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Large-time behavior of solution for generalized Benjamin–Bona–Mahony–Burgers equations [1]

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1. Introduction

In this paper we study the large-time behavior of the global solutions to the Cauchy problem for the generalized Benjamin–Bona–Mahony–Burgers (BBM-B) equations in the form

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0, \quad x \in R^1, \quad t \geq 0, \tag{1.1}$$

with the initial data

$$u|_{t=0} = u_0(x) \to \bar{u} \quad \text{as } x \to \pm\infty, \tag{1.2}$$

where α is a positive constant, β and $\bar{u}$ are any given constants in R^1, and $\phi(u)$ is a C^2−smooth nonlinear function.

Spectial case of Eq. (1.1) is the alternative regularized long-wave equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{1.3}$$

proposed by Peregrine [12] and Benjamin et al. [2]. This equation features a balance between nonlinear and dispersive effects, but takes no account of dissipation. In the physical sense, Eq. (1.1) with the dissipative term $-\alpha u_{xx}$ is proposed if the good predictive power is desired, such problem arises in the phenomena for both the bore

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propagation and the water waves. The temporal decay rates of the solution of Eqs. (1.1) and (1.2) are studied in a number of works, see [1, 3–8, 11, 14, 15] and the references therein. Among them, when $\bar{u} = 0$, and $\int_{-\infty}^{\infty} u_0(y)\,dy = 0$, $xu_0(x) \in L^1$, or $|\hat{u}_0(\xi)| \le C|\xi|$, Amick et al. [1], Bona and Lou [3, 4] and Dix [8] showed the decay rates, like $\|u(t)\|_{L^2} = O(t^{-3/4})$ and $\|u_x(t)\|_{L^2} = O(t^{-5/4})$ as $t \to \infty$, of the solution of Eq. (1.1) for $\phi(u) = u^{p+1}/(p+1)$ and $p \ge 1$. Zhang [14, 15] proved the time decay of the solution for Eqs. (1.1) and (1.2) as $(1+t)^{-1/2}$ in L^∞-norm of u when the nonlinearity $\phi(u)$ satisfies $|\phi'(u)| \le C|u|^{p+1}$, $|\phi''(u)| \le C|u|^p$ for $p \ge 3$. In this note, our purpose is to show further the asymptotic behavior of the solution of Eqs. (1.1) and (1.2). We here improve essentially the previous works [1, 3, 4, 8, 14, 15] with the stronger decay rates and the weaker sufficient conditions. Other new results will also be shown in the present paper. Precisely, dropping the conditions $xu_0 \in L^1$ or $|\hat{u}_0(\xi)| \le C|\xi|$, we will prove the time decay rates of the solution like $\|u(t)\|_{L^\infty} = O(t^{-1})$, $\|u(t)\|_{L^2} = O(t^{-3/4})$ and $\|u_x(t)\|_{L^2} = O(t^{-5/4})$ as $t \to \infty$, for *any* C^2−smooth nonlinearity $\phi(u)$. For details, see Theorem 2.1 and Remark 2.2 below. The scheme of the proof we adopt is based on the method of Fourier transform together with the energy method, which also was used by the author [10] to treat with the Rosenau–Burgers equations.

Our plan in this paper is as follows. After stating some notations in the last part of this section, we give our main theorem on the decay rates of the solution of Eqs. (1.1) and (1.2) in Section 2. Section 3 is the proof of the main theorem.

Notations. We first give some notations for simplicity. C always denotes some positive constants without confusion. H^k ($k \ge 0$ integer) and $W^{k,p}$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$ and $\|\cdot\|_{W^{k,p}}$, respectively. L^2 denotes the square integrable space with the norm $\|\cdot\|$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_\infty$. Suppose that $f(x) \in L^1 \cap L^2(R)$; we define the Fourier transforms of $f(x)$ as follows:

$$F[f](\xi) \equiv \hat{f}(\xi) = \int_R f(x) e^{-ix\xi}\,dx.$$

Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T;B)$ ($k \ge 0$) denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $[0,T]$. The corresponding spaces of B-valued function on $[0,\infty)$ are defined similarly.

2. Main theorem

Throughout this paper, we suppose that

$$\int_{-\infty}^{\infty} (u_0(x) - \bar{u})\,dx = 0. \tag{2.1}$$

Let

$$v_0(x) = \int_{-\infty}^{x} (u_0(y) - \bar{u})\,dy \tag{2.2}$$

and

$$v(t,x) = \int_{-\infty}^{x} (u(t,y) - \bar{u})\,\mathrm{d}y; \tag{2.3}$$

we reformulate the initial-value problem (1.1) and (1.2) as the "integrated" equation

$$v_t - v_{xxt} - \alpha v_{xx} + (\beta + \phi'(\bar{u}))v_x + F(v_x) = 0, \tag{2.4}$$

with the initial data

$$v|_{t=0} = v_0(x), \tag{2.5}$$

where

$$F(v_x) \equiv \phi(\bar{u} + v_x) - \phi(\bar{u}) - \phi'(\bar{u})v_x. \tag{2.6}$$

It is well known that $F(v_x)$ satisfies the following by the Talyor's formula:

$$|F(v_x)| \le C|v_x|^2, \quad |\partial_x F(v_x)| \le C|v_x v_{xx}|. \tag{2.7}$$

We state our main theorem as follows.

