps, we construct the correction functions, so that we can deal with the perturbed system in an L^2-framework. Finally, we show that the solutions of original IBVP time-exponentially converge to their asymptotic profiles by the energy method.

2.1. Asymptotic profiles

As explained before, the asymptotic profile $(\tilde n, \tilde J, \tilde E)(x)$ to the IBVP (1.1), (1.3) and (1.4) satisfies the steady-state system

$$\begin{cases} \tilde J = J_*, \\ \left(p'(\tilde n) - \dfrac{\tilde J^2}{\tilde n^2}\right)\tilde n_x = \tilde n \tilde E - \tilde J, \\ \tilde E_x = \tilde n - D(x), \\ \lim\limits_{x\to\infty} \tilde n(x) = n_+, \ \tilde E(0) = E_*, \ \lim\limits_{x\to\infty} \tilde E(x) = \dfrac{J_*}{n_+}, \end{cases} \qquad x\in\mathbb{R}_+. \tag{2.1}$$

Since $s^2 p'(s)$ is increasing for $s>0$ (see (1.2)), we can conclude that there is a minima value $\check n_1 = \check n_1(|\tilde J| = |J_*|) \ge 0$ such that $\check n_1^2 p'(\check n_1) - J_*^2 = 0$ and for some constant $\check n_1^* > \check n_1$, it holds that

$$\tilde n^2 p'(\tilde n) - J_*^2 \ge C_1^* > 0 \quad \text{for } \tilde n \ge \check n_1^* > \check n_1,$$

for a positive constant C_1^*. This means that $\tilde n$ is the subsonic solution. Especially, $\check n_1(|\tilde J| = 0) = 0$. The existence of solutions to the system (2.1) is submitted in the following Theorem.

Theorem 2.1 (*Asymptotic Profile*)**.** *Assume that (1.2), (1.14) and $D_* \ge \check{n}_1^* > \check{n}_1$ hold. If*

$$\eta_1 := \left|E_* - \frac{J_*}{n_+}\right|^{\frac{1}{2}} + |J_*| + \|D - n_+\|_{H^2} \ll 1,$$

then the system (2.1) possesses a unique solution $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ in the space

$$V_1 := \left\{ (\tilde{n}, \tilde{J}, \tilde{E})(x);\ \tilde{J} \equiv J_*,\ \left(\tilde{n} - n_+, \tilde{E} - \frac{J_*}{n_+}\right) \in H^2(\mathbb{R}_+) \times H^3(\mathbb{R}_+) \right. \\ \left. \text{with } \tilde{n} \ge \check{n}_1^* > \check{n}_1,\ \|n - n_+\|_{H^2} \le \tilde{C}_0\eta_1,\ \text{and } \left\|\tilde{E} - \frac{J_*}{n_+}\right\|_{H^3} \le \tilde{C}_0\eta_1 \right\}, \tag{2.2}$$

where $\tilde{C}_0$ is a positive constant dependent on J_*, D_* and D^*. Moreover, it holds that $(\tilde{n}, \tilde{E}) \in C^1(\mathbb{R}_+) \times C^2(\mathbb{R}_+)$ and

$$\frac{J_*}{n_+} - E_* = \int_0^\infty (\tilde{n} - D)(x)dx, \tag{2.3}$$

$$\|\tilde{n} - n_+\|_\infty + \|\tilde{n}_x\|_\infty \le C\eta_1, \tag{2.4}$$

$$\left\|\tilde{E} - \frac{J_*}{n_+}\right\|_\infty + \|\tilde{E}_x\|_\infty + \|\tilde{E}_{xx}\|_\infty \le C\eta_1. \tag{2.5}$$

In order to obtain the solutions of the system (2.1), we consider the BVP (1.10) with the two well-posed boundary conditions of $\tilde{E}$. To simplify, we set

$$\bar{E}(x) := \tilde{E}(x) - \frac{J_*}{n_+}.$$

Now, the BVP (1.10) with the boundary condition $\tilde{E}(0) = E_*$ and (1.11) is equivalent to the system

$$\begin{cases} P(\bar{E}_x + D, J_*)\bar{E}_{xx} = \bar{E} + \dfrac{J_*}{n_+} - \dfrac{J_*}{\bar{E}_x + D} - P(\bar{E}_x + D, J_*)D', \\ \bar{E}(0) = E_* - \dfrac{J_*}{n_+}, \quad \lim\limits_{x\to\infty} \bar{E}(x) = 0. \end{cases} \tag{2.6}$$

Here, for convenience

$$P(s; J_*) := \frac{s^2 p'(s) - J_*^2}{s^3}.$$

The existence of the solutions to the system (2.6) is stated in the following Lemma.

Lemma 2.1. *Assume that (1.2), (1.14) and $D_* > \check{n}_1$ hold. If $\eta_1 \ll 1$, then the system (2.6) has a unique solution $\bar{E} \in H^3(\mathbb{R}_+)$ satisfying*

$$\|\bar{E}\|_{H^3} \le \tilde{C}_1\eta_1.$$

Proof. We define a set

$$W := \{\varphi \in H^3(\mathbb{R}_+); \lim_{x\to\infty} \varphi(x) = 0 \text{ and } \|\varphi\|_{H^3} \le \tilde{C}_1\eta_1\}$$

and a mapping $T : \mathcal{M} \mapsto \mathcal{E}$ over W by solving the linear problem

$$\begin{cases} P(\mathcal{M}_x + D, J_*)\mathcal{E}_{xx} = \mathcal{E} + \dfrac{J_*}{n_+(\mathcal{M}_x + D)}(\mathcal{E}_x + D - n_+) - P(\mathcal{M}_x + D, J_*)D', \\ \mathcal{E}(0) = E_* - \dfrac{J_*}{n_+}, \quad \lim\limits_{x\to\infty} \mathcal{E}(x) = 0. \end{cases} \tag{2.7}$$

Apparently, the linear system (2.7) possesses a unique solution $\mathcal{E} \in H^3(\mathbb{R}_+)$. Since $D_* > \check{n}_1$ and $\|\mathcal{M}_x\|_\infty \leq \|\mathcal{M}_x\|_{H^1} \leq \tilde{C}_1\eta_1$, we can get $\check{n}_1 < D_* - C\eta_1 \leq \mathcal{M}_x + D < D^* + D_* - \check{n}_1$ by $\eta_1 \ll 1$. Then, from (1.2), we have

$$0 < a_1 \leq P(\mathcal{M}_x + D, J_*) \leq a_2, \tag{2.8}$$

where a_1 and a_2 are two positive constants only dependent on J_*, D_* and D^*. Multiply the equation in (2.7) by $\mathcal{E}$ and integrate the resulting equation over $[0, \infty)$. Integration by parts leads to

$$\begin{aligned} &\int_0^\infty \mathcal{E}^2 dx + \int_0^\infty P(\mathcal{M}_x + D, J_*)\mathcal{E}_x^2 dx \\ &= -P(\mathcal{M}_x + D, J_*)\mathcal{E}\mathcal{E}_x|_{x=0} - \int_0^\infty P(\mathcal{M}_x + D, J_*)_x \mathcal{E}\mathcal{E}_x dx \\ &\quad + \int_0^\infty P(\mathcal{M}_x + D, J_*)D'\mathcal{E} dx - \int_0^\infty \frac{J_*}{n_+(\mathcal{M}_x + D)}(\mathcal{E}_x + D - n_+)\mathcal{E} dx. \end{aligned} \tag{2.9}$$

For convenience, all constants C in the following proof represent different positive constants only dependent on J_*, D_* and D^*. By Sobolev inequality and Cauchy–Schwarz inequality, we can estimate the boundary term as

$$\begin{aligned} -P(\mathcal{M}_x + D, J_*)\mathcal{E}\mathcal{E}_x|_{x=0} &\leq C|\mathcal{E}(0)|^{\frac{1}{2}}(|\mathcal{E}(0)| + \|\mathcal{E}_x\|_\infty^2) \\ &\leq C\left|E_* - \frac{J_*}{n_+}\right|^{\frac{1}{2}}\left(\left|E_* - \frac{J_*}{n_+}\right| + \|\mathcal{E}_x\|\|\mathcal{E}_{xx}\|\right) \\ &\leq C\eta_1^3 + C\eta_1(\|\mathcal{E}_x\|^2 + \|\mathcal{E}_{xx}\|^2). \end{aligned}$$

For the other terms in the right hand side of (2.9), we have

$$\begin{aligned} -\int_0^\infty P(\mathcal{M}_x + D, J_*)_x \mathcal{E}\mathcal{E}_x dx &\leq C\int_0^\infty |(\mathcal{M}_{xx} + D')\mathcal{E}\mathcal{E}_x| dx \\ &\leq C(\|\mathcal{M}_{xx}\|_\infty + \|D'\|_\infty)(\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2) \\ &\leq C\eta_1(\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2), \end{aligned}$$

$$\int_0^\infty P(\mathcal{M}_x + D, J_*)D'\mathcal{E} dx \leq \frac{1}{2}\|\mathcal{E}\|^2 + C\|D'\|^2,$$

and

$$\begin{aligned} -\int_0^\infty \frac{J_*}{n_+(\mathcal{M}_x + D)}(\mathcal{E}_x + D - n_+)\mathcal{E} dx &\leq C\int_0^\infty |J_*(D(x) - n_+ + \mathcal{E}_x)\mathcal{E}| dx \\ &\leq C\eta_1(\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2) + C\eta_1\|D - n_+\|^2. \end{aligned}$$

Thus,

$$\left(\frac{1}{2} - C\eta_1\right)\|\mathcal{E}\|^2 + (a_1 - C\eta_1)\|\mathcal{E}_x\|^2 \leq C\eta_1\|\mathcal{E}_{xx}\|^2 + C\eta_1.$$

If $\eta_1 \ll 1$, we obtain

$$\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2 \leq C\eta_1\|\mathcal{E}_{xx}\|^2 + C\eta_1. \tag{2.10}$$

Multiplying the equation in (2.7) by $\mathcal{E}_{xx}$ and integrating it over $[0, \infty)$, we have

$$\begin{aligned} \int_0^\infty P(\mathcal{M}_x + D, J_*)\mathcal{E}_{xx}^2 dx = &\int_0^\infty \mathcal{E}\mathcal{E}_{xx} dx + \int_0^\infty \frac{J_*}{n_+(\mathcal{M}_x + D)}(\mathcal{E}_x + D - n_+)\mathcal{E}_{xx} dx \\ &- \int_0^\infty P(\mathcal{M}_x + D, J_*)D'\mathcal{E}_{xx} dx. \end{aligned}$$

Here, we get

$$\int_0^\infty \mathcal{E}\mathcal{E}_{xx}dx \le \frac{a_1}{4}\|\mathcal{E}_{xx}\|^2 + C\|\mathcal{E}\|^2,$$

$$\int_0^\infty \frac{J_*}{n_+(\mathcal{M}_x+D)}(\mathcal{E}_x + D - n_+)\mathcal{E}_{xx}dx \le \frac{a_1}{4}\|\mathcal{E}_{xx}\|^2 + C\|\mathcal{E}_x\|^2 + C\|D-n_+\|^2,$$

and

$$\int_0^\infty P(\mathcal{M}_x+D,J_*)D'\mathcal{E}_{xx}dx \le \frac{a_1}{4}\|\mathcal{E}_{xx}\|^2 + C\|D'\|^2.$$

Hence,

$$\|\mathcal{E}_{xx}\|^2 \le C(\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2 + \eta_1^2). \tag{2.11}$$

Combining (2.10) with (2.11), by using $\eta_1 \ll 1$ again, we have

$$\|\mathcal{E}\|^2 + \|\mathcal{E}_x\|^2 + \|\mathcal{E}_{xx}\|^2 \le C\eta_1. \tag{2.12}$$

Differentiating the equation in (2.7) with respect to x, we have

$$\begin{aligned} P(\mathcal{M}_x+D,J_*)\mathcal{E}_{xxx} &= -P'(\mathcal{M}_x+D,J_*)\mathcal{E}_{xx} + \mathcal{E}_x + \frac{J_*}{n_+(\mathcal{M}_x+D)}(\mathcal{E}_{xx}+D') \\ &\quad + \frac{J_*}{n_+(\mathcal{M}_x+D)^2}(\mathcal{M}_{xx}+D')(\mathcal{E}_x+D-n_+) \\ &\quad - P'(\mathcal{M}_x+D,J_*)D' - P(\mathcal{M}_x+D,J_*)D''. \end{aligned} \tag{2.13}$$

Multiply (2.13) by $\mathcal{E}_{xxx}$ and integrate it over $[0,\infty)$, then we can estimate the resultant and get

$$(1-C\eta_1)\|\mathcal{E}_{xxx}\|^2 \le C(\|\mathcal{E}_x\|^2 + \|\mathcal{E}_{xx}\|^2) + \|D-n_+\|^2 + \|D'\|^2 + \|D''\|^2.$$

If $\eta_1 \ll 1$, we get

$$\|\mathcal{E}_{xxx}\|^2 \le C(\|\mathcal{E}_x\|^2 + \|\mathcal{E}_{xx}\|^2) + C\eta_1. \tag{2.14}$$

Combining (2.12) with (2.14), we obtain

$$\|\mathcal{E}\|_{H^3}^2 \le \tilde{C}_1\eta_1, \tag{2.15}$$

where the positive constant $\tilde{C}_1$ just relies on J_*, D_* and D^*. Thus, we have proven that T is a mapping of W into itself.

