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Free boundary value problem for damped Euler equations and related models with vacuum

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Abstract

This paper is concerned with the local well-posedness for the free boundary value problem of smooth solutions to the cylindrical symmetric Euler equations with damping and related models, including the compressible Euler equations and the Euler-Poisson equations. The free boundary is moving in the radial direction with the radial velocity, which will affect the angular velocity but does not affect the axial velocity. However, the compressible Euler equations or Euler-Poisson equations with damping become a degenerate system at the moving boundary. By setting a suitable weighted Sobolev space and using Hardy's inequality, we successfully overcome the singularity at the center point and the vacuum occurring on the moving boundary, and obtain the well-posedness of local smooth solutions. We also summarize the recent related results on the free boundary value problem for the Euler equations with damping, compressible Euler equations and Euler-Poisson equations.

Keywords: Euler equations with damping; Compressible Euler equations; Euler-Poisson equations; Free boundary; Vacuum; Local smooth solutions

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1. Introduction

In this paper, we study the well-posedness of local smooth solutions to the free boundary value problem of the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = -\rho \mathbf{u}, \end{cases} \tag{1.1}$$

where ρ, $\mathbf{u}$, and p denote the density, velocity and pressure, respectively. If the relaxation effect $-\rho\mathbf{u}$ is neglected in the system (1.1), the system (1.1) is reduced to the standard pure Euler equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = 0. \end{cases} \tag{1.2}$$

If the Poisson effect is considered in the system (1.1), for example, concerning the electrostatic potential in semiconductor devices, then the system (1.1) becomes the Euler-Poisson equations with damping relaxation:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \rho \nabla \phi - \rho \mathbf{u}, \\ \Delta \phi = \rho. \end{cases} \tag{1.3}$$

Here, ρ, $\mathbf{u}$, $p(\rho)$ and ϕ denote the electron density, the electron velocity, the pressure and the electrostatic potential for the hydrodynamic models of semiconductors, respectively. For the more general models, the Poisson equation in (1.3) satisfies

$$\Delta \phi = \rho - \mathcal{C}(x), \tag{1.4}$$

where $\mathcal{C}(x)$ is a given background ion density.

Introducing the cylindrical symmetric transformations:

$$\begin{cases} \mathbf{u} = \left(\tilde{u}\frac{x_1}{r} - \tilde{v}\frac{x_2}{r}, \tilde{u}\frac{x_2}{r} + \tilde{v}\frac{x_1}{r}, \tilde{w}\right), & r = \sqrt{x_1^2 + x_2^2}, \\ \tilde{u} = \tilde{u}(r,t),\ \tilde{v} = \tilde{v}(r,t),\ \tilde{w} = \tilde{w}(r,t), & t > 0, \end{cases} \tag{1.5}$$

where the scalar functions $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ represent the radial component, the angular component, and the axial component of the velocity $\mathbf{u}$, the system (1.1) can be transformed into the following form:

$$\begin{cases} \partial_t (r\rho) + \partial_r (r\rho\widetilde{u}) = 0, \\ \rho(\widetilde{u}_t + \widetilde{u}\widetilde{u}_r - \dfrac{\widetilde{v}^2}{r} + \widetilde{u}) + p_r = 0, \\ \widetilde{v}_t + \widetilde{u}\widetilde{v}_r + \dfrac{\widetilde{u}\widetilde{v}}{r} + \widetilde{v} = 0, \\ \widetilde{w}_t + \widetilde{u}\widetilde{w}_r + \widetilde{w} = 0. \end{cases} \tag{1.6}$$

The corresponding cylindrical symmetric compressible Euler equations (1.2) and the Euler-Poisson equations (1.3) respectively satisfy:

$$\begin{cases} \partial_t(r\rho) + \partial_r(r\rho\widetilde{u}) = 0, \\ \rho(\widetilde{u}_t + \widetilde{u}\widetilde{u}_r - \dfrac{\widetilde{\upsilon}^2}{r}) + p_r = 0, \\ \widetilde{\upsilon}_t + \widetilde{u}\widetilde{\upsilon}_r + \dfrac{\widetilde{u}\widetilde{\upsilon}}{r} + \widetilde{\upsilon} = 0, \\ \widetilde{w}_t + \widetilde{u}\widetilde{w}_r + \widetilde{w} = 0, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \partial_t(r\rho) + \partial_r(r\rho\widetilde{u}) = 0, \\ \rho(\widetilde{u}_t + \widetilde{u}\widetilde{u}_r - \dfrac{\widetilde{\upsilon}^2}{r} + \widetilde{u}) + p_r = \rho\dfrac{1}{r}\displaystyle\int_0^r \rho(s)s\,ds, \\ \widetilde{\upsilon}_t + \widetilde{u}\widetilde{\upsilon}_r + \dfrac{\widetilde{u}\widetilde{\upsilon}}{r} + \widetilde{\upsilon} = 0, \\ \widetilde{w}_t + \widetilde{u}\widetilde{w}_r + \widetilde{w} = 0. \end{cases} \tag{1.8}$$

We consider the free boundary value problem of the system (1.6) with the following free boundary condition and initial data in $(0, R(t)) \times [0, T]$:

$$\begin{cases} \rho > 0, \text{ in } [0, R(t)), \\ \rho(R(t), t) = 0, \upsilon(0, t) = 0, \\ \dfrac{dR(t)}{dt} = \widetilde{u}(R(t), t), \;\; R(0) = 1, \\ (\rho, \widetilde{u}, \widetilde{\upsilon}, \widetilde{\omega})(x, 0) = (\rho_0, u_0, \upsilon_0, \omega_0), \;\; \rho_0(x) > 0 \text{ in } [0, 1), \\ -\infty < \dfrac{d\rho_0^{\gamma-1}}{dx} < 0, \;\; \text{on } r = 1, \end{cases} \tag{1.9}$$

where the condition $(1.9)_5$ confirms that $\rho_0^{\gamma-1}$ is equivalent to the distance function $d(x)$ of the boundary near $x = 1$, and also is very important to obtain the regularities of higher order spatial derivatives of velocity, which is called the *physical vacuum condition* (cf. [2,16,18]).

As we know, the free boundary value problem of fluids containing vacuum is one of the most important and difficult problems in the study of partial differential equations from fluid dynamics. In this case, the moving region of the fluid changes with time along the particle path, and the system describing the motions of fluids becomes a degenerate system at the free boundary. Clearly, it is necessary to determine the free boundary, while the solutions of system can be then determined. In particular, the free boundary value problems for the system (1.1), (1.2) and (1.3) with vacuum have been paid more attention and made a breakthrough:

1. *Results on the compressible Euler equations* (1.2). In the pioneering work [1], Coutand, Lindblad and Shkoller established the a priori estimates for the local smooth solution, which was the first result on the free boundary value problem of the three dimensional compressible Euler equations. In [6], Hao also obtained the similar priori estimates of the local smooth solution for the more general class of initial density. Furthermore, in [2,3], Coutand and Shkoller built up the well-posedness of local smooth solutions in Lagrangian coordinates based on Hardy's inequalities and the degenerate parabolic regularization for the one dimensional and the three dimensional cases, respectively. Independent of these works [2,3], Jang and Masmoudi [16,17] also studied the same problem for the one dimensional and the three dimensional cases by the different methods, respectively. Besides the one dimensional and the three dimensional cases, Luo, Xin and Zeng [19] considered the well-posedness of local smooth solutions for the spherically symmetric system of (1.2) and the uniqueness of the three dimensional solutions in the Euler coordinates. For the global smooth solutions, Sideris [24,25] established the spherically symmetric and the three dimensional global affine solutions, respectively. Then, Hadžić and Jang [8] proved the stability and the large time asymptotic behavior of the affine solution obtained in [25]. Regarding as the results on the free boundary value problem to the relativistic motions for the compressible Euler equations (1.2), we refer to the interesting works [11,15,21,22].
2. *Results on the Euler equations with damping* (1.1). Xu and Yang [26] proved the local existence result on the perturbation of a planar wave solution by using Littlewood-Paley theory, and then, Yang [27] summarized the relevant results on the free boundary of the one dimensional Euler equations (1.2) and the Euler equations with damping (1.1). Due to the relaxation term $-\rho\mathbf{u}$ providing the exponential decay rate with respect to times variable for global solutions, it is not too hard to prove the global existence of the solution in the different function spaces, compared to the compressible Euler equations (1.2). In particular, the free boundary value problem of the Euler equations with damping (1.1) admits the Barenblatt solutions as shown in [20,28,29]. Therein, Luo and Zeng [20] and Zeng [28,29] studied the stability and the large time behavior of Barenblatt solutions for the one dimensional, spherically symmetric and three dimensional cases, respectively. In [18], Liu and Yang established a class of particular solutions. It is worthy to point out that the results presented in [7] and [12] show the convergence of weak solutions to Barenblatt solutions for the Cauchy problem of the one-dimensional compressible Euler equations with damping (1.1) involving vacuum for the different exponent γ, respectively.
3. *Results on the Euler-Poisson equations* (1.3). For the model of gravitational interaction, the damping effect $-\rho\mathbf{v}$ is disappeared, and the Euler-Poisson equations (1.3) are reduced to

$$\begin{cases} \partial_t \rho + \operatorname{div} \cdot (\rho\mathbf{v}) = 0, \\ \partial_t (\rho\mathbf{v}) + \operatorname{div} (\rho\mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) = \rho \nabla\varphi, \\ \Delta\varphi = 4\pi\rho. \end{cases} \tag{1.10}$$

In [4] and [5], Gu and Lei investigated the well-posedness of local smooth solutions for the free boundary value problem of the one dimensional and three dimensional system (1.10) with vacuum, respectively. The well-posedness of local smooth solutions for the spherically symmetric system of (1.3) was constructed by Luo, Xin and Zeng [19]. In [13] and [14], Jang showed the instability of Lane-Emden stars for the different $\gamma \in [\frac{6}{5}, \frac{4}{3})$, respectively.

On the other hand, Hadžić and Jang [9] obtained the nonlinear stability of expanding star solutions of the radially symmetric mass-critical Euler-Poisson system. Furthermore, Hadžić and Jang [10] proved the existence of global solutions to the three dimensional Euler-Poisson system (1.3) without any symmetry assumptions in both the gravitational and the plasma case for some exponent γ. Very recently, we [23] first proved the local well-posedness and the relation between the relativistic Euler-Poisson equations for smooth solutions to the free boundary value problem of the Euler-Poisson equations (1.3).

Main goal. In this paper, we mainly study the well-posedness of the local smooth solution for the free boundary value problem (1.6) and (1.9). First of all, by introducing the Lagrangian variable (2.1), we transform the free boundary value problem (1.6) and (1.9) to the initial boundary value problem (2.11) and (2.12). Then, we establish the a priori estimates for the local smooth solution, and prove the existence and uniqueness of local smooth solutions by using a degenerate parabolic approximation and the energy methods (refer to Theorem 2.1 for details), respectively. We also obtain the parallel results on the free boundary value problem for the compressible Euler equations (1.7) and the Euler-Poisson equations (1.8) with the initial boundary condition (1.9), respectively, to see Theorem 2.2 and Remark 2.4.

Technical issue. To prove the main result Theorem 2.1, one of the main difficulties for studying our problem is how to deal with the degenerate of the systems (1.6), (1.7) and (1.8) near the free boundary caused by the pressure term, respectively, because the classical theory of Friedrich-Lax-Kato for quasilinear strictly hyperbolic system can not be directly applied to prove the short time existence of classical solutions. However, we observe that the physical vacuum condition (1.9) confirms that the initial data $\rho_0^{\gamma-1}$ is equivalent to the distance function near the boundary. Thus, the initial data $\rho_0^{\gamma-1}$ plays the weight in the weighted Sobolev embedding inequality (1.11), which is the connection between L^2- norm and the weighted Sobolev spaces. Especially, the initial data $\rho_0^{\gamma-1}$ plays the basic weight in the coefficient of the Lagrangian form (2.11) of the system (1.6). This property helps us to overcome the obstacle by using the Hardy's inequality in a certain weighted Sobolev space. The another difficulty of our problem is caused by the singularity at the center point $r = 0$. We will increase the spatial regularities of fluids velocity near the center point, and apply the Hardy's inequality to establish the desired estimates at the singular point. Compared to the spherically symmetric system of the compressible Euler equations (1.2), at the center point 0, the degeneracy rate $x^{1/2}$ (fractional order) of the cylindrical symmetric system makes the estimates for the higher order derivatives more complicated than the spherically symmetric system (analyzed in [19]), where the rate is x.

