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BEST ASYMPTOTIC PROFILE FOR HYPERBOLIC p-SYSTEM WITH DAMPING*

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Abstract. For the 2×2 hyperbolic p-system with damping, its asymptotic profile has been traditionally regarded as the self-similar solution, the so-called diffusion wave, to the corresponding parabolic equation by Darcy's law, and the convergence of the original solution to the diffusion wave has been intensively studied by many people. However, by a deep observation and a heuristic analysis, we realize that the best asymptotic profile for the solution to the Cauchy problem of the p-system with damping is a particular solution to the corresponding nonlinear parabolic equation with a specified initial data, and we further show the convergence rates to this particular asymptotic profile. These new rates are much better than the existing convergence rates to the diffusion waves. Our results essentially improve and develop the previous studies. Finally, some numerical simulations are carried out, which also confirm our theoretical results.

Key words. p-system of hyperbolic conservation laws, linear damping, porous media equations, diffusion waves, asymptotic behavior, convergence rates

AMS subject classifications. 35L50, 35L60, 35L65, 76R50

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1. Introduction.

For the model of the compressible flow through porous media with dissipative external force field, it can be described in Lagrangian coordinates as the p-system of hyperbolic conservation laws with damping (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]),

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \end{cases} \quad (x,t) \in R \times R_+. \tag{1.1}$$

Here, $v(x,t) > 0$ is the specific volume, $u(x,t)$ is the velocity, and $p(v)$ is the pressure (a smooth function) such that $p(v) > 0$, $p'(v) < 0$. A typical example, in the case of a polytropic gas, is $p(v) = v^{-\gamma}$ with $\gamma \geq 1$. The external term $-\alpha u$ with $\alpha > 0$, called a linear damping, appears in the momentum equation.

In this paper, we consider the damped p-system (1.1) with the following initial data:

$$(v,u)(x,t)|_{t=0} = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty, \tag{1.2}$$

where $v_\pm > 0$ and $u_\pm$ are constant states.

According to Darcy's law, the solutions $(v,u)(x,t)$ of (1.1) and (1.2) are expected to behave time asymptotically as the solutions to the following (parabolic) porous media equations

$$\begin{cases} \bar v_t - \bar u_x = 0, \\ p(\bar v)_x = -\alpha \bar u, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar v_t = -\frac{1}{\alpha} p(\bar v)_{xx}, \\ p(\bar v)_x = -\alpha \bar u, \end{cases} \quad (x,t) \in R \times R_+, \tag{1.3}$$

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with

$$(\bar{v}, \bar{u})(x,t) \to (v_\pm, 0), \quad \text{as } x \to \pm\infty. \tag{1.4}$$

In fact, by setting the following scalings to the variables

$$t = \bar{t}/\varepsilon^2, \quad x = \bar{x}/\varepsilon, \quad v = \bar{v}, \quad u = \varepsilon \bar{u}$$

for $0 < \varepsilon \ll 1$, we then scale the damped p-system (1.1) to the new system (still denoting $\bar{t}$ and $\bar{x}$ as t and x, respectively)

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \varepsilon^2 \bar{u}_t + p(\bar{v})_x = -\alpha \bar{u}. \end{cases}$$

Neglecting the small term $\varepsilon^2 \bar{u}_t$, we derive the asymptotic state equations as (1.3).

For the system (1.3), with a different initial value, the solution $(\bar{v}, \bar{u})(x,t)$ is different. In other words, the asymptotic profiles for the system (1.1) and (1.2) are not unique. The interesting and natural questions are, which particular solution of (1.3) will be the best asymptotic profile, and what will be the better convergence rates for the original solution to the best asymptotic profile? To solve these problems will be our target in the present paper.

It is well known that the asymptotic state equations (1.3) satisfying (1.4) possess self-similar solutions $(\bar{v}, \bar{u})(x,t) = (\phi, \psi)(x/\sqrt{t})$, which are usually called the (nonlinear) diffusion waves for the damped p-system (1.1). The existence of such self-similar solutions (which are unique up to shift) had been proved by Duyn and van Peletier [2] early in 1970. The convergence to the diffusion waves was first studied by Hsiao and Liu [3, 4] in 1992. In these pioneering frameworks, they showed that the solutions of (1.1) and (1.2) converge to a specified diffusion wave in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1/2}, t^{-1/2})$. The correction functions were ingeniously constructed to eliminate the gap yielded by the original solutions and the diffusion waves in L^2-space. Then, by taking more detailed but elegant energy estimates, Nishihara [22] succeeded in improving the convergence rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$, when the initial perturbation is in H^3. Furthermore, when the initial perturbation is in $H^3 \cap L^1$, by constructing an approximate Green function with the energy method together, Nishihara, Wang, and Yang [27] completely improved the rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$, which are optimal in the sense comparing with the decay of the solution to the heat equation. Later then, Wang and Yang [30] further gave a more detailed piecewise form on the convergence rates. These convergence results need the initial perturbation around the specified diffusion wave and the wave strength both to be sufficiently small. Such restrictions were then partially released by Zhao [31], where the initial perturbation in the L^∞ sense can be arbitrarily large, but its first derivative still needs to be small, which implies that the wave must also be weak. For the 2×2 quasi-linear p-system but still with linear damping, the convergence with some decay rates was obtained by Li and Saxton [13]. We note also that, for the 3×3 system with linear damping, the convergence to the corresponding diffusion waves was studied by Hsiao and Luo [5] and by Pan [28], respectively, and then improved by Nishihara and Nishikawa in [25]. Regarding the p-system with linear damping and vacuum, the corresponding convergence has been deeply studied by Huang and Pan [10], and by Huang, Marcati, and Pan [9], respectively. Recently, when the damping effect is nonlinear, the convergence to the diffusion waves has been investigated by Zhu and Jiang [33, 34] for $u_+ = u_- = 0$, and further studied by Mei [20] and C.-K. Lin, C.-T. Lin, and Mei [14] with $u_+ \neq u_-$ for the initial value problem and the initial-boundary value problem, respectively. For other studies related to this topic such as the convergence in weak sense, as well as the convergence with boundary effect, etc., we refer to [1, 6, 7, 8, 11, 15, 16, 17, 23, 26, 29, 32] and the references therein.

We notice that the optimality of the convergence rates obtained in [27, 30] for the Cauchy problem case and in [17] for the initial-boundary condition case is just based on the selection of the self-similar solutions $(\bar{v}, \bar{u})(x/\sqrt{t})$ as the asymptote profile. Namely, for one selected self-similar solution as the asymptotic profile, the convergence rates $O(1)(t^{-1}, t^{-3/2})$ in the L^∞ sense are the best and cannot be improved at all. However, the selected self-similar solutions to the parabolic system (1.3) are not the best asymptotic profiles for the damped p-system (1.1) (intuitively, see the numerical results presented in section 5). As testified also by Nishihara [23] in the special case $v_+ = v_-$, by specifying the location of diffusion waves, the convergence rates can be further improved as $O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$ in the L^∞ sense.

In this paper, we consider the general case $v_+ \neq v_-$. By a heuristic analysis, we realize that the best asymptotic profile of the damped p-system (1.1) and (1.2) is a particular solution $(\bar{v}, \bar{u})(x, t)$ to the corresponding nonlinear parabolic equations (1.3) with specified initial data (see (2.23) below), which satisfies also (1.4). We will further derive the convergence rates $O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$ in the L^∞ sense, which are the same as that obtained in [23] for the case $v_+ = v_-$, but much better than the rates obtained in the previous works for the case $v_+ \neq v_-$. The reason causing us to have an extra $\log t$ is that the decay in the nonlinear perturbation term for $\bar{v}_t$ in L^1 is only as $\|\bar{v}_t(t)\|_{L^1} = O(1)t^{-1}$.

The paper is organized as follows. In section 2, we give a heuristic analysis to find the best asymptotic profile for the system (1.1) and (1.2), and build up the working equations, and then we state our main results. To prove the convergence of the solution for the system (1.1) and (1.2) to its best asymptotic profile, as well as the convergent rates, is the main effort in the present paper, which will be done in section 3. As a supplement to section 3, we prove the existence of the best asymptotic profile and derive its properties of decay rates in section 4. Finally, in section 4, we carry out some numerical computations, which confirm also our theoretical results.