Now, we prove that T is a continuous contraction operator over W. Let $\mathcal{M}_i \in W$ and $T(\mathcal{M}_i) = \mathcal{E}_i$ $(i = 1, 2)$, then

$$\begin{aligned} &P(\mathcal{M}_{1x}+D,J_*)(\mathcal{E}_1-\mathcal{E}_2)_{xx} + [P(\mathcal{M}_{1x}+D,J_*) - P(\mathcal{M}_{2x}+D,J_*)]\mathcal{E}_{2xx} \\ &= \mathcal{E}_1 - \mathcal{E}_2 + \frac{J_*}{n_+(\mathcal{M}_{1x}+D)}(\mathcal{E}_1-\mathcal{E}_2)_x + \frac{J_*(\mathcal{M}_1-\mathcal{M}_2)_x}{n_+(\mathcal{M}_{1x}+D)(\mathcal{M}_{2x}+D)}\mathcal{E}_{2x} \\ &\quad + \frac{J_*(D-n_+)(\mathcal{M}_2-\mathcal{M}_1)_x}{n_+(\mathcal{M}_{1x}+D)(\mathcal{M}_{2x}+D)} - [P(\mathcal{M}_{1x}+D,J_*) - P(\mathcal{M}_{2x}+D,J_*)]D'. \end{aligned} \tag{2.16}$$

Similarly, it is easy to get that

$$\|\mathcal{E}_1 - \mathcal{E}_2\|_{H^3}^2 \le C\eta_1(\|(\mathcal{M}_1-\mathcal{M}_2)_x\|^2 + \|(\mathcal{M}_1-\mathcal{M}_2)_{xx}\|^2). \tag{2.17}$$

By $\eta \ll 1$, we have

$$\|\mathcal{E}_1 - \mathcal{E}_2\|_{H^3}^2 \le \|\mathcal{M}_1 - \mathcal{M}_2\|_{H^3}^2. \tag{2.18}$$

This means that T is a continuous contraction operator over W. By Banach Fix-Point Theorem, there exists a unique $\bar{E} \in W$ such that $T(\bar{E}) = \bar{E}$. $\square$

Proof of Theorem 2.1. We set

$$\tilde{n}(x) := \bar{E}_x(x) + D(x), \quad \tilde{J} = J_* \quad \text{and} \quad E(x) := \bar{E}(x) + \frac{J_*}{n_+}.$$

It is not difficult to verify that $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ is the solution of the system (2.1). Moreover, since $D_* > \check{n}_1$ and $\|\bar{E}_x\|_\infty \leq \|\bar{E}_x\|_{H^2} \leq C\eta_1 \ll 1$, we have $\tilde{n}(x) \geq D_* - C\eta_1 \geq \check{n}_1^* > \check{n}_1$. Integrate $\bar{E}_x = \tilde{n}(x) - D(x)$ over $(0, \infty)$, then we get the equality (2.3). In the last, based on the embedding theorem, $\bar{E} \in H^3$ leads to $(\tilde{n}, \tilde{E}) \in C^1(\mathbb{R}_+) \times C^2(\mathbb{R}_+)$ and the estimates (2.4) and (2.5).

2.2. Correction functions

Let us investigate the behavior of solutions to the IBVP (1.1), (1.3) and (1.4) at $x = \infty$. Denote

$$(n^+, J^+, E^+)(t) := \lim_{x\to\infty}(n, J, E)(x, t).$$

By (1.4) and integrating $(1.1)_3$ with respect to x over $[0, x]$ and letting $t = 0$, we can get

$$E(x, 0) = E_* + \int_0^x [n_0(y) - D(y)]dx. \tag{2.19}$$

Then, taking the limit as $x \to \infty$ to (2.19), we have

$$E^+(0) = E_* + \int_0^\infty [n_0(x) - D(x)]dx := E_+. \tag{2.20}$$

The same as [20], we can solve $(n^+, J^+, E^+)(t)$ through the following ODEs

$$\begin{cases} \dfrac{d}{dt}n^+(t) = 0, \quad \text{i.e. } n^+(t) = n_+, \\ \dfrac{d}{dt}J^+(t) = n_+E^+(t) - J^+(t), \\ \dfrac{d}{dt}E^+(t) = -J^+(t) + J_*, \\ J^+(0) = J_+, \quad E^+(0) = E_+. \end{cases} \tag{2.21}$$

By turning (2.21) into a second order ODE of $J^+(t)$, we can tediously solve $J^+(t)$ and $E^+(t)$ as follows

$$J^+(t) = \begin{cases} e^{-\frac{1}{2}t}\left\{J_+ - J_* + \left[n_+\left(E_+ - \dfrac{J_*}{n_+}\right) - 2n_+(J_+ - J_*)\right]t\right\} + J_*, & \text{for } n_+ = \dfrac{1}{4}, \\ A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + J_*, & \text{for } n_+ < \dfrac{1}{4}, \\ e^{-\frac{1}{2}t}\left[A_3\cos(\lambda_3 t) + A_4\sin(\lambda_3 t)\right] + J_*, & \text{for } n_+ > \dfrac{1}{4}, \end{cases}$$

and

$$E^+(t) = \begin{cases} e^{-\frac{1}{2}t}\left\{E_+ - \dfrac{J_*}{n_+} + \left[2n_+\left(E_+ - \dfrac{J_*}{n_+}\right) - (J_+ - J_*)\right]t\right\} + \dfrac{J_*}{n_+}, & \text{for } n_+ = \dfrac{1}{4}, \\ -\dfrac{A_1}{\lambda_1}e^{\lambda_1 t} - \dfrac{A_2}{\lambda_2}e^{\lambda_2 t} + \dfrac{J_*}{n_+}, & \text{for } n_+ < \dfrac{1}{4}, \\ \dfrac{e^{-\frac{1}{2}t}}{2n_+}\left[(A_4 - 2\lambda_3A_3)\sin(\lambda_3 t) + (A_3 + 2\lambda_3A_4)\cos(\lambda_3 t)\right] + \dfrac{J_*}{n_+}, & \text{for } n_+ > \dfrac{1}{4}. \end{cases}$$

Here $\lambda_1 = \frac{-1+\sqrt{1-4n_+}}{2}$, $\lambda_2 = \frac{-1-\sqrt{1-4n_+}}{2}$ for $n_+ < \frac{1}{4}$, and $\lambda_3 = \frac{\sqrt{4n_+-1}}{2}$ for $n_+ > \frac{1}{4}$,

$$A_1 = -\frac{(\lambda_2 + 1)(J_+ - J_*) + n_+\left(\frac{J_*}{n_+} - E_+\right)}{\lambda_1 - \lambda_2}, \quad A_2 = \frac{(\lambda_1 + 1)(J_+ - J_*) + n_+\left(\frac{J_*}{n_+} - E_+\right)}{\lambda_1 - \lambda_2},$$

$$A_3 = J_+ - J_*, \qquad A_4 = \frac{2n_+\left(E_+ - \frac{J_*}{n_+}\right) - (J_+ - J_*)}{2\lambda_3}.$$

It has been known that the steady-state solution

$$(\tilde{n},\tilde{J},\tilde{E})(+\infty)=\left(n_+,J_*,\frac{J_*}{n_+}\right).$$

Unless

$$J_+=J_* \text{ and } E_+-\frac{J_*}{n_+}=E_*-\frac{J_*}{n_+}+\int_0^\infty (n_0-D)(x)dx=0 \quad \text{(switch-off case)},$$

it must hold that

$$(|n^+(t)-\tilde{n}(+\infty)|,|J^+(t)-\tilde{J}(+\infty)|,|E^+(t)-\tilde{E}(+\infty)|)=(0,Ce^{-\mu_0 t},Ce^{-\mu_0 t})$$

for some constant $0<\mu_0<\frac{1}{2}$. We have found the precise form of gaps between $J(x,t)$ and $\tilde{J}(x)$, $E(x,t)$ and $\tilde{E}(x)$ at the far field $x=\infty$, which leads to

$$J(x,t)-\tilde{J}(x) \quad \text{and} \quad E(x,t)-\tilde{E}(x)\notin L^2(\mathbb{R}_+).$$

In order to delete the gaps, we construct correction functions $(\hat{n},\hat{J},\hat{E})(x,t)$ such that

$$\begin{cases} \hat{n}_t+\hat{J}_x=0, \\ \hat{J}_t=n_+\hat{E}-\hat{J}, \\ \hat{E}_x=\hat{n}, \\ \hat{n}(+\infty,t)=0, \\ \hat{J}(+\infty,t)=J^+(t)-\tilde{J}(+\infty), \\ \hat{E}(+\infty,t)=E^+(t)-\tilde{E}(+\infty). \end{cases} \tag{2.22}$$

We can obtain $(\hat{n},\hat{J},\hat{E})(x,t)$ satisfying (2.22) through the following linear equation

$$\hat{J}_{tt}(x,t)+\hat{J}_t(x,t)+n_+\hat{J}(x,t)=0, \tag{2.23}$$

with the initial data

$$\hat{J}(x,0)=(J_+-J_*)\int_0^x m(y)dy \quad \text{and} \quad \hat{E}(x,0)=\left(E_+-\frac{J_*}{n_+}\right)\int_0^x m(y)dy. \tag{2.24}$$

Here $m(x)\geq 0$ is a smooth function with compact support satisfying

$$\int_0^\infty m(y)dy=1 \text{ and } \mathrm{supp}\, m\subset(0,\infty).$$

Through tedious calculations, we can obtain the $(\hat{n},\hat{J},\hat{E})(x,t)$ satisfying (2.22) as follows

$$\begin{aligned} \hat{J}(x,t)&=(J^+(t)-J_*)\int_0^x m(y)dy, \\ \hat{E}(x,t)&=\left(E^+(t)-\frac{J_*}{n_+}\right)\int_0^x m(y)dy, \\ \hat{n}(x,t)&=\left(E^+(t)-\frac{J_*}{n_+}\right)m(x). \end{aligned}$$

Lemma 2.2. *The correction function $(\hat{n},\hat{J},\hat{E})(x,t)$ to the IBVP (1.1), (1.3) and (1.4) satisfies*

$$\|(\hat{n},\hat{J},\hat{E})(t)\|_\infty\leq C\hat{\delta}_1 e^{-\mu_0 t}, \tag{2.25}$$

$$\mathit{supp}\ \hat{n}=\mathit{supp}\ \partial_x^j\hat{J}=\mathit{supp}\ \partial_x^j\hat{E}=\mathit{supp}\ m \quad \mathit{for} \quad j=1,2, \tag{2.26}$$

and

$$\begin{cases} (J-\hat{J}-\tilde{J})(+\infty,t)=(J-\hat{J}-\tilde{J})(0,t)=0, \\ (E-\hat{E}-\tilde{E})(+\infty,t)=(E-\hat{E}-\tilde{E})(0,t)=0. \end{cases} \tag{2.27}$$

Here $\hat{\delta}_1:=|J_+|+|J_*|+|E_*|+\|n_0-D\|_1$ and $0<\mu_0<\frac{1}{2}$.