The structure of this paper is as follows. In section 2, we transform the free boundary value problem of the cylindrical symmetric Euler with damping into the initial boundary value problem in the fixed domain, and state main results in Lagrangian coordinates. In section 3, we make some a-priori assumptions and computations, which are very crucial to establish the a-priori estimates of solutions. In section 4 and section 5, we mainly show the uniformly a-priori estimates of local smooth solutions. The energy estimates for the higher order time derivatives are obtained in section 4 and the elliptic type estimates are established in section 5, respectively. In section 6, we prove the existence results by a particular degenerate parabolic regularization with the help of Hardy's inequality. Finally, we study the uniqueness in section 7.

Notation and weighted Sobolev spaces. Let $H^k(0, 1)$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$, especially, $\|\cdot\|_0 = \|\cdot\|_{L^2(0,1)}$. For real number l, the Sobolev spaces $H^l(0, 1)$ and the norm $\|\cdot\|_l$ are defined by interpolation. The function space $L^\infty(0,1)$ is simplified by L^∞. The notation C denotes the generic positive constant and the notation M_0 denotes the generic constants depending only on $\max_{x\in[0,1]}\{\rho_0, |\rho_{0x}(x)|, |\rho_{0xx}(x)|\}$.

Let $d(x)$ be the distance function to boundary Γ as $d(x)=dist(x,\Gamma)$ for $x\in\Gamma$. For any $a>0$ and non-negative b, the weighted Sobolev space $H^{a,b}$ is given by

$$H^{a,b}(0,1) := \{d^{\frac{a}{2}}F \in L^2(0,1) : \int_0^1 d^a|\partial_x^k F|^2 dx < \infty, 0 \le k \le b\},$$

with the norm $\|F\|^2_{H^{a,b}} := \sum_{k=0}^b \int_0^1 d^a|\partial_x^k F|dx$. Then, it holds the following embedding: $H^{a,b}(0,1) \hookrightarrow H^{b-a/2}(0,1)$, with the estimate $\|F\|_{b-a/2} \le C_0\|F\|_{H^{a,b}}$. In particular, we have

$$\|F\|_0^2 \le C_0 \int_0^1 d(x)^2(|F(x)|^2 + |F'(x)|^2)dx, \tag{1.11}$$

$$\|F\|_0^2 \le C_0 \int_0^1 d(x)(|F(x)|^2 + |F'(x)|^2)dx. \tag{1.12}$$

2. Working problems and main results

The aim of this section is to transform the free boundary value problem (1.6) and (1.9) into the initial boundary value problem (2.11) and (2.12), and states the main result of this paper in Theorem 2.1.

For this goal, the Lagrangian variable $r(x,t)$ is defined by

$$\partial_t r(x,t) = \widetilde{u}(r(x,t),t) \quad \text{for } t>0 \text{ and } r(x,0)=x. \tag{2.1}$$

Define the Lagrangian density and velocity by

$$\begin{cases} f(x,t) = \rho(r(x,t),t), \\ u(x,t) = \widetilde{u}(r(x,t),t), \\ \upsilon(x,t) = \widetilde{\upsilon}(r(x,t),t), \\ w(x,t) = \widetilde{w}(r(x,t),t). \end{cases} \tag{2.2}$$

The system (1.6) can be transformed into

$$\begin{cases} \partial_t(rf) + rf\dfrac{u_x}{r_x} = 0, \\ f(\partial_t u - \dfrac{\upsilon^2}{r} + u) + \dfrac{(f^\gamma)_x}{r_x} = 0, \\ \partial_t \upsilon + \dfrac{u\upsilon}{r} + \upsilon = 0, \\ \partial_t w + w = 0. \end{cases} \tag{2.3}$$

It follows from $(2.3)_1$ that

$$f = \frac{1}{r_x}\frac{x}{r}\rho_0, \tag{2.4}$$

so the system (2.3) changes to

$$\begin{cases} \dfrac{x}{r}\rho_0\left(\partial_t u - \dfrac{\upsilon^2}{r} + u\right) + \partial_x\left[(\dfrac{1}{r_x}\dfrac{x}{r}\rho_0)^\gamma\right] = 0, \\ \partial_t \upsilon + \dfrac{u\upsilon}{r} + \upsilon = 0, \\ \partial_t w + w = 0. \end{cases} \tag{2.5}$$

Similarly, under the new variables in (2.2), the system (1.7) and (1.8) can be respectively rewritten as

$$\begin{cases} \dfrac{x}{r}\rho_0\left(\partial_t u - \dfrac{\upsilon^2}{r}\right) + \partial_x\left[(\dfrac{1}{r_x}\dfrac{x}{r}\rho_0)^\gamma\right] = 0, \\ \partial_t \upsilon + \dfrac{u\upsilon}{r} + \upsilon = 0, \\ \partial_t w + w = 0, \end{cases} \tag{2.6}$$

and

$$\begin{cases} \dfrac{x}{r}\rho_0\left(\partial_t u - \dfrac{\upsilon^2}{r} + u\right) + \partial_x\left[(\dfrac{x}{r}\dfrac{1}{r_x}\rho_0)^\gamma\right] = \dfrac{x}{r^2}\rho_0\displaystyle\int_0^x \rho_0(y)y\,dy, \\ \partial_t \upsilon + \dfrac{u\upsilon}{r} + \upsilon = 0, \\ \partial_t w + w = 0, \end{cases} \tag{2.7}$$

where we have used

$$\int_0^{r(x,t)} \rho(s,t)s\,ds = \int_0^x \rho_0(y)y\,dy, \tag{2.8}$$

which follows from $(2.3)_1$.

Denoting by

$$\sigma(x) := x\rho_0^{\gamma-1}, \tag{2.9}$$

that is a distance function near the center point $x = 0$ and the boundary point $x = 1$, the system $(2.5)_1$ can be rewritten as

$$\frac{1}{\rho_0^{\gamma-2}}\sigma(\partial_t u - \frac{v^2}{r} + u) + \partial_x\left[\frac{1}{\rho_0^{\gamma-2}}\frac{\sigma^2}{r}(\frac{x}{r})^{\gamma-1}(\frac{1}{r_x})^{\gamma}\right] - \frac{1}{\rho_0^{\gamma-2}}\frac{\sigma^2}{r^2}(\frac{x}{r})^{\gamma-2}(\frac{1}{r_x})^{\gamma-1} = 0. \tag{2.10}$$

In this paper, we mainly analyze the case of $\gamma = 2$, then, the system (2.5) reads

$$\begin{cases} \sigma(\partial_t u - \frac{v^2}{r} + u) + \partial_x\left[\sigma^2\frac{x}{r}(\frac{1}{r_x})^2\right] - \frac{\sigma^2}{x^2}(\frac{x}{r})^2\frac{1}{r_x} = 0, \\ \partial_t v + \frac{uv}{r} + v = 0, \\ \partial_t w + w = 0. \end{cases} \tag{2.11}$$

The initial and boundary conditions (1.9) for $\gamma = 2$ become

$$\begin{cases} \rho_0(x) > 0, \ \ x \ \text{in} \ [0,1), \qquad \rho_0(1) = 0, \\ -\infty < \frac{d}{dx}\rho_0(1) < 0, \\ u(0,t) = 0 \quad \text{on} \ \{x = 0\} \times (0,T], \\ (u,v,w)(x,0) = (u_0,v_0,w_0) \ \text{in} \ (0,1). \end{cases} \tag{2.12}$$

In order to deal with the different singularities at the points $x = 0$ and $x = 1$, the interior and the boundary C^∞ cut-off functions $\xi(x)$ and $\eta(x)$ are respectively given by

$$\xi(x) = 1 \quad \text{on} \ [0,\delta], \quad \xi(x) = 0 \quad \text{on} \ [2\delta, 1], \quad |\xi'(x)| \le \frac{C_0}{\delta}, \tag{2.13}$$

$$\eta(x) = 1 \quad \text{on} \ [\delta, 1], \quad \eta(x) = 0 \quad \text{on} \ [0, \frac{\delta}{2}], \quad |\eta'(x)| \le \frac{C_0}{\delta}, \tag{2.14}$$

where C_0 and δ are positive constants, and δ will be determined later. Thus, we define the higher-order weighted energy functional for smooth solutions (r,u,v,w) by:

$$E(t) := E(u) + E(v), \tag{2.15}$$

with

$$\begin{aligned} E(u) :=& \|\sqrt{\sigma}\partial_t^5 u(t)\|_0^2 + \|\frac{\sigma}{\sqrt{x}}\partial_t^4\partial_x u(t)\|_0^2 + \|\frac{\sigma}{\sqrt{x}}\frac{\partial_t^4 u}{x}(t)\|_0^2 \\ &+ \|\partial_x^4 u(t)\|_0^2 + \|\sigma u(t)\|_3^2 + \|u(t)\|_2^2 + \left\|\frac{u}{x}(t)\right\|_1^2 \end{aligned}$$

$$
\begin{aligned}
&+\|\frac{\sigma}{\sqrt{x}}\partial_t^2 u(t)\|_2^2+\|\partial_t^2 u(t)\|_1^2+\|\frac{\partial_t^2\partial_x u}{\sqrt{x}}(t)\|_0^2+\|\frac{\partial_t^2 u}{x\sqrt{x}}(t)\|_0^2\\
&+\sum_{s=0}^{1}\big(\|\partial_t^{2s+1}u(t)\|_{\frac{3}{2}-s}^2+\|\frac{\partial_t^{2s+1}u}{x}(t)\|_{1-s}^2+\|\sqrt{\sigma}\partial_t^{2s+1}\partial_x^{2-s}u(t)\|_0^2\big)\\
&+\sum_{s=0}^{1}\big(\|\big(\frac{\sigma^3}{x}\big)^{\frac{1}{2}}\partial_t^{2s+1}\partial_x^{3-s}u(t)\|_0^2+\|\xi\sigma\partial_t^{2s+1}u(t)\|_{3-s}^2+\|\xi\partial_t^{2s+1}u(t)\|_{1-s}^2\big),
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
E(\upsilon):=&\|\frac{\sigma}{\sqrt{x}}\partial_t^4\partial_x\upsilon(t)\|_0^2+\|\partial_t^4\upsilon(t)\|_0^2+\|\partial_t^3\upsilon(t)\|_{L^4}^2+\|\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x^2\upsilon(t)\|_0^2\\
&+\|\big(\partial_t\upsilon,\partial_t^2\upsilon,\frac{\partial_t\upsilon}{x},\frac{\partial_t^2\upsilon}{x},\frac{\sigma}{x}\partial_t\partial_x\upsilon,\frac{\sigma}{\sqrt{x}}\partial_t^3\upsilon\big)(t)\|_{L^\infty}^2\\
&+\|(\upsilon_x,\partial_t\partial_x\upsilon,\sqrt{\sigma}\partial_t^2\partial_x\upsilon)(t)\|_0^2.
\end{aligned}
\tag{2.17}
$$

Note that the definition of $E(t)$ in (2.15) doesn't involve the higher order energy of the axial velocity ω. The reason of it is that

$$
\omega=\omega_0 e^{-t},
\tag{2.18}
$$

which can be derived from (2.5).

For obtaining the existence result, the following compatibility conditions should be satisfied for $1\le k\le 5$:

$$
\partial_t^k u(x,0):=\partial_t^{k-1}\big(\frac{\upsilon_0^2}{x}-u_0-2\sigma_x+\frac{\sigma}{x}\big),
\tag{2.19}
$$

$$
\partial_t^k \upsilon(x,0):=\partial_t^{k-1}\big(-\frac{u_0}{x}\upsilon_0-\upsilon_0\big).
\tag{2.20}
$$

From now on, $\mathcal{P}$ denotes a generic polynomial function of its argument.

We state the main result of this paper as follows.

Theorem 2.1 *(Main Theorem). Let the initial data* $(\rho_0,u_0,\upsilon_0)\in C^2([0,1])$ *satisfy* (2.12), (2.19), (2.20) *and*

$$
E(0)<+\infty.
$$

Then, there exists a positive constant T_0 *such that the problem* (2.11) *and* (2.12) *has a unique smooth solution* (r,u,υ) *in* $[0,1]\times[0,T_0]$ *satisfying*

$$
\sup_{t\in[0,T_0]} E(t)\le 2\mathcal{P}_0.
\tag{2.21}
$$

Here and hereafter, $\mathcal{P}_0=\mathcal{P}(E(0))$.