Notations. We give some notations as follows. Throughout the paper, $C > 0$ denotes a generic constant which may change its value from line to line or even in the same line, while $C_i > 0$ $(i = 0, 1, 2, \ldots)$ represents a specific constant. $L^2(R)$ is the space of square integrable functions, and $H^k(R)$ $(k \geq 0)$ is the Sobolev space of L^2-functions $f(x)$ whose derivatives $\frac{d^i}{dx^i}f$, $i = 1, \ldots, k$, also belong to $L^2(R)$. The norms of $L^2(R)$ and $H^k(R)$ are denoted as $\|f\|_{L^2(R)}$ and $\|f\|_{H^k(R)}$, respectively. For the sake of simplicity, we also denote $\|(f, g, h)\|^2_{L^2(R)} = \|f\|^2_{L^2(R)} + \|g\|^2_{L^2(R)} + \|h\|^2_{L^2(R)}$. Let $T > 0$ and let $\mathcal{B}$ be a Banach space. We denote by $C^0([0, T]; \mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0, T]$, and $L^2([0, T]; \mathcal{B})$ as the space of $\mathcal{B}$-valued L^2 functions on $[0, T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0, \infty)$ are defined similarly.

2. Best asymptotic profile and main theorems. In what follows, we are first going to make a heuristic analysis, and then we will show how to find the best asymptotic profile and what will be the best asymptotic profile. Finally, we will establish the working equations and state our main convergence results with the improved decay rates.

We first investigate the asymptotic behavior of $(v,u)(x,t)$ at $x=\pm\infty$. Let us take the limits as $x\to\pm\infty$ to the damped p-system (1.1), and note that u_x and $p(v)_x$ will vanish at $x=\pm\infty$ due to the boundedness of $(v,u)(x,t)$, and then we have

$$\begin{cases} \frac{d}{dt}v(\pm\infty,t)=0, \\ \frac{d}{dt}u(\pm\infty,t)=-\alpha u(\pm\infty,t), \\ (v,u)(\pm\infty,0)=(v_0,u_0)(\pm\infty)=(v_\pm,u_\pm), \end{cases} \tag{2.1}$$

which can be exactly solved as

$$\begin{cases} v(\pm\infty,t)=v_\pm, \\ u(\pm\infty,t)=u_\pm e^{-\alpha t}, \end{cases} \qquad t\geq 0. \tag{2.2}$$

As shown in section 1, the expected asymptotic profile of (1.1) is the (parabolic) porous media equation (1.3) with the restriction (1.4). In the same way as (2.1), it can be easily verified that the solution $(\bar v,\bar u)$ of (1.3) satisfies

$$(\bar v,\bar u)(\pm\infty,t)=(v_\pm,0). \tag{2.3}$$

From $(1.1)_1$ and $(1.3)_1$, we have

$$(v-\bar v)_t=(u-\bar u)_x.$$

Integrating the above equation with respect to x over $(-\infty,\infty)$ and noting (2.2) and (2.3), we then get

$$\frac{d}{dt}\int_{-\infty}^{\infty}(v-\bar v)(x,t)dx=u(+\infty,0)-u(-\infty,t)=(u_+-u_-)e^{-\alpha t}\neq 0.$$

In order to eliminate the gap $u(+\infty,0)-u(-\infty,t)=(u_+-u_-)e^{-\alpha t}$, we need to construct a pair of correction functions $(\hat v,\hat u)(x,t)$, which was first introduced by Hsiao and Liu in [3]. Namely, let $\hat u(x,t)$ be the solution to the following equation:

$$\frac{d}{dt}\hat u(x,t)=-\alpha\hat u(x,t)\quad\text{with}\quad \hat u(\pm\infty,t)=u_\pm e^{-\alpha t},$$

and then it can easily be solved as

$$\hat u(x,t)=m(x)e^{-\alpha t}, \tag{2.4}$$

where $m(x)$ needs to be $m(\pm\infty)=u_\pm$. For this, we construct it as

$$m(x)=u_-+(u_+-u_-)\int_{-\infty}^{x}m_0(y)dy, \tag{2.5}$$

and

$$m_0(x)\in C_0^\infty(R)\quad\text{with}\quad\int_{-\infty}^{\infty}m_0(x)dx=1. \tag{2.6}$$

Here, $m_0(x)$ may be chosen in many different ways. One example for $m_0(x)$ is

$$m_0(x)=\frac{A}{\sqrt{\pi}}e^{-A^2x^2} \tag{2.7}$$

for some constant $A > 0$. Now setting $\hat{v}(x,t)$ such that

$$\hat{v}_t = \hat{u}_x,$$

one then immediately obtains

$$\hat{v}(x,t) = -\frac{u_+ - u_-}{\alpha} m_0(x) e^{-\alpha t}. \tag{2.8}$$

Thus, the correction functions $(\hat{v}, \hat{u})(x,t)$ satisfy

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha \hat{u}, \\ (\hat{v}, \hat{u})|_{x=\pm\infty} = (0,\ u_\pm e^{-\alpha t}). \end{cases} \tag{2.9}$$

Now we are going to look for the best asymptotic profile $(\bar{v}, \bar{u})(x,t)$. Traditionally, we take the self-similar solution $(\bar{v}, \bar{u}) = (\phi, \psi)((x+\bar{x})/\sqrt{1+t})$ as the asymptotic profile for the solution $(v,u)(x,t)$. Here, in order to avoid the singularity, we use $(\phi,\psi)(x/\sqrt{1+t})$ to replace $(\phi,\psi)(x/\sqrt{t})$, and $\bar{x}$ is a shift constant. However, this is not the best asymptotic profile. In fact, as shown in [3, 22, 27], one can expect only

$$\int_{-\infty}^{\infty} (v - \bar{v} - \hat{v})(x,t)dx = 0,$$

but

$$\int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx \neq 0.$$

This implies that the selected asymptotic profile $(\bar{v}, \bar{u}) = (\phi, \psi)((x+\bar{x})/\sqrt{1+t})$ is not optimal. In order to get the best asymptotic profile $(\bar{v}, \bar{u})$, we need technically to construct a particular solution $(\bar{v}, \bar{u})(x,t)$ such that

$$\int_{-\infty}^{\infty} (v - \bar{v} - \hat{v})(x,t)dx = 0, \tag{2.10}$$

$$\int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx = 0, \qquad \text{for all } \ t \geq 0. \tag{2.11}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dy = 0, \tag{2.12}$$

Let $(\bar{v}, \bar{u})(x,t)$ be the expected particular solution of the Cauchy problem

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \end{cases} \tag{2.13}$$

where the initial data $\bar{v}_0(x)$ satisfy

$$\bar{v}_0(x) \to v_\pm, \qquad \text{as } \ x \to \pm\infty,$$

and will be specified later. Furthermore, let us consider the correction function $(\hat{v}, \hat{u}) = (\hat{v}, \hat{u})(x + x_0, t)$ with a shift x_0 which will also be determined later. Namely,

$$(\hat{v}, \hat{u})(x + x_0, t) = \Big(-\frac{u_+ - u_-}{\alpha} m_0(x + x_0) e^{-\alpha t},\ m(x + x_0) e^{-\alpha t} \Big), \tag{2.14}$$

and satisfies

$$\|(\hat{v},\hat{u})(t)\|_{L^\infty(R)} = O(1)|u_+ - u_-|(e^{-\alpha t}, e^{-\alpha t}). \tag{2.15}$$

From (1.1), (2.9), and (2.13), we have

$$\begin{cases} (v-\bar{v}-\hat{v})_t - (u-\bar{u}-\hat{u})_x = 0, \\ (u-\bar{u}-\hat{u})_t + (p(v)-p(\bar{v}))_x = -\alpha(u-\bar{u}-\hat{u}) + \frac{1}{\alpha}p(\bar{v})_{xt}. \end{cases} \tag{2.16}$$

Integrating $(2.16)_2$ with respect to x over R, and noting that $p(v)\to p(v_\pm)$, $p(\bar{v})\to p(v_\pm)$ as $x\to\pm\infty$ (see (2.2) and (2.3)), we have

$$\begin{aligned} &\frac{d}{dt}\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx \\ &= \Big\{-[p(v)-p(\bar{v})] + \frac{1}{\alpha}p(\bar{v})_t\Big\}\Big|_{x=-\infty}^{\infty} - \alpha\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx \\ &= -\alpha\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx, \end{aligned}$$

which can be solved as (by using $(1.3)_2$, (2.14), and (2.3))

$$\begin{aligned} &\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx \\ &= e^{-\alpha t}\int_{-\infty}^{\infty}[u_0(x)-\bar{u}(x,0)-\hat{u}(x+x_0,0)]dx \\ &= e^{-\alpha t}\int_{-\infty}^{\infty}\Big[u_0(x)+\frac{1}{\alpha}p(\bar{v}(x,0))_x - m(x+x_0)\Big]dx \\ &= e^{-\alpha t}\Big\{\int_{-\infty}^{\infty}[u_0(x)-m(x+x_0)]dx + \frac{1}{\alpha}\int_{-\infty}^{\infty}p(\bar{v}(x,0))_x dx\Big\} \\ &= e^{-\alpha t}\Big\{\Big[\int_{-\infty}^{\infty}[u_0(x)-m(x+x_0)]dx\Big] + \frac{1}{\alpha}[p(v_+)-p(v_-)]\Big\} \\ &=: e^{-\alpha t}I(x_0). \end{aligned} \tag{2.17}$$

