# Nonlinear Stability of Traveling Wave Solutions for Non-Convex Viscous Conservation Laws

# 非凸性を持つ粘性的保存則に対する進行波解の非線形漸近安定性

Ming MEI（梅　茗）

1996年3月

# 博　士　論　文

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# KANAZAWA UNIVERSITY

## Doctoral Thesis

# Nonlinear Stability of Traveling Wave Solutions for Non-Convex Viscous Conservation Laws

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# Abstract

This thesis is to study the nonlinear asymptotic stability of "traveling wave solutions with shock profile" (simply saying, "viscous shock profiles" in what follows) for non-convex viscous conservation laws. There are three main goals in this thesis.

The first is to investigate the exponential decay rate of asymptotics for one-dimensional scalar viscous conservation laws with general non-convexity in the form $u_t + f(u)_x = \mu u_{xx}, \quad x \in R^1, t > 0$. We show that if the viscous shock profile is non-degenerate, the solution converges to it in maximum norm, at some exponential time decay rates, when the initial perturbation has some exponential spatial decay order.

The second is to improve the results on the nonlinear stability of viscous shock profile for systems in the provious works by Kawashima and Matsumura (Commun. Pure Appl. Math. **47**, 1547–1569 (1994)) and by Nishihara (J. Diff. Eqns. **120**, 304-318 (1995)). With some weaker conditions on nonlinearity, initial disturbance and weight, we prove the asymptotic stability of viscous shock profiles for one-dimensional non-convex system of viscoelasticity in the form $v_t - u_x = 0$, $u_t - \sigma(v)_x = \mu u_{xx}$ with the non-convexity $\sigma''(v) \lessgtr 0$ for $v \lessgtr 0$.

The third is to study the nonlinear stability of non-degenerate shock profile for above system with an opposite non-convexity condition $\sigma''(v) \lessgtr 0$ for $v \gtrless 0$, which is an open problem proposed by Kawashima and Matsumura (Commun. Pure Appl. Math. **47**, 1547–1569 (1994)). We prove the asymptotic stability for any non-degenerate shock profile under some restrictions. Thus, we partly answer the open question as above.

The approach is an elementary but technical weighted energy method. To select the weight functions and transform functions plays a key role in our proofs.

# Contents

# Chapter 1. Introduction

## 1.1. Background and Purpose

The purpose of this thesis is to study the nonlinear asymptotic stability of "traveling wave solutions with shock profile" ( we simply say "viscous shock profiles" in what follows) for non-convex viscous conservation laws. We have three main aims in this thesis.

1). We study the exponential decay rate of asymptotics to the viscous shock profile for the solution of one-dimensional scalar viscous conservation laws with general non-convexity in the form

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in R^1, t > 0, \tag{1.1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad x \in R^1, \tag{1.1.2}$$

where $\mu > 0$ is a constant, and the initial data tends toward some given constants $u_\pm$ as $x \to \pm\infty$, $f \in C^2$. We shall show that if the viscous shock profile $U(x - st)$ of (1.1.1), $s$ is the speed of shock profile, is non-degenerate, the solution of (1.1.1),(1.1.2) converges to it in maximum norm, at some exponential time decay rate, when the initial perturbation has some exponential decay order.

2). We improve the results on the nonlinear stability of viscous shock profile for system in the previous works [7,17]. Roughly speaking, with some weaker conditions on nonlinearity, initial disturbance and weight function than ones in [7,17], we shall prove the asymptotic stability of viscous shock profiles for one-dimensional non-convex system of viscoelasticity in the form

$$v_t - u_x = 0, \tag{1.1.3}$$

$$u_t - \sigma(v)_x = \mu u_{xx}, \tag{1.1.4}$$

with the initial data

$$(v, u)|_{t=0} = (v_0, u_0)(x) \to (v_\pm, u_\pm) \qquad \text{as} \quad x \to \pm\infty. \tag{1.1.5}$$

Here, $x \in R^1$ and $t \geq 0$, $v$ is the strain, $u$ the velocity, $\mu > 0$ the viscous constant, $\sigma(v)$ is the smooth stress function satisfying, as considered in [7,17]

$$\sigma'(v) > 0 \quad \text{for all} \quad v \quad \text{under consideration}, \tag{1.1.6}$$

$$\sigma''(v) \lesseqgtr 0 \quad \text{for} \quad v \lesseqgtr 0 \quad \text{under consideration}, \tag{1.1.7}$$

so that $\sigma(v)$ is neither convex nor concave, and has an inflection point at $v = 0$.

3). We partly answer an open problem on nonlinear stability of viscous shock profile proposed by Kawashima and Matsumura [7]. Namely, we study the nonlinear asymptotic stability of viscous shock profile for the system (1.1.3)-(1.1.5) with an opposite non-convexity to (1.1.7)

$$\sigma''(v) \gtreqless 0 \quad \text{for} \quad v \lesseqgtr 0 \quad \text{under consideration}, \tag{1.1.8}$$

which Kawashima and Matsumura [7] proposed as an unsolved case.