Theorem 2.2. *Under the same condition of the initial data* $(\rho_0, u_0, \upsilon_0)$, *there exists a positive constant* $\overline{T}_0$ *such that the problem* (2.6) *(with* $\gamma = 2$*) and* (2.12) *has a unique smooth solution* (r, u, υ) *in* $[0, 1] \times [0, \overline{T}_0]$ *satisfying* (2.21).

Remark 2.3. It is worth mentioning that the energy $E(t)$ in (2.15) is as same as the definition of energy functionals for the radial velocity and the angular velocity energy studying the relativistic Euler equations in [22], but the energy for the relativistic system involves the axial velocity energy, which makes the estimates for the relativistic system more complicated.

Remark 2.4. Under the same condition of the initial data $(\rho_0, u_0, \upsilon_0)$, there exists a positive constant $\hat{T}_0$ such that the problem (2.7) (with $\gamma = 2$) and (2.12) has a unique smooth solution (r, u, υ) in $[0, 1] \times [0, \hat{T}_0]$ satisfying (2.6). Moreover, we can generalize the similar result to the Euler-Poisson system (1.10) and (1.3) (with (1.4)).

Remark 2.5. As mentioned above, due to the physical vacuum condition $(1.9)_5$, the value of γ confirms the rate of degeneracy near the vacuum boundary $x = 1$, but it will not affect the rate of degeneracy near the original point $x = 0$, since $\rho_0 \sim (1 - x)^{\frac{1}{\gamma - 1}}$ as $x \to 1$. In fact, the rate of degeneracy is more strong for the smaller value of γ. Thus, we divide γ into the two cases $1 < \gamma < 2$ and $\gamma > 2$. Inspired [19], we can prove the well-posedness of local smooth solutions of system (2.10) by the similar argument to the case for $\gamma = 2$.

3. Some preliminaries

It is assumed that there exists a smooth solution (r, u, υ) to the problem (2.11) and (2.12) on $[0, 1] \times [0, T]$, which satisfies the a-priori assumptions below

$$\sup_{t \in [0,T]} \left\| \left(u_x, \frac{u}{x}\right)(t) \right\|_{L^\infty} \leq M_0, \tag{3.1}$$

for some constant $M_0 > 0$ determined later. Then, a straightforward calculation gives that there exists a small enough time $0 < T_1 < T$ such that for any $(x, t) \in (0, t) \times (0, T_1]$, it holds that

$$\frac{1}{2} \leq \frac{r(x,t)}{x} \leq 2, \qquad \frac{1}{2} \leq r'(x,t) \leq 2. \tag{3.2}$$

Lemma 3.1. *Let* $T > 0$ *and* (r, u, υ) *be a smooth solution to the free boundary problem* (2.11) *satisfying* (3.1) *on* $[0, 1] \times [0, T]$. *Then, for any* $1 < p < \infty$, *the following estimates hold:*

$$\begin{aligned}
&\left\| \left(\frac{u}{x}, u_x, \sigma(x) u_{xx}, \partial_t u, \frac{\partial_t u}{x}, \frac{\sigma}{x} \partial_t \partial_x u, \partial_t^2 u, \frac{\sigma}{\sqrt{x}} \partial_t^2 \partial_x u, \frac{\sigma}{\sqrt{x}} \partial_t^3 u \right)(t) \right\|_{L^\infty} \\
&\quad + \left\| \left(\frac{\sigma}{\sqrt{x}} \partial_t^3 \partial_x u, \frac{\sigma}{\sqrt{x}} \partial_t \partial_x^2 u \right)(t) \right\|_{\frac{1}{2}} \\
&\qquad + \left\| \left(\partial_t \partial_x u, \frac{\sigma(x)}{\sqrt{x}} \partial_t \partial_x^2 u, \frac{\sigma}{\sqrt{x}} \partial_t^3 \partial_x u \right)(t) \right\|_{L^p} \leq C\sqrt{E(u)}.
\end{aligned} \tag{3.3}$$

By the fundamental theorem of calculus,

$$
\begin{aligned}
&\left\| \left(\frac{u}{x}, \frac{\sigma}{x}u_x, \partial_t u, \frac{\sigma}{\sqrt{x}}\partial_t\partial_x u, \frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x u \right)(t) \right\|_{L^\infty} \\
&+\left\| \left(u_x, \frac{\sigma}{\sqrt{x}}\partial_x^2 u, \partial_t^2 u, \sqrt{\sigma}\partial_t\partial_x^2 u, \partial_t^3 u \right)(t) \right\|_{L^p} \\
&\le \mathcal{P}_0 + C\int_0^t \sqrt{E(u)(\tau)}d\tau. \qquad (3.4)
\end{aligned}
$$

Proof. Using $H^1(0,1) \hookrightarrow L^\infty(0,1)$, $H^{\frac{1}{2}}(0,1) \hookrightarrow L^p(0,1)(1<p<\infty)$ and the weighted energy estimates (1.11) and (1.12), we can prove the estimates in (3.3) and (3.4), with the help of (2.16) and (2.17), respectively. $\square$

4. Energy estimates

This section is devoted to proof of a higher-order energy estimate of local smooth solutions to the problem (2.11)-(2.12) on $[0,1]\times[0,T]$ under the assumption (3.1).

Lemma 4.1. *Assume that (r,u,υ) is a smooth solution to the problem (2.11)-(2.12) satisfying (3.1) on $[0,1]\times[0,T]$. Then, there exists a small time $0<\bar{T}_1\le T_1$, such that for any $t\in(0,\bar{T}_1]$, we have that*

$$
\left\|\left(\sqrt{\sigma}\partial_t^5 u, \frac{\sigma}{\sqrt{x}}\partial_t^4 u_x, \frac{\sigma}{\sqrt{x}}\frac{\partial_t^4 u}{x}\right)(\tau)\right\|_0^2 + \int_0^t\int_0^1 \sigma(\partial_t^5 u)^2 dx d\tau \le \mathcal{P}_0 + Ct\mathcal{P}\Big(\sup_{\tau\in[0,t]} E(\tau)\Big). \qquad (4.1)
$$

Proof. It follows that, by taking ∂_t^{k+1} over $(2.11)_1$,

$$
\begin{aligned}
&\sigma\partial_t^{k+2}u - \partial_t^{k+1}(\sigma\frac{\upsilon^2}{r}) + \sigma\partial_t^{k+1}u - \Big[\frac{\sigma^2}{x}(\frac{x^2}{r^2r_x^2}\frac{\partial_t^k u}{x} + 2\frac{x}{rr_x^3}\partial_t^k\partial_x u)\Big]_x \\
&+\frac{\sigma^2}{x^2}(2\frac{x^3}{r^3r_x}\frac{\partial_t^k u}{x} + \frac{x^2}{r^2r_x^2}\partial_t^k\partial_x u) + \sum_{l=1}^{2} A_l^k = 0, \qquad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^k &:= -\sum_{i=0}^{k-1} C_k^i\Big[\frac{\sigma^2}{x}\partial_t^{k-i}(\frac{x^2}{r^2r_x^2})\frac{\partial_t^i u}{x} + 2\frac{\sigma^2}{x}\partial_t^{k-i}(\frac{x}{rr_x^3})\partial_t^i\partial_x u\Big]_x, \\
A_2^k &:= \sum_{i=0}^{k-1} C_k^i\Big[2\frac{\sigma^2}{x^2}\partial_t^{k-i}(\frac{x^3}{r^3r_x})\frac{\partial_t^i u}{x} + \frac{\sigma^2}{x^2}\partial_t^{k-i}(\frac{x^2}{r^2r_x^2})\partial_t^i\partial_x u\Big], \qquad (4.3)
\end{aligned}
$$

here and thereafter, $C_k^i = \frac{k!}{i!(k-i)!}$.

In the case of $k=4$, multiplying (4.2) by $\partial_t^5 u$, integrating the resulting equations over $(0,t)\times(0,1)$, then the integration by parts gives

$$
\begin{aligned}
&\int_0^1 \Big\{\frac{1}{2}\sigma(\partial_t^5 u)^2 + \frac{\sigma^2}{x}\frac{x}{rr_x}\Big[\frac{1}{r_x^2}(\partial_t^4\partial_x u)^2 + \frac{x}{rr_x}\partial_t^4\partial_x u\frac{\partial_t^4 u}{x} + \frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2}\Big]\Big\}dx\Big|_0^t \\
&\quad - \int_0^t\int_0^1 \partial_t^5(\sigma\frac{v^2}{r})\partial_t^5 u dxd\tau + \int_0^t\int_0^1 \sigma(\partial_t^5 u)^2 dxd\tau \\
&= \int_0^t\int_0^1 \frac{\sigma^2}{x}\Big[\partial_t(\frac{x^2}{r^2r_x^2})\frac{\partial_t^4 u}{x}\partial_t^4\partial_x u + \partial_t(\frac{x}{rr_x^3})(\partial_t^4\partial_x u)^2 + \partial_t(\frac{x^3}{r^3r_x})\frac{(\partial_t^4 u)^2}{x^2}\Big]dxd\tau \\
&\quad - \sum_{l=1}^2\int_0^t\int_0^1 A_l^4\partial_t^5 u dxd\tau \\
&=: B_1 - B_2.
\end{aligned}
\tag{4.4}
$$

Before the estimate of the right hand side (4.4), due to (3.2) we notice the following facts, for any nonnegative integers m and n,

$$
\Big|\partial_t^{k+1}(\frac{x^m}{r^m r_x^n})\Big| \le C\mathfrak{A}_k, \quad k = 0, \cdots, 4, \tag{4.5}
$$

where

$$
\begin{aligned}
&\mathfrak{A}_0 = \Big|\frac{u}{x}\Big| + |u_x|, \\
&\mathfrak{A}_k = \Big|\frac{\partial_t^k u}{x}\Big| + |\partial_t^k u_x| + \mathfrak{A}_{k-1}\mathfrak{A}_0 + \mathfrak{A}_{k-2}\mathfrak{A}_1, \ \ k = 1, \cdots, 4,
\end{aligned}
$$

with $\mathfrak{A}_i = 0 (i < 0)$. Similar to (3.3), we conclude that, for any $1 < p < \infty$,

$$
\begin{aligned}
&\|\mathfrak{A}_0\|_{L^\infty} + \|\mathfrak{A}_1\|_{L^p} + \|\mathfrak{A}_2\|_0 + \|\mathfrak{A}_3\|_0 + \|\sigma\mathfrak{A}_2\|_{L^p} + \|\sigma\mathfrak{A}_4\|_0 \le C\mathcal{P}(E^{\frac{1}{2}}), \\
&\|\mathfrak{A}_0(t)\|_{L^p}^2 + \|\mathfrak{A}_1(t)\|_0^2 + \|\sigma\mathfrak{A}_2(t)\|_{L^p}^2 + \|\sigma\mathfrak{A}_3(t)\|_0^2 \le M_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big).
\end{aligned}
\tag{4.6}
$$

Thus, it is easy to observe that

$$
\begin{aligned}
|B_1| &\le C\int_0^t \|\mathfrak{A}_0(\tau)\|_{L^\infty}\big(\|\frac{\sigma}{\sqrt{x}}\partial_t^4 u_x\|_0^2 + \|\frac{\sigma}{\sqrt{x}}\frac{\partial_t^4 u}{x}\|_0^2\big)d\tau \\
&\le Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big).
\end{aligned}
\tag{4.7}
$$

The second term B_2 in (4.4) can be written as

$$B_2 = \int_0^t \int_0^1 A_1^4 \partial_t^5 u dx d\tau + \int_0^t \int_0^1 A_2^4 \partial_t^5 u dx d\tau. \tag{4.8}$$