Theorem 2.1. *Suppose that* (2.1) *and* $v_0(x) \in W^{3,1}$ *hold; then there exists a positive constant* δ_1 *such that when* $\|v_0\|_{W^{3,1}} < \delta_1$, *then Eqs.* (2.4) *and* (2.5) *have a unique global solution* $v(x,t)$ *satisfying*

$$v(t,x) \in C(0,\infty; H^2 \cap W^{1,\infty}) \cap C^1(0,\infty; L^2 \cap L^\infty).$$

Moreover, the asymptotic decay rates of the solution $v(t,x)$

$$\|v(t)\| \le C(1+t)^{-1/4}, \quad \|v(t)\|_\infty \le C(1+t)^{-1/2}, \tag{2.8}$$

$$\|v_x(t)\| \le C(1+t)^{-3/4}, \quad \|v_x(t)\|_\infty \le C(1+t)^{-1}, \tag{2.9}$$

$$\|v_t(t)\| \le C(1+t)^{-5/4}, \quad \|v_t(t)\|_\infty \le C(1+t)^{-3/2}, \tag{2.10}$$

$$\|v_{xx}(t)\| \le C(1+t)^{-5/4} \tag{2.11}$$

hold for all $t \ge 0$.

Remark 2.1. Going back to the original problem (1.1) and (1.2), from Theorem 2.1, we see that, when $\bar{u} = 0$, $u_0 \in W^{2,1}$ and $\int_{-\infty}^{\infty} u_0(y)\mathrm{d}y = 0$, we have $\|u(t)\| \le C(1+t)^{-3/4}$, $\|u_x(t)\| \le C(1+t)^{-5/4}$ and $\|u(t)\|_\infty \le C(1+t)^{-1}$ for *any* nonlinear function $\phi \in C^2$. We improve the decay rate $O(t^{-1/2})$ in L^∞-norm of u by Zhang [14, 15] for $|\phi'(u)| \le C|u|^{p+1}$, $|\phi''(u)| \le C|u|^p$ and $p \ge 3$. We also get the same decay rates in L^2-norm by dropping the sufficient conditions $xu_0 \in L^1$ or $|\hat{u}_0(\xi)| \le C|\xi|$ used by Amick *et al.* [1], Dix [8], and Bona and Luo [3, 4] for $\phi(u) = u^{p+1}/(p+1)$ and $p \ge 1$.

3. The proof of the main theorem

To prove Theorem 2.1, we need the preparations below. We take the Fourier transform to Eq. (2.4) to yield

$$\hat{v}_t - (\mathrm{i}\xi)^2\hat{v}_t - \alpha(\mathrm{i}\xi)^2\hat{v} + (\beta + \phi'(\bar{u}))\mathrm{i}\xi\hat{v} + \widehat{F(v_x)} = 0, \tag{3.1}$$

which gives us

$$\hat{v}(t,\xi) = \mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi) - \int_0^t \mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}s, \tag{3.2}$$

where

$$A(\xi) = B(\xi) + \mathrm{i}\frac{(\beta + \phi'(\bar{u}))\xi}{1+\xi^2}, \quad B(\xi) = \frac{\alpha\xi^2}{1+\xi^2}. \tag{3.3}$$

Taking the inverse Fourier transform to Eq. (3.2) yields

$$\begin{aligned} v(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \\ &\quad - \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

Differentiating Eq. (3.4) on x and t, respectively, we obtain

$$\begin{aligned} v_x(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{i}\xi\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \\ &\quad - \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} \mathrm{i}\xi\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}\xi\,\mathrm{d}s, \end{aligned} \tag{3.5}$$

$$\begin{aligned} v_t(t,x) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\,\mathrm{d}\xi, \end{aligned} \tag{3.6}$$

$$\begin{aligned} v_{xx}(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\xi)^2\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \\ &\quad - \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} (\mathrm{i}\xi)^2\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{3.7}$$

For a positive constant δ, define a Banach space as the solution space

$$X_\delta = \{v \in C(0,\infty; H^2 \cap W^{1,\infty}) \cap C^1(0,\infty; L^2 \cap L^\infty) | M(v) \le \delta\}$$

with the distance

$$M(v) = \sup_{0\le t<\infty} \{(1+t)^{1/4}\|v(t)\| + (1+t)^{1/2}\|v(t)\|_\infty$$
$$+(1+t)^{3/4}\|v_x(t)\| + (1+t)\|v_x(t)\|_\infty$$
$$+(1+t)^{5/4}\|v_t(t)\| + (1+t)^{3/2}\|v_t(t)\|_\infty$$
$$+(1+t)^{5/4}\|v_{xx}(t)\|\}. \tag{3.8}$$

Rewriting Eq. (3.4) as the operatoral form $v = Sv$, we will prove that there exists the positive constant δ_1, such that the operator S maps X_{δ_1} into itself and has a unique fixed point in X_{δ_1}. Thus, such a fixed point $v(t,x)$ is the solution of Eq. (2.4) in X_{δ_1}, namely $u(t,x) = \bar{u} + v_x(t,x)$ is the unique solution of Eq. (1.1) globally in time. To prove these, the following lemmas are available.

