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Convergence rates to travelling waves for a nonconvex relaxation model

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In this paper we study the asymptotic behaviour of the solution for a nonconvex relaxation model. The time decay rates in both the exponential and algebraic forms of the travelling wave solutions are shown by the weighted energy method. Our results develop and improve the stability theory in [**8**, **9**].

1. Introduction

Relaxation is a phenomenon which occurs in water waves, thermo-nonequilibrium gases, rarefield gas dynamics, traffic flow, viscoelasticity with memory and magneto-hydrodynamics. Hyperbolic conservation laws with relaxation also serve as kinetic models. The relaxation is usually stiff when the relaxation time is much shorter than the scales of other physical quantities. The 2×2 relaxation hyperbolic equations were first analysed by T.-P. Liu [**10**] when considering nonlinear stability criteria for diffusion waves, expansion waves and travelling waves. Since then, there has been much work carried out on this subject, see [**1–4**, **7–10**, **12**, **16**] and the references therein.

This paper is concerned with the simplest model of two equations which captures the basic features of those physical models in the form

$$\begin{cases} u_t + v_x = 0, \quad x \in R^1, \, t \in R_+, \\ v_t + a u_x = \dfrac{f(u) - v}{\varepsilon}. \end{cases} \tag{1.1}$$

Such a model is included in [**3**, **10**], and was also introduced by Jin and Xin [**4**] for numerical analysis, and studied by H. Liu and Wang, Woo and Yang [**8**, **9**] as well as Mascia and Natalini [**12**] for the stability of travelling waves.

We consider the Cauchy problem of model (1.1) with initial data

$$(u, v)(x, 0) = (u_0, v_0)(x), \tag{1.2}$$

where $(u_0, v_0)(x) \to (u_\pm, v_\pm)$ as $x \to \pm\infty$, $(u_\pm, v_\pm)$ are the given state constants, and

$v_\pm = f(u_\pm)$. Here u is some conserved physical quantity, v is some rate variable that measures the departure of the relaxation from the local equilibrium, $0<\varepsilon\ll 1$ is the relaxation time and a is a positive constant satisfying

$$-\sqrt{a}<f'(u)<\sqrt{a}, \quad \text{for all } u \text{ under consideration,} \tag{1.3}$$

which is the subcharacteristic condition introduced by T.-P. Liu [**10**].

By scaling the variable (x, t) to a new one $(\varepsilon x, \varepsilon t)$, equation (1.1) is reduced to

$$\begin{cases} u_t + v_x = 0, \quad x\in R^1,\, t\in R_+, \\ v_t + au_x = f(u) - v. \end{cases} \tag{1.4}$$

The behaviour of the solution (u, v) for (1.1) and (1.2) at fixed time as $\varepsilon\to 0^+$ is equivalent to the long-time behaviour of the solution (u, v) in (1.4) and (1.2) as $t\to\infty$, see [**7**].

The travelling waves, which are the viscous shock waves of (1.4) and (1.2), are the solutions of (1.4) in the form

$$\begin{cases} (u, v)(x, t) = (U, V)(x-st) = (U, V)(z), \\ (U, V)(z)\to(u_\pm, v_\pm), \quad \text{as } z\to\pm\infty; \end{cases}$$

s is the propagation speed of waves. The stability and the time decay rates are shown in [**8, 9**], and the L^1-stability is given in [**12**]. Applying the L^2-weighted energy method used in [**13**], see also [**5, 6, 14**], the authors in [**8, 9**] proved that the travelling wave solutions of strong shocks are stable for both the nondegenerate and degenerate cases, when $a\gg 1$. More precisely, if the initial data $(u_0, v_0)(x)$ approach a travelling wave solution $(U, V)(x)$ with a spatial decay rate $O(|x|^{-\alpha})$ for any given $\alpha>0$, then the solution (u, v) approaches a shifted travelling wave solution $(U, V)(x-st+x_0)$ with a shift constant x_0 in the algebraic time decay rate $t^{-\alpha+\eta}$, for any $\eta\geqq 0$, but $\eta=0$ only as α is an integer, i.e. $\alpha=[\alpha]$, in the case of nondegenerate shock $f'(u_+)<s<f'(u_-)$; while the algebraic time decay rate $t^{-(\alpha/2)+\eta}$ for any $\eta>0$ even if $\alpha=[\alpha]$, in the case of degenerate shock $f'(u_+)=s<f'(u_-)$. However, the decay rates they obtained are not optimal. As shown in [**17**] for the scalar viscous conservation laws, we also expect to get the algebraic time-decay rates by removing η for both the degenerate and nondegenerate cases. Roughly speaking, the aim of one part of this paper is to improve the time decay rates as $t^{-\alpha}$ in the case of nondegenerate shock $f'(u_+)<s<f'(u_-)$, as well as $t^{-\alpha/2}$ for the case of degenerate shock $f'(u_+)=s<f'(u_-)$, even for noninteger α. The other part of this paper aims to show the time exponential decay rate $e^{-\theta t}$ for some constant $\theta>0$, when the initial data $(u_0, v_0)(x)$ approach a travelling wave solution $(U, V)(x)$ in the spatial decay rate $O(e^{-\alpha|x|})$ for some given $\alpha>0$. To prove this, we make use of the weighted energy method in the first author's work [**15**] for the single equation of conservation laws. We also point out that the stability and the time decay rates hold, when a is large for the strong shock profiles, but without the assumption that $a\gg 1$ for the weak shock profiles.

This paper is organised as follows. After stating some notation below, we give some preliminaries and main theorems in Section 2. In Section 3, we reformulate the original problem. Section 4 is the proof of the exponential time decay rate. In Section 5, we give the proof of the improved algebraic time decay rates. Finally, we remark on the stability and time-decay rates of the strong detonation travelling waves for a viscous combustion model in Section 6.