Notice that

$$\begin{aligned} I'(x_0) &= \frac{d}{dx_0}\Big\{\Big[\int_{-\infty}^{\infty}[u_0(x)-m(x+x_0)]dx\Big] + \frac{1}{\alpha}[p(v_+)-p(v_-)]\Big\} \\ &= -\int_{-\infty}^{\infty}m'(x+x_0)dx = -[m(\infty)-m(-\infty)] \\ &= -(u_+-u_-), \end{aligned}$$

and then integrating the above equation with respect to x_0 yields

$$I(x_0) - I(0) = -(u_+-u_-)x_0. \tag{2.18}$$

Since we expect

$$\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u})(x,t)dx = 0,$$

which is, due to (2.17), equivalent to $I(x_0) = 0$, then we get from (2.18) that

$$x_0 := \frac{1}{u_+ - u_-} I(0) = \frac{1}{u_+ - u_-} \left\{ \int_{-\infty}^{\infty} [u_0(x) - m(x)]dx + \frac{1}{\alpha}[p(v_+) - p(v_-)] \right\}. \tag{2.19}$$

Thus, we arrive at

$$\int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx = 0, \qquad t \geq 0. \tag{2.20}$$

Now we return to $(2.16)_1$. Integrating it over $(-\infty, x]$ yields

$$\frac{d}{dt} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dy = (u - \bar{u} - \hat{u})(x,t).$$

Again, integrating the above equation over $(-\infty, \infty)$ with respect to x and noting (2.20), we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dydx = \int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})(x,t)dx = 0.$$

Then, integrating the above equation with respect to t, we further obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dydx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x} [v_0(y) - \bar{v}_0(y) - \hat{v}(y,0)]dydx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x} \left[v_0(y) - \bar{v}_0(y) + \frac{u_+ - u_-}{\alpha} m_0(y + x_0) \right] dydx. \end{aligned} \tag{2.21}$$

Now we select the particular initial data $\bar{v}_0(x)$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} \left[v_0(y) - \bar{v}_0(y) + \frac{u_+ - u_-}{\alpha} m_0(y + x_0) \right] dydx = 0; \tag{2.22}$$

as a particular example, we may take

$$\bar{v}_0(x) := v_0(x) + \frac{u_+ - u_-}{\alpha} m_0(x + x_0), \tag{2.23}$$

and then we can expect, from (2.21), that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(y,t)dydx = 0, \quad t \geq 0. \tag{2.24}$$

Thus, we have found such a particular solution $(\bar{v}, \bar{u})(x,t)$ satisfying (2.10)–(2.12).

Defining

$$V(x,t) := \int_{-\infty}^{x} \int_{-\infty}^{y} (v - \bar{v} - \hat{v})(z,t)dzdy, \tag{2.25}$$

$$U(x,t) := \int_{-\infty}^{x} (u - \bar{u} - \hat{u})(y,t)dy, \tag{2.26}$$

$$V_0(x) := \int_{-\infty}^{x} \int_{-\infty}^{y} [v_0(z) - \bar{v}_0(z) - \hat{v}(z,0)]dzdy, \tag{2.27}$$

$$U_0(x) := \int_{-\infty}^{x} [u_0(y) - \bar{u}(y,0) - \hat{u}(y,0)]dy, \tag{2.28}$$

namely,

$$V_{xx} = v - \bar{v} - \hat{v}, \qquad U_x = u - \bar{u} - \hat{u},$$

and applying them to (2.16), we finally establish a new working system of equations

$$\begin{cases} V_t - U = 0, \\ U_t + p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) = -\alpha U + \frac{1}{\alpha} p(\bar{v})_t, \\ (V, U)|_{t=0} = (V_0, U_0)(x), \end{cases} \tag{2.29}$$

namely,

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v}) V_x)_x = -\alpha U - F_1 - F_2, \\ (V, U)|_{t=0} = (V_0, U_0(x)), \end{cases} \tag{2.30}$$

where

$$F_1 := -\frac{1}{\alpha} p(\bar{v})_t, \tag{2.31}$$

$$F_2 := [p(\bar{v} + \hat{v} + V_{xx}) - p(\bar{v}) - p'(\bar{v}) V_{xx}] - p'(\bar{v})_x V_x, \tag{2.32}$$

which implies, by Taylor's formula, that

$$|F_1| = O(1)|\bar{v}_t|, \tag{2.33}$$

$$|F_2| = O(1)[|\hat{v}| + |\hat{v} + V_{xx}|^2 + |\bar{v}_x V_x|]. \tag{2.34}$$

Now we are going to state our convergence results. First of all, we have the following existence of the solution $(\bar{v}, \bar{u})(x, t)$ (the best asymptotic profile) for the system (2.13) and (2.23).

PROPOSITION 2.1 (property of the best asymptotic profile). *Suppose $\bar{v}_0 - \bar{\bar{v}}_0 \in H^m(R)$, $\bar{v}_0'(x) \in H^{m-1}(R)$, where $m \geq 3$ is an integer, and*

$$\bar{\bar{v}}_0(x) = \begin{cases} v_+, & x \geq 0, \\ v_-, & x < 0. \end{cases} \tag{2.35}$$

Then the unique solution $(\bar{v}, \bar{u})(x, t)$ for (2.13) *and* (2.23) *globally exists and satisfies*

$$\|\partial_x^k \bar{v}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} - \frac{1}{4})}, \qquad k = 1, 2, 3, \ldots, m, \tag{2.36}$$

$$\|\partial_x^k \bar{u}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{1}{4})}, \qquad k = 0, 1, 2, \ldots, m-1, \tag{2.37}$$

$$\|\partial_x^k \bar{v}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{3}{4})}, \qquad k = 0, 1, 2, \ldots, m-2, \tag{2.38}$$

$$\|\partial_x^k \bar{u}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{5}{4})}, \qquad k = 0, 1, 2, \ldots, m-3. \tag{2.39}$$

Furthermore, if $\bar{v}_0 - \bar{\bar{v}}_0 \in L^1(R) \cap H^m(R)$ and $\bar{v}_0'(x) \in H^{m-1}(R)$, then

$$\|\partial_x^k \bar{v}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{1}{4})}, \qquad k = 1, 2, 3, \ldots, m, \tag{2.40}$$

$$\|\partial_x^k \bar{u}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{3}{4})}, \qquad k = 0, 1, 2, \ldots, m-, \tag{2.41}$$

$$\|\partial_x^k \bar{v}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{5}{4})}, \qquad k = 0, 1, 2, \ldots, m-2, \tag{2.42}$$

$$\|\partial_x^k \bar{u}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2} + \frac{7}{4})}, \qquad k = 0, 1, 2, \ldots, m-3. \tag{2.43}$$

THEOREM 2.2 (convergence). *Let $\bar v_0(x)$ be chosen such that* (2.22) *holds, and let $(V_0, U_0) \in H^3(R)\times H^2(R)$. There exists a number $\varepsilon_1 > 0$, when the initial perturbation $(V_0, U_0)(x)$ and the wave strength $\delta := |v_+ - v_-| + |u_+ - u_-|$ are suitably small such that $\delta + \|V_0\|_{H^3(R)} + \|U_0\|_{H^2(R)} \le \varepsilon_1$, and then the global solution $(V, U)(x, t)$ of* (2.29) *(or* (2.30)*) uniquely exists and satisfies*

$$V(x,t) \in C^k(0,\infty; H^{3-k}(R)),\ k = 0,1,2,3,\ U(x,t) \in C^k(0,\infty; H^{2-k}(R)),\ k = 0,1,2,$$

and

$$\begin{aligned} &\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k U(t)\|^2_{L^2(R)} \\ &+ \int_0^t \left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right] ds \\ &\le C(\|V_0\|^2_{H^3(R)} + \|U_0\|^2_{H^2(R)} + \delta). \end{aligned} \tag{2.44}$$

Moreover, if $(V_0, U_0) \in (H^3(R) \cap L^1(R)) \times (H^2(R) \cap L^1(R))$, then the rates can be further improved as follows:

$$\|\partial_x^k V(t)\|_{L^2(R)} \le C\delta_1(1+t)^{-\frac14 - \frac k2}\log(2+t), \qquad k = 0,1,2,3, \tag{2.45}$$

$$\|\partial_x^k U(t)\|_{L^2(R)} \le C\delta_1(1+t)^{-\frac54 - \frac k2}\log(2+t), \qquad k = 0,1,2, \tag{2.46}$$

where $\delta_1 := \|V_0\|^2_{H^3(R)} + \|U_0\|^2_2 + \delta$.