Lemma 3.1. *Suppose that* $a > 0$ *and* $b > 0$, *and* $\max(a,b) > 1$, *then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}\,\mathrm{d}s \le C(1+t)^{-\min(a,b)}. \tag{3.9}$$

Here, this lemma is more accurate than the Gronwall's inequality to get the time decay rates, we believe, although the Gronwall's inequality are effectively used in the previous works [14, 15]. Lemma 3.1 can be found in Ref. [13], see also Ref. [9].

Lemma 3.2. *The following*

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-B(\xi)t}}{1+\xi^2}\,\mathrm{d}\xi \le C(1+t)^{-1/2}, \tag{3.10}$$

$$\int_{-\infty}^{\infty} \frac{|\xi|\mathrm{e}^{-B(\xi)t}}{(1+\xi^2)(1+|\xi|)}\,\mathrm{d}\xi \le C(1+t)^{-1}, \tag{3.11}$$

$$\int_{-\infty}^{\infty} \frac{|\xi|^2\mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)^2}\,\mathrm{d}\xi \le C(1+t)^{-3/2}, \tag{3.12}$$

$$\int_{-\infty}^{\infty} \frac{\xi^4\mathrm{e}^{-2B(\xi)t}}{(1+\xi^2)^3}\,\mathrm{d}\xi \le C(1+t)^{-5/2}, \tag{3.13}$$

hold for all $t \ge 0$.

Proof. *Proof of* (3.10). We first have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-B(\xi)t}}{1+\xi^2}\,\mathrm{d}\xi = 2\left(\int_0^1 + \int_1^\infty\right) \frac{\mathrm{e}^{-B(\xi)t}}{1+\xi^2}\,\mathrm{d}\xi. \tag{3.14}$$

Since $\frac{1}{2} \le 1/(1+\xi^2) \le 1$ and $e^{-B(\xi)t} \le e^{-\alpha\xi^2 t/2}$ for $\xi \in [0,1]$, we have

$$\begin{aligned}\int_0^1 \frac{e^{-B(\xi)t}}{1+\xi^2}\, d\xi &\le \int_0^1 e^{-\alpha\xi^2/2t}\, d\xi \\ &= \int_0^1 e^{-(\alpha\xi^2/2)(1+t)} e^{\alpha\xi^2/2}\, d\xi \le e^{(\alpha/2)} \int_0^1 e^{-(\alpha\xi^2/2)(1+t)}\, d\xi \\ &\le e^{\alpha/2} \int_0^\infty e^{-(\alpha\xi^2/2)(1+t)}\, d\xi \le C(1+t)^{-1/2}. \end{aligned} \tag{3.15}$$

For the second term on the right-hand side of (3.14), noting $\frac{1}{2} \le \xi^2/(1+\xi^2) \le 1$ for $\xi \in [1,\infty)$, we have

$$\int_1^\infty \frac{e^{-B(\xi)t}}{1+\xi^2}\, d\xi \le \int_1^\infty \frac{e^{-\alpha t/2}}{1+\xi^2}\, d\xi \le Ce^{-\alpha t/2}. \tag{3.16}$$

Thus, plugging (3.15) and (3.16) into (3.14) implies (3.10).

Proof of (3.11). We have similarly

$$\int_{-\infty}^\infty \frac{|\xi| e^{-B(\xi)t}}{(1+\xi^2)(1+|\xi|)}\, d\xi = 2\left(\int_0^1 + \int_1^\infty\right) \frac{\xi e^{-B(\xi)t}}{1+\xi^2}\, d\xi. \tag{3.17}$$

First, by $\dfrac{1}{2} \le 1/(1+\xi^2) \le 1$ and $1/((1+\xi^2)(1+\xi)) \le 1$ for $\xi \in [0,1]$, we get

$$\begin{aligned}\int_0^1 \frac{\xi e^{-B(\xi)t}}{1+\xi^2}\, d\xi &\le \int_0^1 \xi e^{-(\alpha\xi^2/2)t}\, d\xi \\ &= \frac{1}{2}\int_0^1 e^{-(\alpha\xi^2/2)(1+t)} e^{\alpha\xi^2/2}\, d\xi^2 \le \frac{e^{\alpha/2}}{2} \int_0^1 e^{-(\alpha\xi^2/2)(1+t)}\, d\xi^2. \\ &\le C(1+t)^{-1}. \end{aligned} \tag{3.18}$$

Second, using the facts (3.16) and $\xi/(1+\xi) \le 1$ for $\xi \in [1,\infty)$ yields

$$\int_1^\infty \frac{\xi e^{-B(\xi)t}}{(1+\xi^2)(1+\xi)}\, d\xi \le \int_1^\infty \frac{e^{-B(\xi)t}}{1+\xi^2}\, d\xi \le Ce^{-\alpha t/2}. \tag{3.19}$$

Therefore, (3.17)–(3.19) give us the desired estimate (3.11).