We only give the estimate of the first term of (4.8), while the second term of it can be similarly estimated. Thanks to (4.3), the integration by parts shows

$$\begin{aligned}
&-\int_0^t \int_0^1 A_1^4 \partial_t^5 u dx d\tau \\
&= -\sum_{i=0}^{3} \mathrm{C}_4^i \int_0^1 \Big[\frac{\sigma^2}{x}\partial_t^{4-i}\big(\frac{x^2}{r^2 r_x^2}\big)\frac{\partial_t^i u}{x} + 2\frac{\sigma^2}{x}\partial_t^{4-i}\big(\frac{x}{r r_x^3}\big)\partial_t^i \partial_x u\Big]\partial_t^4 \partial_x u dx\Big|_0^t \\
&\quad + \sum_{i=0}^{3} \mathrm{C}_4^i \int_0^t \int_0^1 \partial_t\Big[\frac{\sigma^2}{x}\partial_t^{4-i}\big(\frac{x^2}{r^2 r_x^2}\big)\frac{\partial_t^i u}{x} + 2\frac{\sigma^2}{x}\partial_t^{4-i}\big(\frac{x}{r r_x^3}\big)\partial_t^i u_x\Big]\partial_t^4 \partial_x u dx d\tau \\
&=: B_{21}|_0^t + B_{22}.
\end{aligned} \tag{4.9}$$

The Young's inequality shows, for any positive constant ε,

$$|B_{21}| \le \varepsilon \|\frac{\sigma}{\sqrt{x}}\partial_t^4 u_x\|_0^2 + C(\varepsilon)\sum_{i=0}^{3}\Big(\|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_{3-i}(t)\frac{\partial_t^i u}{x}(t)\|_0^2 + \|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_{3-i}(t)\partial_t^i u_x(t)\|_0^2\Big). \tag{4.10}$$

Moreover, it holds that

$$\begin{aligned}
&\Big\|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_0(t)\frac{\partial_t^3 u}{x}(t)\Big\|_0 + \Big\|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_0(t)\partial_t^3 u_x(t)\Big\|_0 \\
&\le \|\mathfrak{A}_0(t)\|_{L^4}\left(\Big\|\frac{\partial_t^3 u}{x}(0)\Big\|_{L^4} + \|\sigma\partial_t^3 u_x(0)\|_{L^4}\right) \\
&\quad + \|\mathfrak{A}_0(t)\|_{L^\infty}\int_0^t \big(\|\frac{\sigma}{\sqrt{x}}\frac{\partial_t^4 u}{x}(\tau)\|_0 + \|\frac{\sigma}{\sqrt{x}}\partial_t^4 u_x(\tau)\|_0\big)d\tau \\
&\le Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big).
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
&\sum_{i=0}^{2}\left(\Big\|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_{3-i}(t)\frac{\partial_t^i u}{x}(t)\Big\|_0 + \Big\|\frac{\sigma}{\sqrt{x}}\mathfrak{A}_{3-i}(t)\partial_t^i u_x(t)\Big\|_0\right) \\
&\le \|\mathfrak{A}_1\|_0\left(\Big\|\frac{\partial_t^2 u}{x}(0)\Big\|_{L^\infty} + \Big\|\frac{\sigma}{\sqrt{x}}\partial_t^2 u_x(0)\Big\|_{L^\infty}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\|\mathfrak{A}_1\|_{L^4}\int_0^t\left(\left\|\frac{\partial_t^3 u}{x}(\tau)\right\|_{L^4}+\left\|\sigma\partial_t^3 u_x(\tau)\right\|_{L^4}\right)d\tau\\
&+\|\sigma\mathfrak{A}_2\|_{L^4}\left(\left\|\frac{\partial_t u}{x}(0)\right\|_{L^4}+\|\partial_t u_x(0)\|_{L^4}\right)\\
&+\|\sigma\mathfrak{A}_2\|_{L^\infty}\int_0^t\left(\|\frac{\sigma}{\sqrt{x}}\partial_t^2 u(\tau)\|_0+\|\frac{\partial_t^2 u_x}{\sqrt{x}}(\tau)\|_0\right)d\tau\\
&+\|\sigma\mathfrak{A}_3\|_0\left(\left\|\frac{u}{x}(0)\right\|_{L^\infty}+\|\partial_x u(0)\|_{L^\infty}\right)\\
&+\|\sigma\mathfrak{A}_3\|_0\int_0^t\left(\left\|\frac{\partial_t u}{x}(\tau)\right\|_{L^\infty}+\|\sigma\partial_t\partial_x u(\tau)\|_{L^\infty}\right)d\tau\\
\le\ &\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
\end{aligned}
$$

Thus, it derives from (4.10)

$$
|B_{21}|\le\varepsilon\int_0^t\|\frac{\sigma}{\sqrt{x}}\partial_t^4 u_x(\tau)\|_0^2d\tau+C(\varepsilon)\big[\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big)\big].
$$

By the similar analysis, we have for the estimate of B_{22} in (4.9), with the help of (4.6), as

$$
B_{22}\le Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
$$

Thus, we obtain from (4.4)

$$
\begin{aligned}
&\int_0^1\left\{\frac{1}{2}\sigma(\partial_t^5 u)^2+\frac{\sigma^2}{x}\frac{x}{rr_x}\big[\frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2}+\frac{x}{rr_x}\frac{\partial_t^4 u}{x}\partial_t^4 u_x+\frac{1}{r_x^2}(\partial_t^4 u_x)^2\big]\right\}dx\Big|_0^t\\
&-\int_0^t\int_0^1\partial_t^5(\sigma\frac{v^2}{r})\partial_t^5 u\,dx\,d\tau+\int_0^t\int_0^1\sigma(\partial_t^5 u)^2dx\,d\tau\\
\le\ &\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big)+\varepsilon\left(\left\|\frac{\sigma}{\sqrt{x}}\partial_t^4 u_x\right\|_0^2+\left\|\frac{\sigma}{\sqrt{x}}\frac{\partial_t^4 u}{x}\right\|_0^2\right).
\end{aligned}
\tag{4.11}
$$

Since

$$
\frac{\sigma^2}{x}\frac{x}{rr_x}\big[\frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2}+\frac{x}{rr_x}\frac{\partial_t^4 u}{x}\partial_t^4 u_x+\frac{1}{r_x^2}(\partial_t^4 u_x)^2\big]
$$

$$
\begin{aligned}
&= \frac{\sigma^2}{x}\frac{x}{rr_x}\Big[\frac{1}{2}\frac{1}{r_x^2}(\partial_t^4 u_x)^2 + \frac{1}{2}\frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2} + \Big(\frac{1}{\sqrt{2}}\frac{1}{r_x}\partial_t^4 u_x + \frac{1}{\sqrt{2}}\frac{x}{r}\frac{\partial_t^4 u}{x}\Big)^2\Big] \\
&\geq \frac{\sigma^2}{x}\frac{1}{8}\Big[\frac{1}{r_x^2}(\partial_t^4 u_x)^2 + \frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2}\Big],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_0^1 \frac{1}{2}\sigma(\partial_t^5 u)^2 dx + \frac{1}{16}\int_0^1 \frac{\sigma^2}{x}\Big[\frac{1}{r_x^2}(\partial_t^4 u_x)^2 + \frac{x^2}{r^2}\frac{(\partial_t^4 u)^2}{x^2}\Big]dx \\
&\quad - \int_0^t\int_0^1 \partial_t^5\Big(\sigma\frac{\upsilon^2}{r}\Big)\partial_t^5 u\,dxd\tau + \int_0^t\int_0^1 \sigma(\partial_t^5 u)^2 dxd\tau \\
&\leq \mathcal{P}_0 + Ct\mathcal{P}\Big(\sup_{\tau\in[0,t]} E(\tau)\Big).
\end{aligned} \tag{4.12}
$$

Now we turn to deal with the term of υ in (4.12). Taking ∂_t^k over (2.11), one has

$$
\begin{aligned}
&\partial_t^{k+1}\upsilon + \Big(1 + \frac{x}{r}\frac{u}{x}\Big)\partial_t^k\upsilon + \upsilon\frac{x}{r}\frac{\partial_t^k u}{x} \\
&= -\upsilon\sum_{i=0}^{k-1} \mathrm{C}_k^i \partial_t^{k-i}\Big(\frac{x}{r}\Big)\frac{\partial_t^i u}{x} - \sum_{i=1}^{k-1} \mathrm{C}_k^i \partial_t^{k-i}\upsilon\,\partial_t^i\Big(\frac{x}{r}\frac{u}{x}\Big).
\end{aligned}
$$

Taking $k = 4$, it concludes that

$$
\begin{aligned}
&\partial_t^5\upsilon + \Big(1 + \frac{x}{r}\frac{u}{x}\Big)\partial_t^4\upsilon + \upsilon\frac{x}{r}\frac{\partial_t^4 u}{x} \\
&= -\upsilon\sum_{i=0}^{3} \mathrm{C}_4^i \partial_t^{4-i}\Big(\frac{x}{r}\Big)\frac{\partial_t^i u}{x} - \sum_{i=1}^{3} \mathrm{C}_4^i \partial_t^{4-i}\upsilon\,\partial_t^i\Big(\frac{x}{r}\frac{u}{x}\Big).
\end{aligned}
$$

Thus, it holds that

$$
\begin{aligned}
&- \int_0^t\int_0^1 \partial_t^5\Big(\frac{\sigma}{x}\frac{x}{r}\upsilon^2\Big)\partial_t^5 u\,dxd\tau \\
&= -\int_0^t\int_0^1 \frac{\sigma}{x}\Big[\frac{x}{r}\partial_t^5(\upsilon^2) + \sum_{i=0}^{4} \mathrm{C}_5^i \partial_t^{5-i}\Big(\frac{x}{r}\Big)\partial_t^i(\upsilon^2)\Big]\partial_t^5 u\,dxd\tau \\
&= -\int_0^t\int_0^1 \frac{\sigma}{x}\Big[2\frac{x}{r}\upsilon\partial_t^5\upsilon + 30\frac{x}{r}\partial_t^5\upsilon + 2\sum_{i=0}^{4} \mathrm{C}_5^i \partial_t^{5-i}\Big(\frac{x}{r}\Big)(\upsilon\partial_t^i\upsilon + \partial_t^i\upsilon)\Big]\partial_t^5 u\,dxd\tau
\end{aligned}
$$

$$\le \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big),$$

which in combination with (4.4) yields (4.1) for small enough time T. This is the end of proof. □

5. Elliptic estimates

This section is to establish the high-order spatial derivative estimates of the local smooth solution for the problem (2.11)-(2.12) on $[0, 1] \times [0, T]$ under the assumption (3.1). As mentioned above, we need to separate the estimate of each term into the internal estimates and boundary estimates near the point $x = 0$ and $x = 1$, respectively. More precisely, the estimates of u, $\partial_t u$ in the Subsection 5.1, the estimate of $\partial_t^2 u$ in the Subsection 5.2 and the estimate of $\partial_t^3 u$ in the Subsection 5.3 are constructed, respectively. Finally, we obtain the estimate of $E(t)$ in the Subsection 5.4.

To obtain the desired estimates, we rewrite the system (4.2) as

$$\sigma\partial_t^k\partial_x^2 u + 2\sigma'\partial_t^k\partial_x u - \sigma'\frac{\partial_t^k u}{x} = -\rho_{0x}x\partial_t^k\partial_x u - 2\rho_{0x}\partial_t^k u + \mathfrak{B}^k, \tag{5.1}$$

where

$$\mathfrak{B}^k := A_5^k + \frac{x}{2\sigma}\sum_{l=1}^{4} A_l^k, \tag{5.2}$$

with $A_l^k (l = 1, 2)$ given by (4.3),

$$\begin{aligned} A_3^k &:= \sigma\partial_t^{k+2}u - \partial_t^{k+1}(\sigma\frac{v^2}{r}) + \sigma\partial_t^{k+1}u,\\ A_4^k &:= -\partial_x\left\{\frac{\sigma^2}{x}\big[(\frac{x^2}{r^2r_x^2}-1)\frac{\partial_t^k u}{x} + 2(\frac{x}{rr_x^3}-1)\partial_t^k\partial_x u\big]\right\},\\ A_5^k &:= \frac{\sigma}{x}\big[(\frac{x^3}{r^3r_x}-1)\frac{\partial_t^k u}{x} + \frac{1}{2}(\frac{x^2}{r^2r_x^2}-1)\partial_t^k\partial_x u\big]. \end{aligned} \tag{5.3}$$

We first determine the constant δ in (2.13) and (2.14). Because $\rho(0) > 0$ and $\sigma'(0) > 0$, then there exists a positive constant δ_0 such that for any $x \in (0, \delta_0)$,

$$\frac{\rho_0(0)}{2} \le \sigma'(x) \le \frac{3\rho_0(0)}{2}. \tag{5.4}$$

Then, we take δ as $0 < 2\delta \le \delta_0$.