NOTATION. L^2 denotes the space of measurable functions on R which are square integrable, with the norm

$$\|f\| = \left(\int |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

$H^l(l \geqq 0)$ denotes the Sobolev space of L^2-functions f on R whose derivatives $\partial_x^j f$, $j = 1, \ldots, l$, are also L^2-functions, with the norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{\frac{1}{2}}.$$

L_w^2 denotes the space of measurable functions on R which satisfy $w(x)^{\frac{1}{2}} f \in L^2$, where $w(x) > 0$ is called a weight function, with the norm

$$|f|_w = \left(\int w(x) |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

H_w^l $(l \geqq 0)$ denotes the weighted Sobolev space of L_w^2-functions f on R whose derivatives $\partial_x^j f$, $j = 1, \ldots, l$, are also L_w^2-functions, with the norm

$$|f|_{l,w} = \left(\sum_{j=0}^{l} |\partial_x^j f|_w^2 \right)^{\frac{1}{2}}.$$

Denoting $\langle x \rangle = \sqrt{1 + x^2}$ and

$$\langle x \rangle_+ = \begin{cases} \sqrt{1 + x^2}, & \text{if } x \geqq 0, \\ 1, & \text{if } x < 0, \end{cases}$$

we will make use of the spaces $L^2_{\langle x \rangle_+}$ and $H^l_{\langle x \rangle_+}$ $(l = 1, 2)$ later. If $w(x) = \langle x \rangle^\alpha$, we denote $L_w^2 = L_\alpha^2$. The weighted space L_w^2 for such weight function $w = \langle x \rangle^\alpha \langle x \rangle_+$ is denoted as $L^2_{\alpha \langle x \rangle_+}$, and the corresponding norm is $|\cdot|_{\alpha \langle x \rangle_+}$. We denote also $f(x) \sim g(x)$ as $x \to x_0$ when $C^{-1} g \leqq f \leqq Cg$ in a neighbourhood of x_0, and $|(f_1, f_2, f_3)|_X \sim |f_1|_X + |f_2|_X + |f_3|_X$, where $|\cdot|_X$ is the norm of space X. Without any ambiguity, we denote several constants by C_i, or c_i, $i = 1, 2, \ldots$, or by C. When $C^{-1} \leqq \mathrm{w}(x) \leqq C$ for $x \in R$, we note that $L^2 = H^0 = L_w^2 = H_w^0$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$.

Let T and B be a positive constant and a Banach space, respectively. We denote $C^k(0, T; B)$ $(k \geqq 0)$ as the space of B-valued k-times continuously differentiable functions on $[0, T]$, and L^2 $(0, T; B)$ as the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued functions on $[0, \infty)$ are defined similarly.

2. Preliminaries and main results

The travelling wave solution of system (1.4) is a solution $(U, V)(z)$, $(z = x - st)$, satisfying equations (1.4) and $(U, V)(\pm \infty) = (u_\pm, v_\pm)$, namely,

$$\begin{cases} -sU_z + V_z = 0, \\ -sV_z + aU_z = f(U) - V, \\ (U, V)(\pm \infty) = (u_\pm, v_\pm), \end{cases} \tag{2.1}$$

which implies

$$(a-s^2)U_z = f(U) - V. \tag{2.2}$$

Integrating the first equation of (2.1) over $(\pm\infty, z)$ and noting $(U, V)(\pm\infty) = (u_\pm, v_\pm)$, as well as $v_\pm = f(u_\pm)$, yields

$$-sU + V = -su_\pm + v_\pm = -su_\pm + f(u_\pm). \tag{2.3}$$

Substituting (2.3) into (2.1), we have

$$(a-s^2)U_z = f(U) - f(u_\pm) - s(U - u_\pm) \equiv: h(U). \tag{2.4}$$

From (2.3), we see that the speed s and the state constants $(u_\pm, v_\pm)$ satisfy the so-called Rankine–Hugoniot condition

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.5}$$

It is well known that the ordinary differential equation (2.4) has a solution if and only if the Oleinik entropy condition

$$h(u) = f(u) - f(u_\pm) - s(u - u_\pm) \begin{cases} <0, & u_+ < u_-, \\ >0, & u_+ > u_-, \end{cases} \tag{2.6}$$

holds. This entropy condition implies that

$$f'(u_+) < s < f'(u_-) \tag{2.7}$$

or

$$f'(u_+) = s < f'(u_-) \quad \text{or} \quad f'(u_+) < s = f'(u_-) \quad \text{or} \quad f'(u_\pm) = s. \tag{2.8}$$

The entropy condition (2.7) is the well-known Lax shock condition. We call it the *nondegenerate* shock condition. For each case in (2.8), we call it the *degenerate* shock condition, or the *contact* shock condition. If the viscous shocks $(U, V)(x - st)$ are degenerate, we restrict ourself to the case

$$f'(u_+) = s < f'(u_-), \tag{2.9}$$

since other cases in (2.8) can be treated similarly. Furthermore, we assume that for the case (2.9),

$$h^{(n)}(u_+) = 0 \quad \text{and} \quad h^{(n+1)}(u_+) \neq 0 \quad \text{for } n \geqq 1. \tag{2.10}$$

In this paper, without loss of generality, we focus on the case

$$u_+ < u_-.$$

Regarding the existence of the travelling wave solutions, by a similar proof in [**6**] for the scalar viscous conservation laws, see also [**13, 14**], the existence result is given in [**9**] as follows:

PROPOSITION 2.1 [**9**]. *Under Oleinik shock condition* (2.6) *and the Rankine–Hugoniot condition* (2.5), *there exists a travelling wave solution* $(U, V)(x - st)$ *of* (1.4) *with* $(U, V)(\pm\infty) = (u_\pm, v_\pm)$, *unique up to a shift, and the speed satisfies*

$$s^2 < a. \tag{2.11}$$

Moreover, it holds that

$$(a-s^2)U_z = h(U) < 0 \quad \text{for } u_+ < u_- \tag{2.12}$$

and, as $z \to \pm\infty$,

$$\begin{cases} |h(U)| \sim |(U-u_\pm, V-v_\pm)(z)| \sim \exp(-c_\pm|z|), & \text{if } f'(u_+) < s < f'(u_-), \\ |h(U)|^{1/(1+n)} \sim |(U-u_+, V-v_+)(z)| \sim |z|^{-(1/n)}, & \text{if } f'(u_+) = s < f'(u_-), \end{cases} \tag{2.13}$$

where $c_\pm = |f'(u_\pm) - s|/(a-s^2) > 0$.