2.3. Convergence results

Now we can reasonably make a perturbation of the solution $(n,J,E)(x,t)$ to the IBVP (1.1), (1.3) and (1.4) around the steady solution $(\tilde{n},\tilde{J},\tilde{E})(x)$ of the system (2.1) corrected by the corresponding correction function $(\hat{n},\hat{J},\hat{E})(x,t)$:

$$\begin{cases} (n-\hat{n}-\tilde{n})_t+(J-\hat{J}-\tilde{J})_x=0, \\ (J-\hat{J}-\tilde{J})_t+(p(n)-p(\tilde{n}))_x=nE-\tilde{n}\tilde{E}-n_+\hat{E}-(J-\hat{J}-\tilde{J})-\left(\dfrac{J^2}{n}-\dfrac{\tilde{J}^2}{\tilde{n}}\right)_x, \\ (E-\hat{E}-\tilde{E})_x=n-\hat{n}-\tilde{n}. \end{cases} \tag{2.28}$$

By integrating $(2.28)_1$ over $[0,\infty)$ and using $(2.27)_1$ we have

$$\int_0^\infty [n(x,t)-\hat{n}(x,t)-\tilde{n}(x)]dx=\int_0^\infty [n_0(x)-\hat{n}(x,0)-\tilde{n}(x)]dx=0, \tag{2.29}$$

where we have used (2.3) to get the last equality. Hence, we reach the setting of perturbation

$$\begin{cases} \phi(x,t):=\displaystyle\int_0^x [n(y,t)-\hat{n}(y,t)-\tilde{n}(y)]dy, \\ \psi(x,t):=(J-\hat{J}-\tilde{J})(x,t), \\ \omega(x,t):=(E-\hat{E}-\tilde{E})(x,t). \end{cases} \tag{2.30}$$

By (2.29), we get $\phi(0,t)=\phi(+\infty,t)=0$. Let

$$\begin{cases} \phi_0(x):=\displaystyle\int_0^x [n_0(y)-\hat{n}(y,0)-\tilde{n}(y)]dy, \\ \psi_0(x):=J_0(x)-\hat{J}(x,0)-\tilde{J}(x). \end{cases} \tag{2.31}$$

After integrating $(2.28)_1$ and $(2.28)_3$ over $(0,x)$, we obtain the reformulated problem

$$\begin{cases} \phi_t+\psi=0, \\ \psi_t+\psi-\tilde{n}\omega=-(p(n)-p(\tilde{n}))_x-\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x+(\phi_x+\hat{n})(\omega+\hat{E}+\tilde{E})+(\tilde{n}-n_+)\hat{E}, \\ \omega=\phi. \end{cases} \tag{2.32}$$

Apparently, $\omega=\phi$ and $\psi=-\phi_t$. Thus, we can reduce (2.32) into

$$\begin{cases} \phi_{tt}+\phi_t+\tilde{n}\phi-(p'(\tilde{n})\phi_x)_x=(p(\tilde{n}+\hat{n}+\phi_x)-p(\tilde{n})-p'(\tilde{n})\phi_x)_x+F_{1x}-F_2, \\ (\phi,\phi_t)(x,0)=(\phi_0,-\psi_0)(x), \\ \phi(0,t)=\phi(+\infty,t)=0, \end{cases} \tag{2.33}$$

where

$$F_1=\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}} \quad \text{and} \quad F_2=(\tilde{E}+\hat{E})\phi_x+\hat{n}\phi+\phi\phi_x+(\hat{n}+\tilde{n}-n_+)\hat{E}+\hat{n}\tilde{E}. \tag{2.34}$$

Now we are ready to state the convergence results as follows.

Theorem 2.2. *Let $\delta_1=\eta_1+|J_+|+|E_*|+\|n_0-D\|_1$ and $\Phi_0:=\|\phi_0\|_{H^3}+\|\psi_0\|_{H^2}$. Then there is a small $\tau_1>0$ such that when $\delta_1+\Phi_0<\tau_1$, the system (2.33) has a unique time-global solution $\phi(x,t)$ satisfying*

$$\phi\in C^i([0,\infty);H^{3-i}(\mathbb{R}_+)), \quad i=0,1,2,3. \tag{2.35}$$

Moreover,

$$\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt},\phi_{ttt})(t)\|^2\le C(\delta_1+\Phi_0^2)e^{-\alpha_1 t}, \tag{2.36}$$

where α_1 is a positive constant.

Corollary 2.1 (*Convergence to Asymptotic Profile*). *Under the conditions of Theorem 2.2, it holds that*

$$\|(n-\tilde{n}, J-\tilde{J}, E-\tilde{E})(t)\|_\infty \le Ce^{-\sigma_1 t}, \tag{2.37}$$

for $\sigma_1 = \min\{\alpha_1, \mu_0\} > 0$.

2.4. Proof of Theorem 2.2

It is well known that Theorem 2.2 can be proved by the classical energy method with the continuation argument based on the local existence and the a priori estimates. The crucial step is to establish the a priori estimates for the solutions, which will be our main target in this section.

For $T \in (0, +\infty]$, set the a priori assumption

$$N(T)^2 =: \sup_{0\le t\le T}\{\|\phi(t)\|^2_{H^3} + \|\phi_t(t)\|^2_{H^2}\} \le \varepsilon^2, \tag{2.38}$$

where ε is sufficiently small which will be determined later. It should be noted that (2.38) with Sobolev inequality $\|\partial_x^k f\|_\infty \le C\|\partial_x^k f\|^{\frac{1}{2}}\|\partial_x^{k+1} f\|^{\frac{1}{2}}$ gives

$$\sum_{k=0}^{2}\|\partial_x^k\phi(t)\|_\infty + \sum_{k=0}^{1}\|\partial_x^k\phi_t(t)\|_\infty \le C\varepsilon. \tag{2.39}$$

Then, it is easy to verify that there exist two positive constants $\overline{N}$ and $\underline{N}$ such that

$$0 < \underline{N} \le n = \phi_x + \hat{n} + \tilde{n} \le \overline{N}.$$

Furthermore, there exists a positive $N_* > 0$ such that

$$\frac{n^2 p'(n) - J^2}{n^2} \ge N_* > 0, \tag{2.40}$$

by the subsonic property $\tilde{n}(x) \ge \check{n}_1^* > \check{n}_1$ and the smallness of ε and δ_1.

Based on the properties of the correction functions in Lemma 2.2 and the results to the stationary solutions in Lemma 2.1, we obtain the following Lemmas.

Lemma 2.3. *If* $\varepsilon + \delta_1 \ll 1$*, it holds that*

$$\|(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{xt}, \phi_{tt})(t)\|^2 \le C(\delta_1 + \Phi_0^2)e^{-\mu_1 t}, \tag{2.41}$$

for some $0 < \mu_1 \le \mu_0$.

Proof. Multiplying the equation in (2.33) by $\phi + 2\phi_t$ and integrating the resultant equation over $[0, +\infty)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \left(\phi_t\phi + \frac{1}{2}\phi^2 + \phi_t^2 + p(\tilde{n})\phi_x^2 + \tilde{n}\phi^2\right)dx + \int_0^\infty (\phi_t^2 + p'(\tilde{n})\phi_x^2 + \tilde{n}\phi^2)dx \\
&= -\int_0^\infty (p(\tilde{n}+\hat{n}+\phi_x) - p(\tilde{n}) - p'(\tilde{n})\phi_x)(\phi_x + 2\phi_{xt})dx \\
&\quad - \int_0^\infty F_1(\phi_x + 2\phi_{xt})dx - \int_0^\infty F_2(\phi + 2\phi_t)dx.
\end{aligned} \tag{2.42}$$

Let us estimate the terms appearing in the right hand of (2.42). By Taylor's formula, Cauchy–Schwarz inequality and the properties of $\hat{n}$ in Lemma 2.2, we have

$$\begin{aligned}
&\int_0^\infty [p(\tilde{n}+\hat{n}+\phi_x) - p(\tilde{n}) - p'(\tilde{n})\phi_x](\phi_x + 2\phi_{xt})dx \\
&\le C(\delta_1+\varepsilon)\|(\phi_x, \phi_{xt})(t)\|^2 + C\delta_1 e^{-\mu_0 t}.
\end{aligned} \tag{2.43}$$

Note that

$$F_1 = -\frac{\tilde{J}^2}{n\tilde{n}}(\phi_x + \hat{n}) + \frac{\phi_t^2 - 2\phi_t(\hat{J} + \tilde{J})}{n} + \frac{\hat{J}^2 + 2\tilde{J}\hat{J}}{n}.$$

Since $\|\hat{J}(t)\|_\infty \|\tilde{n}_x\|^2 \le C\delta_1 e^{-\mu_0 t}$ and $\|\hat{J}_x\| \le C\delta_1 e^{-\mu_0 t}$, we can get

$$\begin{aligned}
&\int_0^\infty \frac{\hat{J}^2 + 2\tilde{J}\hat{J}}{n}(\phi_x + 2\phi_{xt})dx \\
&= -\int_0^\infty \left(\frac{\hat{J}^2 + 2\tilde{J}\hat{J}}{n}\right)_x (\phi + 2\phi_t)dx \\
&= \int_0^\infty \left[\frac{2\tilde{J}\hat{J} + \hat{J}^2}{n^2}\phi_{xx} + \frac{2\tilde{J}\hat{J} + \hat{J}^2}{n^2}(\hat{n}_x + \tilde{n}_x) - \frac{2(\tilde{J} + \hat{J})\hat{J}_x}{n}\right](\phi + 2\phi_t)dx \\
&\le -\int_0^\infty \left(\frac{2\tilde{J}\hat{J} + \hat{J}^2}{n^2}\right)_x \phi_x(\phi + 2\phi_t)dx - \int_0^\infty \frac{2\tilde{J}\hat{J} + \hat{J}^2}{n^2}\phi_x(\phi_x + 2\phi_{xt})dx \\
&\quad + C\int_0^\infty \left[(\hat{n}_x + \tilde{n}_x)\hat{J} + (\tilde{J} + \hat{J})\hat{J}_x\right](\phi + 2\phi_t)dx \\
&\le C(\delta_1 + \varepsilon)\|(\phi, \phi_x, \phi_t, \phi_{xt})(t)\|^2 + C\delta_1 e^{-\mu_0 t}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\int_0^\infty F_1(\phi_x + 2\phi_{xt})dx \le& \int_0^\infty \frac{\hat{J}^2 + 2\tilde{J}\hat{J}}{n}(\phi_x + 2\phi_{xt})dx \\
&+ \int_0^\infty |(\tilde{J}\phi_x + \tilde{J}\hat{n} + \phi_t^2 + \phi_t\hat{J} + \phi_t\tilde{J})(\phi_x + 2\phi_{xt})|dx \\
\le& \ C(\delta_1 + \varepsilon)\|(\phi, \phi_x, \phi_t, \phi_{xt})(t)\|^2 + C\delta_1 e^{-\mu_0 t}.
\end{aligned} \tag{2.44}$$

By (2.5) and $\|\hat{E}(t)\|_\infty \|\tilde{n} - n_+\|^2 \le C\delta_1 e^{-\mu_0 t}$, we can get

$$\begin{aligned}
\int_0^\infty F_2(\phi + 2\phi_t)dx =& \int_0^\infty \left[(\tilde{E} + \hat{E})\phi_x + \hat{n}\phi + \phi\phi_x + (\hat{n} + \tilde{n} - n_+)\hat{E} + \hat{n}\tilde{E}\right](\phi + 2\phi_t)dx \\
\le& \ C(\delta_1 + \varepsilon)\|(\phi, \phi_x, \phi_t, \phi_{xt})(t)\|^2 + C\delta_1 e^{-\mu_0 t}.
\end{aligned} \tag{2.45}$$