5.1. Estimates for u, $\partial_t u$

Lemma 5.1. *Assume that (r,u,υ) is a smooth solution to the problem (2.11)-(2.12) satisfying (3.1) on $[0,1]\times[0,T]$. Then, there exists a small time $0<\bar{T}_3\le\bar{T}_1$, such that for any $t\in(0,\bar{T}_3]$, it holds that*

$$\begin{aligned}&\|\xi\sigma\partial_t\partial_x^3(\frac{u}{x})\|_0^2+\|\sigma\partial_t\partial_x^2(\frac{u}{x})\|_0^2+\|\sigma_x\partial_t\partial_x(\frac{u}{x})\|_0^2\\&\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big),\end{aligned}\tag{5.5}$$

$$\begin{aligned}&\|\eta\sigma^{\frac{3}{2}}\partial_t\partial_x^3u\|_0^2+\|\eta\sigma^{\frac{1}{2}}\sigma_x\partial_t\partial_x^2u\|_0^2\\&\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big)+M_0\|\eta\sigma^{1/2}\partial_t^3\partial_xu\|_0^2,\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}&\|\eta\sigma\partial_x^3u\|_0^2+\|\eta\sigma_x\partial_x^2u\|_0^2\\&\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big)+M_0\|\eta\partial_t^2\partial_xu\|_0^2.\end{aligned}\tag{5.7}$$

Proof. We divide the proof into two steps.

Step 1. Interior estimates of u and $\partial_t u$.

In this step, we prove the interior estimate in (5.5). Note that

$$\partial_t^k\partial_x^ju:=x\partial_x^j\big(\frac{\partial_t^ku}{x}\big)+j\partial_x^{j-1}\big(\frac{\partial_t^ku}{x}\big),\qquad j=1,2,\cdots,\tag{5.8}$$

the equations in (5.1) can be transformed into

$$\begin{aligned}&x\sigma\partial_t^k\partial_x^3(\frac{u}{x})+5\sigma\partial_t^k\partial_x^2(\frac{u}{x})+3\sigma_x\partial_t^k\partial_x(\frac{u}{x})\\&=-2x^2\rho_{0x}\partial_t^k\partial_x^2(\frac{u}{x})-\sigma_{xx}\partial_t^k\partial_x(\frac{u}{x})-\sigma_{xx}\frac{\partial_t^ku}{x}\\&\quad-\partial_x\big(\rho_{0x}x\partial_t^k\partial_xu\big)-\partial_x\big(2\rho_{0x}\partial_t^ku\big)+\mathfrak{B}_x^k.\end{aligned}\tag{5.9}$$

From now on, M_0 denotes the general positive constant depending only on

$$\max_{x\in[0,1]}\{\rho_0,|\rho_{0x}(x)|,|\rho_{0xx}(x)|\}.$$

Multiplying (5.9) by ξ and taking L^2-norm, for $k=1$, one has

$$\begin{aligned}&\Big\|\xi\big[x\sigma\partial_t\partial_x^3(\frac{u}{x})+5\sigma\partial_t\partial_x^2(\frac{u}{x})+3\sigma_x\partial_t\partial_x(\frac{u}{x})\big]\Big\|_0^2\\&\le\Big\|\xi\big[-2x^2\rho_{0x}\partial_t\partial_x^2(\frac{u}{x})-\sigma_{xx}\partial_t\partial_x(\frac{u}{x})-\sigma_{xx}\frac{\partial_tu}{x}\big]\Big\|_0^2\end{aligned}$$

$$+\left\|\xi\left\{\partial_x\left(\rho_{0x}x\partial_t\partial_x u\right)+\partial_x\left(2\rho_{0x}\partial_t u\right)\right\}\right\|_0^2+\|\xi\mathfrak{B}_x^1\|_0^2, \tag{5.10}$$

where a straightforward calculation gives

$$\begin{aligned}
&\left\|\xi\left[-2x^2\rho_{0x}\partial_t\partial_x^2\left(\frac{u}{x}\right)-\sigma_{xx}\partial_t\partial_x\left(\frac{u}{x}\right)-\sigma_{xx}\frac{\partial_t u}{x}\right]\right\|_0^2\\
&\quad+\left\|\xi\left\{\partial_x\left[\rho_{0x}x\partial_t\partial_x u\right]+\partial_x\left[2\rho_{0x}\partial_t u\right]\right\}\right\|_0^2\\
&\le\mathcal{P}_0+Ct\mathcal{P}\Big(\sup_{\tau\in[0,t]}E(\tau)\Big).
\end{aligned} \tag{5.11}$$

The left-hand side of (5.10) can be estimated as follows,

$$\begin{aligned}
&\left\|\xi\left[x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)+5\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)+3\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)\right]\right\|_0^2\\
&=\left\|\xi x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)\right\|_0^2+25\left\|\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)\right\|_0^2+9\left\|\xi\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)\right\|_0^2\\
&\quad+10\int_0^1\xi x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)dx+6\int_0^1\xi x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)\xi\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)dx\\
&\quad+30\int_0^1\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)\xi\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)dx.
\end{aligned} \tag{5.12}$$

The integration by parts shows

$$\begin{aligned}
&10\int_0^1\xi x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)dx\\
&=-15\left\|\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)\right\|_0^2-10\int_0^1\xi\xi_x x\sigma^2\left|\partial_t\partial_x^2\left(\frac{u}{x}\right)\right|^2dx\\
&\quad-10\int_0^1\xi^2\sigma(x\sigma_x-\sigma)\left|\partial_t\partial_x^2\left(\frac{u}{x}\right)\right|^2dx,
\end{aligned}$$

and

$$\begin{aligned}
&6\int_0^1\xi x\sigma\partial_t\partial_x^3\left(\frac{u}{x}\right)\xi\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)dx+30\int_0^1\xi\sigma\partial_t\partial_x^2\left(\frac{u}{x}\right)\xi\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)dx\\
&=-12\int_0^1\xi\xi_x x\sigma\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)\partial_t\partial_x^2\left(\frac{u}{x}\right)dx+24\int_0^1\xi^2\sigma\sigma_x\partial_t\partial_x\left(\frac{u}{x}\right)\partial_t\partial_x^2\left(\frac{u}{x}\right)dx
\end{aligned}$$

$$-6\int_0^1 \xi^2 x\sigma_x^2\partial_t\partial_x(\frac{u}{x})\partial_t\partial_x^2(\frac{u}{x})dx-6\int_0^1 \xi^2 x\sigma\sigma_{xx}\partial_t\partial_x(\frac{u}{x})\partial_t\partial_x^2(\frac{u}{x})dx$$
$$-6\int_0^1 \xi^2\sigma(x\sigma'-\sigma)\left|\partial_t\partial_x^2(\frac{u}{x})\right|^2dx-6\left\|\xi\sigma\partial_t\partial_x^2(\frac{u}{x})\right\|_0^2.$$

Similarly, we can get

$$\begin{aligned}
&\int_0^1 \xi^2\sigma\sigma'\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&=-\int_0^1 \xi^2\big(\sigma'\big)^2\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&\quad-2\int_0^1 \xi\xi_x\sigma\sigma'\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&\quad-\int_0^1 \xi^2\sigma\sigma''\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx.
\end{aligned}$$

Thus, we can obtain from (5.12), for any positive constant ε, that

$$\begin{aligned}
&\left\|\xi\left[x\sigma\partial_t\partial_x^3(\frac{u}{x})+5\sigma\partial_t\partial_x^2(\frac{u}{x})+3\sigma_x\partial_t\partial_x(\frac{u}{x})\right]\right\|_0^2\\
&\geq\left\|\xi x\sigma\partial_t\partial_x^3(\frac{u}{x})\right\|_0^2+3\left\|\xi\sigma\partial_t\partial_x^2(\frac{u}{x})\right\|_0^2-\varepsilon\left\|\xi\sigma_x\partial_t\partial_x(\frac{u}{x})\right\|_0^2\\
&\quad-\mathcal{P}_0-Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
\end{aligned}\tag{5.13}$$

In particular, we have used by (5.4)

$$\begin{aligned}
&-\int_0^1 \xi^2\sigma\sigma''\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&=-\int_0^{\delta_0} \xi^2\sigma\frac{\sigma''}{\sigma'}\sigma'\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&\quad-\int_{\delta_0}^1 \xi^2\sigma\sigma''\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx
\end{aligned}$$

$$\leq \varepsilon \int_0^1 \xi^2 (\sigma')^2 \big(\partial_t \partial_x (\frac{u}{x})\big)^2 dx + M_0(\varepsilon) \left(\mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big) \right). \tag{5.14}$$

We turn to handle the last term $\|\xi \mathfrak{B}_x^1\|_0^2$ on the right-hand side of (5.10). Due to (4.3) and (5.3), A_1^1 and A_2^1 are equivalent to 0, the highest-order terms with respect to t and x are $\|\xi \partial_x (\frac{x}{2\sigma} A_3^1)\|_0^2$ and $\|\xi \partial_x (\frac{x}{2\sigma} A_4^1)\|_0^2$, respectively. Thus, we only give the estimates for these two terms, while the other terms in $\|\xi \mathfrak{B}_x^1\|_0^2$ can be similarly estimated. From (5.3), it follows

$$\begin{aligned}
&\|\xi \partial_x (\frac{x}{2\sigma} A_3^1)\|_0^2 \\
&= \|\xi \frac{1}{2}\big[\partial_t^3 u + x\partial_t^3 \partial_x u - \partial_t^2 (\frac{\upsilon^2}{r}) - x\partial_t^2 (\frac{2\upsilon\upsilon_x}{r} - \frac{\upsilon^2}{r^2} r_x) + \partial_t^2 u + \partial_t^2 \partial_x u\big]\|_0^2 \\
&\leq M_0 \left(\|\xi \partial_t^3 u\|_0^2 + \|\xi\sigma \partial_t^3 \partial_x u\|_0^2 + \|\xi \partial_t^2 u\|_0^2 + \|\xi \sigma \partial_t^2 \partial_x u\|_0^2 \right) \\
&\quad + M_0(\|\frac{u}{x}\|_{L^\infty}^2 + \|\frac{\upsilon}{x}\|_{L^\infty}^2)(\|(\partial_t u, \partial_t \upsilon)\|_0^2 + \|(\sigma \partial_t \partial_x u, \sigma \partial_t^2 \partial_x \upsilon)\|_0^2) \\
&\quad + M_0(\|\partial_t^2 u\|_0^2 \|\frac{u}{x}\|_{L^\infty}^2 + \|\partial_t u\|_0^2 \|\frac{\partial_t u}{x}\|_{L^\infty}^2 + \|\partial_t^2 \upsilon\|_0^2 \|\frac{\upsilon}{x}\|_{L^\infty}^2 + \|\partial_t \upsilon\|_0^2 \|\frac{\partial_t \upsilon}{x}\|_{L^\infty}^2),
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
&\|\xi \partial_x (\frac{x}{2\sigma} A_4^1)\|_0^2 \\
&= \| -\xi \partial_x \Big\{ (\sigma_x - \frac{\sigma}{2x})\big[(\frac{x^2}{r^2 r_x^2} - 1)\frac{\partial_t u}{x} + 2(\frac{x}{r r_x^3} - 1)\partial_t \partial_x u\big] \\
&\qquad + \frac{\sigma}{2}\big[(\frac{x^2}{r^2 r_x^2} - 1)\frac{\partial_t u}{x} + 2(\frac{x}{r r_x^3} - 1)\partial_t \partial_x u\big]_x \Big\} \|_0^2 \\
&\leq M_0 \|\xi \sigma (\partial_t \partial_x^3 u, (\frac{\partial_t u}{x})_{xx})\|_0^2 \int_0^t \|(\partial_x u, \frac{u}{x})(\tau)\|_{L^\infty}^0 d\tau \\
&\quad + M_0 \|\xi \sigma (\partial_t \partial_x^2 u, (\frac{\partial_t u}{x})_x)\|_{L^\infty}^2 \|(r_{xx}, \partial_x (\frac{x}{r}))\|_0^2 \\
&\quad + M_0 \|\xi (\partial_t \partial_x u, \frac{\partial_t u}{x})\|_{L^\infty}^2 \|\sigma (r_{xxx}, \partial_{xx} (\frac{x}{r}))\|_0^2 \\
&\quad + M_0 \|\xi (\partial_t \partial_x u, \frac{\partial_t u}{x})\|_{L^\infty}^2 \|(r_{xx}, \partial_x (\frac{x}{r}))\|_0^2 \\
&\quad + M_0 \|\xi (\partial_t \partial_x^2 u, (\frac{\partial_t u}{x}))_x\|_0^2 \int_0^t \|(\partial_x u, \frac{u}{x})(\tau)\|_{L^\infty}^2 d\tau \\
&\quad + M_0 \|\xi (\partial_t \partial_x u, \frac{\partial_t u}{x})\|_{L^\infty}^2 \int_0^t \|(\partial_x u, \frac{u}{x})(\tau)\|_{L^\infty}^2 d\tau.
\end{aligned} \tag{5.16}$$

Finally, we obtain (5.5) from (5.13)-(5.16).