Defining the following weight functions, cf. [**13**, **15**],

$$w_1(U) = \frac{(U-u_+)(U-u_-)}{h(U)}, \quad w_2(U) = -\frac{(U-u_+)^{\frac{1}{2}}(u_- - U)^{\frac{1}{2}}}{h(U)} \tag{2.14}$$

for $U \in (u_+, u_-)$, which are positive due to $u_+ < u_-$ and $h(U) < 0$, we give the properties of the travelling wave solutions (U, V) as follows:

Lemma 2.2. *Let* $(U, V)(x-st)$ *be the travelling wave solution of* (1.4). *Then*:

$$\begin{cases} w_1(U) \sim O(1), \quad w_2(U) \sim e^{c_\pm|z|/2}, & \text{if } f'(u_+) < s < f'(u_-), \\ w_1(U) \sim \langle z \rangle_+, & \text{if } f'(u_+) = s < f'(u_-), \end{cases} \tag{2.15}$$

as $z \to \pm\infty$, *and*

$$(w_1 h)''(U) = 2, \quad \left|\frac{w_i(U)_z}{w_i(U)}\right| = O(1)\frac{|u_+ - u_-|}{a-s^2}, \quad i = 1, 2, \tag{2.16}$$

$$-h(U)(w_2 h)''(U) = O(1) w_2(U), \quad \text{for } f'(u_+) < s < f'(u_-). \tag{2.17}$$

Proof. It is easy to check (2.15) and $(w_1 h)''(U) = 2$ by (2.13) and (2.14).

For (2.16) and (2.17), (2.14) and (2.13) give

$$\begin{aligned} (w_2 h)''(U) &= -((U-u_+)^{\frac{1}{2}}(u_- - U)^{\frac{1}{2}})'' \\ &= \frac{1}{4}(u_+ - u_-)^2 (U-u_+)^{-3/2}(u_- - U)^{-3/2} = O(1) w_2(U)/|h(U)| \end{aligned}$$

for $f'(u_+) < s < f'(u_-)$. This proves (2.17).

We also have, by (2.12),

$$\frac{w_i(U)_z}{w_i(U)} = \frac{w_i'(U)}{w_i(U)} \frac{h(U)}{a-s^2}, \quad i = 1, 2,$$

and by (2.4), see also (2.14),

$$\frac{w_i'(U)}{w_i(U)} = k_i \frac{1}{h(U)} \left\{ \frac{h(U)}{U-u_+} + \frac{h(U)}{U-u_-} - \frac{1}{k_i} h'(U) \right\}, \quad i = 1, 2,$$

where $k_1 = 1$ and $k_2 = \frac{1}{2}$. Since

$$0 = h(u_\pm) = h(U) + h'(U)(u_\pm - U) + \frac{h''(\tilde{u}_\pm)}{2}(u_\pm - U)^2$$

for some $\tilde{u}_\pm \in (u_+, u_-)$ and $|h'(U)| = |f'(U) - s| \leqq C|u_+ - u_-|$, we can conclude that

$$\left|\frac{h(U)}{U-u_+} + \frac{h(U)}{U-u_-} - \frac{1}{k_i} h'(U)\right| \leqq \left|\frac{h(U)}{U-u_+} + \frac{h(U)}{U-u_-} - h'(U)\right| + \frac{k_i - 1}{k_i} |h'(U)|$$
$$\leqq C|u_+ - u_-|, \quad \text{for } i = 1, 2,$$

where $C > 0$ is independent of $|u_+ - u_-|$ and a. Thus, we show that

$$\left|\frac{w_i(U)_z}{w_i(U)}\right| = \frac{k_i}{a - s^2} \left|\frac{h(U)}{U-u_+} + \frac{h(U)}{U-u_-} - \frac{1}{k_i} h'(U)\right| \leqq C \frac{|u_+ - u_-|}{a - s^2}.$$

This completes the proof of Lemma 2.2. □

REMARK 2.3. It is well known that if $|u_+ - u_-| \ll 1$ or $a \gg 1$, then $|w_i(U)_z / w_i(U)| \ll 1$, $i = 1, 2$. This will be used to get the *a-priori* estimates below.

After assuming

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))\, dx = 0, \tag{2.18}$$

also letting $x_0 = 0$ for simplicity, and denoting

$$\varphi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)\, dy, \quad \psi_0(x) = (v_0 - V)(x), \tag{2.19}$$

the authors in [9] proved the time decay rates to the travelling wave solutions as follows.

THEOREM 2.4. [9]. *Under the assumptions of* (1.3), (2.5), (2.6) *and* (2.18), *and letting a be suitably large*:

(i) *the case $f'(u_+) < s < f'(u_-)$: suppose that $(\varphi_0, \psi_0)(x) \in L^2_\alpha \cap H^2$ holds. Then there exists a positive constant ε_1 such that if $|(\varphi_0, \psi_0)|_\alpha + \|(\varphi_0, \psi_0)\|_2 < \varepsilon_1$, then the system* (1.4) *and* (1.2) *has a unique global solution $(u, v)(x, t)$ satisfying*

$$\sup_{x \in \boldsymbol{R}} |(u, v)(x, t) - (U, V)(x - st)| \leqq C(1 + t)^{-(\alpha/2) + \varepsilon}(|(\varphi_0, \psi_0)|_\alpha + \|(\varphi_0, \psi_0)\|_2) \tag{2.20}$$

for any constant $\varepsilon \geqq 0$ and $\varepsilon = 0$ only as α is integer.

(ii) *The case $f'(u_+) = s < f'(u_-)$: suppose that $(\varphi_0, \psi_0)(x) \in L^2_{\alpha\langle x\rangle_+} \cap H^2$ holds, where $(0 < \alpha < 2/n)$. Then there exists a positive constant ε_2 such that if $|(\varphi_0, \psi_0)|_{\alpha\langle x\rangle_+} + \|(\varphi_0, \psi_0)\|_2 < \varepsilon_2$, then the system* (1.4) *and* (1.2) *has a unique global solution $(u, v)(x, t)$ satisfying*

$$\sup_{x \in \boldsymbol{R}} |(u, v)(x, t) - (U, V)(x - st)| \leqq C(1 + t)^{-(\alpha/4) + \varepsilon}(|(\varphi_0, \psi_0)|_{\alpha\langle x\rangle_+} + \|(\varphi_0, \psi_0)\|_2) \tag{2.21}$$

for any constant $\varepsilon > 0$ whether or not α is an integer.