Notice that $V_{xx} = v - \bar v - \hat v$, $U_x = u - \bar u - \hat u$, and use (2.14) and (2.15), i.e., $|\hat v(x,t)|, |\hat u(x,t)| \sim O(1)e^{-\alpha t}$, and Sobolev's embedding inequalities $\|f\|_{L^\infty(R)} \le \sqrt{2}\|f\|^{1/2}_{L^2(R)}\|f_x\|^{1/2}_{L^2(R)}$, and we immediately obtain the following convergence rates.

COROLLARY 2.3 (convergence to the best asymptotic profile). *Under the conditions in Theorem* 2.2*, the system* (1.1) *and* (1.2) *possesses a uniquely global solution $(v, u)(x, t)$, which converges to its best asymptotic profile $(\bar v, \bar u)(x, t)$ defined in* (2.13) *and* (2.23) *in the form of*

$$\|(v - \bar v)(t)\|_{L^\infty(R)} = O(1)(1+t)^{-3/2}\log(2+t), \tag{2.47}$$

$$\|(u - \bar u)(t)\|_{L^\infty(R)} = O(1)(1+t)^{-2}\log(2+t). \tag{2.48}$$

Remark 2.4.

1. In Theorem 2.2, the restriction on the initial perturbation $(V_0, U_0)(x) \in H^3(R) \cap H^2(R)$ with the facts $V_{0xx} = v_0(x) - \bar v_0(x) - \hat v(0, x)$ and $U_{0x} = u_0(x) - \bar u_0(x) - \hat u(0, x) = u_0(x) + \frac{1}{\alpha}p'(\bar v_0(x))\bar v_0'(x) - \hat u(0, x)$ implies that the initial perturbation is most like

$$\begin{aligned} v_0(x) - \bar v_0(x) - \hat v(0,x) &= O(1)(1+|x|)^{-2-\gamma}, \qquad \gamma > \frac12, \\ u_0(x) - \bar u_0(x) - \hat u(0,x) &= O(1)(1+|x|)^{-1-\gamma}, \qquad \gamma > \frac12. \end{aligned}$$

2. The rates shown in (2.47) and (2.48) are the same as that shown by Nishihara [23] for the case $v_+ = v_-$, and much better than all existing convergence rates obtained in the previous works. However, it seems hard for us to remove $\log t$

in (2.47) and (2.48), because the nonlinear source term $||F_1 = O(1)|\bar{v}_t|$ (see (2.31) and (2.33) above) decays in the L^1 sense as $\|F_1\|_{L^1} = O(1)\|\bar{v}_t(t)\|_{L^1} = O(1)t^{-1}$, after integration which leads us to have an extra $\log t$ (see (3.33) below).

3. Convergence to the best asymptotic profile. Substituting $U = V_t$ (the first equation of (2.30)) into the second equation of (2.30), we obtain

$$\begin{cases} V_{tt} + \alpha V_t + (p'(\bar{v})V_x)_x = -F_1 - F_2, & (x,t) \in R \times R_+, \\ (V, V_t)|_{t=0} = (V_0, U_0)(x), & x \in R. \end{cases} \tag{3.1}$$

It is well known that Theorem 2.2 can be proved by the classical continuation method based on the local existence and the a priori estimates. The local existence of the solution for (3.1) can be obtained by the standard iteration method (cf. [19, 21]). To establish the a priori estimates for the solution usually is technical, which will be the main effort in this section.

We are going to prove Theorem 2.2 in two steps. The first step is to establish the a priori estimates for (2.44) by the basic energy method. The second step is to improve the decay rates to that given in (2.45) by using the approximate Green function method.

3.1. Proof of the a priori estimates (2.44). Let $T \in [0, \infty]$; we define

$$N(T)^2 := \sup_{0 \le t \le T} \left\{ \sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(t)\|^2_{L^2(R)} + \sum_{k=0}^{2} (1+t)^{k+2} \|\partial_x^k V_t(t)\|^2_{L^2(R)} \right\}. \tag{3.2}$$

We first establish the following basic energy estimate.

LEMMA 3.1 (basic energy estimates). *It follows that*

$$\begin{aligned} &\|(V, V_x, V_t)(t)\|^2_{L^2(R)} + \int_0^t \|(V_x, V_t)(s)\|^2_{L^2(R)} ds \\ &\le C[\|V_0\|^2_{H^3(R)} + \|U_0\|^2_{H^2(R)} + N(T)\delta + N(T)^3] := \bar{C}_0. \end{aligned} \tag{3.3}$$

Proof. Multiplying (3.1) by $\lambda V + V_t$ with small constant $0 < \lambda \ll 1$, we have

$$\{E_1(V, V_x, V_t)\}_t + E_2(V_x, V_t) + \{E_3(x,t)\}_x = -(F_1 + F_2)(\lambda V + V_t), \tag{3.4}$$

where

$$E_1(V, V_x, V_t) := \frac{1}{2}V_t^2 + \lambda V V_t + \frac{\alpha\lambda}{2}V^2 - \frac{1}{2}p'(\bar{v})V_x^2, \tag{3.5}$$

$$E_2(V_x, V_t) := (\alpha - \lambda)V_t^2 + \left[-\lambda p'(\bar{v}) + \frac{1}{2}p''(\bar{v})\bar{v}_t\right] V_x^2, \tag{3.6}$$

$$E_3(x,t) := p'(\bar{v})V_x(\lambda V + V_t). \tag{3.7}$$

Notice that $-p'(\bar{v}) \ge C_0 > 0$ for some positive constant C_0. When $\lambda \ll 1$ and $|v_+ - v_-| \ll 1$, then the following estimates hold:

$$C_1(V^2 + V_x^2 + V_t^2) \le E_1(V, V_x, V_t) \le C_2(V^2 + V_x^2 + V_t^2), \tag{3.8}$$

$$C_3(V_x^2 + V_t^2) \le E_2(V_x, V_t), \tag{3.9}$$

for some positive constants C_i $(i = 1, 2, 3)$. Integrating (3.4) over $R \times [0, t]$ with respect to x and t, we have

$$\begin{aligned}&\|(V, V_x, V_t)(t)\|^2_{L^2(R)} + \int_0^t \|(V_x, V_t)(s)\|^2_{L^2(R)} ds\\ &\leq C(\|V_0\|^2_{H^1(R)} + \|U_0\|^2_{L^2(R)}) + C\int_0^t\int_R (|F_1| + |F_2|)(\lambda|V| + |V_t|)dxds.\end{aligned} \tag{3.10}$$

Noting (2.33) and (2.34), as well as the decay rates of $\hat{v}$ and $\bar{v}$ shown in (2.15), and (2.40)–(2.43), respectively, in particular, $\|V(s)\|_{L^\infty(R)} \leq \sqrt{2}\|V(s)\|^{1/2}_{L^2(R)}\|V_x(s)\|^{1/2}_{L^2(R)}$ $\leq CN(T)(1+s)^{-1/2}$, and $\|V_t(s)\|_{L^\infty(R)} \leq CN(T)(1+s)^{-3/2}$, we can estimate

$$\begin{aligned}&\int_0^t\int_R (|F_1| + |F_2|)(\lambda|V| + |V_t|)dxds\\ &\leq C\int_0^t\int_R (|\bar{v}_t| + |\hat{v}| + |\hat{v}|^2 + |V_{xx}|^2 + |\bar{v}_x V_x|)(\lambda|V| + |V_t|)dxds\\ &\leq C\int_0^t [\|V(s)\|_{L^2(R)} + \|V_t(s)\|_{L^2(R)}][\|\bar{v}_t(s)\|_{L^2(R)} + \|\hat{v}(s)\|_{L^2(R)}\\ &\qquad + \|\hat{v}(s)\|^2_{L^4(R)}]ds\\ &\quad + C\int_0^t [\|V(s)\|_{L^\infty(R)} + \|V_t(s)\|_{L^\infty(R)}][\|V_{xx}(s)\|^2_{L^2(R)}\\ &\qquad + \|\bar{v}_x(s)\|^2_{L^2(R)} + \|V_x(s)\|^2_{L^2(R)}]ds\\ &\leq CN(t)|v_+ - v_-|\int_0^t [(1+s)^{-5/4} + e^{-\alpha s} + e^{-2\alpha s}]ds\\ &\quad + CN(t)\int_0^t [(1+s)^{-1/2} + (1+s)^{-3/2}]|v_+ - v_-|(1+s)^{-3/2}ds\\ &\quad + CN(t)\int_0^t [(1+s)^{-1/2} + (1+s)^{-3/2}][\|V_{xx}(s)\|^2_{L^2(R)} + \|V_x(s)\|^2_{L^2(R)}]ds\\ &\leq CN(t)\delta + CN(t)^3\int_0^t [(1+s)^{-1/2} + (1+s)^{-3/2}][(1+s)^{-2} + (1+s)^{-1}]ds\\ &\leq CN(t)\delta + CN(t)^3.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10) yields

$$\begin{aligned}&\|(V, V_x, V_t)(t)\|^2_{L^2(R)} + \int_0^t \|(V_x, V_t)(s)\|^2_{L^2(R)} ds\\ &\leq C[\|V_0\|^2_{H^1(R)} + \|U_0\|^2_{L^2(R)} + N(t)\delta + N(t)^3].\end{aligned} \tag{3.12}$$

The proof is complete. □

Similarly, using (3.3) and multiplying (3.1) by $(1+t)V_t$, we can further obtain the following energy estimate.