Step 2. Boundary estimates of u and $\partial_t u$.

In this step, we prove the boundary estimate in (5.6), while the estimate in (5.7) can be similarly treated. For this goal, we write (5.1), with $k=1$, as

$$\sigma\partial_t\partial_x^2 u+2\sigma_x\partial_t\partial_x u=\sigma_x\frac{\partial_t u}{x}+\frac{\sigma}{x}\partial_t\partial_x u-2\rho_{0x}\frac{\partial_t u}{x}+\mathfrak{B}^1. \tag{5.17}$$

Taking ∂_x over (5.17) and multiplying it by $\eta\sqrt{\sigma}$, one has

$$\begin{aligned}&\|\eta\sqrt{\sigma}(\sigma\partial_t\partial_x^3 u+3\sigma_x\partial_t\partial_x^2 u)\|_0^2\\&\le\left\|\eta\sqrt{\sigma}\left\{-2\sigma_{xx}\partial_t\partial_x u+\partial_x\big[\sigma_x\frac{\partial_t u}{x}+\frac{\sigma}{x}\partial_t\partial_x u-2(\sigma_x-\frac{\sigma}{x})\frac{\partial_t u}{x}\big]\right\}\right\|_0^2\\&\quad+\|\eta\sqrt{\sigma}\mathfrak{B}_x^1\|_0^2.\end{aligned}\tag{5.18}$$

Due to $\sigma'(1)<0$, there exists a positive constant $\sigma_1>0$ such that for any $\delta_1\ge\frac{\delta}{2}$, $-\infty<\sigma_0'(x)<0$, $\forall x\in(\delta,1]$, then the integration by parts shows

$$\begin{aligned}&\|\eta\sqrt{\sigma}(\sigma\partial_t\partial_x^3 u+3\sigma_x\partial_t\partial_x^2 u)\|_0^2\\&=\|\eta\sigma^{\frac32}\partial_t\partial_x^3 u\|_0^2+\|3\eta\sigma^{\frac12}\sigma_x\partial_t\partial_x^2 u\|_0^2+6\int_0^1\eta\sigma^{\frac32}\partial_t\partial_x^3 u\,\eta\sigma^{\frac12}\sigma_x\partial_t\partial_x^2 u\,dx\\&\ge\|\eta\sigma^{\frac32}\partial_t\partial_x^3 u\|_0^2+2\|\eta\sigma^{\frac12}\sigma_x\partial_t\partial_x^2 u\|_0^2-\big[\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big)\big].\end{aligned}\tag{5.19}$$

The first term on the right-hand side of (5.18) can be bounded by

$$\begin{aligned}&\left\|\eta\sqrt{\sigma}\left\{-2\sigma_{xx}\partial_t\partial_x u+\partial_x\big[\sigma_x\frac{\partial_t u}{x}+\frac{\sigma}{x}\partial_t\partial_x u-2(\sigma_x-\frac{\sigma}{x})\frac{\partial_t u}{x}\big]\right\}\right\|_0^2\\&=\Bigg\|\eta\sigma^{\frac12}\frac{\sigma}{x}\partial_t\partial_x u-2\eta\sigma^{\frac12}\sigma_{xx}\partial_t\partial_x u+\eta\sigma^{\frac12}\frac{\sigma_x-\sigma/x}{x}\partial_t\partial_x u+\eta\sigma^{\frac12}\sigma_x\partial_x(\frac{\partial_t u}{x})\\&\quad-2\eta\sigma^{\frac12}(\sigma_x-\frac{\sigma}{x})\partial_x(\frac{\partial_t u}{x})-\eta\sigma^{\frac12}\sigma_{xx}\frac{\partial_t u}{x}+2\eta\sigma^{\frac12}\frac{\sigma_x-\sigma/x}{x}\frac{\partial_t u}{x}\Bigg\|_0^2\\&\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).\end{aligned}\tag{5.20}$$

For the estimate of $\|\eta\sqrt{\sigma}\mathfrak{B}_x^1\|_0^2$ in (5.18), the main difficult terms are $\|\eta\sqrt{\sigma}\partial_x(\frac{x}{2\sigma}A_3^1)\|_0^2$ and $\|\eta\sqrt{\sigma}\partial_x(\frac{x}{2\sigma}A_4^1)\|_0^2$. Similar to (5.15) and (5.16), we have

$$
\begin{aligned}
&\|\eta\sqrt{\sigma}\partial_x(\frac{x}{2\sigma}A_3^1)\|_0^2\\
&=\|\eta\sqrt{\sigma}\frac{1}{2}\big[\partial_t^3 u+x\partial_t^3\partial_x u-\partial_t^2(\frac{\upsilon^2}{r})-x\partial_t^2(\frac{2\upsilon\upsilon_x}{r}-\frac{\upsilon^2}{r^2}r_x)+\partial_t^2 u+x\partial_t^2\partial_x u\big]\|_0^2\\
&\le M_0\|\Big(\eta\sqrt{\sigma}\partial_t^3\partial_x u,\eta\sqrt{\sigma}\partial_t^3 u,\eta\sqrt{\sigma}\partial_t^2\partial_x u,\sigma^{\frac{3}{2}}\partial_t^2\partial_x\upsilon\Big)\|_0^2\\
&\quad+M_0\|(\eta\sigma\partial_t^3 u,\partial_t^2\upsilon,\partial_t\upsilon)\|_{L^\infty}^2[1+\|(r_{xx},\partial_x(\frac{x}{r}))\|_0^2]\\
&\quad+M_0(\|(\partial_t u,\partial_t\upsilon)\|_{L^\infty}^2\|(\partial_t\partial_x u,\partial_t\partial_x\upsilon)\|_0^2+\|\partial_x\upsilon\|_0^2\|\partial_t\upsilon\|_{L^\infty}^2+\|\sqrt{\sigma}\partial_t\partial_x\upsilon\|_0^2),
\end{aligned}
\tag{5.21}
$$

and

$$
\begin{aligned}
&\|\eta\sqrt{\sigma}\partial_x(\frac{x}{2\sigma}A_4^1)\|_0^2\\
&\le M_0\|\eta\sigma^{\frac{3}{2}}(\partial_t\partial_x^3 u,\partial_t\partial_x^2 u,\partial_t\partial_x u,\partial_t u)\|_0^2\int_0^t\|(\partial_x u,\frac{u}{x})(\tau)\|_{L^\infty}^2 d\tau\\
&\quad+M_0\Big\|\eta\sigma^{\frac{1}{2}}(\partial_t\partial_x^2 u,\partial_t\partial_x u,\partial_t u)+\eta(\partial_t\partial_x u,\partial_t u)\Big\|_0^2\int_0^t\|(\partial_x u,\frac{u}{x})(\tau)\|_{L^\infty}^2 d\tau\\
&\quad+M_0\Big\|\eta\sigma^{\frac{1}{2}}(\partial_t\partial_x u,\partial_t u)\Big\|_{L^\infty}^2\|\sigma(r_{xxx},(\frac{x}{r})_{xx})\|_0^2+M_0\|(\partial_t\partial_x u,\frac{\partial_t u}{x})\|_0^2\|\sigma r_{xx}\|_{L^\infty}^2\\
&\quad+M_0\Big\|\eta\sigma^{\frac{1}{2}}(\partial_t\partial_x u,\partial_t u)\Big\|_{L^\infty}^2\|(r_{xx},(\frac{x}{r})_x)\|_0^2.
\end{aligned}
\tag{5.22}
$$

From (5.19) to (5.22), we can conclude the estimate in (5.6). □

5.2. *Estimates for $\partial_t^2 u$*

Lemma 5.2. *Let (r,u,υ) be a smooth solution to the problem (2.11)-(2.12) satisfying (3.1) on $[0,1]\times[0,T]$. Then, there exists a small time $0<\bar{T}_4\le T_1$, such that for any $t\in(0,T_1]$, it holds that*

$$
\Big\|\xi\frac{1}{\sqrt{x}}(\sigma\partial_t^2\partial_x^2 u,\sigma_x\partial_t^2\partial_x u,\sigma_x\frac{\partial_t^2 u}{x})\Big\|_0^2\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big),
\tag{5.23}
$$

and

$$
\|\eta\sigma\partial_t^2\partial_x^2 u\|_0^2+\|\eta\sigma_x\partial_t^2\partial_x u\|_0^2\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
\tag{5.24}
$$

Proof. Clearly, the boundary estimate in (5.24) can be obtained by the same fashion in (5.7). Thus, we omit it in details. But the interior estimate of $\partial_t^2 u$ is more complicated than its boundary estimate.

By analogy with (4.5), for any nonnegative integers m and n, it holds that

$$\left|\partial_t^{k+1}(\frac{x^m}{r^m r_x^n})_x\right| \leq C\mathfrak{F}_k, \quad k=0,1,2, \tag{5.25}$$

where

$$\begin{aligned}
&\mathfrak{F}_0 = \left|(\frac{u}{x})_x\right| + |u_{xx}| + \mathfrak{A}_0\mathcal{H}_0, \\
&\mathfrak{F}_k = \left|(\frac{\partial_t^k u}{x})_x\right| + |\partial_t^k\partial_x^2 u| + \mathfrak{A}_k\mathcal{H}_0 + \mathfrak{A}_{k-1}\mathfrak{F}_0 + \mathfrak{A}_{k-2}\mathfrak{F}_1, \quad k=1,2,
\end{aligned} \tag{5.26}$$

with $\mathfrak{A}_i = 0 (i<0)$ given by (4.5) and $\mathcal{H}_0 = \left|(\frac{x}{r})_x\right| + |r_{xx}|$. Similar to the estimate (4.6), we can obtain that

$$\|\mathcal{H}_0\|_0 + \|\sigma\mathcal{H}_0\|_{L^\infty} + \|\xi\sigma\mathfrak{F}_1\|_{L^\infty}^2 + \|\sigma\mathfrak{F}_2\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big). \tag{5.27}$$

Multiplying (5.1) by $\xi\frac{1}{\sqrt{x}}$, we have, for $k=2$, that

$$\begin{aligned}
&\xi\frac{1}{\sqrt{x}}\sigma\partial_t^2\partial_x^2 u + \xi\frac{1}{\sqrt{x}}\sigma_x\partial_t^2\partial_x u - \xi\frac{1}{\sqrt{x}}\sigma_x\frac{\partial_t^2 u}{x} \\
&= -\xi\frac{1}{\sqrt{x}}\rho_{0x}x\partial_t^2\partial_x u - 2\xi\frac{1}{\sqrt{x}}\rho_{0x}\partial_t^2 u + \xi\frac{1}{\sqrt{x}}\mathfrak{B}^2.
\end{aligned} \tag{5.28}$$

Using (5.8), in the similar way carried in (5.10), one has

$$\begin{aligned}
&\|\xi\frac{\sigma}{\sqrt{x}}x\partial_t^2\partial_x^2(\frac{u}{x}) + 3\xi\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x(\frac{u}{x})\|_0^2 \\
&\leq \| -2\xi\frac{1}{\sqrt{x}}x\rho_{0x}\partial_t^2\partial_x(\frac{u}{x}) - 3\xi\frac{1}{\sqrt{x}}\rho_{0x}\frac{\partial_t^2 u}{x}\|_0^2 + \|\xi\frac{1}{\sqrt{x}}\mathfrak{B}^2\|_0^2,
\end{aligned} \tag{5.29}$$

where the first term has a bound as

$$\| -2\xi\frac{1}{\sqrt{x}}x\rho_{0x}\partial_t^2\partial_x(\frac{u}{x}) - 3\xi\frac{1}{\sqrt{x}}\rho_{0x}\frac{\partial_t^2 u}{x}\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big). \tag{5.30}$$