However, these decay rates in [9] are not optimal. The ε in Theorem 2.4 can be dropped for both the nondegenerate and degenerate cases. The method of proof we adopt follows from [17] for scalar equations. One of our main results can be stated as follows:

THEOREM 2.5 (algebraic rates). *Under the same assumptions as in Theorem 2.4, let a be suitably large or $|u_+ - u_-|$ be suitably small. Then the solution $(u, v)(x, t)$ of* (1.4) *and* (1.2) *satisfies the following*:

(i) *the case $f'(u_+) < s < f'(u_-)$*:

$$\sup_{x \in R} |(u, v)(x, t) - (U, V)(x - st)| \leqq C(1 + t)^{-(\alpha/2)}(|(\varphi_0, \psi_0)|_\alpha + \|(\varphi_0, \psi_0)\|_2); \tag{2.22}$$

(ii) *the case $f'(u_+) = s < f'(u_-)$*:

$$\sup_{x \in R} |(u, v)(x, t) - (U, V)(x - st)| \leqq C(1 + t)^{-(\alpha/4)}(|(\varphi_0, \psi_0)|_{\alpha\langle x\rangle_+} + \|(\varphi_0, \psi_0)\|_2). \tag{2.23}$$

On the other hand, when the initial perturbation $(\varphi_0, \psi_0)(x)$ has an exponentially spatial decay rate as $x \to \pm\infty$, we will expect that the solution $(u, v)(x, t)$ of (1.4) and (1.2) converges to the travelling wave solution $(U, V)(x - st)$ at an exponential time decay rate, too. This is our other aim in the paper. The method of proof we use is the weighted energy method developed by the first author in **[15]** for scalar equations. The exponential time decay is stated in the following theorem:

THEOREM 2.6 (exponential rates). *Let a be suitably large or $|u_+ - u_-|$ be suitably small. If $f'(u_+) < s < f'(u_-)$ and $\varphi_0 \in H^3_{w_2(U)}$, $\psi_0 \in H^2_{w_2(U)}$, then there exist positive constants ε_3 and $\theta = \theta(|u_+ - u_-|, a)$ such that if $|(\varphi_0, \psi_0)|_{2,w_2} \leqq \varepsilon_3$, then the Cauchy problem* (1.4) *and* (1.2) *has a unique global solution $(u, v)(x, t)$ satisfying*

$$u - U \in C^0(0, \infty; H^3_{w_2}) \cap L^2(0, \infty; H^3_{w_2}),$$
$$v - U \in C^0(0, \infty; H^2_{w_2}) \cap L^2(0, \infty; H^2_{w_2})$$

and

$$\sup_{x \in R} |(u, v)(x, t) - (U, V)(x - st)| \leqq Ce^{-\theta t/2}|(\varphi_0, \psi_0)|_{2,w_2}. \tag{2.24}$$

REMARK 2.7. As in **[8, 9]**, we also need the condition $a \gg 1$ for the stability of the strong viscous shock waves. However, as shown in Theorems 2.5 and 2.6, if the shock is weaker, i.e. the strength $|u_+ - u_-| \ll 1$, the condition $a \gg 1$ can be dropped.

3. Reformulation of the original problem

Letting $(U, V)(z)$ be the travelling wave solution, and putting

$$(u, v)(x, t) = (U, V)(z) + (\varphi_z, \psi)(z, t), \quad z = x - st, \tag{3.1}$$

we substitute (3.1) into the original problem (1.4) and (1.2), and integrate the first equation once with respect to z, to yield a new system as follows:

$$\begin{cases} \varphi_t - s\varphi_z + \psi = 0, \\ \psi_t - s\psi_z + a\varphi_{zz} = f(U + \varphi_z) - f(U) - \psi, \\ (\varphi, \psi)(z, 0) = (\varphi_0, \psi_0)(x). \end{cases} \tag{3.2}$$

The first equation of (3.2) gives $\psi = -(\varphi_t - s\varphi_z)$. Substituting it into the second equation of (3.2) yields

$$L(\varphi) :\equiv (\varphi_t - s\varphi_z)_t - s(\varphi_t - s\varphi_z)_z - a\varphi_{zz} + \varphi_t + h'(U)\varphi_z = F, \tag{3.3}$$

where

$$F=-(f(U+\varphi_z)-f(U)-f'(U)\varphi_z) \quad \text{and} \quad |F|=O(1)|\varphi^2|. \tag{3.4}$$

The corresponding initial data for the scalar differential equation (3.3) are

$$\varphi(z,0)=\varphi_0(z), \quad \varphi_t(z,0)=s\varphi_0'(z)-\psi_0(z)\equiv:\varphi_1(z). \tag{3.5}$$

We now state the theorems corresponding to Theroems 2.5 and 2.6.

THEOREM 3.1 (exponential rates). *Under the same assumptions as in Theorem 2.6, if $\varphi_0\in H^3_{w_2(U)}$, $\varphi_1\in H^2_{w_2(U)}$, then there exist positive constants ε_4 and θ such that if $|\varphi_0|_{3,w_2}+|\varphi_1|_{2,w_2}\leqq\varepsilon_4$, then the Cauchy problem (3.3) and (3.5) has a unique global solution $\varphi(z,t)$ satisfying*

$$\varphi(z,t)\in C^0(0,\infty;H^3_{w_2})\cap\cap L^2(0,\infty;H^3_{w_2})$$
$$\varphi_t(z,t)\in C^0(0,\infty;H^2_{w_2})\cap L^2(0,\infty;H^2_{w_2})$$

and

$$|\varphi(t)|^2_{3,w_2}+|\varphi_t(t)|^2_{2,w_2}\leqq Ce^{-\theta t}(|\varphi_0|^2_{3,w_2}+|\varphi_1|^2_{2,w_2}). \tag{3.6}$$

THEOREM 3.2 (algebraic rates). *Under the same assumptions as in Theorem 2.6, let a be suitably large or $|u_+-u_-|$ be suitably small. Then the solution $(u,v)(x,t)$ of (1.4) and (1.2) satisfies the following*:

(i) *the case $f'(u_+)<s<f'(u_-)$*:

$$\sup_{z\in R}|(\varphi,\varphi_z,\varphi_t)(z,t)|\leqq C(1+t)^{-(\alpha/2)}(|(\varphi_0,\psi_0)|_\alpha+\|(\varphi_0,\psi_0)\|_2). \tag{3.7}$$

(ii) *the case $f'(u_+)=s<f'(u_-)$*:

$$\sup_{z\in R}|(\varphi,\varphi_z,\varphi_z)(z,t)|\leqq C(1+t)^{-(\alpha/4)}(|(\varphi_0,\psi_0)|_{\alpha\langle x\rangle_+}+\|(\varphi_0,\psi_0)\|_2). \tag{3.8}$$

4. Exponential time decay rate

In this section, we investigate the exponential time decay rate for the stability problem of the travelling wave. The local existence and uniqueness of the solution φ of (3.3) and (3.5) is well known by a standard argument. Our intention in this section is to establish the *a-priori* estimate.