Proofs of (3.12) *and* (3.13). In the same way as above, we can prove (3.12) and (3.13). In fact, since

$$\begin{aligned}\int_0^1 \frac{\xi^2 e^{-2B(\xi)t}}{(1+\xi^2)^2}\, d\xi &\le \int_0^1 \xi^2 e^{-2\alpha\xi^2 t}\, d\xi \\ &\le \frac{e^\alpha}{3}\int_0^1 e^{-\alpha\xi^2(1+t)}\, d\xi^3 \le C(1+t)^{-3/2}, \end{aligned} \tag{3.20}$$

$$\int_0^1 \frac{\xi^4 e^{-2B(\xi)t}}{(1+\xi^2)^3}\, d\xi \le \int_0^1 \xi^4 e^{-2\alpha\xi^2 t}\, d\xi$$
$$\le \frac{e^\alpha}{5}\int_0^1 e^{-\alpha\xi^2(1+t)}\, d\xi^5 \le C(1+t)^{-5/2}, \tag{3.21}$$

and by (3.16)

$$\int_1^\infty \frac{\xi^2 e^{-2B(\xi)t}}{(1+\xi^2)^2}\, d\xi \le \int_1^\infty \frac{e^{-2B(\xi)t}}{1+\xi^2}\, d\xi \le Ce^{-\alpha t}, \tag{3.22}$$

$$\int_1^\infty \frac{\xi^4 e^{-2B(\xi)t}}{(1+\xi^2)^3}\, d\xi \le \int_1^\infty \frac{e^{-2B(\xi)t}}{1+\xi^2}\, d\xi \le Ce^{-\alpha t}, \tag{3.23}$$

we get the desired estimates (3.12) and (3.13) from (3.20) to (3.23). We now have completed the proof of the lemma. □

Lemma 3.3. *Suppose that $v_0 \in W^{3,1}$, then we have*

$$\left\| \int_{-\infty}^\infty e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\| \le C\|v_0\|_{W^{1,1}}(1+t)^{-1/4}, \tag{3.24}$$

$$\left\| \int_{-\infty}^\infty e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_\infty \le C\|v_0\|_{W^{2,1}}(1+t)^{-1/2}, \tag{3.25}$$

$$\left\| \int_{-\infty}^\infty i\xi e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\| \le C\|v_0\|_{W^{2,1}}(1+t)^{-3/4}, \tag{3.26}$$

$$\left\| \int_{-\infty}^\infty i\xi e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_\infty \le C\|v_0\|_{W^{3,1}}(1+t)^{-1}, \tag{3.27}$$

$$\left\| \int_{-\infty}^\infty (i\xi)^2 e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\| \le C\|v_0\|_{W^{3,1}}(1+t)^{-5/4}, \tag{3.28}$$

$$\left\| \int_{-\infty}^\infty A(\xi) e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\| \le C\|v_0\|_{W^{2,1}}(1+t)^{-5/4}, \tag{3.29}$$

$$\left\| \int_{-\infty}^\infty A(\xi) e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|_\infty \le C\|v_0\|_{W^{2,1}}(1+t)^{-3/2}. \tag{3.30}$$

Proof. Making use of the Parseval's equality and (3.10), we have

$$\left\| \frac{1}{2\pi}\int_{-\infty}^\infty e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|$$
$$= \|e^{-A(\xi)t}\hat{v}_0(\xi)\|$$

$$= \left\{ \int_{-\infty}^{\infty} |e^{-A(\xi)t}\hat{v}_0(\xi)|^2 \, d\xi \right\}^{1/2}$$

$$\le \sup_{\xi\in R}\{(1+|\xi|)|\hat{v}_0(\xi)|\} \left\{ \int_{-\infty}^{\infty} \frac{e^{-2B(\xi)t}}{1+\xi^2} \, d\xi \right\}^{1/2}$$

$$\le C\|v_0\|_{W^{1,1}}(1+t)^{-1/4},$$

where we have used the fact

$$\sup_{\xi\in R}\{(1+|\xi|)|\hat{v}_0(\xi)|\}$$

$$\le \sup_{\xi\in R}\left| \int_{-\infty}^{\infty} e^{-ix\xi} v_0(x)\, dx \right| + \sup_{\xi\in R}\left| \int_{-\infty}^{\infty} e^{-ix\xi} v_{0x}(x)\, dx \right|$$

$$\le \int_{-\infty}^{\infty} |v_0(x)|\, dx + \int_{-\infty}^{\infty} |v_{0x}(x)|\, dx$$

$$= \|v_0\|_{W^{1,1}}. \tag{3.31}$$

So, we have proved (3.24).