Substituting (2.43)–(2.45) into (2.42) and noticing the smallness of $\delta_1 + \varepsilon$, for some $C_0 > 0$, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \left(\phi_t\phi + \frac{1}{2}\phi^2 + \phi_t^2 + p(\tilde{n})\phi_x^2 + \tilde{n}\phi^2\right)dx + C_0\int_0^\infty (\phi^2 + \phi_x^2 + \phi_t^2)dx \\
&\le C(\delta_1 + \varepsilon)\|\phi_{xt}(t)\|^2 + C\delta_1 e^{-\mu_0 t}.
\end{aligned} \tag{2.46}$$

Multiplying the equation in (2.33) by ϕ_{xx} and integrating it over $[0, +\infty)$, we obtain

$$\begin{aligned}
&\int_0^\infty \left(p'(n) - \frac{J^2}{n^2}\right)\phi_{xx}^2 dx \\
&= -\int_0^\infty \left(p'(n) - \frac{J^2}{n^2}\right)\hat{n}_x\phi_{xx}dx - \int_0^\infty \left[p'(n) - p'(\tilde{n}) - \left(\frac{J^2}{n^2} - \frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]\tilde{n}_x\phi_{xx}dx \\
&\quad + \int_0^\infty (\phi_{tt} + \phi_t + \tilde{n}\phi)\phi_{xx}dx + \int_0^\infty F_2\phi_{xx}dx + \int_0^\infty \frac{2J}{n}(\hat{J}_x - \phi_{xt})\phi_{xx}dx.
\end{aligned}$$

Then we can get

$$\|\phi_{xx}(t)\|^2 \le C\|(\phi, \phi_t, \phi_x, \phi_{xt}, \phi_{tt})(t)\|^2 + C\delta_1 e^{-\mu_0 t}, \tag{2.47}$$

because

$$\begin{aligned}
\int_0^\infty \left[p'(n) - p'(\tilde{n}) - \left(\frac{J^2}{n^2} - \frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]\tilde{n}_x\phi_{xx}dx \le& \ C\int_0^\infty (\phi_x + \phi_t + \hat{n} + \hat{J})\tilde{n}_x\phi_{xx}dx \\
\le& \ C\delta_1\|(\phi_x, \phi_t, \phi_{xx})(t)\|^2 + C\delta_1 e^{-\mu_0 t},
\end{aligned}$$

and

$$\int_0^\infty F_2\phi_{xx}dx \le C(\delta_1+\varepsilon)\|(\phi,\phi_x,\phi_{xx})(t)\|^2 + C\delta_1 e^{-\mu_0 t}.$$

Differentiating the equation in (2.33) with respect t leads to

$$\phi_{ttt}+\phi_{tt}+\tilde{n}\phi_t-(p'(\tilde{n})\phi_{xt})_x = \left[\left(p(n)+\frac{J^2}{n}\right)_t - p'(\tilde{n})\phi_{xt}\right]_x - F_{2t}. \tag{2.48}$$

Multiplying (2.48) by $\phi_t+2\phi_{tt}$ and integrating it over $[0,+\infty)$, we can obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty\left(\phi_{tt}\phi_t+\frac{1}{2}\phi_t^2+\phi_{tt}^2+p'(\tilde{n})\phi_{xt}^2+\tilde{n}\phi_t^2\right)dx+\int_0^\infty(\phi_{tt}^2+\tilde{n}\phi_t^2+p'(\tilde{n})\phi_{xt}^2)dx\\
&=-\int_0^\infty\left[\left(p(n)+\frac{J^2}{n}\right)_t-p'(\tilde{n})\phi_{xt}\right](\phi_{xt}+2\phi_{xtt})dx-\int_0^\infty F_{2t}(\phi_t+2\phi_{tt})dx.
\end{aligned} \tag{2.49}$$

Through integrating by parts, we get

$$\begin{aligned}
&-\int_0^\infty\left[\left(p(n)+\frac{J^2}{n}\right)_t-p'(\tilde{n})\phi_{xt}\right](\phi_{xt}+2\phi_{xtt})dx\\
&=-\int_0^\infty\left[\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xt}+\left(p'(n)-\frac{J^2}{n^2}\right)\hat{n}_t+\frac{2JJ_t}{n}\right](\phi_{xt}+2\phi_{xtt})dx\\
&=-\frac{d}{dt}\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xt}^2dx-\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xt}^2dx+I_1+I_2\\
&\le-\frac{d}{dt}\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xt}^2dx-\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xt}^2dx\\
&\quad+C(\delta_1+\varepsilon)\|(\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2+C\delta_1e^{-\mu_0t},
\end{aligned} \tag{2.50}$$

where

$$\begin{aligned}
I_1 :=& \int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)(\hat{n}_{tx}\phi_{tt}-\hat{n}_t\phi_{xt})dx\\
&+\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)_x\hat{n}_t\phi_{tt}dx+\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)_t\phi_{xt}^2dx\\
\le&\ C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{tt})(t)\|^2+C\delta_1e^{-\mu_0t},
\end{aligned}$$

and

$$\begin{aligned}
I_2 :=& -\int_0^\infty\frac{2JJ_t}{n}(\phi_{xt}+2\phi_{xtt})dx\\
=& -\int_0^\infty\frac{2J}{n}\phi_{xtt}(\phi_t+2\phi_{tt})dx+\int_0^\infty\left[\frac{2}{n}(J\hat{J}_{xt}+J_xJ_t)-\frac{2}{n^2}JJ_tn_x\right](\phi_t+2\phi_{tt})dx\\
\le&\ \int_0^\infty\frac{2J}{n}\phi_{tt}\phi_{xt}dx+\int_0^\infty\left(\frac{2J}{n}\right)_x(\phi_{tt}\phi_t+\phi_{tt}^2)dx\\
&+C\int_0^\infty(J\hat{J}_{xt}+J_xJ_t+J_tn_x)(\phi_t+\phi_{tt})dx\\
\le&\ C(\delta_1+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt})(t)\|^2+C\delta_1e^{-\mu_0t}.
\end{aligned}$$

Analogous to (2.45), it is easy to estimate the last term in (2.49) as

$$\int_0^\infty F_{2t}(\phi_t+2\phi_{tt})dx \le C(\delta_1+\varepsilon)\|(\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2+C\delta_1e^{-\mu_0t}. \tag{2.51}$$

Substituting (2.50)–(2.51) into (2.49) and combining (2.47), by using the smallness of $\delta_1+\varepsilon$ again, we finally get that

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty\left[\phi_{tt}\phi_t+\frac{1}{2}\phi_t^2+\phi_{tt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xt}^2+\tilde{n}\phi_t^2\right]dx+C_1\int_0^\infty(\phi_t^2+\phi_{tt}^2+\phi_{xt}^2)dx\\&\le C(\delta_1+\varepsilon)\|(\phi,\phi_x)(t)\|^2+C\delta_1e^{-\mu_0t},\end{aligned}\tag{2.52}$$

for some constant $C_1>0$. Let

$$\begin{aligned}Q_1(t):=\int_0^\infty&\Big[\phi_t\phi+\left(\frac{1}{2}+\tilde{n}\right)\phi^2+\left(\frac{3}{2}+\tilde{n}\right)\phi_t^2+\phi_{tt}\phi_t+\phi_{tt}^2+p'(\tilde{n})\phi_x^2\\&+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xt}^2\Big]dx,\end{aligned}$$

then from (2.40) and (2.47), there exist two positive constants d_{11} and d_{12} such that

$$d_{11}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{xx})(t)\|^2-C\delta_1e^{-\mu_0t}\le Q_1(t)\le d_{12}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2.\tag{2.53}$$

In fact, from (2.40), for two positive constants $\tilde{d}_{11}$ and $\tilde{d}_{12}$, it holds that

$$\tilde{d}_{11}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2\le Q_1(t)\le\tilde{d}_{12}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2.$$

Then for a constant $\hat{d}_{11}$ satisfying $0<\hat{d}_{11}<\tilde{d}_{11}$, by (2.47) we have

$$\begin{aligned}&\tilde{d}_{11}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{tt})(t)\|^2\\&=(\tilde{d}_{11}-\hat{d}_{11})\|(\phi,\phi_t,\phi_x,\phi_{xt},\phi_{tt})(t)\|^2+\hat{d}_{11}\|(\phi,\phi_t,\phi_x,\phi_{xt},\phi_{tt})(t)\|^2\\&\ge\bar{d}_{11}\|\phi_{xx}(t)\|^2-C\delta_1e^{-\mu_0t}+\hat{d}_{11}\|(\phi,\phi_t,\phi_x,\phi_{xt},\phi_{tt})(t)\|^2\\&\ge\min\{\bar{d}_{11},\hat{d}_{11}\}\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{xx})(t)\|^2-C\delta_1e^{-\mu_0t},\end{aligned}$$

for some $\bar{d}_{11}>0$. Let $d_{11}:=\min\{\bar{d}_{11},\hat{d}_{11}\}$, we finally obtain (2.53). Now by taking (2.46)+(2.52) and using the smallness of $\delta_1+\varepsilon$, we have

$$\frac{d}{dt}Q_1(t)+C_2Q_1(t)\le C\delta_1e^{-\mu_0t},$$

for some positive constant C_2. The Gronwall's inequality implies that

$$Q_1(t)\le C(\delta_1+\Phi_0^2)e^{-\mu_1t},\tag{2.54}$$

for some $0<\mu_1\le\mu_0$. Here, we have used

$$\begin{aligned}\|\phi_{tt}(0)\|^2&\le C\|(\phi,\phi_x,\phi_t,\phi_{xx})(0)\|^2+C\delta_1\\&\le C(\delta_1+\Phi_0^2),\end{aligned}\tag{2.55}$$

which can be easily get from the original equation in (2.33). Finally, by using (2.53) again, we have

$$\|(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{xx},\phi_{tt})(t)\|^2\le CQ_1(t)+C\delta_1e^{-\mu_1t}\le C(\delta_1+\Phi_0^2)e^{-\mu_1t}.\quad\square$$

Lemma 2.4. *If $\varepsilon+\delta_1\ll1$, it holds that*

$$\|(\phi_{tt},\phi_{xtt},\phi_{xxt},\phi_{ttt},\phi_{xxx})(t)\|^2\le C(\delta_1+\Phi_0^2)e^{-\mu_2t},\tag{2.56}$$

for some $0<\mu_2\le\mu_1$.