LEMMA 3.2 (decay rate for V_x). *It holds that*

$$(1+t)\|(V_x, V_t)(t)\|^2_{L^2(R)} + \int_0^t (1+s)\|V_t(s)\|^2_{L^2(R)} ds \leq \bar{C}_0. \tag{3.13}$$

Furthermore, differentiating (3.1) with respect to x and multiplying it by $(1+t)(\lambda V_x + V_{xt})$ for $0 < \lambda \ll 1$, and integrating the resultant equation over $R \times [0,t]$ with respect to x and t, noting also the boundedness obtained in (3.3), we then have

$$(1+t)\|(V_x, V_{xt}, V_{xx})(t)\|^2_{L^2(R)} + \int_0^t (1+s)\|(V_{xt}, V_{xx})(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.14}$$

Again, by taking $\int_0^t \int_R \partial_x(3.1) \cdot (1+s)^k V_{xt}\ dx\ ds$ ($k = 0, 1, 2$), and using (3.13) and (3.14), we obtain the decay rate for V_{xx} as follows.

LEMMA 3.3 (decay rate for V_{xx}). *It holds that*

$$(1+t)^2\|(V_{xx}, V_{xt})(t)\|^2_{L^2(R)} + \int_0^t (1+s)^2\|V_{xt}(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.15}$$

Similarly, differentiating (3.1) with respect to x, multiplying it by $-(1+t)^k(V_{xxxt} + \lambda V_{xxx})$, $k = 0, 1, 2$, and then integrating the resultant equation over $R \times [0,t]$ with respect to x and t, we can prove

$$(1+t)^2\|(V_{xx}, V_{xxx}, V_{xxt})(t)\|^2_{L^2(R)} + \int_0^t \|(V_{xxt}, V_{xxx})(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.16}$$

Thus, by taking $\int_0^t \int_R \partial_x(3.1) \cdot [-(1+s)^k V_{xxxt}]\ dx\ ds$ ($k = 0, 1, 2, 3$) and using (3.16), we further obtain the decay rates for V_{xxx}.

LEMMA 3.4 (decay rate for V_{xxx}). *It holds that*

$$(1+t)^3\|(V_{xxx}, V_{xxt})(t)\|^2_{L^2(R)} + \int_0^t (1+s)^2\|V_{xxt}(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.17}$$

Now we are going to establish the energy decay estimates for V_t, V_{xt}, and V_{xxt}. Differentiating (3.1) with respect to t yields

$$V_{ttt} + \alpha V_{tt} - (p'(\bar{v})V_x)_{xt} = -F_{1t} - F_{2t}. \tag{3.18}$$

Multiplying (3.18) by $(1+t)^k(V_{tt} + \lambda V_t)$, $0 < \lambda \ll 1$, and $k = 0, 1, 2$, and integrating it with respect to x and t, then we can obtain the decay rate for V_t as follows.

LEMMA 3.5 (decay rate for V_t). *It holds that*

$$(1+t)^2\|(V_t, V_{xt}, V_{tt})(t)\|^2_{L^2(R)} + \int_0^t (1+s)^2\|(V_{xt}, V_{tt})(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.19}$$

Next, multiplying (3.18) by $-(1+t)^k V_{tt}$, $k = 0, 1, 2, 3$, and integrating the resultant equation with respect to x and t, we obtain the decay rate for V_{xt}.

LEMMA 3.6 (decay rate for V_{xt}). *It holds that*

$$(1+t)^3\|(V_{xt}, V_{tt})(t)\|^2_{L^2(R)} + \int_0^t (1+s)^3\|V_{tt}(s)\|^2_{L^2(R)} ds \le \bar{C}_0. \tag{3.20}$$

Finally, in a similar way as before, differentiating (3.1) with respect to x, multiplying it by $(1+t)^4 V_{xtt}$, and then integrating the resultant equation over $R \times [0,t]$ with respect to x and t, also using Lemmas 3.2–3.6, we can get the following decay rate for V_{xxt}.

LEMMA 3.7 (decay rate for V_{xxt}). *It holds that*

$$(1+t)^4\|(V_{xxt},V_{xtt})(t)\|^2_{L^2(R)} + \int_0^t [(1+s)^3\|V_{xxt}(s)\|^2_{L^2(R)} + (1+s)^4\|V_{xtt}(s)\|^2_{L^2(R)}]ds \le \bar{C}_0. \tag{3.21}$$

Based on Lemmas 3.1–3.7, adding (3.3), (3.13), (3.15), (3.17), (3.19), (3.20), and (3.21), we have proved that

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k U(t)\|^2_{L^2(R)}\\ &+\int_0^t\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds\\ &\le C(\|U_0\|^2_{H^2(R)} + N(T)\delta + N(T)^3).\end{aligned} \tag{3.22}$$

So, for all $0 \le t \le T$, we have

$$\begin{aligned}&\sup_{0\le t\le T}\left\{\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k U(t)\|^2_{L^2(R)}\right\}\\ &+\int_0^T\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds\\ &\le C(\|U_0\|^2_{H^2(R)} + N(T)\delta + N(T)^3),\end{aligned} \tag{3.23}$$

namely,

$$\begin{aligned}&N(T)^2 + \int_0^T\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds\\ &\le C(\|U_0\|^2_{H^2(R)} + N(T)\delta + N(T)^3),\end{aligned} \tag{3.24}$$

which is equivalent to

$$\begin{aligned}&(1-CN(T))N(T)^2 + \int_0^T\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)}\right.\\ &\qquad\left.+\sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds\\ &\le C(\|U_0\|^2_{H^2(R)} + N(T)\delta).\end{aligned} \tag{3.25}$$

Let $N(T) \le 1$ be small enough such that

$$1 - CN(T) > 0,$$

and then (3.25) implies

$$\begin{aligned}&N(T)^2 + \int_0^T\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)} + \sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds\\ &\le C(\|U_0\|^2_{H^2(R)} + \delta).\end{aligned} \tag{3.26}$$

This implies (2.44), namely, the following lemma.

LEMMA 3.8 (the a priori estimates). *It holds that*

$$\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(t)\|^2_{L^2(R)}+\sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k U(t)\|^2_{L^2(R)}$$
$$+\int_0^t\left[\sum_{k=0}^{3}(1+s)^{k-1}\|\partial_x^k V(s)\|^2_{L^2(R)}+\sum_{k=0}^{2}(1+s)^{k+1}\|\partial_x^k U(s)\|^2_{L^2(R)}\right]ds$$
$$\le C[\|V_0\|^2_{H^3(R)}+\|U_0\|^2_{H^2(R)}+\delta] \tag{3.27}$$

provided $N(T)\ll 1$.

3.2. Proof of the decay rates (2.45) and (2.46). Now, we are going to adopt the technique of approximating the Green function to obtain the improved decay rates (2.45) and (2.46).

As in [27], we rewrite Eq. (3.1) as

$$\alpha V_t-(a(x,t)V_x)_x=-F_1-F_2-V_{tt}, \tag{3.28}$$

where $a(x,t)=-p'(\bar v(x,t))\ge C_0>0$, and construct a minimizing Green function as (4.23), i.e.,

$$G(x,t;y,s)=\Big(\frac{\alpha}{4\pi a(x,t)(t-s)}\Big)^{1/2}\exp\Big(\frac{-\alpha(x-y)^2}{4a(y,s)(t-s)}\Big). \tag{3.29}$$

Then the solution of (3.28) can be written in the integral form

$$V(x,t)=\int_{-\infty}^{\infty}G(x,t;y,0)V_0(y)dy$$
$$+\alpha^{-1}\int_0^t\int_{-\infty}^{\infty}G(x,t;y,s)[-F_1(y,s)-F_2(y,s)-V_{ss}(y,s)]dyds$$
$$+\int_0^t\int_{-\infty}^{\infty}R_G(x,t;y,s)V(y,s)dyds, \tag{3.30}$$

where

$$R_G(x,t;y,s):=G_s(x,t;y,s)+\alpha^{-1}\{a(y,s)G_y(x,t;y,s)\}_y.$$

Differentiating (3.30) with respect to x and t, we have, for $l\le 1, k+l\le 3$,

$$\begin{aligned}\partial_t^l\partial_x^k V(x,t)\ =\ &\partial_t^l\partial_x^k\int_{-\infty}^{\infty}G(x,t;y,0)V_0(y)dy\\ &-\alpha^{-1}\partial_t^l\partial_x^k\int_0^t\int_{-\infty}^{\infty}G(x,t;y,s)F_1(y,s)dyds\\ &-\alpha^{-1}\partial_t^l\partial_x^k\int_0^t\int_{-\infty}^{\infty}G(x,t;y,s)F_2(y,s)dyds\\ &-\alpha^{-1}\partial_t^l\partial_x^k\int_0^t\int_{-\infty}^{\infty}G(x,t;y,s)V_{ss}(y,s)dyds\\ &+\partial_t^l\partial_x^k\int_0^t\int_{-\infty}^{\infty}R_G(x,t;y,s)V(y,s)dyds\\ =:\ &I_1^{l,k}+I_2^{l,k}+I_3^{l,k}+I_4^{l,k}+I_5^{l,k}.\end{aligned} \tag{3.31}$$