To prove (3.25), using (3.10) and the fact proved similar to (3.31)

$$\sup_{\xi\in R}\{(1+\xi^2)|\hat{v}_0(\xi)|\} \le \|v_0\|_{W^{2,1}}, \tag{3.32}$$

we obtain

$$\left| \int_{-\infty}^{\infty} e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right|$$

$$\le \int_{-\infty}^{\infty} e^{-B(\xi)t} |\hat{v}_0(\xi)|\, d\xi$$

$$\le \sup_{\xi\in R}\{(1+\xi^2)|\hat{v}_0(\xi)|\} \int_{-\infty}^{\infty} \frac{e^{-B(\xi)t}}{1+\xi^2}\, d\xi$$

$$\le C\|v_0\|_{W^{2,1}}(1+t)^{-1/2}.$$

Here we have proved (3.25).

The Parseval's equality and (3.11) and (3.32) give us the estimate (3.26) as follows:

$$\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi)\, d\xi \right\|$$

$$= \|i\xi e^{-A(\xi)t}\hat{v}_0(\xi)\|$$

$$= \left\{ \int_{-\infty}^{\infty} |i\xi e^{-A(\xi)t}\hat{v}_0(\xi)|^2 \, d\xi \right\}^{1/2}$$

$$\leq \sup_{\xi\in R}\{(1+\xi^2)|\hat{v}_0(\xi)|\}\left\{\int_{-\infty}^{\infty}\frac{e^{-2B(\xi)t}}{1+\xi^2}\,d\xi\right\}^{1/2}$$
$$\leq C\|v_0\|_{W^{2,1}}(1+t)^{-3/4}.$$

Similarly, (3.27) can be proved by (3.11) as follows:

$$\left|\int_{-\infty}^{\infty} i\xi e^{ix\xi}e^{-A(\xi)t}\hat{v}_0(\xi)\,d\xi\right|$$
$$\leq \int_{-\infty}^{\infty}|\xi|e^{-B(\xi)t}|\hat{v}_0(\xi)|\,d\xi$$
$$\leq \sup_{\xi\in R}\{(1+\xi^2)(1+|\xi|)|\hat{v}_0(\xi)|\}\int_{-\infty}^{\infty}\frac{|\xi|e^{-B(\xi)t}}{(1+\xi^2)(1+|\xi|)}\,d\xi$$
$$\leq C\|v_0\|_{W^{3,1}}(1+t)^{-1},$$

provided that

$$\sup_{\xi\in R}\{(1+\xi^2)(1+|\xi|)|\hat{v}_0(\xi)|\}\leq \|v_0\|_{W^{3,1}}, \tag{3.33}$$

which can be easily proved as (3.31) and (3.32).

Similar to above, we can also prove (3.28)–(3.30) by Lemma 3.2 and (3.31)–(3.33). We omit here the details. □

Lemma 3.4. *Suppose that $v(x,t)\in X_\delta$; then we have*

$$\left\|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-1/4}, \tag{3.34}$$

$$\left\|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|_{\infty}\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-1/2}, \tag{3.35}$$

$$\left\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-3/4}, \tag{3.36}$$

$$\left\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|_{\infty}\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-1}, \tag{3.37}$$

$$\left\|\int_{-\infty}^{\infty} (i\xi)^2 e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-5/4}, \tag{3.38}$$

$$\left\|\int_{-\infty}^{\infty} A(\xi) e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-5/4}, \tag{3.39}$$

$$\left\| \int_{-\infty}^{\infty} A(\xi) e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2} \, d\xi \right\|_{\infty} \le C\delta^2 (1+s)^{-3/2}(1+t-s)^{-3/2}. \tag{3.40}$$

Proof. By the Parseval's equality and (3.10) we have

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2} \, d\xi \right\| \\
&= \left\| e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}}{1+\xi^2} \right\| \\
&= \left\{ \int_{-\infty}^{\infty} \left| e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}}{1+\xi^2} \right|^2 d\xi \right\}^{1/2} \\
&\le \sup_{\xi\in R} |\widehat{F(v_x)}(\xi,s)| \left\{ \int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} \, d\xi \right\}^{1/2} \\
&\le \sup_{\xi\in R} |\widehat{F(v_x)}(\xi,s)| \left\{ \int_{-\infty}^{\infty} \frac{e^{-B(\xi)(t-s)}}{1+\xi^2} \, d\xi \right\}^{1/2} \\
&\le C(1+t-s)^{-1/4} \sup_{\xi\in R} |\widehat{F(v_x)}(\xi,s)|.
\end{aligned} \tag{3.41}$$