By the analogy with (5.13), it yields that

$$\begin{aligned}
&\left\|\xi\frac{\sigma}{\sqrt{x}}x\partial_t^2\partial_x^2(\frac{u}{x}) + 3\xi\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x(\frac{u}{x})\right\|_0^2 \\
&\geq \left\|\xi\frac{\sigma}{\sqrt{x}}x\partial_t^2\partial_x^2(\frac{u}{x})\right\|_0^2 + \left\|\xi\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x(\frac{u}{x})\right\|_0^2 - \big[\mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big)\big].
\end{aligned} \tag{5.31}$$

For the estimate of $\|\xi\frac{1}{\sqrt{x}}\mathfrak{B}^2\|_0^2$ on the right side of (5.29), the main difficult term is $\|\xi\frac{1}{\sqrt{x}}\frac{x}{2\sigma}A_1^2\|_0^2$, which can be estimated as, from (4.3),

$$
\begin{aligned}
\xi\frac{1}{\sqrt{x}}\frac{x}{2\sigma}A_1^2 = &-\xi\frac{1}{2\sqrt{x}}\big[2\rho_{0x}\partial_t^2(\frac{x^2}{r^2r_x^2})\frac{u}{x}+\sigma\partial_t^2\partial_x(\frac{x^2}{r^2r_x^2})\frac{u}{x}+\frac{\sigma}{x}\partial_t^2(\frac{x^2}{r^2r_x^2})u_x\big]\\
&-\xi\frac{1}{\sqrt{x}}\big[2\rho_{0x}\partial_t^2(\frac{x^2}{r^2r_x^2})u_x+\sigma\partial_t^2\partial_x(\frac{x^2}{r^2r_x^2})u_x+\sigma\partial_t^2(\frac{x^2}{r^2r_x^2})\partial_t^2u\big]\\
&-\xi\frac{1}{\sqrt{x}}\big[2\rho_{0x}\partial_t(\frac{x^2}{r^2r_x^2})\frac{\partial_t u}{x}+\frac{\sigma}{x}\partial_t\partial_x(\frac{x^2}{r^2r_x^2})\frac{\partial_t u}{x}+\sigma\partial_t(\frac{x^2}{r^2r_x^2})\partial_t\partial_x u\big]\\
&-2\xi\frac{1}{\sqrt{x}}\big[2\rho_{0x}\partial_t(\frac{x}{rr_x^3})\partial_t\partial_x u+\sigma\partial_t\partial_x(\frac{x}{rr_x^3})\partial_t\partial_x u+\sigma\partial_t(\frac{x}{rr_x^3})\partial_t\partial_x^2u\big].
\end{aligned}
\tag{5.32}
$$

Furthermore, $\|\xi\frac{1}{\sqrt{x}}\sigma\partial_t^2\partial_x(\frac{x^2}{r^2r_x^2})u_x\|_0^2$ and $\|\xi\frac{1}{\sqrt{x}}\sigma\partial_t\partial_x(\frac{x}{rr_x^3})\partial_t^2\partial_x u\|_0^2$ can be estimated as

$$
\begin{aligned}
&\|\xi\frac{1}{\sqrt{x}}\sigma\partial_t^2\partial_x(\frac{x^2}{r^2r_x^2})u_x\|_0^2\\
&\le C\|\xi\sigma\mathfrak{F}_1(t)\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\\
&\le M_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\xi\frac{1}{\sqrt{x}}\sigma\partial_t(\frac{x}{rr_x^3})\partial_t^2\partial_x u\|_0^2\\
&\le C\|\mathfrak{A}_0(t)\|_{L^\infty}^2\|\xi\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x u\|_0^2\\
&\le M_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
\end{aligned}
$$

Similarly, we can estimate the other terms of $\|\xi\frac{1}{\sqrt{x}}\frac{x}{2\sigma}A_1^2\|_0^2$ and have

$$
\|\xi\frac{1}{\sqrt{x}}\frac{x}{2\sigma}A_1^2\|_0^2\le\mathcal{P}_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big).
$$

Finally, we can obtain from (5.29)

$$
\Big\|\xi\frac{\sigma}{\sqrt{x}}x\partial_t^2\partial_x^2(\frac{u}{x})\Big\|_0^2+\Big\|3\xi\frac{\sigma}{\sqrt{x}}\partial_t^2\partial_x(\frac{u}{x})\Big\|_0^2\le M_0+Ct\mathcal{P}\big(\sup_{\tau\in[0,t]}E(\tau)\big). \tag{5.33}
$$

Because of $\sigma'(x)=x\rho_{0x}+\rho_0=x\rho_{0x}+\frac{\sigma}{x}$, we have

$$
\begin{aligned}
&\Big\|\xi\frac{\sigma}{\sqrt{x}}x\partial_t^2\partial_x(\frac{u}{x})\Big\|_0^2\\
&=\Big\|\xi(\sigma_x-x\rho_{0x})(\frac{\partial_t^2\partial_x u}{\sqrt{x}}-\frac{\partial_t^2 u}{x\sqrt{x}})\Big\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \xi\sigma_x(\frac{\partial_t^2\partial_x u}{\sqrt{x}} - \frac{\partial_t^2 u}{x\sqrt{x}}) - \xi x\rho_{0x}(\frac{\partial_t^2\partial_x u}{\sqrt{x}} - \frac{\partial_t^2 u}{x\sqrt{x}}) \right\|_0^2 \\
&\geq C \left\| \xi\sigma_x(\frac{\partial_t^2\partial_x u}{\sqrt{x}} - \frac{\partial_t^2 u}{x\sqrt{x}}) \right\|_0^2,
\end{aligned}
$$

and

$$
\left\| \xi \frac{\sigma}{\sqrt{x}} x\partial_t^2\partial_x^2(\frac{u}{x}) \right\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big),
$$

which in combination with (5.33) shows

$$
\begin{aligned}
&-18\int_0^1 \xi^2\sigma_x^2 \frac{\partial_t^2\partial_x u}{\sqrt{x}} \frac{\partial_t^2 u}{x\sqrt{x}} dx \\
&\leq \left\| 3\xi \frac{\sigma}{\sqrt{x}} \partial_t^2\partial_x u \right\|_0^2 + M_0\|(\xi\sigma\partial_t^2\partial_x u, \xi\partial_t^2 u)\|_0^2 \\
&\leq M_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big).
\end{aligned} \tag{5.34}
$$

Thus, we can obtain (5.23). This completes the proof. □

5.3. *Estimates for $\partial_t^3 u$*

Lemma 5.3. *Let (r, u, υ) be a smooth solution to the problem (2.11)-(2.12) satisfying (3.1) on $[0, 1] \times [0, T]$. Then, there exists a small time $0 < \bar{T}_5 \leq T_1$, such that for any $t \in (0, \bar{T}_5]$, it holds that*

$$
\left\| \xi(\sigma\partial_t^3\partial_x^2 u, \sigma_x\partial_t^3\partial_x u, \sigma_x \frac{\partial_t^3 u}{x}) \right\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big), \tag{5.35}
$$

and

$$
\|\eta\sigma^{\frac{3}{2}}\partial_t^2\partial_x^3 u\|_0^2 + \|\eta\sigma^{\frac{1}{2}}\sigma_x\partial_t^3\partial_x^2 u\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big). \tag{5.36}
$$

Proof. Similar to the estimates for $\partial_t^2 u$, we only give the interior estimate (5.35), while the boundary estimate in (5.36) can be obtained by the same way in (5.7). For this goal, we have from (5.1) for $k = 3$ that

$$
\begin{aligned}
&\left\| \xi(\sigma\partial_t^3\partial_x^2 u + \sigma_x\partial_t^3\partial_x u - \sigma_x \frac{\partial_t^3 u}{x}) \right\|_0^2 \\
&\leq M_0 \left[\|\xi(\sigma_x - \frac{\sigma}{x})\partial_t^3\partial_x u\|_0^2 + \|2\xi(\sigma_x - \frac{\sigma}{x})\frac{\partial_t^3 u}{x}\|_0^2 + \|\xi\mathfrak{B}^3\|_0^2 \right].
\end{aligned} \tag{5.37}
$$

Similar to (5.13), we have

$$
\begin{aligned}
&\left\| \xi\sigma\partial_t^3\partial_x^2 u + \xi\sigma_x\partial_t^3\partial_x u - \xi\sigma_x\frac{\partial_t^3 u}{x}\right\|_0^2 \\
&\geq \|\xi\sigma\partial_t^3\partial_x^2 u\|_0^2 + \frac{1}{4}\|\xi\sigma_x\partial_t^3\partial_x u\|_0^2 + \frac{1}{6}\|\xi\sigma_x\frac{\partial_t^3 u}{x}\|_0^2 - \big[\mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big)\big]. \qquad (5.38)
\end{aligned}
$$

For the estimate of $\|\xi\mathfrak{B}^3\|_0^2$ on the right side of (5.37), we only give the estimate of $\|\xi\frac{x}{2\sigma}A_1^3\|_0^2$, while the others can be similarly bounded. Thus, we can obtain

$$
\|\xi\mathfrak{B}^3\|_0^2 \leq \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big).
$$

By (4.5) and (5.27), it holds that

$$
\begin{aligned}
&\|\xi\frac{x}{2\sigma}A_1^3\|_0^2 \\
&= \left\| -\sum_{i=0}^{2}\xi\frac{x}{2\sigma}C_3^i\left[\frac{\sigma^2}{x}\partial_t^{3-i}(\frac{x^2}{r^2r_x^2})\frac{\partial_t^i u}{x} + 2\frac{\sigma^2}{x}\partial_t^{3-i}(\frac{x}{rr_x^3})\partial_t^i u_x\right]_x\right\|_0^2 \\
&\leq M_0\|(\sigma\mathfrak{A}_2, \sigma\mathfrak{F}_2)\|_0^2\|\xi(\partial_x^2 u, u_x, \frac{u}{x})(0)\|_{L^\infty}^2 \\
&\quad + M_0\|(\sigma\mathfrak{A}_2, \sigma\mathfrak{F}_2)\|_0^2\int_0^t \|\xi(\partial_t\partial_x^2 u, \partial_t\partial_x u, \frac{\partial_t u}{x})(\tau)\|_0^2 d\tau \\
&\quad + M_0\|(\sigma\mathfrak{A}_1, \sigma\mathfrak{F}_1)\|_0^2\|\xi(\partial_t\partial_x^2 u, \partial_t\partial_x u, \frac{\partial_t u}{x})(0)\|_{L^\infty}^2 \\
&\quad + M_0\|(\sigma\mathfrak{A}_1, \sigma\mathfrak{F}_1)\|_0^2\int_0^t \|\xi(\partial_t^2\partial_x^2 u, \frac{\partial_t^2 u}{x})(\tau)\|_0^2 d\tau + \mathcal{S}, \qquad (5.39)
\end{aligned}
$$

where $\mathcal{S}$ is

$$
\begin{aligned}
S :=& M_0\|(\sigma\mathfrak{A}_0, \sigma\mathfrak{F}_0)\|_0^2\|\xi\sigma(\partial_t^2\partial_x^2 u, \partial_t^2\partial_x u, \frac{\partial_t^2 u}{x})(0)\|_{L^\infty}^2 \\
&+ M_0\|(\sigma\mathfrak{A}_0, \sigma\mathfrak{F}_0)\|_0^2\int_0^t \|\xi\sigma(\partial_t^3\partial_x^2 u, \partial_t^3\partial_x u, \frac{\partial_t^3 u}{x})(\tau)\|_0^2 d\tau \\
&+ M_0\|(\sigma\mathfrak{F}_2, \sigma\mathfrak{F}_1)\|_0^2\left(\|\xi(u_x, \frac{\partial_t u}{x})(0)\|_{L^\infty}^2 + \int_0^t \|\xi(\partial_t\partial_x u, \frac{\partial_t^2 u}{x})(\tau)\|_0^2 d\tau\right) \\
&+ M_0\|(\mathfrak{F}_0, \sigma\mathfrak{F}_1)\|_0^2\left(\|\xi(\sigma\frac{\partial_t^2\partial_x u}{\sqrt{x}}, \frac{\partial_t u}{x})(0)\|_{L^\infty}^2 + \int_0^t \|\xi(\sigma\partial_t^3\partial_x u, \frac{\partial_t^2 u}{x})(\tau)\|_0^2 d\tau\right),
\end{aligned}
$$

where $\mathfrak{A}_i$ and $\mathfrak{F}_i$, $i = 0, 1, 2$, are given by (5.25) and (5.26), respectively. From (5.38) and (5.39), the estimate (5.35) can be obtained. □