Define the solution space of (3.3) and (3.5) as

$$X_1(0,T)=\{\varphi\in C^0(0,T;H^3_{w_2})\cap L^2(0,T;H^3_{w_2}),\ \varphi_t\in C^1(0,T;H^2_{w_2})\cap L^2(0,T;H^2_{w_2})\}$$

and let

$$N_1(T)=\sup_{0\leqq t\leqq T}\{|\varphi(t)|_{3,w_2}+|\varphi_t(t)|_{2,w_2}\}$$

for $T\in[0,\infty]$.

Firstly, we have the following lemma:

LEMMA 4.1 (basic energy estimate). *For any $T>0$, let $\varphi\in X_1(0,T)$ be a solution of (3.3) and (3.5), and assume that $a\gg 1$ or $|u_+-u_-|\ll 1$, and (1.3) hold. Then there* *exists a positive constant θ dependent on $|u_+ - u_-|$ and a such that*

$$|\varphi(t)|_{1,w_2}^2 + |\varphi_t(t)|_{w_2}^2 + \theta \int_0^t (|\varphi(\tau)|_{1,w_2}^2 + |\varphi_t(\tau)|_{w_2}^2)\, d\tau \leqq C(|\varphi_0|_{1,w_2}^2 + |\varphi_1|_{w_2}^2) \quad (4.1)$$

holds for $t \in [0, T]$ provided $N_1(T) \ll 1$.

Proof. We multiply (3.3) by $2w_2(U)\varphi$ to obtain

$$2w_2(U)\varphi \cdot L(\varphi) = -2Fw_2(U)\varphi. \quad (4.2)$$

By a simple but tedious computation, we have from (4.2)

$$\begin{aligned}&[w_2(U)\varphi^2 + 2w_2(U)\varphi(\varphi_t - s\varphi_z) + sw_2(U)_z\varphi^2]_t - 2w_2(U)(\varphi_t - s\varphi_z)^2 \\ &\quad + 2aw_2(U)\varphi_z^2 - (w_2h)''(U)U_z\varphi^2 + \{\cdots\}_z = -2Fw_2(U)\varphi, \end{aligned} \quad (4.3)$$

where we used $(a - s^2)U_z = h(U)$; $\{\cdots\}$ denotes the terms which will disappear after integration with respect to $z \in R$.

On the other hand, we multiply (3.3) by $2w_2(U)(\varphi_t - s\varphi_z)$ to obtain

$$2w_2(U)(\varphi_t - s\varphi_z) \cdot L(\varphi) = -2Fw_2(U)(\varphi_t - s\varphi_z), \quad (4.4)$$

which gives

$$\begin{aligned}&[aw_2\varphi_z^2 + w_2(\varphi_t - s\varphi_z)^2]_t + (2w_2 + sw_{2z})(\varphi_t - s\varphi_z)^2 \\ &\quad + saw_{2z}\varphi_z^2 + 2f'(U)w_2\varphi_z(\varphi_t - s\varphi_z) + 2aw_{2z}\varphi_z(\varphi_t - s\varphi_z) \\ &\quad - [sw_2(\varphi_t - s\varphi_z)^2 + 2aw_2\varphi_z(\varphi_t - s\varphi_z) + asw_2\varphi_z^2]_z \\ &= -2Fw_2(\varphi_t - s\varphi_z). \end{aligned} \quad (4.5)$$

Hence, the combination $(4.3) \times \frac{1}{2} + (4.5)$ yields

$$\begin{aligned}&\{E_1(\varphi, (\varphi_t - s\varphi_z)) + E_3(\varphi_z)\}_t + E_2(\varphi_z, (\varphi_t - s\varphi_z)) \\ &\quad + E_4(\varphi) + \{\cdots\}_z = -2Fw_2\left[\frac{1}{2}\varphi + (\varphi_t - s\varphi_z)\right], \end{aligned} \quad (4.6)$$

where

$$E_1(\varphi, (\varphi_t - s\varphi_z)) = w_2\left[(\varphi_t - s\varphi_z)^2 + \varphi(\varphi_t - s\varphi_z) + \frac{1}{2}\left(1 + s\frac{w_{2z}}{w_2}\right)\varphi^2\right], \quad (4.7)$$

$$\begin{aligned}E_2(\varphi, (\varphi_t - s\varphi_z)) = w_2\Bigg[&\left(1 + s\frac{w_{2z}}{w_2}\right)(\varphi_t - s\varphi_z)^2 \\ &+ 2\left(f'(U) + a\frac{w_{2z}}{w_2}\right)\varphi_z(\varphi_t - s\varphi_z) + a\left(1 + s\frac{w_{2z}}{w_2}\right)\varphi_z^2\Bigg], \end{aligned} \quad (4.8)$$

$$E_3(\varphi_z) = aw_2\varphi_z^2, \quad (4.9)$$

$$E_4(\varphi) = -\frac{1}{2}(w_2h)''(U)U_z\varphi^2. \quad (4.10)$$

Since

$$\left|\frac{w_{2z}}{w_2}\right| = O(1)\frac{|u_+ - u_-|}{a - s^2} \ll 1 \quad \text{as } a \gg 1 \text{ or } |u_+ - u_-| \gg 1,$$

see (2.16), we have

$$1 + 2s\frac{w_2(U)_z}{w_2(U)} \geqq c_1 > 0, \tag{4.11}$$

for some positive constant c_1, and also

$$a\left(1 + s\frac{w_2(U)_z}{w_2(U)}\right)^2 - \left(f'(U) + a\frac{w_2(U)_z}{w_2(U)}\right)^2 \geqq c_2 > 0, \tag{4.12}$$

for some positive constant c_2 provided $a \gg 1$ or $|u_+ - u_-| \ll 1$ and the fact (1.3) $a > f'(U)^2$. Thus, the discriminants D_i $(i = 1, 2)$ of E_i $(i = 1, 2)$ are negative, that is,

$$D_1 :\equiv 1 - 4 \times \frac{1}{2}\left(1 + s\frac{w_{2z}}{w_2}\right) = -\left(1 + 2s\frac{w_2(U)_z}{w_2(U)}\right) \leqq -c_1 < 0, \tag{4.13}$$

and

$$D_2 :\equiv 4\left(f'(U) + a\frac{w_2(U)_z}{w_2(U)}\right)^2 - 4a\left(1 + s\frac{w_2(U)_z}{w_2(U)}\right)^2 \leqq -4c_2 < 0. \tag{4.14}$$