According to the definations of $\widehat{F(v_x)}$ and X_δ, and noting $|F| \le C|v_x|^2$, see (2.6) and (2.7), and $v \in X_\delta$, we have

$$\begin{aligned}
\sup_{\xi\in R} |\widehat{F(v_x)}(\xi,s)| &= \sup_{\xi\in R} \left| \int_{-\infty}^{\infty} e^{-ix\xi} F(v_x)(x,s) dx \right| \\
&\le \int_{-\infty}^{\infty} |F(v_x)| \, dx \le C\|v_x(s)\|^2 \le C\delta^2(1+s)^{-3/2}.
\end{aligned} \tag{3.42}$$

Therefore, substituting (3.42) into (3.41) yields (3.34). Similarly, using the Parseval's equality and (3.12) and (3.13) and (3.42), we can prove (3.36) and (3.39) without any difficulty. (3.35) and (3.40) can also be easily proved by (3.10) and (3.12) and (3.42); we omit the details, too.

Now we focus on the proofs of (3.37) and (3.38). Since $|F(v_x)| \le C|v_x|^2$, $|\partial_x F(v_x)| \le C|v_x v_{xx}|$, see (2.7), and $v \in X_\delta$, i.e., $\|v_x(t)\| \le \delta(1+t)^{-3/4}$, $\|v_{xx}(t)\| \le \delta(1+t)^{-5/4}$, we have

$$\begin{aligned}
&\sup_{\xi\in R} \{(1+|\xi|)|\widehat{F(v_x)}(s,\xi)|\} \\
&\le \sup_{\xi\in R} \left| \int_{-\infty}^{\infty} e^{-ix\xi} F(v_x)(s,x) \, dx \right| + \sup_{\xi\in R} \left| \int_{-\infty}^{\infty} e^{-ix\xi} \partial_x F(v_x)(s,x) \, dx \right|
\end{aligned}$$

$$\begin{aligned}
&\le \int_{-\infty}^{\infty} |F(v_x)|\,\mathrm{d}x + \int_{-\infty}^{\infty} |\partial_x F(v_x)|\,\mathrm{d}x \\
&\le C(\|v_x(s)\|^2 + \|v_x(s)\|\|v_{xx}(s)\|) \\
&\le C\delta^2[(1+s)^{-3/2} + (1+s)^{-3/4}(1+s)^{-5/4}] \\
&\le C(1+s)^{-3/2}.
\end{aligned} \tag{3.43}$$

Then, (3.43) and (3.11) imply (3.37) by the following computation:

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \mathrm{i}\xi \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,\mathrm{d}\xi \right| \\
&\le \int_{-\infty}^{\infty} \frac{|\xi| \mathrm{e}^{-B(\xi)(t-s)} |\widehat{F(v_x)}(\xi,s)|}{1+\xi^2}\,\mathrm{d}\xi \\
&\le \sup_{\xi\in R}\{(1+|\xi|)|\widehat{F(v_x)}(\xi,s)|\} \int_{-\infty}^{\infty} \frac{|\xi| \mathrm{e}^{-B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)}\,\mathrm{d}\xi \\
&\le C(1+s)^{-3/2}(1+t-s)^{-1}.
\end{aligned}$$

Similarly, using the Parseval's equality and (3.13) and (3.43), we prove (3.38) as follows:

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^2 \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(s,\xi)}{1+\xi^2}\,\mathrm{d}\xi \right\| \\
&= \left\| \frac{(\mathrm{i}\xi)^2 \mathrm{e}^{-A(\xi)(t-s)} \widehat{F(v_x)}(s,\xi)}{1+\xi^2} \right\| \\
&= \left\{ \int_{-\infty}^{\infty} \frac{\xi^4 \mathrm{e}^{-2B(\xi)(t-s)} |\widehat{F(v_x)}(s,\xi)|^2}{(1+\xi^2)^2}\,\mathrm{d}\xi \right\}^{1/2} \\
&\le \sup_{\xi\in R}\{(1+|\xi|)|\widehat{F(v_x)}(s,\xi)|\} \left\{ \int_{-\infty}^{\infty} \frac{|\xi|^4 \mathrm{e}^{-2B(\xi)(t-s)}}{(1+\xi^2)^3}\,\mathrm{d}\xi \right\}^{1/2} \\
&\le C\delta^2(1+s)^{-3/2}(1+t-s)^{-5/4}.
\end{aligned}$$

We have completed the proof of Lemma 3.4. □

Proof of Theorem 2.1. To prove Theorem 2.1, we need to prove that there exists the positive constant δ_1 such that the operator S is a contraction mapping from X_{δ_1} into X_{δ_1}.