Proof. Differentiating the equation in (2.33) with respect to t twice, we get that

$$\phi_{tttt}+\phi_{ttt}-(p'(\tilde{n})\phi_{xtt})_x+\tilde{n}\phi_{tt}=\left[\left(p(n)+\frac{J^2}{n}\right)_{tt}-p'(\tilde{n})\phi_{xtt}\right]_x-F_{2tt}. \tag{2.57}$$

Multiplying (2.57) by $\phi_{tt}+2\phi_{ttt}$ and integrating it over $[0,+\infty)$ leads that

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty\left(\phi_{ttt}\phi_{tt}+\frac{1}{2}\phi_{tt}^2+\phi_{ttt}^2+p'(\tilde{n})\phi_{xtt}^2+\tilde{n}\phi_{tt}^2\right)dx+\int_0^\infty(\phi_{ttt}^2+p'(\tilde{n})\phi_{xtt}^2+\tilde{n}\phi_{tt}^2)dx\\&=-\int_0^\infty\left[\left(p(n)+\frac{J^2}{n}\right)_{tt}-p'(\tilde{n})\phi_{xtt}\right](\phi_{xtt}+2\phi_{xttt})dx-\int_0^\infty F_{2tt}(\phi_{tt}+2\phi_{ttt})dx.\end{aligned} \tag{2.58}$$

For the first term in the right hand of (2.58), we can get that

$$\begin{aligned}&-\int_0^\infty\left[\left(p(n)+\frac{J^2}{n}\right)_{tt}-p'(\tilde{n})\phi_{xtt}\right](\phi_{xtt}+2\phi_{xttt})dx\\&=-\frac{d}{dt}\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xtt}^2dx-\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xtt}^2dx+I_3+I_4\\&\le-\frac{d}{dt}\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xtt}^2dx-\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}-p'(\tilde{n})\right)\phi_{xtt}^2dx\\&\quad+C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{tt},\phi_{xxt},\phi_{xtt},\phi_{ttt})(t)\|^2+C\delta_1e^{-\mu_0t},\end{aligned} \tag{2.59}$$

where

$$\begin{aligned}I_3:=&\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)(2\hat{n}_{ttx}\phi_{ttt}-\hat{n}_{tt}\phi_{xtt}+2\phi_{xt}\phi_{ttt}+2\tilde{n}_t\phi_{ttt})dx\\&+\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)_t(\phi_{xtt}^2-\phi_{xt}\phi_{xtt}-\hat{n}_t\phi_{xtt}+2\phi_{xxt}\phi_{ttt}+2\hat{n}_t\phi_{ttt})dx\\&+\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)_x2\hat{n}_{tt}\phi_{ttt}dx+\int_0^\infty\left(p'(n)-\frac{J^2}{n^2}\right)_{tx}(2\phi_{xt}\phi_{ttt}+2\hat{n}_t\phi_{ttt})\phi_{ttt}dx\\\le&\ C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{xtt},\phi_{xxt},\phi_{ttt})(t)\|^2+C\delta_1e^{-\mu_0t},\end{aligned}$$

and

$$\begin{aligned}I_4:=&\int_0^\infty\left(\frac{2JJ_t}{n}\right)_{tx}2\phi_{ttt}dx-\int_0^\infty\left(\frac{2JJ_t}{n}\right)_t\phi_{xtt}dx\\=&\int_0^\infty\left[\frac{2}{n}(JJ_{tt}+J_t^2)\right]_x(\phi_{tt}+\phi_{ttt})dx+\int_0^\infty\frac{2}{n^2}JJ_tn_t\phi_{xtt}dx-\int_0^\infty\left(\frac{2}{n^2}JJ_tn_t\right)_x\phi_{ttt}dx\\\le&\ C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{xtt},\phi_{xxt},\phi_{ttt})(t)\|^2+C\delta_1e^{-\mu_0t}.\end{aligned}$$

Similarly,

$$\int_0^\infty F_{2tt}(\phi_{tt}+2\phi_{ttt})dx\le C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{tt},\phi_{xtt},\phi_{ttt})(t)\|^2+C\delta_1e^{-\mu_0t}. \tag{2.60}$$

Substituting (2.59)–(2.60) into (2.58), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty\left[\phi_{ttt}\phi_{tt}+\frac{1}{2}\phi_{tt}^2+\phi_{ttt}^2+\left(p(n)-\frac{J^2}{n^2}\right)\phi_{xtt}^2+\tilde{n}\phi_{tt}^2\right]dx\\&+\int_0^\infty\left[\phi_{ttt}^2+\left(p(n)-\frac{J^2}{n^2}\right)\phi_{xtt}^2+\tilde{n}\phi_{tt}^2\right]dx\\&\le C(\delta_1+\varepsilon)\|(\phi_{xt},\phi_{tt},\phi_{xxt},\phi_{xtt},\phi_{ttt})(t)\|^2+C\delta_1e^{-\mu_1t}.\end{aligned} \tag{2.61}$$

Multiplying (2.48) by ϕ_{xxt} and integrating it over $[0,+\infty)$, we can get

$$
\begin{aligned}
&\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2dx\\
=&-\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)\hat{n}_{tx}\phi_{xxt}dx-\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)_x(\phi_{xt}+\hat{n}_t)\phi_{xxt}dx\\
&+\int_0^\infty \left(\frac{2JJ_t}{n}\right)_x\phi_{xxt}dx+\int_0^\infty \phi_{ttt}\phi_{xxt}dx+\int_0^\infty \phi_{tt}\phi_{xxt}dx\\
&+\int_0^\infty \tilde{n}\phi_t\phi_{xxt}dx+\int_0^\infty F_{2t}\phi_{xxt}dx.
\end{aligned}
$$

Analogous to (2.47), by using (2.41), it is easy to get

$$
\|\phi_{xxt}(t)\|^2\le C\|(\phi_{xtt},\phi_{ttt})(t)\|^2+C(\delta_1+\Phi_0^2)e^{-\mu_1 t}. \tag{2.62}
$$

Differentiating the equation in (2.33) with respect to x, we have

$$
\phi_{ttx}+\phi_{tx}+\tilde{n}\phi_x+\tilde{n}_x\phi=\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_{xx}-F_{2x}. \tag{2.63}
$$

Multiplying (2.63) by ϕ_{xxx} and integrating it over $[0,+\infty)$, we can get

$$
\begin{aligned}
&\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxx}^2dx\\
=&-\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)\hat{n}_{xx}\phi_{xxx}dx-\int_0^\infty \left[p'(n)-p'(\tilde{n})-\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]\tilde{n}_{xx}\phi_{xxx}dx\\
&-\int_0^\infty \left[p'(n)-p'(\tilde{n})-\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]_x\tilde{n}_x\phi_{xxx}dx-\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)_x(\phi_{xx}+\hat{n}_x)\phi_{xxx}dx\\
&-\int_0^\infty \left(\frac{2JJ_x}{n}\right)_x\phi_{xxx}dx+\int_0^\infty F_{2x}\phi_{xxx}dx+\int_0^\infty (\phi_{xtt}+\phi_{xt}+\tilde{n}\phi_x+\tilde{n}_x\phi)\phi_{xxx}dx.
\end{aligned}
$$

Here, we have known that

$$
\int_0^\infty \left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxx}^2dx\ge N_*\|\phi_{xxx}(t)\|^2.
$$

Since the Sobolev inequality and the property of $\tilde{n}_{xx}$ in Theorem 2.1 leads

$$
(\|\phi_x(t)\|_\infty+\|\phi_t(t)\|_\infty)^2\|\tilde{n}_{xx}\|^2\le C\delta_1\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2,
$$

and the properties of the correction functions in Lemma 2.2 leads to

$$
(\|\hat{n}(t)\|_\infty+\|\hat{J}(t)\|_\infty)\|\tilde{n}_{xx}\|^2\le C\delta_1e^{-\mu_1 t},
$$

we can get

$$
\begin{aligned}
&\int_0^\infty \left[(p'(n)-p'(\tilde{n}))-\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]\tilde{n}_{xx}\phi_{xxx}dx\\
&\le C\int_0^\infty (\hat{n}+\phi_x+\phi_t+\hat{J})\tilde{n}_{xx}\phi_{xxx}dx\\
&\le (\|\hat{n}(t)\|_\infty+\|\hat{J}(t)\|_\infty)(\|\tilde{n}_{xx}\|^2+\|\phi_{xxx}(t)\|^2)\\
&\quad+(\|\phi_x(t)\|_\infty+\|\phi_t(t)\|_\infty)\|\tilde{n}_{xx}\|\|\phi_{xxx}(t)\|\\
&\le \frac{N_*}{4}\|\phi_{xxx}(t)\|^2+C\delta_1\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{xxx})(t)\|^2+C\delta_1e^{-\mu_1 t}.
\end{aligned}
$$

Similarly, it is easy to get that

$$\begin{aligned}&\int_0^\infty \left[(p'(n)-p'(\tilde n))-\left(\frac{J^2}{n^2}-\frac{\tilde J^2}{\tilde n^2}\right)\right]_x \tilde n_x\phi_{xxx}dx\\ &\le C(\delta_1+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{xxx})\|^2+C\delta_1 e^{-\mu_1 t},\end{aligned}$$

$$\begin{aligned}&\int_0^\infty \left(\frac{2JJ_x}{n}\right)_x \phi_{xxx}dx\\ &\le C\int_0^\infty [J(\hat J_{xx}-\phi_{xxt})+(\hat J_x-\phi_{xt})^2+(\hat J_x-\phi_{xt})(\phi_x+\hat n+\tilde n)]\phi_{xxx}dx\\ &\le \frac{N_*}{4}\|\phi_{xxx}(t)\|^2+C(\delta_1+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{xxx},\phi_{xxt})\|+C\delta_1 e^{-\mu_1 t},\end{aligned}$$

and

$$\int_0^\infty F_{2t}\phi_{txx}dx\le C(\delta_1+\varepsilon)\|(\phi_x,\phi_{xt},\phi_{xxt})(t)\|^2+C\delta_1 e^{-\mu_1 t}.$$

By (2.41), we have

$$\|\phi_{xxx}(t)\|^2\le C\|(\phi_{xtt},\phi_{ttt})(t)\|^2+C\delta_1 e^{-\mu_1 t}. \tag{2.64}$$

Let

$$Q_2(t)=\int_0^\infty\left[\phi_{ttt}\phi_{tt}+\frac12\phi_{tt}^2+\phi_{ttt}^2+\left(p(n)-\frac{J^2}{n^2}\right)\phi_{xtt}^2+\tilde n\phi_{tt}^2\right]dx.$$

The same as (2.53), from (2.62) and (2.64), there exist two positive constants d_{13} and d_{14} such that

$$d_{13}\|(\phi_{xxt},\phi_{xxx})(t)\|^2-C\delta_1 e^{-\mu_1 t}\le Q_2(t)\le d_{14}\|(\phi_{tt},\phi_{xtt},\phi_{ttt})(t)\|^2. \tag{2.65}$$

Substituting (2.62) into (2.61) and noticing the smallness of $\delta_1+\varepsilon$, by (2.41) again, we have

$$\frac{d}{dt}Q_2(t)+C_3Q_2(t)\le C\delta_1 e^{-\mu_1 t}, \tag{2.66}$$

for some constant $C_3>0$. The Gronwall's inequality implies that

$$Q_2(t)\le C(\delta_1+\Phi_0^2)e^{-\mu_2 t}, \tag{2.67}$$

where $0<\mu_2\le\mu_1$. Here, we have also used

$$\begin{aligned}\|(\phi_{xtt},\phi_{ttt})(0)\|^2&\le C\|(\phi_{xxt},\phi_{xxx})(0)\|^2+C\delta_1\\ &\le C(\delta_1+\Phi_0^2),\end{aligned} \tag{2.68}$$

which can be easily get from Eqs. (2.33) and (2.63) by the same method as (2.55). By using (2.65) again, we obtain

$$\|(\phi_{xxt},\phi_{xxx},\phi_{xtt},\phi_{ttt})(t)\|^2\le C(\delta_1+\Phi_0^2)e^{-\mu_2 t}. \quad\square$$

Proof of Theorem 2.2. Let $\alpha_1=\min\{\mu_1,\mu_2\}$, then Lemmas 2.3 and 2.4 imply Theorem 2.2. $\square$

3. The insulating problem

In this section, we consider the system (1.1) with the initial data (1.3) and the insulating boundary (1.5). Similarly, we first look for the asymptotic profile of the original solutions, the corresponding steady-states subjected to certain boundary conditions. Secondly, we carefully look at what kind gaps between the original IBVP solutions and the expected asymptotic profiles at the far field, then we construct some suitable correction functions to delete the gaps. Finally, we prove the convergence of the original IBVP solutions to the corresponding asymptotic profiles time-exponentially or algebraically, according to the decay situation on the boundary at the far field.

3.1. Asymptotic profiles

Different from the inflow/ourflow/impermeable problem, the asymptotic profile $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ to the IBVP (1.1), (1.3) with the insulating boundary (1.4) satisfies the system

$$\begin{cases} \tilde{J} = n_+ E^*, \\ \left(p'(\tilde{n}) - \dfrac{\tilde{J}^2}{\tilde{n}^2}\right) \tilde{n}_x = \tilde{n}\tilde{E} - \tilde{J}, \\ \tilde{E}_x = \tilde{n} - D(x), \\ \tilde{E}_x(0) = n_0(0) - D(0), \quad \lim\limits_{x\to\infty} \tilde{E}(x) = E^*, \end{cases} \qquad x \in \mathbb{R}_+. \tag{3.1}$$

Here, as showed in Section 2, for $\tilde{J} \equiv n_+E^*$, we can also conclude that there is a minima value $\check{n}_2 = \check{n}_2(|\tilde{J}| = |n_+E^*|) \geq 0$ such that $\check{n}_2^2 p'(\check{n}_2) - |n_+E^*|^2 = 0$, and for some constant $\check{n}_2^* > \check{n}_2$, it holds that

$$\tilde{n}^2 p'(\tilde{n}) - |n_+E^*|^2 \geq C_2^* > 0 \quad \text{for } \tilde{n} \geq \check{n}_2^* > \check{n}_2,$$

for a positive constant C_2^*. The existence of solutions to system (3.1) is submitted in the following Theorem.