These facts imply that

$$E_1(\varphi, (\varphi_t - s\varphi_z)) \geqq c_3 w_2(U)\varphi^2 + c_4 w_2(U)(\varphi_t - s\varphi_z)^2 \tag{4.15}$$

and

$$E_2(\varphi_z, (\varphi_t - s\varphi_z)) \geqq c_5 w_2(U)\varphi_z^2 + c_6\, w_2(U)(\varphi_t - s\varphi_z)^2 \tag{4.16}$$

for some positive constants c_i, $i = 3, 4, 5, 6$.

On the other hand, (2.17) gives us that

$$-(w_2 h)''(U)U_z \geqq c_7 w_2(U)$$

for some positive constant c_7, which implies

$$E_4(\varphi) \geqq c_7 w_2(U)\varphi^2. \tag{4.17}$$

Hence, we have by (4.15)–(4.17)

$$E_1(\varphi, (\varphi_t - s\varphi_z)) + E_3(\varphi_z) \geqq c_8(|\varphi(t)|^2_{1,w_2} + |\varphi_t(t)|^2_{w_2}) \tag{4.18}$$

and

$$E_2(\varphi_z, (\varphi_t - s\varphi_z)) + E_4(\varphi) \geqq c_9(|\varphi(t)|^2_{1,w_2} + |\varphi_t(t)|^2_{w_3}) \tag{4.19}$$

for some positive constants c_8 and c_9.

Integrating (4.6) over $[0, t] \times R$, and making use of (4.18), (4.19) and

$|F| = O(1)|\varphi_z|^2$, we have

$$|\varphi(t)|_{1,w_2}^2 + |\varphi_t(t)|_{w_2}^2 + \frac{c_9}{c_8}\int_0^t (|\varphi(\tau)|_{1,w_2}^2 + |\varphi_t(\tau)|_{w_2}^2)\,d\tau$$
$$\leqq C\left(|\varphi_0|_{1,w_2}^2 + |\varphi_1|_{w_2}^2 + N_1(t)\int_0^t |(\varphi_z, \varphi_t)(\tau)|_{w_2}^2\,d\tau\right). \tag{4.20}$$

Letting $0 < \theta < c_9/c_8$ and $N_1(t) < (c_9 - c_9\theta)/c_8 C$, we obtain (4.1) from (4.20). This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Assume that* $a \gg 1$ *or* $|u_+ - u_-| \ll 1$, *and* (1.3) *holds. Then*:

$$|\partial_z\varphi(t)|_{1,w_2}^2 + |\partial_z\varphi_t(t)|_{w_2}^2 + \theta\int_0^t (|\partial_z\varphi(\tau)|_{1,w_2}^2 + |\partial_z\varphi_t(\tau)|_{w_2}^2)\,d\tau$$
$$\leqq C(|\varphi_0|_{2,w_2}^2 + |\varphi_1|_{1,w_2}^2) \tag{4.21}$$

for $t \in [0, T]$ *provided* $N_1(t) \ll 1$.

Proof. We first differentiate the equation (3.3) once with respect to z, and multiply it by $2w_2(U)\varphi_z$ and $2w_2(U)\,\partial_z(\varphi_t - s\varphi_z)$ respectively, that is,

$$2w_2(U)\varphi_z \cdot \partial_z L(\varphi) = -2w_2(U)\varphi_z F_z \tag{4.22}$$

and

$$2w_2(U)\,\partial_z(\varphi_t - s\varphi_z) \cdot \partial_z L(\varphi) = -2w_2(U)\,\partial_z(\varphi_t - s\varphi_z)F_z. \tag{4.23}$$

Combining $(4.22) \times \frac{1}{2} + (4.23)$, and integrating it over $[0, T] \times R$ and using (4.1) gives us the desired estimate (4.21) in the same way as in Lemma 4.2. The details are omitted here. □

In a similar way, differentiating the equation (3.3) twice with respect to z, and multiplying it by $2w_2(U)\,\partial_z^2\varphi$ and $2w_2(U)\,\partial_z^2(\varphi_t - s\varphi_z)$, respectively, we then integrate it over $[0, t] \times R$ and make use of Lemmas 4.1 and 4.2 to obtain the higher derivatives estimate as follows:

LEMMA 4.3. *Assume that* $a \gg 1$ *or* $|u_+ - u_-| \ll 1$, *and* (1.3) *hold. Then*:

$$|\partial_{zz}\varphi(t)|_{1,w_2}^2 + |\partial_z^2\varphi_t(t)|_{w_2}^2 + \theta\int_0^t (|\partial_z^2\varphi(\tau)|_{1,w_2}^2 + |\partial_z^2\varphi_t(\tau)|_{w_2}^2)\,d\tau$$
$$\leqq C(|\varphi_0|_{3,w_2}^2 + |\varphi_1|_{2,w_2}^2) \tag{4.24}$$

for $t \in [0, T]$ *provided* $N_1(t) \ll 1$.

Thus, we have the *a-priori* estimate as follows:

LEMMA 4.4. *Assume that* $a \gg 1$ *or* $|u_+ - u_-| \ll 1$, *and* (1.3) *holds. Then*:

$$|\varphi(t)|_{3,w_2}^2 + |\varphi_t(t)|_{2,w_2}^2 + \theta\int_0^t (|\varphi(\tau)|_{3,w_2}^2 + |\varphi_t(\tau)|_{2,w_2}^2)\,d\tau \leqq C(|\varphi_0|_{3,w_2}^2 + |\varphi_1|_{2,w_2}^2) \tag{4.25}$$

and

$$|\varphi(t)|^2_{3,w_2}+|\varphi_t(t)|^2_{2,w_2}\leqq Ce^{-\theta t}(|\varphi_0|^2_{3,w_2}+|\varphi_1|^2_{2,w_2}) \tag{4.26}$$

for $t\in[0,T]$, *provided* $N_1(t)\ll 1$.