Based on the estimates obtained in (2.44), and by applying the decay of the approximating Green function $G(x,t;y,s)$, as exactly shown in [27], we can further prove, for $l \le 1, k+l \le 3$,

$$\|I_1^{l,k}\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.32}$$

$$\|I_2^{l,k}\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}\log(2+t), \tag{3.33}$$

$$\|I_3^{l,k}\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.34}$$

$$\|I_4^{l,k}\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.35}$$

$$\|I_5^{l,k}\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}. \tag{3.36}$$

Here, since $\|F_1\|_{L^1} = O(1)\|\bar{v}_t(t)\|_{L^1} = O(1)t^{-1}$, after integration, which leads us to have an extra $\log t$ in (3.33). The detail of proof for (3.32)–(3.35) is omitted.

Combining (3.31)–(3.36), we then obtain the improved rates (2.45) and (2.46).

LEMMA 3.9. *It holds that*

$$\|\partial_x^k V(t)\|_{L^2(R)} = O(1)(1+t)^{-\frac{1}{4}-\frac{k}{2}}\log(2+t), \qquad k=0,1,2,3,$$
$$\|\partial_x^k V_t(t)\|_{L^2(R)} = O(1)(1+t)^{-\frac{5}{4}-\frac{k}{2}}\log(2+t), \qquad k=0,1,2.$$

4. Property of the best asymptotic profile. This section is devoted to the proof of Proposition 2.1; namely, we are going to prove the existence and uniqueness of the particular solution $(\bar{v},\bar{u})(x,t)$ to (2.13), as well as its optimal decay rates (2.40)–(2.43).

Let $\phi(x/\sqrt{1+t})$ be the self-similar solution to the nonlinear equation (1.3) and (1.4), i.e.,

$$\begin{cases} \phi_t = -\frac{1}{\alpha}p(\phi)_{xx}, \\ \phi(\frac{x}{\sqrt{1+t}}) \to v_\pm \text{ as } x \to \pm\infty. \end{cases} \tag{4.1}$$

As shown in [2], the self-similar solution $\phi(x/\sqrt{1+t})$ exists uniquely (up to shift) and satisfies

$$\|\partial_x^k \phi(t)\|_{L^p(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k}{2}-\frac{1}{2p})}, \quad k=1,2,\dots, 1\le p\le\infty, \tag{4.2}$$

$$\|\partial_x^k \phi_t(t)\|_{L^p(R)} = O(1)|v_+ - v_-|(1+t)^{-(\frac{k+2}{2}-\frac{1}{2p})}, \quad k=0,1,2,\dots, 1\le p\le\infty. \tag{4.3}$$

Let $\bar{v}(x,t)$ be the solution for the following initial value problem:

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx}, \\ \bar{v}|_{t=0} = \bar{v}_0(x) \to v_\pm \text{ as } x \to \pm\infty, \end{cases} \tag{4.4}$$

with the specified initial value $\bar{v}_0(x)$ given in (2.23), and consider the perturbation of $\bar{v}(x,t)$ around $\phi(\frac{x+x_1}{\sqrt{1+t}})$ for some shift x_1:

$$(\bar{v}-\phi)_t = -\frac{1}{\alpha}[p(\bar{v}) - p(\phi)]_{xx}. \tag{4.5}$$

Integrating it with respect to (x,t) over $R\times[0,t]$, we get

$$\begin{aligned}
&\int_R \left[\bar v(x,t)-\phi\left(\frac{x+x_1}{\sqrt{1+t}}\right)\right]dx\\
&=\int_R[\bar v_0(x)-\phi(x+x_1)]dx-\frac{1}{\alpha}\int_0^t\int_R[p(\bar v)-p(\phi)]_{xx}dxds\\
&=\int_R[\bar v_0(x)-\phi(x+x_1)]dx\\
&=0 \qquad (4.6)
\end{aligned}$$

by selecting x_1 as

$$x_1=\frac{1}{v_+-v_-}\int_R[\bar v_0(x)-\phi(x)]dx. \tag{4.7}$$

Define

$$\Phi(x,t):=\int_{-\infty}^x\left[\bar v(x,t)-\phi\left(\frac{x+x_1}{\sqrt{1+t}}\right)\right]dx, \tag{4.8}$$

$$\Phi_0(x):=\int_{-\infty}^x[\bar v_0(x)-\phi(x+x_1)]dx, \tag{4.9}$$

namely,

$$\Phi_x=\bar v-\phi,\qquad \Phi_0'(x)=\bar v_0(x)-\phi(x+x_1).$$

Then $\Phi(x,t)$ satisfies

$$\begin{cases}\Phi_t=-\frac{1}{\alpha}[p(\Phi_x+\phi)-p(\phi)]_x,\\ \Phi|_{t=0}=\Phi_0(x),\end{cases}$$

i.e.,

$$\begin{cases}\alpha\Phi_t+(p'(\phi)\Phi_x)_x=-F_0,\\ \Phi|_{t=0}=\Phi_0(x),\end{cases} \tag{4.10}$$

where

$$F_0:=p(\Phi_x+\phi)-p(\phi)-p'(\phi)\Phi_x. \tag{4.11}$$

The existence and uniqueness of $\Phi(x,t)$ for (4.10) are given as follows.

LEMMA 4.1. *Suppose that* $\Phi_0\in H^m(R)$. *When* $\|\Phi_0\|_{H^m(R)}\ll 1$, *then the solution* $\Phi(x,t)$ *of* (4.10) *uniquely exists and satisfies*

$$\Phi\in C^0(0,+\infty;H^m(R)),\qquad \Phi_x\in L^2(0,+\infty;H^m(R)),$$

and

$$\sum_{k+2j=0}^m(1+t)^{k+2j}\|\partial_t^j\partial_x^k\Phi(t)\|^2_{L^2(R)}\le C\|\Phi_0\|^2_{H^m(R)}. \tag{4.12}$$

Proof. This can also be proved by the classical continuity-extension method based on the local existence and the a priori estimates (cf. [18, 19]). The local existence for (4.10) can be obtained by the iteration technique, which yields a Cauchy sequence and leads the existence of the solution within a local time t_0. The detail is omitted. Here, we give the proof of the a priori estimates.

For $T > 0$, we denote

$$N_0(T)^2 := \sup_{0\le t\le T} \sum_{k+2j=0}^{m} (1+t)^{k+2j} \|\partial_t^j \partial_x^k \Phi(t)\|^2_{L^2(R)}.$$

Multiplying (4.10) by Φ and integrating it with respect to (x,t) over $R\times[0,t]$, noting $-p'(\phi) \ge C_0 > 0$ for some positive constant C_0 and $|F_0| = O(1)|\Phi_x|^2$, we have

$$\|\Phi(t)\|^2_{L^2(R)} + \int_0^t \|\Phi_x(s)\|^2_{L^2(R)} ds \le C\|\Phi_0\|^2_{L^2(R)} + CN_0(t)\int_0^t \|\Phi_x(s)\|^2_{L^2(R)} ds,$$

which gives

$$\|\Phi(t)\|^2_{L^2(R)} + \int_0^t \|\Phi_x(s)\|^2_{L^2(R)} ds \le C\|\Phi_0\|^2_{L^2(R)} \tag{4.13}$$

provided $N_0(t) \ll 1$.

Similarly, by taking $\int_0^t \int_R \partial_x^k(4.10) \cdot \partial_x^k \Phi dxds$, $k \le m$, we can further prove

$$\|\partial_x^k\Phi(t)\|^2_{L^2(R)} + \int_0^t \|\partial_x^k\Phi_x(s)\|^2_{L^2(R)} ds \le C\|\Phi_0\|^2_{H^k(R)}, \qquad k\le m, \tag{4.14}$$

provided $N_0(t) \ll 1$.