Theorem 3.1 (*Asymptotic Profiles*)**.** *Assume that (1.2), (1.14) and $D_* > \check{n}_2$ hold. If*

$$\eta_2 := \|n_0 - D\|_\infty^{\frac{1}{2}} + |E^*| + \|D - n_+\|_{H^2} \ll 1,$$

then the system (3.1) possesses a unique solution $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ in the solution space

$$V_2 := \Big\{ (\tilde{n}, \tilde{J}, \tilde{E})(x);\ \tilde{J} \equiv n_+E^*,\ (\tilde{n} - n_+, \tilde{E} - E^*) \in H^2(\mathbb{R}_+) \times H^3(\mathbb{R}_+) \\ \text{with } \tilde{n} \geq \check{n}_2^* > \check{n}_2,\ \|n - n_+\|_{H^2} \leq \tilde{C}_2\eta_2 \text{ and } \|\tilde{E} - E^*\|_{H^3} \leq \tilde{C}_2\eta_2 \Big\}, \tag{3.2}$$

where $\tilde{C}_2$ is a positive constant dependent on E^*, D_* and D^*. Moreover, it holds that $(\tilde{n}, \tilde{E}) \in C^1(\mathbb{R}_+) \times C^2(\mathbb{R}_+)$ and

$$\|\tilde{n} - n_+\|_\infty + \|\tilde{n}_x\|_\infty \leq C\eta_2, \tag{3.3}$$

$$\|\tilde{E} - E^*\|_\infty + \|\tilde{E}_x\|_\infty + \|\tilde{E}_{xx}\|_\infty \leq C_2\eta_2. \tag{3.4}$$

Similarly, we construct the solutions satisfying (3.1) by proving that there exists a solution to the well-posed BVP (1.13) with the boundary (1.9). For this purpose, we set

$$\mathbb{E}(x) := \tilde{E}(x) - E^*,$$

then the BVP (1.13) with (1.9) is equivalent to the system

$$\begin{cases} \left[\dfrac{p'(\mathbb{E}_x + D(x))}{\mathbb{E}_x + D(x)} - \dfrac{(n_+E^*)^2}{(\mathbb{E}_x + D(x))^3}\right](\mathbb{E}_{xx} + D'(x)) = \mathbb{E} + E^* - \dfrac{n_+E^*}{\mathbb{E}_x + D(x)}, \\ \mathbb{E}_x(0) - n_0(0) - D(0), \quad \lim\limits_{x\to\infty} \mathbb{E}(x) = 0. \end{cases} \tag{3.5}$$

By the same method as Lemma 2.1, we can get the existence of the solutions to the system (3.5) in the following Lemma.

Lemma 3.1. *Assume that (1.2), (1.14) and $D_* > \check{n}_2$ hold. If $\eta_2 \ll 1$, then the system (3.5) has a unique solution $\mathbb{E} \in H^3(\mathbb{R}_+)$ satisfying*

$$\|\mathbb{E}\|_{H^3} \leq \tilde{C}_2\eta_2.$$

Proof of Theorem 3.1. The method is analogous to the proof of Theorem 2.1, where the only difference is that we set

$$\tilde{n}(x) := \mathbb{E}_x(x) + D(x),\ \ \tilde{J} = n_+E^*,\ \text{and } \tilde{E}(x) := \mathbb{E}(x) + E^*. \quad \square$$

3.2. Correction functions

We also denote

$$(n^+, J^+, E^+)(t) := \lim_{x\to\infty}(n, J, E)(x,t).$$

By the boundary condition (1.5), we already have $E^+(t) = f(t)$. From the original IBVP, we have the following first order ODE

$$\begin{cases} \dfrac{d}{dt}J^+(t) = n_+ f(t) - J^+(t), \\ J^+(0) = J_+. \end{cases} \tag{3.6}$$

It is easy to solve the system (3.6) as

$$J^+(t) = J_+ e^{-t} + n_+ \int_0^t e^{-(t-s)} f(s)ds.$$

It has been known that the corresponding stationary solution

$$(\tilde{n}, \tilde{J}, \tilde{E})(\infty) = (n_+, n_+E^*, E^*).$$

By (1.6), it must hold that

$$\left| J_+ e^{-t} + n_+ \int_0^t e^{-(t-s)} f(s)ds - n_+E^* \right| = \begin{cases} O(e^{-\theta_1 t}) \le Ct^{-1} & \text{for } \theta_1 > 0, \\ O((1+t)^{-\theta_2}) \le Ct^{-1} & \text{for } \theta_2 > 1. \end{cases}$$

So, we get

$$(|n^+(t) - \tilde{n}(+\infty)|, |J^+(t) - \tilde{J}(+\infty)|, |E^+(t) - \tilde{E}(+\infty)|) = (0, Ct^{-1}, Ct^{-1}).$$

Now we have found the precise forms of the gaps between $J(x,t)$ and $\tilde{J}(x)$, $E(x,t)$ and $\tilde{E}(x)$ at $x = \infty$, which leads to

$$J(x,t) - \tilde{J}(x) \quad \text{and} \quad E(x,t) - \tilde{E}(x) \notin L^2(\mathbb{R}_+).$$

In order to delete the gaps, we construct correction functions $(\hat{n}, \hat{J}, \hat{E})(x,t)$ such that

$$\begin{cases} \hat{n}_t + \hat{J}_x = 0, \\ \hat{J}_t = n_+\hat{E} - \hat{J}, \\ \hat{E}_x = \hat{n}, \\ \hat{n}(+\infty, t) = 0, \\ \hat{J}(+\infty, t) = J^+(t) - \tilde{J}(+\infty), \\ \hat{E}(+\infty, t) = E^+(t) - \tilde{E}(+\infty). \end{cases} \tag{3.7}$$

To get $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfying the system (3.7), we solve the following linear equation

$$\hat{n}_{tt}(x,t) + \hat{n}_t(x,t) + n_+\hat{n}(x,t) = 0, \tag{3.8}$$

with the selected initial data as

$$\hat{n}(x,0) = n_+ m(x) \quad \text{and} \quad \hat{J}_x(x,0) = (J_+ - n_+E^*)m(x). \tag{3.9}$$

It can be easily solved (3.8) with (3.9) that

$$\hat{n}(x,t) = \begin{cases} e^{-\frac{1}{2}t}\left[n_+ + \left(\dfrac{1}{2}n_+ - A_5\right)t\right] m(x), & \text{for } n_+ = \dfrac{1}{4}, \\ \left(\dfrac{\lambda_2 n_+ + A_5}{\lambda_2 - \lambda_1}e^{\lambda_1 t} + \dfrac{\lambda_1 n_+ + A_5}{\lambda_1 - \lambda_2}e^{\lambda_2 t}\right) m(x), & \text{for } n_+ < \dfrac{1}{4}, \\ e^{-\frac{1}{2}t}\left[n_+\cos(\lambda_3 t) + \dfrac{n_+ - 2A_5}{2\lambda_3}\sin(\lambda_3 t)\right] m(x), & \text{for } n_+ > \dfrac{1}{4}, \end{cases}$$

$$
\hat{J}(x,t) = J^+(t) - n_+E^* - \begin{cases} e^{-\frac{1}{2}t}\left[A_5 + \frac{1}{2}\left(\frac{1}{2}n_+ - A_5\right)t\right]\int_x^{+\infty} m(y)dy, & \text{for } n_+ = \frac{1}{4}, \\ \left[\frac{\lambda_1(\lambda_2 n_+ + A_5)}{\lambda_1-\lambda_2}e^{\lambda_1 t} + \frac{\lambda_2(\lambda_1 n_+ + A_5)}{\lambda_2-\lambda_1}e^{\lambda_2 t}\right]\int_x^{+\infty} m(y)dy, & \text{for } n_+ < \frac{1}{4}, \\ e^{-\frac{1}{2}t}\left[A_5\cos(\lambda_3 t) + \frac{4n_+^2 - 2A_5}{4\lambda_3}\sin(\lambda_3 t)\right]\int_x^{+\infty} m(y)dy, & \text{for } n_+ > \frac{1}{4}, \end{cases}
$$

and

$$
\hat{E}(x,t) = f(t) - E^* - \begin{cases} e^{-\frac{1}{2}t}\left[n_+ + \left(\frac{1}{2}n_+ - A_5\right)t\right]\int_x^{+\infty} m(y)dy, & \text{for } n_+ = \frac{1}{4}, \\ \left(\frac{\lambda_2 n_+ + A_5}{\lambda_2-\lambda_1}e^{\lambda_1 t} + \frac{\lambda_1 n_+ + A_5}{\lambda_1-\lambda_2}e^{\lambda_2 t}\right)\int_x^{+\infty} m(y)dy, & \text{for } n_+ < \frac{1}{4}, \\ e^{-\frac{1}{2}t}\left[n_+\cos(\lambda_3 t) + \frac{n_+ - 2A_5}{2\lambda_3}\sin(\lambda_3 t)\right]\int_x^{+\infty} m(y)dy, & \text{for } n_+ > \frac{1}{4}, \end{cases}
$$

where λ_i $(i = 1,2,3)$ is the same as before and $A_5 = J_+ - n_+E^*$.

Lemma 3.2. *The correction functions* $(\hat{n},\hat{J},\hat{E})(x,t)$ *to the IBVP* (1.1), (1.3) and (1.5) *satisfy*

$$
\|\hat{n}(t)\|_\infty \le C\hat{\delta}_2 e^{-\nu_0 t}, \tag{3.10}
$$

$$
\|(\hat{J},\hat{E})(t)\|_\infty \le \begin{cases} C\hat{\delta}_2 e^{-\nu_1 t} & \textit{as } |f(t)-E^*| = O(e^{-\theta_1 t}), \quad \theta_1 > 0, \\ C\hat{\delta}_2(1+t)^{-\theta_2} & \textit{as } |f(t)-E^*| = O((1+t)^{-\theta_2}), \quad \theta_2 > 1, \end{cases} \tag{3.11}
$$

and

$$
supp\ \hat{n} = supp\ \partial_x^j\hat{J} = supp\ \partial_x^j\hat{E} = supp\ m_1, \quad for \quad j = 1,2. \tag{3.12}
$$

Moreover,

$$
\begin{cases} n(0,t) - \hat{n}(0,t) - \tilde{n}(0) = 0, \\ (J - \hat{J} - \tilde{J})(+\infty,t) = (J_x - \hat{J}_x - \tilde{J}_x)(0,t) = 0, \\ (E - \hat{E} - \tilde{E})(+\infty,t) = 0. \end{cases} \tag{3.13}
$$

Here $\hat{\delta}_2 = |J_+| + |E^*|$, $0 < \nu_0 \le \frac{1}{2}$ *and* $\nu_1 = \min\{\theta_1,\nu_0\}$.