Proof. Combining Lemmas 4.1–4.3 yields (4.25). Based on (4.25) and by applying Gronwall's inequality, we can obtain the decay rate (4.26). □

Proof of Theorem 3.1. Applying the method of continuous extension, the *a-priori* estimate of Lemma 4.4 together with the local existence result then implies Theorem 3.1. □

5. Algebraic time decay rates

In this section, we intend to improve the algebraic time decay rates for both the nondegenerate and degenerate cases in [**9**], by a method similar to that used in [**17**]. We focus here only on the nondegenerate case $f'(u_+)<s<f'(u_-)$, since the degenerate case $f'(u_+)=s<f'(u_-)$ can be treated similarly.

Let

$$\bar{u}=\frac{u_++u_-}{2}\in(u_+,u_-)$$

and z^* be a unique number in R such that $U(z^*)=\bar{u}$. Denote also

$$K(z,t):\equiv(1+t)^\gamma\langle(z-z^*)/a\rangle^\beta w_1(U),$$

where $\langle(z-z^*)/a\rangle=\sqrt{1+(z-z^*)^2/a^2}$.

By multiplying (3.3) by $2K(z,t)\varphi$ and $2K(z,t)(\varphi_t-s\varphi_z)$, respectively, to give

$$2K(z,t)\varphi\cdot L(\varphi)=2FK(z,t)\varphi \tag{5.1}$$

and

$$2K(z,t)(\varphi_t-s\varphi_z)\cdot L(\varphi)=2FK(z,t)(\varphi_t-s\varphi_z), \tag{5.2}$$

and combining $(5.1)\times\frac{1}{2}+(5.2)$, the authors in [**9**] showed the following lemma:

LEMMA 5.1 [**9**]. *There is a positive constant* ε_5 *such that*

$$N_2(T):\equiv\sup_{t\in[0,T]}|(\varphi,\varphi_z,\varphi_t)(t)|_\alpha<\varepsilon_5.$$

Then it holds that, for $t\in[0,T]$,

$$(1+t)^\gamma|(\varphi,\varphi_z,\varphi_t)(t)|^2_\beta+\int_0^t\{\beta(1+\tau)^\gamma|\varphi(\tau)|^2_{\beta-1}+(1+\tau)^\gamma|(\varphi_z,\varphi_t)(\tau)|^2_\beta\}\,d\tau$$
$$\leqq C\left\{|(\varphi,\varphi_z,\varphi_t)(0)|^2_\beta+\gamma\int_0^t(1+\tau)^{\gamma-1}|(\varphi,\varphi_z,\varphi_t)(\tau)|^2_\beta\,d\tau+\beta\int_0^t(1+\tau)^\gamma\|\varphi_z(\tau)\|^2\,d\tau\right\} \tag{5.3}$$

for any $\gamma \geqq 0$ *and* $\beta \in [0, \alpha]$, *and*

$$(1+t)^{\gamma}|(\varphi, \varphi_z, \varphi_t)(t)|^2_{\alpha-\gamma} + (\alpha-\gamma)\int_0^t (1+\tau)^{\gamma}|\varphi(\tau)|^2_{\alpha-\gamma-1}\,d\tau$$
$$+\int_0^t (1+\tau)^{\gamma}|(\varphi_z, \varphi_t)(\tau)|^2_{\alpha-\gamma}\,d\tau \leqq C|(\varphi, \varphi_z, \varphi_t)(0)|^2_{\alpha} \tag{5.4}$$

for γ *being an integer in* $[0, \alpha]$.

If α is integral, then by taking $\gamma = \alpha$ (5.4) gives

$$(1+t)^{\alpha}\|(\varphi, \varphi_z, \varphi_t)(t)\|^2 + \int_0^t (1+\tau)^{\alpha}\|(\varphi_z, \varphi_t)(\tau)\|^2\,d\tau \leqq C|(\varphi, \varphi_z, \varphi_t)(0)|^2_{\alpha},$$

which reduces to our desired decay rate.

If α is not integral, then taking $\beta = 0$ in (5.3) yields

$$(1+t)^{\gamma}\|(\varphi, \varphi_z, \varphi_t)(t)\|^2 + \int_0^t (1+\tau)^{\gamma}\|(\varphi_z, \varphi_t)(\tau)\|^2\,d\tau$$
$$\leqq C\left\{\|(\varphi, \varphi_z, \varphi_t)(0)\|^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|(\varphi, \varphi_z, \varphi_t)(\tau)\|^2\,d\tau\right\}. \tag{5.5}$$

Also, taking $\gamma = [\alpha]$ in (5.4) yields

$$(1+t)^{[\alpha]}|(\varphi, \varphi_z, \varphi_t)(t)|^2_{\alpha-[\alpha]} + (\alpha-[\alpha])\int_0^t (1+\tau)^{[\alpha]}|\varphi(\tau)|^2_{\alpha-[\alpha]-1}\,d\tau$$
$$+\int_0^t (1+\tau)^{[\alpha]}|(\varphi_z, \varphi_t)(\tau)|^2_{\alpha-[\alpha]}\,d\tau \leqq C\{|(\varphi, \varphi_z, \varphi_t)(0)|^2_{\alpha}\}. \tag{5.6}$$

We are going to estimate the last term in (5.5) by making use of (5.6) and Hölder's inequality:

$$\int_0^t (1+t)^{\gamma-1}\|(\varphi, \varphi_z, \varphi_t)(\tau)\|^2\,d\tau$$
$$= \int_0^t (1+t)^{\gamma-1}\int_R \langle (z-z^*)/a\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)-(\alpha-[\alpha])([\alpha]+1-\alpha)}$$
$$\times ((\varphi, \varphi_z, \varphi_t)(z,\tau))^{2(([\alpha]+1-\alpha)+(\alpha-[\alpha]))}\,dz\,d\tau$$
$$\leqq \int_0^t (1+\tau)^{\gamma-1}\left(\int_R \langle (z-z^*)/a\rangle^{(\alpha-[\alpha])}((\varphi, \varphi_z, \varphi_t)(z,\tau))^2\,dz\right)^{[\alpha]+1-\alpha}$$
$$\times\left(\int_R \langle (z-z^*)/a\rangle^{-([\alpha]+1-\alpha)}((\varphi, \varphi_z, \varphi_t)(z,\tau))^2\,dz\right)^{\alpha-[\alpha]}d\tau$$
$$= \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)}\{(1+\tau)^{[\alpha]}|(\varphi, \varphi_z, \varphi_t)(\tau)|^2_{\alpha-[\alpha]}\}^{[\alpha]+1-\alpha}$$
$$\times\{(1+\tau)^{[\alpha]}|(\varphi, \varphi_z, \varphi_t)(\tau)|^2_{\alpha-[\alpha]-1}\}^{\alpha-[\alpha]}\,d\tau$$