Moreover, by taking $\int_0^t \int_R \partial_t^j\partial_x^k(4.10)\cdot(1+s)^{k+2j}\partial_t^j\partial_x^k\Phi dxds$, $0\le k+2j\le m$, $k = 0,1,2,\ldots,m$, $j = 0,1,2,\ldots,[\frac{m}{2}]$, and using (4.14), we obtain the following a priori estimates:

$$(1+t)^{k+2j}\|\partial_t^j\partial_x^k\Phi(t)\|^2_{L^2(R)} + \int_0^t (1+s)^{k+2j}\|\partial_t^j\partial_x^k\Phi_x(s)\|^2_{L^2(R)} ds \le C\|\Phi_0\|^2_{H^m(R)}, \tag{4.15}$$

provided $N_0(t) \ll 1$. Here, $0\le k+2j\le m$, $k = 0,1,2,\ldots,m$, $j = 0,1,2,\ldots,[\frac{m}{2}]$. Thus, the proof is complete. ∎

Notice that $\Phi_x = \bar{v} - \phi$ and

$$\|\partial_x^k \bar{v}_x(t)\|_{L^2(R)} \le \|\partial_x^k\Phi_{xx}(t)\|_{L^2(R)} + \|\partial_x^k\phi_x(t)\|_{L^2(R)},$$

as well as Lemma 4.1, (4.2), and (4.3), and we immediately prove the global existence of the particular solution $\bar{v}(x,t)$ for (4.4) with the same decay rates of $\phi(x/\sqrt{1+t})$:

$$\|\partial_x^k\bar{v}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|t^{-(\frac{k}{2}-\frac{1}{4})}, \qquad k = 1,2,\ldots,m, \tag{4.16}$$

$$\|\partial_x^k\bar{v}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|t^{-(\frac{k}{2}+\frac{3}{4})}, \qquad k = 0,1,2,\ldots,m-2. \tag{4.17}$$

Now we are going to improve the above rates to the optimal ones as follows.

LEMMA 4.2. *If $\bar{v}_0 - \bar{\bar{v}}_0 \in L^1(R)\cap H^2(R)$ and $\bar{v}_0'(x) \in H^{m-1}(R)$, then*

$$\|\partial_x^k\bar{v}(t)\|_{L^2(R)} = O(1)|v_+ - v_-|t^{-(\frac{k}{2}+\frac{1}{4})}, \qquad k = 1,2,\ldots,m, \tag{4.18}$$

$$\|\partial_x^k\bar{v}_t(t)\|_{L^2(R)} = O(1)|v_+ - v_-|t^{-(\frac{k}{2}+\frac{5}{4})}, \qquad k = 0,1,2,\ldots,m-2. \tag{4.19}$$

Proof. We use the approximate Green function method to prove (4.18) and (4.19).

Let $w(x,t) := \bar{v}_x(x,t)$. Then, differentiating (4.4) with respect to x yields

$$\begin{cases} \alpha w_t + (p'(\bar{v})w)_{xx} = 0, \\ w|_{t=0} = \bar{v}_0'(x), \end{cases} \tag{4.20}$$

namely,

$$\begin{cases} \alpha w_t - (a(x,t)w_x)_x = -H_x, \\ w|_{t=0} = \bar{v}_0'(x), \end{cases} \tag{4.21}$$

where $a(x,t) := -p'(\bar{v}(x,t)) \geq C_0 > 0$, and $H(x,t) := p''(\bar{v})w^2 = p''(\bar{v})\bar{v}_x^2$.

As shown in [27], we can construct an approximate Green function

$$G(x,t;y,s) = \Big(\frac{\alpha}{4\pi a(x,t)(t-s)}\Big)^{1/2} \exp\Big(-\frac{\alpha(x-y)^2}{4a(y,s)(t-s)}\Big). \tag{4.22}$$

Thus, the solution of (4.21) can be expressed as

$$\begin{aligned} w(x,t) = &\int_R G(x,t;y,0)\bar{v}_0'(y)dy - \frac{1}{\alpha}\int_0^t\int_R G(x,t;y,s)H_y(y,s)dyds \\ &+ \int_0^t\int_R R_G(x,t;y,s)w(y,s)dyds, \end{aligned} \tag{4.23}$$

where

$$R_G(x,t;y,s) := G_s(s,t;y,s) + \alpha^{-1}\{a(y,s)G_y(x,t;y,s)\}_y. \tag{4.24}$$

Let $M_\varepsilon(x)$ be the mollifier, and $\bar{\bar{v}}_\varepsilon(x) = (M_\varepsilon * \bar{v})(x)$. By using the smooth approximation technique and integrating (4.23) by parts, we have

$$\begin{aligned} w(x,t) &= \lim_{\varepsilon\to 0}\int_R G(x,t;y,0)[\bar{v}_0'(y) - \bar{\bar{v}}_\varepsilon'(y)]dy \\ &\quad + \frac{1}{\alpha}\int_0^t\int_R \partial_y G(x,t;y,s)H(y,s)dyds \\ &\quad + \int_0^t\int_R R_G(x,t;y,s)w(y,s)dyds \\ &= -\lim_{\varepsilon\to 0}\int_R \partial_y G(x,t;y,0)[\bar{v}_0(y) - \bar{\bar{v}}_\varepsilon(y)]dy \\ &\quad + \frac{1}{\alpha}\int_0^t\int_R \partial_y G(x,t;y,s)H(y,s)dyds \\ &\quad + \int_0^t\int_R R_G(x,t;y,s)w(y,s)dyds \\ &= -\int_R \partial_y G(x,t;y,0)[\bar{v}_0(y) - \bar{\bar{v}}]dy \\ &\quad + \frac{1}{\alpha}\int_0^t\int_R \partial_y G(x,t;y,s)H(y,s)dyds \\ &\quad + \int_0^t\int_R R_G(x,t;y,s)w(y,s)dyds. \end{aligned} \tag{4.25}$$

Differentiating $w(x,t)$ with respect to x and t, we have

$$\begin{aligned}\partial_t^j\partial_x^{k-1}\bar{v}_x(x,t) &= \partial_t^j\partial_x^{k-1}w(x,t)\\ &= -\partial_t^j\partial_x^{k-1}\int_R \partial_y G(x,t;y,0)[\bar{v}_0(y)-\bar{\bar{v}}]dy\\ &\quad+\frac{1}{\alpha}\partial_t^j\partial_x^{k-1}\int_0^t\int_R \partial_y G(x,t;y,s)H(y,s)dyds\\ &\quad+\partial_t^j\partial_x^{k-1}\int_0^t\int_R R_G(x,t;y,s)\bar{v}_y(y,s)dyds\\ &=: J_1^{l,k-1}+J_2^{l,k-1}+J_3^{l,k-1},\quad k=1,2,\dots,\ l=0,1,2,\dots.\end{aligned}\tag{4.26}$$

In the same way as [27], we can obtain

$$\|J_1^{l,k-1}\|_{L^2(R)} = O(1)|v_+-v_-|(1+t)^{-\frac{1}{4}-l-\frac{k}{2}},\tag{4.27}$$

$$\|J_2^{l,k-1}\|_{L^2(R)} = O(1)|v_+-v_-|(1+t)^{-\frac{1}{4}-l-\frac{k}{2}},\tag{4.28}$$

$$\|J_3^{l,k-1}\|_{L^2(R)} = O(1)|v_+-v_-|(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}.\tag{4.29}$$

Since the proof is really tedious but can be exactly followed as [27], we omit it all. Substituting (4.27)–(4.29) to (4.26), we obtain (4.18) and (4.19). The proof is completed. ∎

Finally, from (2.13), i.e., $\bar{u} = -\frac{1}{\alpha}p'(\bar{v})\bar{v}_x$, we then obtain from Lemma 4.2 the decay rates for $\bar{u}(x,t)$.

LEMMA 4.3. *It holds that*

$$\|\partial_x^k\bar{u}(t)\|_{L^2(R)} = O(1)|v_+-v_-|(1+t)^{-(\frac{k}{2}+\frac{3}{4})},\qquad k=0,1,2,\dots,m-1,\tag{4.30}$$

$$\|\partial_x^k\bar{u}_t(t)\|_{L^p(R)} = O(1)|v_+-v_-|(1+t)^{-(\frac{k}{2}+\frac{7}{4})},\qquad k=0,1,2,\dots,m-3.\tag{4.31}$$

Proof of Theorem. 2.1 Combining Lemmas 4.1–4.3, we have proved Proposition 2.1. ∎

5. Numerical computations. In this section, we present some numerical simulations, which also confirm our theoretical results.