3.3. Convergence results

Similarly, we can make a perturbation of the solution $(n,J,E)(x,t)$ to the IBVP (1.1), (1.3) and (1.5) around the steady solution $(\tilde{n},\tilde{J},\tilde{E})(x)$ to the system (3.1) corrected by the corresponding correction function $(\hat{n},\hat{J},\hat{E})(x,t)$ as follows.

$$
\begin{cases} (n - \hat{n} - \tilde{n})_t + (J - \hat{J} - \tilde{J})_x = 0, \\ (J - \hat{J} - \tilde{J})_t + (p(n) - p(\tilde{n}))_x = nE - \tilde{n}\tilde{E} - n_+\hat{E} - (J - \hat{J} - \tilde{J}) - \left(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}\right)_x, \\ (E - \hat{E} - \tilde{E})_x = n - \hat{n} - \tilde{n}. \end{cases} \tag{3.14}
$$

We define the new unknown functions as

$$\begin{cases} \phi(x,t) := -\displaystyle\int_x^{\infty} [n(y,t) - \hat{n}(y,t) - \tilde{n}(y)]dy, \\ \psi(x,t) := (J - \hat{J} - \tilde{J})(x,t), \\ \omega(x,t) := (E - \hat{E} - \tilde{E})(x,t). \end{cases} \tag{3.15}$$

Here, different from the inflow/outflow/impermeable problem, we have $\phi_x(0,t) = \phi(+\infty,t) = 0$ from (3.13)$_1$. Let

$$\begin{cases} \phi_0(x) := -\displaystyle\int_x^{+\infty} [n_0(y) - \hat{n}(y,0) - \tilde{n}(y)]dy, \\ \psi_0(x) := J_0(x) - \hat{J}(x,0) - \tilde{J}(x). \end{cases} \tag{3.16}$$

Then, the same as the inflow/outflow/impermeable problem, we can reduce the problem into

$$\begin{cases} \phi_{tt} + \phi_t + \tilde{n}\phi = (p(\tilde{n} + \hat{n} + \phi_x) - p(\tilde{n}))_x + \left(\dfrac{J^2}{n} - \dfrac{\tilde{J}^2}{\tilde{n}}\right)_x - F_2, \\ (\phi, \phi_t)(x,0) = (\phi_0, -\psi_0)(x), \\ \phi_x(0,t) = \phi(+\infty,t) = 0. \end{cases} \tag{3.17}$$

Here F_2 is the same with (2.34). Now the convergence results are stated as follows.

Theorem 3.2. *Let $\delta_2 = \eta_2 + |J_+|$ and $\Phi_0 := \|\phi_0\|_{H^3} + \|\psi_0\|_{H^2}$. Then there is a small $\tau_2 > 0$ such that when $\delta_2 + \Phi_0 < \tau_2$, the system (3.17) has a unique time-global solution $\phi(x,t)$ satisfying*

$$\phi \in C^i([0,\infty); H^{3-i}(\mathbb{R}_+)), \quad i = 0,1,2. \tag{3.18}$$

Moreover,

$$\begin{aligned} &\|(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{xt}, \phi_{tt}, \phi_{xxx}, \phi_{xxt})(t)\|^2 \\ &\le \begin{cases} C\delta_2 e^{-\alpha_2 t} & \text{as } |f(t) - E^*| = Ce^{-\theta_1 t}, \quad \theta_1 > 0, \\ C\delta_2 (1+t)^{-\theta_2} & \text{as } |f(t) - E^*| = C(1+t)^{-\theta_2}, \quad \theta_2 > 1, \end{cases} \end{aligned} \tag{3.19}$$

where α_2 is a positive constant.

Corollary 3.1 (*Convergence to Asymptotic Profiles*)**.** *Under the conditions of Theorem 3.2, it holds that*

$$\begin{aligned} &\|(n - \tilde{n}, J - \tilde{J}, E - \tilde{E})(t)\|_\infty \\ &= \begin{cases} C\delta_2 e^{-\sigma_2 t} & \text{as } |f(t) - E^*| = Ce^{-\theta_1 t}, \quad \theta_1 > 0, \\ C\delta_2 (1+t)^{-\theta_2} & \text{as } |f(t) - E^*| = C(1+t)^{-\theta_2}, \quad \theta_2 > 1, \end{cases} \end{aligned} \tag{3.20}$$

for $\sigma_2 = \min\{\alpha_2, \theta_1, \nu_1\} > 0$.

3.4. Proof of Theorem 3.2

Because the methods of establishing the a priori estimates are same for the two situations of $f(t)$, we just give the proof for the case

$$|f(t) - E^*| = O((1+t)^{-\theta_2}) \quad \text{as } t \to \infty.$$

Set the same a priori assumption as the inflow/outflow/impermeable problem, by the properties of the corresponding correction functions in Lemma 3.2 and the results to the steady-state solutions on Theorem 3.1, we can obtain the a priori estimates in the following Lemmas.

Lemma 3.3. *If $\varepsilon+\delta_2 \ll 1$, it holds that*

$$\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt})(t)\|^2 \le C(\delta_2+\Phi_0^2)(1+t)^{-\theta_2}. \tag{3.21}$$

Proof. Multiplying the equation in (3.17) by $\phi+2\phi_t$ and integrating it over $[0,+\infty)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int\left[\phi_t\phi+\left(\frac{1}{2}+\tilde{n}\right)\phi^2+\phi_t^2\right]dx+\int(\phi_t^2+\tilde{n}\phi^2)dx \\ &=\int\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x(\phi+2\phi_t)dx-\int F_2(\phi+2\phi_t)dx. \end{aligned} \tag{3.22}$$

Notice that

$$\begin{aligned} &\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x \\ &=\left(p'(n)-\frac{J^2}{n^2}\right)(\phi_{xx}+\hat{n}_x)+\left[p'(n)-p'(\tilde{n})-\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)\right]\tilde{n}_x+\frac{2J}{n}J_x. \end{aligned}$$

We have

$$\begin{aligned} &\int\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x(\phi+2\phi_t)dx \\ &\le\int\left[\left(p'(n)-\frac{J^2}{n^2}\right)(\phi_{xx}+\hat{n}_x)+O(1)(\phi_x+\hat{n}+\hat{J}+\phi_t)\tilde{n}_x+\frac{2J}{n}(\hat{J}_x-\phi_{xt})\right](\phi+2\phi_t)dx \\ &\le C(\delta_2+\varepsilon)\|(\phi,\phi_x,\phi_t,\phi_{xt})(t)\|^2+C\|\phi_{xx}(t)\|^2+C\delta_2(1+t)^{-\theta_2}. \end{aligned}$$

It is easy to get

$$\int F_2(\phi+2\phi_t)dx\le C\delta_2\|(\phi,\phi_x,\phi_t)(t)\|^2+C\delta_2(1+t)^{-\theta_2}.$$

Because of the smallness of $\delta_2+\varepsilon$, there exist two constants $C_4,C_5>0$ such that

$$\begin{aligned} &\frac{d}{dt}\int\left[\phi_t\phi+\left(\frac{1}{2}+\tilde{n}\right)\phi^2+\phi_t^2\right]dx+C_4\int(\phi^2+\phi_t^2)dx \\ &\le C(\delta_2+\varepsilon)\|(\phi_x,\phi_{xt})(t)\|^2+C_5\|\phi_{xx}(t)\|^2+C\delta_2(1+t)^{-\theta_2}. \end{aligned} \tag{3.23}$$

Differentiating the equation in (3.17) in x leads to

$$\phi_{ttx}+\phi_{tx}+\tilde{n}\phi_x+\tilde{n}_x\phi=\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_{xx}-F_{2x}. \tag{3.24}$$

Multiplying (3.24) by $\phi_x+2\phi_{xt}$ and integrating the resultant equation over $[0,+\infty)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int\left[\phi_{xt}\phi_x+\left(\frac{1}{2}+\tilde{n}\right)\phi_x^2+\phi_{xt}^2\right]dx+\int(\tilde{n}\phi_x^2+\phi_{xt}^2)dx+\int\tilde{n}_x\phi(\phi_x+2\phi_{xt})dx \\ &=-\int\left(p(n)-p(\tilde{n})+\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x(\phi_{xx}+2\phi_{xxt})dx-\int F_{2x}(\phi_x+2\phi_{xt})dx. \end{aligned} \tag{3.25}$$

It is easy to get

$$-\int\tilde{n}_x\phi(\phi_x+2\phi_{xt})dx\le C(\delta_2+\varepsilon)\|(\phi,\phi_x,\phi_{xt})(t)\|^2. \tag{3.26}$$

For the right hand side of (3.25), we can estimate that

$$
\begin{aligned}
&-\int\left(p(n)-p(\tilde n)+\frac{J^2}{n}-\frac{\tilde J^2}{\tilde n}\right)_x(\phi_{xx}+2\phi_{xxt})dx\\
&=-\int\left\{\left[\left(p'(n)-\frac{J^2}{n^2}\right)-\left(p'(\tilde n)-\frac{\tilde J^2}{\tilde n^2}\right)\right]\tilde n_x+\frac{2J}{n}(\hat J_x-\phi_{xt})\right\}(\phi_{xx}+2\phi_{xxt})dx\\
&=-\frac{d}{dt}\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2dx-\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2dx+K_1+K_2+K_3\\
&\le-\frac{d}{dt}\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2dx-\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2dx\\
&\quad+C(\delta_2+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}\tag{3.27}
$$

where

$$
\begin{aligned}
K_1:=&\int\left(p'(n)-\frac{J^2}{n^2}\right)(2\hat n_{xx}\phi_{xt}-\hat n_x\phi_{xx})dx\\
&+2\int\left(p'(n)-\frac{J^2}{n^2}\right)_x\hat n_x\phi_{xt}dx+\int\left(p'(n)-\frac{J^2}{n^2}\right)_t\phi_{xx}^2dx\\
\le\ &C(\delta_2+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}
$$

$$
\begin{aligned}
K_2:=&\int\left[p(n)-p'(\tilde n)-\left(\frac{J^2}{n^2}-\frac{\tilde J^2}{\tilde n^2}\right)\right](2\tilde n_{xx}\phi_{xt}-\tilde n_x\phi_{xx})dx\\
&+2\int\left[p(n)-p'(\tilde n)-\left(\frac{J^2}{n^2}-\frac{\tilde J^2}{\tilde n^2}\right)\right]_x\tilde n_x\phi_{xt}dx\\
\le\ &C(\delta_2+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_3:=&-\int\frac{2J}{n}(\hat J_x-\phi_{xt})\phi_{xx}dx+2\int\left[\frac{2J}{n}(\hat J_x-\phi_{xt})\right]_x\phi_{xt}dx\\
\le\ &C(\delta_2+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
\end{aligned}
$$

The second term in the right hand of (3.25) can be done as

$$
\begin{aligned}
&\int F_{2x}(\phi_x+2\phi_{xt})dx\\
&\le C(\delta_2+\varepsilon)\|(\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}\tag{3.28}
$$

where we have used

$$
\begin{aligned}
\int_x^{+\infty}\phi_x\tilde E_x(\phi_x+\phi_{xt})dx&\le\|\phi_x(t)\|_\infty\|\tilde E_x\|(\|\phi_x(t)\|+\|\phi_{xt}(t)\|)\\
&\le C\delta_2\|(\phi_x,\phi_{xx},\phi_{xt})(t)\|^2.
\end{aligned}
$$

Substituting (3.26)–(3.28) into (3.25) and noticing the smallness of $\delta_1+\varepsilon$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int\left[\phi_{xt}\phi_x+\left(\frac12+\tilde n\right)\phi_x^2+\phi_{xt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2\right]dx+C_6\int(\phi_{xt}^2+\phi_x^2+\phi_{xx}^2)dx\\
&\le C(\delta_2+\varepsilon)\|(\phi,\phi_t)(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}\tag{3.29}
$$

for some constant $C_6>0$. By choosing $\lambda'C_6>C_5$ and taking (3.23)+$\lambda'\times$(3.29), we can get that the term

$$
\begin{aligned}
Q_3(t):=&\int\left[\phi_t\phi+\left(\frac12+\tilde n\right)\phi^2+\phi_t^2+\lambda'\phi_{xt}\phi_x+\lambda'\left(\frac12+\tilde n\right)\phi_x^2+\lambda'\phi_{xt}^2\right.\\
&\left.+\lambda'\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xx}^2\right]dx
\end{aligned}
$$

satisfying

$$d_{21}\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2 \le Q_3(t) \le d_{22}\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2, \tag{3.30}$$

for two positive constants d_{21} and d_{22}. Moreover, by the smallness of $\delta_2+\varepsilon$, we obtain

$$\frac{d}{dt}Q_3(t) + C_7 Q_3(t) \le C\delta_2(1+t)^{-\theta_2},$$

for some constant C_7. The Gronwall's inequality and (3.30) imply that

$$\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt})(t)\|^2 \le Q_3(t) \le C(\delta_2+\Phi_0^2)(1+t)^{-\theta_2}. \tag{3.31}$$