5.4. Estimates for $E(t)$

Lemma 5.4. *Let (r, u, υ) be a smooth solution of the free boundary value problem (2.11)-(2.12) on $[0, 1] \times [0, T]$ under the assumption (3.1). Then, for any $t \in (0, T)$, (2.21) is satisfied.*

Proof. According to (5.5)-(5.7), (5.23), (5.35) and (5.36), we can get

$$E(u) \le \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big). \tag{5.40}$$

By analogy with the estimate of u, we can conclude that the estimate of υ

$$E(\upsilon) \le \mathcal{P}_0 + Ct\mathcal{P}\big(\sup_{\tau\in[0,t]} E(\tau)\big). \tag{5.41}$$

Combining (5.40) with (5.41) shows (2.21), where we have used a polynomial-type inequality introduced in [1]. This ends the proof. □

6. Existence results

This section is to investigate the existence of smooth solutions to the problem (2.11)-(2.12) by applying a degenerate parabolic regularization based on the priori estimate in (3.1). First, the degenerate parabolic approximation system can be constructed by the approach similar to the analysis of the cylindrical symmetric relativistic Euler system in [22] and the spherical symmetric system in [19]. Then, the existence of solutions of the regularized problem can be obtained by the similar analysis in [2,4,22] using the fixed point theorem. Then the estimate of solutions independent of μ be similarly obtained by applying the Lemma 3.2 on the page 336 in [2] due to our estimates in (2.21).

7. Uniqueness results

This section is to show the uniqueness of smooth solutions to the problem (2.11)-(2.12) on $[0, 1] \times [0, T]$ obtained in Theorem (2.1).

Lemma 7.1. *Let (r, u, υ) be smooth solutions of the problem (2.11)-(2.12) on $[0, 1] \times [0, T]$ given by Theorem (2.1) satisfying (2.21) with*

$$r_i = x + \int_0^t u_i(x, \tau)d\tau, \qquad i = 1, 2. \tag{7.1}$$

Then, there exists a positive time $0 < \tilde{T} < T$ such that, for any $[0, 1] \times [0, \tilde{T}]$, the solution (r, u, υ) is unique corresponding to $(\rho_0, u_0, \upsilon_0)$.

Proof. Set

$$R := r_2 - r_1, \quad R_t = U := u_2 - u_1, \quad V := \upsilon_2 - \upsilon_1.$$

From the system (2.11), a direct calculation gives

$$\sigma(x)U_t - \Big[\frac{\sigma^2}{x}(\mathcal{L}_1 R_x + \mathcal{L}_2 \frac{R}{x})\Big]_x + \frac{\sigma^2}{x^2}(\mathcal{L}_3 R_x + \mathcal{L}_4 \frac{R}{x}) + H(U, V, \frac{R}{x}) = 0, \tag{7.2}$$

where

$$\begin{aligned}
&\mathcal{L}_1 := \frac{x}{r_2}(\frac{1}{r_{1x}} + \frac{1}{r_{2x}})\frac{1}{r_{1x}}\frac{1}{r_{2x}}, \qquad \mathcal{L}_2 := (\frac{1}{r_{1x}})^2 \frac{x}{r_1}\frac{x}{r_2},\\
&\mathcal{L}_3 := (\frac{x}{r_2})^2 \frac{1}{r_{1x}}\frac{1}{r_{2x}}, \qquad \mathcal{L}_4 := \frac{1}{r_{1x}}(\frac{x}{r_1} + \frac{x}{r_1})\frac{x}{r_1}\frac{x}{r_2},\\
&H(U, V, \frac{R}{x}) := \sigma\Big[U - (\frac{\upsilon_2^2}{r_2} - \frac{\upsilon_1^2}{r_1})\Big] = \sigma(U + \frac{\upsilon_2^2}{r_1 r_2}R - \frac{\upsilon_1 + \upsilon_2}{r_1}V).
\end{aligned} \tag{7.3}$$

Multiplying (7.2) by U and integrating over $(0, 1) \times (0, t)$, we have

$$\begin{aligned}
&\int_0^1 \sigma(x)\frac{U^2}{2}dx + \int_0^1 \frac{\sigma^2}{x}\Big[\mathcal{L}_1 R_x^2 + (\mathcal{L}_2 + \mathcal{L}_3)\frac{R}{x}R_x + \mathcal{L}_4 \frac{R^2}{x^2}\Big]dx\\
&= \int_0^t \int_0^1 \frac{\sigma^2}{x}\big(\partial_t \mathcal{L}_1 R_x^2 + \mathcal{L}_1 R_x U_x + \partial_t \mathcal{L}_2 \frac{R}{x}R_x + \mathcal{L}_2 \frac{U}{x}R_x\big)dxd\tau\\
&\quad + \int_0^t \int_0^1 \frac{\sigma^2}{x}\big(\partial_t \mathcal{L}_3 R_x \frac{R}{x} + \mathcal{L}_3 U_x \frac{R}{x} + \partial_t \mathcal{L}_4 \frac{R^2}{x^2} + \mathcal{L}_4 \frac{U}{x}\frac{R}{x}\big)dxd\tau\\
&\quad - \int_0^t \int_0^1 H(U, V, \frac{R}{x})U dxd\tau,
\end{aligned} \tag{7.4}$$

where

$$\begin{aligned}
&\left|H(U, V, \frac{R}{x})U\right|\\
&= \left|\sigma(U + \frac{\upsilon_2^2}{r_1 r_2}R - \frac{\upsilon_1 + \upsilon_2}{r_1}V)U\right|\\
&\le \sigma\left|U + \frac{x}{r_1}\frac{x}{r_2}\frac{R^2}{x^2}\upsilon_2^2 + \frac{2\upsilon_1 + V}{r_1}V\right| \cdot \left|\frac{U}{x}\right|.
\end{aligned}$$

From (2.1), there exists a positive constant K_0, depending on $\mathcal{P}_0$, such that

$$|\partial_t \mathcal{L}_i| + |\mathcal{L}_i| \le K_0, \quad i = 2, 3, 4. \tag{7.5}$$

Thus,

$$\begin{aligned}
&\int_0^1 \sigma(x)\frac{U^2}{2}dx + \int_0^1 \frac{\sigma^2}{x}(\mathcal{L}_1 R_x^2 + \mathcal{L}_2\frac{R}{x}R_x + \mathcal{L}_4\frac{R^2}{x^2})dx \\
&\le C(K_0)\int_0^t\int_0^1 \frac{\sigma^2}{x}(R_x^2 + \frac{R^2}{x^2} + U_x^2 + \frac{U^2}{x^2})dxd\tau \\
&+ C(K_0)\int_0^t\int_0^1 \sigma(x)\big[U^2 + \big(\frac{U}{x}\big)^2 + R_x^2 + \big(\frac{R}{x}\big)^2\big]dxd\tau.
\end{aligned} \tag{7.6}$$

Differentiating (7.2) with respect to t and multiplying it by U_t, the similar procedure of (7.6) shows

$$\begin{aligned}
&\int_0^1 \sigma(x)\frac{U_t^2}{2}dx + \int_0^1 \frac{\sigma^2}{x}(\mathcal{L}_1 U_x^2 + 2\mathcal{L}_2\frac{U}{x}U_x + \mathcal{L}_4\frac{U^2}{x^2})dx \\
&\le \int_0^t\int_0^1 \sigma^2(\mathcal{L}_2 - \mathcal{L}_3)\frac{U_t}{x}U_x dxd\tau \\
&+ C(K_0)\int_0^t\int_0^1 \sigma(x)(U^2 + V^2 + \frac{R^2}{x^2})dxd\tau \\
&+ C(K_0)\int_0^t\int_0^1 \frac{\sigma^2}{x}(U^2 + U_x^2 + \frac{U^2}{x^2} + R_x^2 + \frac{R^2}{x^2})dxd\tau.
\end{aligned} \tag{7.7}$$

Moreover, a straightforward computation yields, with the help of (7.3),

$$|\mathcal{L}_2 - \mathcal{L}_3| \le C(K_0)(|R| + |U|), \tag{7.8}$$

which implies

$$\int_0^t\int_0^1 \sigma^2(\mathcal{L}_2 - \mathcal{L}_3)\frac{U_t}{x}U_x dxd\tau$$

$$\leq C(K_0)\int_0^t\int_0^1 \sigma^2\big(R_x^2+U_x^2+\frac{U^2}{x^2}\big)dxd\tau. \tag{7.9}$$

Based on some straightforward calculations, there exist positive constants T_1^* and $\varepsilon_{T_1^*}$, such that $0<T_1^*<T$ and for any $0<t\leq T_1^*$,

$$\frac{1}{1+\varepsilon_{T_1^*}}\leq\frac{x}{r_i}\leq\frac{1}{1-\varepsilon_{T_1^*}},\qquad 1-\varepsilon_{T_1^*}\leq r_{ix}\leq 1+\varepsilon_{T_1^*},\quad i=1,2, \tag{7.10}$$

where $\lim_{T_1^*\to 0}\varepsilon_{T_1^*}=0$, then for any $r=r_2+\mu(r_1-r_2)$ and $r_x=r_{2x}+\mu(r_{1x}-r_{2x})$

$$\begin{gathered}\frac{-2\varepsilon_{T_1^*}}{(1-\varepsilon_{T_1^*})(1+\varepsilon_{T_1^*})}+\frac{x}{r}\leq\frac{x}{r_1}\leq\frac{x}{r}+\frac{2\varepsilon_{T_1^*}}{(1-\varepsilon_{T_1^*})(1+2\varepsilon_{T_1^*})},\\ r_{2x}-\varepsilon_{T_1^*}\leq r_x\leq r_{2x}+2\varepsilon_{T_1^*}.\end{gathered} \tag{7.11}$$

For the second term on the left side of (7.7). From (7.3), $\mathcal{L}_1>G(\mu,\varepsilon_{T_1^*})$, where

$$G(\mu,\varepsilon_{T_1^*})=\int_0^1\left(\frac{x}{r}-\frac{2\varepsilon_{T_1^*}}{(1-\varepsilon_{T_1^*})(1+\varepsilon_{T_1^*})}\right)\frac{1}{(r_{2x}+2\varepsilon_{T_1^*})^3}\Big|_{r=r_2+\mu(r_1-r_2)}d\mu,$$

which implies

$$\lim_{T_1^*\to 0}G(\mu,\varepsilon_{T_1^*})=\mathcal{L}_1^*,$$

with

$$\mathcal{L}_1^*=\int_0^1\frac{x}{rr_{2x}^3}\Big|_{r=r_2+\mu(r_1-r_2)}d\mu.$$

So, there is a positive constant T_2^* satisfying $0<T_2^*<T_1^*$, such that for $t\in(0,T_2^*]$

$$\mathcal{L}_1U_x^2+2\mathcal{L}_2\frac{U}{x}U_x+\mathcal{L}_4\frac{U^2}{x^2}\geq\mathcal{L}_1^*U_x^2+2\mathcal{L}_2\frac{U}{x}U_x+\mathcal{L}_4\frac{U^2}{x^2}.$$

Due to (7.3), there exists a positive constant M^* such that

$$\mathcal{L}_1^*U_x^2+2\mathcal{L}_2\frac{U}{x}U_x+\mathcal{L}_4\frac{U^2}{x^2}\geq M^*(U_x^2+\frac{U^2}{x^2}).$$

Finally, for any $0<t<\min\{T_1^*,T_2^*\}$, it follows that

$$\mathcal{L}_1U_x^2+2\mathcal{L}_2\frac{U}{x}U_x+\mathcal{L}_4\frac{U^2}{x^2}\geq M^*(U_x^2+\frac{U^2}{x^2}). \tag{7.12}$$

From (7.6)-(7.12), we can obtain $U = R = 0$ by using Gronwall's inequality. The proof is complete. $\square$

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