$$
\begin{aligned}
&\leqq C|(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^{2([\alpha]+1-\alpha)} \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)} \\
&\quad \times \{(1+\tau)^{[\alpha]}|(\varphi, \varphi_z, \varphi_t)(\tau)|^2_{\alpha-[\alpha]-1}\}^{\alpha-[\alpha]}\, d\tau \\
&\leqq C|(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^{2([\alpha]+1-\alpha)} \left\{ \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)/([\alpha]+1-\alpha)}\, d\tau \right\}^{[\alpha]+1-\alpha} \\
&\quad \times \left\{ \int_0^t (1+\tau)^{[\alpha]}|(\varphi, \varphi_z, \varphi_t)(\tau)|^2_{\alpha-[\alpha]-1}\, d\tau \right\}^{\alpha-[\alpha]} \\
&\leqq C|(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2 \left\{ \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)/([\alpha]+1-\alpha)}\, d\tau \right\}^{[\alpha]+1-\alpha}.
\end{aligned}
\tag{5.7}
$$

Now putting

$$
\gamma = \alpha + \varepsilon \tag{5.8}
$$

for any $\varepsilon > 0$, a simple computation yields the last integral in (5.7) as

$$
\begin{aligned}
\int_0^t (1+\tau)^{-([\alpha]+1-\gamma)/([\alpha]+1-\alpha)}\, d\tau &= \frac{[\alpha]+1-\alpha}{\varepsilon}[(1+t)^{\varepsilon/([\alpha]+1-\alpha)} - 1] \\
&\leqq C(1+t)^{\varepsilon/([\alpha]+1-\alpha)}.
\end{aligned}
\tag{5.9}
$$

Applying (5.8), (5.9) and (5.7) to (5.5) gives us

$$
\begin{aligned}
&(1+t)^{\alpha+\varepsilon}\|(\varphi, \varphi_z, \varphi_t)(t)\|^2 + \int_0^t (1+\tau)^{\alpha+\varepsilon}\|(\varphi_z, \varphi_t)(\tau)\|^2\, d\tau \\
&\quad \leq C\bigg\{ \|(\varphi, \varphi_z, \varphi_t)(0)\|^2 \\
&\qquad + (\alpha+\varepsilon)|(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2 \left(\int_0^t (1+\tau)^{-([\alpha]+1-\gamma)/([\alpha]+1-\alpha)} d\tau \right)^{[\alpha]+1-\alpha} \bigg\} \\
&\quad \leq C\{\|(\varphi, \varphi_z, \varphi_t)(0)\|^2 + (1+t)^\varepsilon |(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2\} \\
&\quad \leq C(1+t)^\varepsilon\{\|(\varphi, \varphi_z, \varphi_t)(0)\|^2 + |(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2\}.
\end{aligned}
$$

Since

$$
\|(\varphi, \varphi_z, \varphi_t)(0)\| \leq |(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2,
$$

we have the following lemma:

LEMMA 5.2. *For any* $\varepsilon > 0$,

$$
\begin{aligned}
(1+t)^{\alpha+\varepsilon}\|(\varphi, \varphi_z, \varphi_t)(t)\|^2 &+ \int_0^t (1+\tau)^{\alpha+\varepsilon}\|(\varphi_z, \varphi_t)(\tau)\|^2\, d\tau \\
&\leqq C(1+t)^\varepsilon|(\varphi, \varphi_z, \varphi_t)(0)|_\alpha^2;
\end{aligned}
\tag{5.10}
$$

namely,

$$(1+t)^{\alpha}\|(\varphi,\varphi_z,\varphi_t)(t)\|^2+(1+t)^{-\varepsilon}\int_0^t(1+\tau)^{\alpha+\varepsilon}\|(\varphi_z,\varphi_t)(\tau)\|^2\,d\tau$$
$$\leqq C|(\varphi,\varphi_z,\varphi_t)(0)|_{\alpha}^2. \tag{5.11}$$

Using Lemma 5.2, in a similar way as in Lemma 5.2 we can show the following estimate. The details are omitted.

LEMMA 5.3. *For any* $\varepsilon>0$,

$$(1+t)^{\alpha}\|\partial_z^l(\varphi,\varphi_z,\varphi_t)(t)\|^2+(1+t)^{-\varepsilon}\int_0^t(1+\tau)^{\alpha+\varepsilon}\|\partial_z^l(\varphi_z,\varphi_t)(\tau)\|^2\,d\tau$$
$$\leqq C(\|(\varphi,\varphi_z,\varphi_t)(0)\|_2^2+|(\varphi,\varphi_z,\varphi_t)(0)|_{\alpha}^2),\quad l=1,2. \tag{5.12}$$

Thus, we can easily prove the *a-priori* estimate as follows by Lemmas 5.2 and 5.3:

LEMMA 5.4. *For any* $\varepsilon>0$,

$$(1+t)^{\alpha}\|\varphi(t)\|_3^2+\|\varphi_t(t)\|_2^2+(1+t)^{-\varepsilon}\int_0^t(1+\tau)^{\alpha+\varepsilon}(\|\varphi(\tau)\|_3^2+\|\varphi_t(\tau)\|_2^2)\,d\tau$$
$$\leqq C(\|\varphi_0\|_3^2+\|\varphi_1\|_2^2+|\varphi_0|_{1,\alpha}^2+|\varphi_1|_{\alpha}^2). \tag{5.13}$$

This lemma implies Theorem 3.2.

6. Remark

The same method can be applied to the study of convergence rates in both algebraic and exponential forms for the strong detonation travelling waves for a viscous combustion model

$$(u+qz)_t+f(u)_x=\beta u_{xx},$$
$$z_t=-K\varphi(u)z,$$

where q, β and K are given positive constants. The stability of travelling waves for this model was studied in [**11**] and a convergence rate was given in [**18**].

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