From the equation in (3.17), it can be easy to get that

$$\|\phi_{tt}(t)\|^2 \le C(\delta_2+\Phi_0^2)(1+t)^{-\theta_2}. \quad \square$$

Lemma 3.4. *If $\varepsilon+\delta_2 \ll 1$, it holds that*

$$\|(\phi_{xxt},\phi_{xtt},\phi_{xxx})(t)\|^2 \le C(\delta_2+\Phi_0^2)(1+t)^{-\theta_2}. \tag{3.32}$$

Proof. Differentiating (3.24) in t leads to

$$\phi_{tttx}+\phi_{ttx}+\tilde{n}\phi_{xt}+\tilde{n}_x\phi_t = \left(p(n)+\frac{J^2}{n}\right)_{xxt} - F_{2xt}. \tag{3.33}$$

Multiplying (3.33) by $\phi_{xt}+2\phi_{xtt}$ and integrating it over $[0,+\infty)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\left[\phi_{xt}\phi_{xtt}+\left(\frac{1}{2}+\tilde{n}\right)\phi_{xt}^2+\phi_{xtt}^2\right]dx+\int(\phi_{xtt}^2+\tilde{n}\phi_{xt}^2)dx+\int\tilde{n}_x\phi_t(\phi_{xt}+2\phi_{xtt})dx\\
&=-\int\left(p(n)+\frac{J^2}{n}\right)_{xt}(\phi_{xxt}+2\phi_{xxtt})dx-\int F_{2xt}(\phi_{xt}+2\phi_{xtt})dx.
\end{aligned} \tag{3.34}$$

It is easy to get

$$\int\tilde{n}_x\phi_t(\phi_{xt}+2\phi_{xtt})dx \le C\delta_2\|(\phi_t,\phi_{xt},\phi_{xtt})(t)\|^2. \tag{3.35}$$

Notice that

$$\left(p(n)+\frac{J^2}{n}\right)_{xt} = \left(p'(n)-\frac{J^2}{n^2}\right)(\phi_{xxt}+\hat{n}_{xt}) + \left(p'(n)-\frac{J^2}{n^2}\right)_t(\phi_{xx}+\hat{n}_x+\tilde{n}_x) + \left(\frac{2JJ_x}{n}\right)_t.$$

After integrating by parts, we can get

$$\begin{aligned}
&-\int\left(p(n)+\frac{J^2}{n}\right)_{xt}(\phi_{xxt}+2\phi_{xxtt})dx\\
&=-\frac{d}{dt}\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2dx-\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2dx+K_4+K_5\\
&\le-\frac{d}{dt}\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xtt}^2dx-\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xtt}^2dx\\
&\quad+C(\delta_2+\varepsilon)\|(\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}$$

where

$$\begin{aligned}
K_4 :=& \int\left(p'(n)-\frac{J^2}{n^2}\right)(2\hat{n}_{xxt}\phi_{xxt}-\hat{n}_{xt}\phi_{xxt})dx+2\int\left(p'(n)-\frac{J^2}{n^2}\right)_x\hat{n}_{xt}\phi_{xtt}dx\\
&+\int\left(p'(n)-\frac{J^2}{n^2}\right)_t[(\phi_{xxt}-\phi_{xx}-\hat{n}_x+\tilde{n}_x)\phi_{xxt}+(2\phi_{xx}+2\hat{n}_{xx}+2\tilde{n}_{xx})\phi_{xtt}]dx
\end{aligned}$$

$$
\begin{aligned}
&+\int\left(p'(n)-\frac{J^2}{n^2}\right)_{tx}(2\phi_{xx}\phi_{xtt}+2\hat{n}_x\phi_{xtt}+\tilde{n}_x\phi_{xtt})dx\\
&\le C(\delta_2+\varepsilon)\|(\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_5 :=& \int\left(\frac{2JJ_x}{n}\right)_t(\phi_{xxt}+2\phi_{xxtt})dx\\
=& \int\frac{2J}{n}J_{xt}\phi_{xxt}dx+\int\left(\frac{2J}{n}\right)_tJ_x\phi_{xxt}dx-\int\left[\frac{2J}{n}J_{xt}\right]_x\phi_{xtt}dx+\int\left[\left(\frac{2J}{n}\right)_tJ_x\right]_x\phi_{xtt}dx\\
\le& C(\delta_2+\varepsilon)\|(\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
\end{aligned}
$$

The last term in (3.34) can be estimated as

$$
-\int F_{2xt}(\phi_{xt}+2\phi_{xtt})dx\le C(\delta_2+\varepsilon)\|(\phi_{xx},\phi_{xt},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
$$

Thus, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\left[\phi_{xt}\phi_{xtt}+\left(\frac{1}{2}+\tilde{n}\right)\phi_{xt}^2+\phi_{xtt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2\right]dx\\
&+\int\left[\phi_{xtt}^2+\tilde{n}\phi_{xt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2\right]dx\\
&\le C(\delta_2+\varepsilon)\|(\phi_t,\phi_{xx},\phi_{xt},\phi_{tt},\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
\end{aligned}
\tag{3.36}
$$

Multiplying (3.24) by ϕ_{xxx} and integrating the resultant equation over $[0,+\infty)$, we obtain

$$
\begin{aligned}
&\int\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxx}^2dx\\
&\le C(\delta_2+\varepsilon)\|\phi_{xxx}(t)\|^2+C\|(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
\end{aligned}
$$

By using smallness of $\delta_2+\varepsilon$ and (3.21), we have

$$
\|\phi_{xxx}(t)\|^2\le C\|(\phi_{xxt},\phi_{xtt})(t)\|^2+C\delta_2(1+t)^{-\theta_2}.
\tag{3.37}
$$

Let

$$
Q_4(t)=\int\left[\phi_{xt}\phi_{xtt}+\left(\frac{1}{2}+\tilde{n}\right)\phi_{xt}^2+\phi_{xtt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)\phi_{xxt}^2\right]dx.
$$

Then from (3.37), there exist two positive constants d_{23} and d_{24} such that

$$
d_{23}\|(\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2-C\delta_2(1+t)^{-\theta_2}\le Q_4(t)\le d_{24}\|(\phi_{xt},\phi_{xxt},\phi_{xtt})(t)\|^2.
\tag{3.38}
$$

Substituting (3.37) into (3.36) and noticing the smallness of $\delta_2+\varepsilon$, by (3.21), we get

$$
\frac{d}{dt}Q_4(t)+C_8Q_4(t)\le C\delta_2(1+t)^{-\theta_2},
\tag{3.39}
$$

for some positive constant C_8. The Gronwall's inequality and (3.38) imply that

$$
\|(\phi_{xxx},\phi_{xxt},\phi_{xtt})(t)\|^2\le Q_4(t)\le C(\delta_2+\varPhi_0^2)(1+t)^{-\theta_2}.\quad\Box
\tag{3.40}
$$

Proof of Theorem 3.2. Lemmas 3.3 and 3.4 imply Theorem 3.2.

4. Numerical simulations

In this section, we are going to carry out some numerical simulations according to the different boundary cases. As showed before, the solutions $(n, J, E)(x, t)$ for the dynamical system (1.1) with the initial value (1.3) subjected to either the inflow/outflow/impermeable boundary (1.4) or the insulating boundary (1.5) time-asymptotically converge to the corresponding steady-states (asymptotic profiles), so we are mainly interested in what shapes of the asymptotic profiles. Namely, we numerically calculate the corresponding steady-state system

$$\begin{cases} \tilde{J}_x = 0, \\ \left(\dfrac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n})\right)_x = \tilde{n}\tilde{E} - \tilde{J}, \quad x \in \mathbb{R}_+, \\ \tilde{E}_x = \tilde{n} - D(x), \end{cases} \tag{4.1}$$

with the inflow/outflow/impermeable boundary:

$$\tilde{J}(0) = J_*, \quad \tilde{E}(0) = E_*, \tag{4.2}$$

or the insulating boundary:

$$\tilde{E}_x(0) = n_0(0) - D(0), \quad \lim_{x\to\infty} \tilde{E}(x) = E^*. \tag{4.3}$$

For numerical calculations, let us simply take the initial data as

$$\begin{aligned} n_0(x) &= \begin{cases} 2 + e^{-x}, & \text{for inflow/outflow/impermeable case;} \\ \cos(\pi x/200) + 0.1, & \text{for insulating case,} \end{cases} \\ J_0(x) &= J_* + e^{-0.5x} - 1, \end{aligned}$$

the doping profile as

$$D(x) = 2 + \frac{1}{1 + x^2},$$

and the pressure function as

$$p(\tilde{n}) = \tilde{n}^3.$$

For the insulating boundary conditions, we take

$$E^* = 0.1, \quad n_+ = 2,$$

and for the inflow/outflow/impermeable boundary conditions, we take

$$E_* = 0.1, \quad n_+ = 2, \quad J_* = \begin{cases} 0.5, & \text{for the inflow case;} \\ -0.5, & \text{for the outflow case;} \\ 0, & \text{for the impermeable case.} \end{cases}$$

By using difference scheme, we get the numerical solutions of the electronic density $\tilde{n}(x)$ and the electric field $\tilde{E}(x)$ for all boundary cases as follows (see Fig. 1 for $\tilde{n}(x)$ and Fig. 2 for $\tilde{E}(x)$). Here, the Curve 1, Curve 2, Curve 3 and Curve 4 are corresponding for the insulating boundary case, the inflow boundary case, the outflow boundary case, and the impermeable boundary case, respectively. Note that, the graphs for $\tilde{n}(x)$ and $\tilde{E}(x)$ in different boundary cases are significantly distinct.

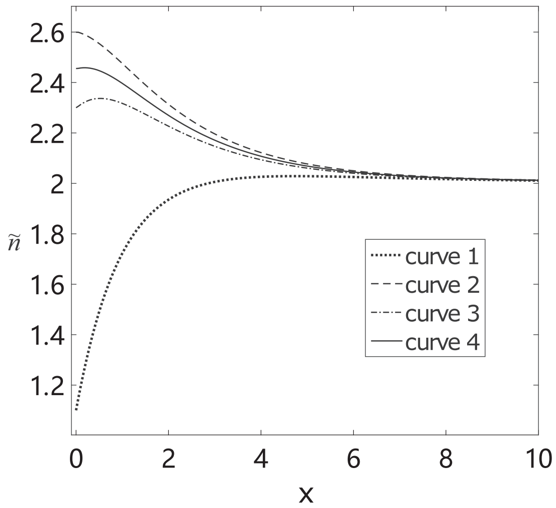

Fig. 1. The graphs for the solution $\tilde{n}(x)$: Curve 1 is for the insulating boundary case, Curve 2 is for the inflow boundary case, Curve 3 is for the outflow boundary case, and Curve 4 is for the impermeable boundary case.

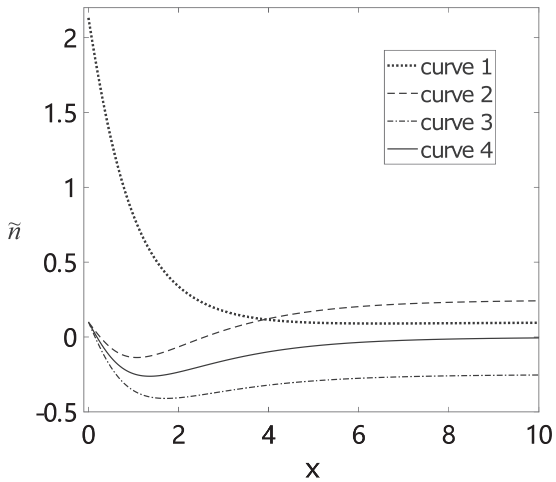

Fig. 2. The graphs for the solution $\tilde{E}(x)$: Curve 1 is for the insulating boundary case, Curve 2 is for the inflow boundary case, Curve 3 is for the outflow boundary case, and Curve 4 is for the impermeable boundary case.

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