Let $v_+ = 4$, $v_- = 2$, $u_+ = 1$, $u_- = -1$, $\alpha = 1$, and $p(v) = v^{-2}$. We choose the initial data as follows:

$$v_0(x) = \begin{cases}2, & \text{for } x<-10,\\ \frac{1}{10}x+3, & \text{for } -10\le x<10,\\ 4, & \text{for } x\ge 10,\end{cases}\tag{5.1}$$

and

$$u_0(x) = \begin{cases}-1, & \text{for } x<-10,\\ -\frac{9}{40000}x^2+\frac{1}{10}x+\frac{9}{400}, & \text{for } -10\le x\le 10,\\ 1, & \text{for } x\ge 10.\end{cases}\tag{5.2}$$

Then we select $m_0(x)$ as

$$m_0(x) = \begin{cases}0, & \text{for } x<-10,\\ \frac{1}{100}x+\frac{1}{10}, & \text{for } -10\le x<0,\\ -\frac{1}{100}x+\frac{1}{10}, & \text{for } 0\le x<10,\\ 0, & \text{for } x\ge 10,\end{cases}\tag{5.3}$$

which is continuous and compact supported and satisfies $\int_{-\infty}^{\infty} m_0(x)dx = 1$. Then

$$m(x) := u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)dy = \begin{cases} -1, & \text{for } x < -10, \\ \frac{1}{100}x^2 + \frac{1}{5}x, & \text{for } -10 \le x < 0, \\ -\frac{1}{100}x^2 + \frac{1}{5}x, & \text{for } 0 \le x < 10, \\ 1, & \text{for } x > 10. \end{cases} \tag{5.4}$$

By a straightforward calculation, it can be verified that (see (2.19))

$$x_0 := \frac{1}{u_+ - u_-} I(0) = \frac{1}{u_+ - u_-}\left\{\int_{-\infty}^{\infty} [u_0(x) - m(x)]dx + \frac{1}{\alpha}[p(v_+) - p(v_-)]\right\} = 0.$$

So, according to (2.23), we choose the initial data $\bar{v}_0(x)$ for the particular solution $\bar{v}(x,t)$ as follows:

$$\bar{v}_0(x) := v_0(x) + \frac{u_+ - u_-}{\alpha} m_0(x + x_0) = \begin{cases} 2, & \text{for } x < -10, \\ \frac{3}{25}x + \frac{16}{5}, & \text{for } -10 \le x < 0, \\ \frac{2}{25}x + \frac{16}{5}, & \text{for } 0 \le x < 10, \\ 4, & \text{for } x \ge 10. \end{cases} \tag{5.5}$$

It can also be verified that

$$\int_{-\infty}^{\infty} [u_0(x) - \bar{u}(x,0) - \hat{u}(x,0)]dx = \left[\int_{-\infty}^{\infty} [u_0(x) - m(x + x_0)]dx\right] + \frac{1}{\alpha}[p(v_+) - p(v_-)] = 0,$$

and

$$v_0(x) - \bar{v}_0(x) - \hat{v}(x,0) = v_0(x) - \bar{v}_0(x) - \frac{u_+ - u_-}{\alpha} m_0(x) = 0,$$

which ensure (2.17) and (2.24).

By using the central finite-difference scheme to the equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \\ (v,u)|_{t=0} = (\bar{v}_0, \bar{u}_0)(x), \end{cases} \tag{5.6}$$

and

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x), \end{cases} \tag{5.7}$$

and to the equations for the self-similar solution $(\phi, \psi)(\xi)$ with $\xi = x/\sqrt{1+t}$, i.e.,

$$\begin{cases} \alpha \xi \phi' - 2(p(\phi))'' = 0, \\ \psi(\xi) = -\alpha^{-1}\sqrt{1+t}(p(\phi(\xi)))', \\ \phi(\xi) \to v_\pm \quad \text{as } \xi \to \pm\infty, \end{cases} \tag{5.8}$$

respectively, where the initial data are chosen as in (5.1), (5.2), and (5.5), then we obtain the following numerical results (see Figure 5.1 below).

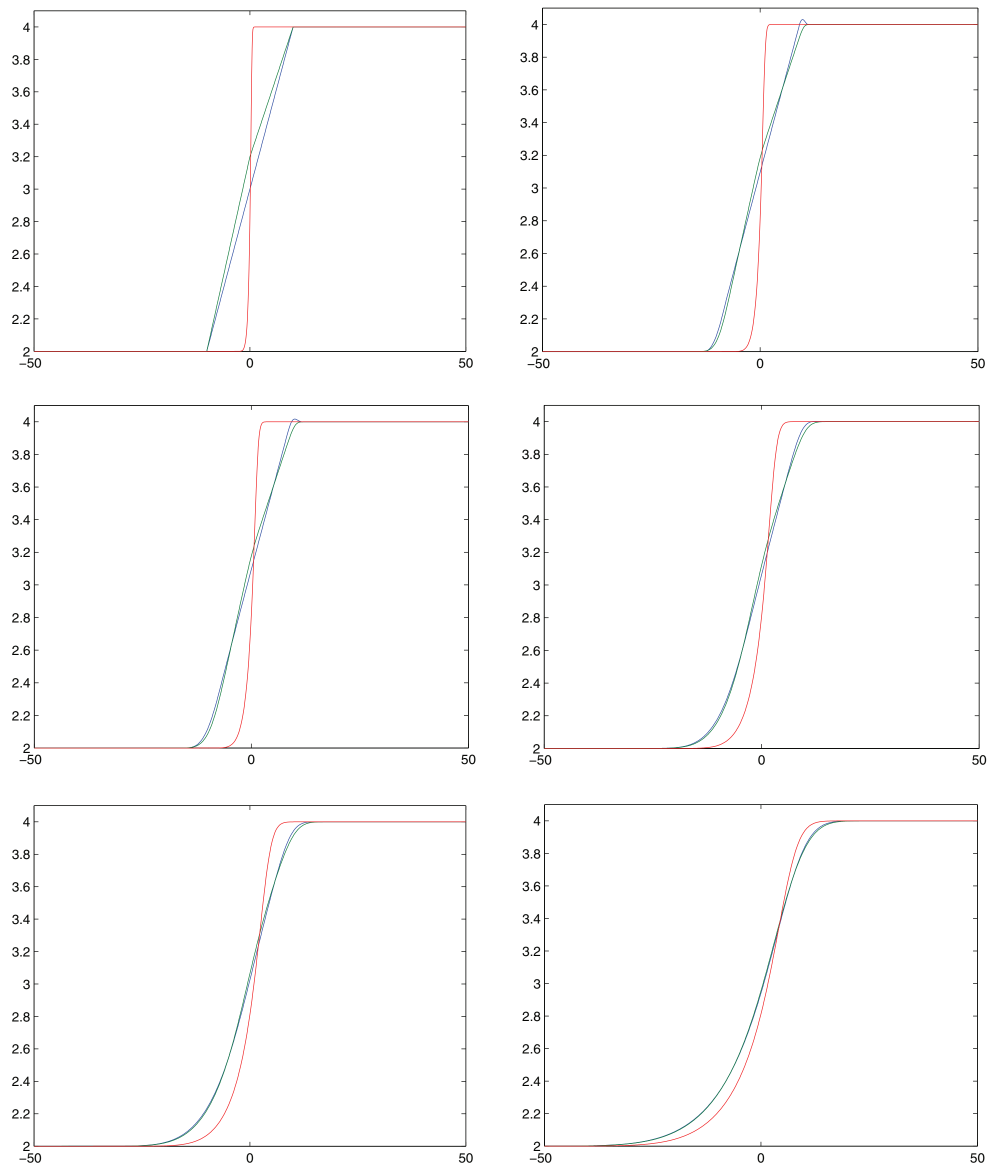

FIG. 5.1. *The graphs of* $v(x,t)$ = *"the middle graph of three graphs counted in the half bottom of each box,"* $\bar{v}(x,t)$ = *"the left graph of the graphs counted in the half bottom of each box," and* $\phi(x/\sqrt{1+t})$ = *"the right graph of three graphs counted in the half bottom of each box" at time* $t = 0, 5, 10, 50, 100, 300$.

In Figure 5.1, we show the graphs for $v(x,t)$, $\bar{v}(x,t)$, and $\phi(x/\sqrt{1+t})$ at time $t = 0$, 5, 10, 50, 100, 300. Here, the graphs of $v(x,t)$ are plotted as the middle graph of three graphs counted in the half bottom of each box, while the graphs of $\bar{v}(x,t)$ and $\phi(x/\sqrt{1+t})$ are plotted as the left and right graphs of three graphs counted in the half bottom of each box, respectively. It can be seen that, comparing with the diffusion wave $\phi(x/\sqrt{1+t})$, the particular parabolic solution $\bar{v}(x,t)$ is much closer to the original solution $v(x,t)$. So, as the asymptotic profiles of $v(x,t)$, the particular parabolic solution $\bar{v}(x,t)$ is better than the diffusion wave $\phi(x/\sqrt{1+t})$. In fact, when $t=300$, the graph of $\bar{v}(x,t)$ is almost the same as the graph of $v(x,t)$, but $\phi(x/\sqrt{1+t})$ is still a bit away from $v(x,t)$. This also confirms numerically our theoretical results obtained in Theorem 2.2 and Corollary 2.3.

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