Step 1. $S : X_\delta \to X_\delta$. For any $v_1(x,t) \in X_\delta$, and denoting $v = Sv_1$, we will prove $v = Sv_1 \in X_\delta$ for some small $\delta > 0$. Indeed, using (3.24), (3.34) and (3.9) we have

$$\begin{aligned}\|v(t)\| &= \|Sv_1\| \\ &\le \frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)\,d\xi\right\| \\ &\quad+\frac{1}{2\pi}\int_0^t\left\|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|ds \\ &\le C\|v_0\|_{W^{1,1}}(1+t)^{-1/4}+C\delta^2\int_0^t(1+s)^{-3/2}(1+t-s)^{-1/4}\,ds \\ &\le C\|v_0\|_{W^{1,1}}(1+t)^{-1/4}+C\delta^2(1+t)^{-1/4}. \end{aligned} \tag{3.44}$$

Similarly, we have due to (3.26), (3.36) and (3.9),

$$\begin{aligned}\|v_x(t)\| &= \|\partial_x Sv_1\| \\ &\le \frac{1}{2\pi}\left\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)\,d\xi\right\| \\ &\quad+\frac{1}{2\pi}\int_0^t\left\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,d\xi\right\|ds \\ &\le C\|v_0\|_{W^{2,1}}(1+t)^{-3/4}+C\delta^2\int_0^t(1+s)^{-3/2}(1+t-s)^{-\frac{3}{4}}ds \\ &\le C\|v_0\|_{W^{2,1}}(1+t)^{-3/4}+C\delta^2(1+t)^{-3/4}. \end{aligned} \tag{3.45}$$

By the same way, we can prove that

$$\|Sv_1(t)\|_\infty \le C\|v_0\|_{W^{2,1}}(1+t)^{-1/2}+C\delta^2(1+t)^{-1/2} \tag{3.46}$$

from (3.25), (3.35) and (3.9), and

$$\|\partial_x Sv_1(t)\|_\infty \le C\|v_0\|_{W^{3,1}}(1+t)^{-1}+C\delta^2(1+t)^{-1} \tag{3.47}$$

from (3.27), (3.37) and (3.9), as well as

$$\|\partial_x^2 Sv_1(t)\|_\infty \le C\|v_0\|_{W^{3,1}}(1+t)^{-5/4}+C\delta^2(1+t)^{-5/4} \tag{3.48}$$

from (3.7), (3.9), (3.28) and (3.38).

Before estimating $\|\partial_t Sv_1(t)\|$ and $\|\partial_t Sv_1(t)\|_\infty$, we first note that

$$\begin{aligned}\left\|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\,d\xi\right\| &= \left\|\frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\right\| \\ &= \left\{\int_{-\infty}^{\infty}\frac{|\widehat{F(v_x)}(t,\xi)|^2}{(1+\xi^2)^2}\,d\xi\right\}^{1/2}\end{aligned}$$

$$\le \sup_{\xi\in R} |\widehat{F(v_x)}(t,\xi)| \left\{ \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^2}\,\mathrm{d}\xi \right\}^{1/2}$$

$$\le C\|v_x(t)\|^2 \le C\delta^2(1+t)^{-3/2}, \tag{3.49}$$

by the Parseval's equality, (2.7) and $v \in X_\delta$, as well as note

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} \frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\,\mathrm{d}\xi \right|$$

$$\le \sup_{\xi\in R} |\widehat{F(v_x)}(t,\xi)| \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)}\,\mathrm{d}\xi$$

$$\le C\|v_x(t)\|^2 \le C\delta^2(1+t)^{-3/2}. \tag{3.50}$$

Then, from Lemma 3.1 and (3.29), (3.30), (3.39), (3.40) and (3.49) and (3.50), we have

$$\begin{aligned}
\|v_t(t)\| &= \|\partial_t S v_1\| \\
&\le \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \right\| \\
&\quad + \frac{1}{2\pi} \int_0^t \left\| \int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,\mathrm{d}\xi \right\| \mathrm{d}s \\
&\quad + \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} \frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\,\mathrm{d}\xi \right\| \\
&\le C\|v_0\|_{W^{2,1}}(1+t)^{-5/4} + C\delta^2(1+t)^{-3/2} \\
&\quad + C\delta^2 \int_0^t (1+s)^{-3/2}(1+t-s)^{-5/4}\,\mathrm{d}s \\
&\le C\|v_0\|_{W^{2,1}}(1+t)^{-5/4} + C\delta^2(1+t)^{-5/4}.
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
\|v_t(t)\|_\infty &= \|\partial_t S v_1\|_\infty \\
&\le \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)t}\hat{v}_0(\xi)\,\mathrm{d}\xi \right\| \\
&\quad + \frac{1}{2\pi} \int_0^t \left\| \int_{-\infty}^{\infty} A(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^2}\,\mathrm{d}\xi \right\| \mathrm{d}s \\
&\quad + \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} \frac{\widehat{F(v_x)}(t,\xi)}{1+\xi^2}\,\mathrm{d}\xi \right\|_\infty
\end{aligned}$$

$$\le C\|v_0\|_{W^{2,1}}(1+t)^{-3/2} + C\delta^2(1+t)^{-3/2}$$
$$+C\delta^2 \int_0^t (1+s)^{-3/2}(1+t-s)^{-3/2}\,\mathrm{d}s$$
$$\le C\|v_0\|_{W^{2,1}}(1+t)^{-3/2} + C\delta^2(1+t)^{-3/2}. \tag{3.52}$$

Thus, combining (3.44)–(3.48) and (3.51) and (3.52) implies that

$$M(v) \le c_1(\|v_0\|_{W^{3,1}} + \delta^2). \tag{3.53}$$

Then there exists some small $\delta_2 > 0$, such that $\delta_2 \le 1/2c_1$. Let $\|v_0\|_{W^{3,1}} \le \delta_2/2c_1$, and $\delta \le \delta_2$, we have proved $M(v) \le \delta$ for some small δ, namely, $S : X_\delta \to X_\delta$ for some small $\delta < \delta_2$.

Step 2. *S is a contraction in* X_δ. Suppose that $v_1(x,t), v_2(x,t) \in X_\delta$ $(\delta < \delta_2)$, and noting the facts

$$\sup_{\xi\in R} |\widehat{F(v_{1x})} - \widehat{F(v_{2x})}|$$
$$\le \int_{-\infty}^{\infty} |F(v_{1x}) - F(v_{2x})|\,\mathrm{d}x$$
$$\le C(\|v_{1x}(s)\| + \|v_{2x}(s)\|)\|(v_{1x} - v_{2x})(s)\|$$
$$\le C\delta M(v_1 - v_2)(1+s)^{-3/2}, \tag{3.54}$$

$$\sup_{\xi\in R} |\partial_x(\widehat{F(v_{1x})} - \widehat{F(v_{2x})})|$$
$$\le \int_{-\infty}^{\infty} |\partial_x(F(v_{1x}) - F(v_{2x}))|\,\mathrm{d}x$$
$$\le C[\|(v_{1x} - v_{2x})(s)\|\,\|v_{1xx}(s)\| + \|v_{2x}(s)\|\,\|(v_{1xx} - v_{2xx})(s)\|]$$
$$\le C\delta M(v_1 - v_2)(1+s)^{-3/2}, \tag{3.55}$$

we have by the Parseval's equality and Lemma 3.1

$$\|Sv_1(t) - Sv_2(t)\|$$
$$\le \frac{1}{2\pi}\int_0^t \left\| \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-A(\xi)(t-s)} \frac{\widehat{F(v_{1x})} - \widehat{F(v_{2x})}}{1+\xi^4}\,\mathrm{d}\xi \right\| \mathrm{d}s$$
$$\le C\int_0^t (1+t-s)^{-1/4} \sup_{\xi\in R} |(\widehat{F(v_{1x})} - \widehat{F(v_{2x})})(s,\xi)|\,\mathrm{d}s$$
$$\le C\delta M(v_1 - v_2)\int_0^t (1+s)^{-3/2}(1+t-s)^{-1/4}\,\mathrm{d}s$$
$$\le C\delta M(v_1 - v_2)(1+t)^{-1/4}. \tag{3.56}$$

Making use of (3.54) and (3.55), we also have in the same way in (3.56),

$$\|(Sv_1(t) - Sv_2)_x(t)\| \le C\delta M(v_1 - v_2)(1+t)^{-3/4}, \tag{3.57}$$

$$\|(Sv_1(t) - Sv_2)(t)\|_\infty \le C\delta M(v_1 - v_2)(1+t)^{-1/2}, \tag{3.58}$$

$$\|(Sv_1(t) - Sv_2)_x(t)\|_\infty \le C\delta M(v_1 - v_2)(1+t)^{-1}, \tag{3.59}$$

$$\|(Sv_1(t) - Sv_2)_t(t)\| \le C\delta M(v_1 - v_2)(1+t)^{-5/4}, \tag{3.60}$$

$$\|(Sv_1(t) - Sv_2)_t(t)\|_\infty \le C\delta M(v_1 - v_2)(1+t)^{-3/2}, \tag{3.61}$$

$$\|(Sv_1(t) - Sv_2)_{xx}(t)\| \le C\delta M(v_1 - v_2)(1+t)^{-5/4}, \tag{3.62}$$

where the details are omitted. Therefore, from (3.56) to (3.62), we obtain

$$M(Sv_1 - Sv_2) \le c_2 \delta M(v_1 - v_2). \tag{3.63}$$

Let us choose $\delta \le \delta_3 < 1/c_2$; we have proved

$$M(Sv_1 - Sv_2) < M(v_1 - v_2),$$

i.e. $S : X_\delta \to X_\delta$ is contraction for some small $\delta < \delta_3$.

Thanks to Steps 1 and 2, let $\delta_1 < \min\{\delta_2, \delta_3\}$, we have proved that the operator S is contraction from X_{δ_1} to X_{δ_1}. By the Banach's fixed point theorem, we see that S has a unique fixed point $v(x,t)$ in X_{δ_1}. This means the integral equation (3.4) has a unique global solution $v(t,x) \in X_{\delta_1}$. Thus, we have completed the proof of Theorem 2.1. □

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