The study of the asymptotic stability of shock profile is very important from both physical and mathematical point of view, and so it becomes one of hot spots in the mathematical physics. Many important progresses have been made by a number of mathematicians in the last 40 years (see [1-23] and references therein). The first work in this research field is due to Il'in and Oleinik [4] in 1960, in which they investigated the asymptotic stability of shock profile for Cauchy problem (1.1.1),(1.1.2) with the assumption of *convexity, i.e.,* $f'' > 0$. Based on the maximum principle, they also showed the solution of (1.1.1) tends toward, in maximum norm, the shock profile $U(x-st)$ of (1.1.1) with an exponential time decay rate $e^{-\beta t}$ (for some $\beta > 0$), when the integral of the initial disturbance over $(-\infty, x]$ decays in an exponential order $O(e^{-\alpha|x|})$ (with some $\alpha > 0$). A same result was showed by Sattinger [21] by using the spectral analysis method. After that, as an important example of (1.1.1) for $f = u^2/2$, which is called as Burgers' equation, using an explicit formula of solution, Nishihara [18] obtained a precise estimate of solution for the decay rates. This estimate shows that if the integral of the initial disturbance over $(-\infty, x]$ has an polynomial order $O(|x|^{-\alpha})$ (with some $\alpha > 0$) as $|x| \to \infty$, then the solution converges, in the maximum norm, to the shock profile at the same algebraic rate $t^{-\alpha}$ as $t \to \infty$. These time decay rates are optimal in general. An asymptotic stability with a fixed time decay rate also was showed in [18]. In 1985, a new different approach based on an energy method which can be applied to systems was introduced independently by Matsumura and Nishihara [13] and by Goodman [1]. This has led a great of progresses in this field. Among them, Kawashima and Matsumura [6] generalized Nishihara's result [18] to the case of *general flux function with convexity* $f'' > 0$, in which the polynomial time decay rate is obtained and is almost optimal in $L^2$-framework from the arguments in Nishihara [18]. Recently, the research interest is focused on the case of non-convexity. In the non-convex case, when $f(u)$ only has one infelection piont somehow corresponding to (1.1.7) or (1.1.8), Kawashima and Matsumura [7] proved the stability of non-degenerate viscous shock profile by applying an technical weighted energy method. Later then, Mei [15] obtained the time decay rates of polynomial or exponential order, corresponding to the initial data with polynomial or exponential spatial decay order respectively. The author also proved the stability of degenerate viscous shock profile *at the first time*. Furthermore, for *general non-convex* flux function $f \in C^2$, Matsumura and Nishihara [14] in 1994 made an important contribution to the stability and the polynomial time decay rates not only for non-degenerate viscous shock profile but also for degenerate one. Although an asymptotic stability of non-degenerate viscous shock profile was obtained by Jones et al [5] by means of spectral analysis method in 1993, but their result is less sufficient than Matsumura and Nishihara's one [14]. The polynomial time decay rates showed in [14,15] are also optimal in $L^2-$framework. However, the exponential time decay rate of asymptotics for the *general non-convex case* is unknown yet. To solve this problem is one of our purposes in this thesis. We will see our results are much wider, also the proofs are much simpler, than the previous ones in [4,15,18,21].

On the other hand, the stability for the non-convex system of viscoelasticity is another interesting problem, but has not been studied well yet. In [7], Kawashima-Matsumura also treated the system case and proved the stability of non-degenerate viscous shock profile for system (1.1.3),(1.1.5) with the non-convexity condition (1.1.7). Although it seemed hardly to solve the stability in the case of the degenerate shock. Nishihara [17] successfully showed the stability provided that the integral of the initial disturbance over $(-\infty, x]$ have a polynomial decay $O(|x|^{-\frac{1+\alpha}{2}})$ (for some $0 < \alpha < 1$) as $x \to +\infty$. In the both papers [7,17], the authers supposed as sufficient conditions that the third derivative of the stress function $\sigma'''(v) > 0$ and the shock strength $|(v_+ - v_-, u_+ - u_-)|$ is suitably small. Here, we expect these stability results can be improved by making weaker conditions on nonlinear stress function, initial disturbance and weight function. This is our another purpose in the present thesis. Exectly saying, we have two goals here. One is to show the stability of traveling wave solutions without the condition $\sigma'''(v) > 0$. Another is to improve the weight function used in [17] in the degenerate shock case. The stability theorems will be shown even in the degenerate shock case with the improved weight function, provided that both the shock strength and the initial disturbance are suitably small. In the degenerate case, we shall impose the initial disturbances just have the decay order $O(|x|^{-\frac{1}{2}})$ as $x \to +\infty$. Thus, we improve the results in both [7] and [17].

When the nonlinear stress function $\sigma(v)$ satisfies the opposite non-convex condition (1.1.8), remarkably different from the scalar case, the procedures in the previous works [6,7,12-19] can not be applied to our problem (1.1.3)-(1.1.5). So the stability remains still open as is stated in Kawashima-Matsumura [7] (also cf.[12]).

To answer this open problem is our last purpose in this thesis. In order to overcome this difficulty, we shall introduce a suitable transform function depending on the viscous shock profile of (1.1.3),(1.1.4) to transfer the original system into a new one, and then, following the technique in [7], choose a desired weight function to establish a basic energy estimate. Thus, under some restrictions, we shall get the asymptotic stability for the non-degenerate shock profile even for non-convex condition (1.1.8). We call this scheme as the transform-weighted energy method. To select the desired transform and weight functions plays a key role.

Our plan in this paper is as follows. After stating the notations and an embedding theorem in the next section, we shall study the exponential time decay rate of asymptotics of non-degenerate viscous shock profile for scalar conservation laws in Chapter 2. In Chapter 3, we shall improve the stability results in [7,17] with weaker conditions on nonlinear stress function, initial disturbance and weight function. In Chapter 4, we shall partly answer the open problem on the stability of (1.1.3)-(1.1.5) with the non-convex condition (1.1.8).

## 1.2. Notations and Embedding Theorem

$L^2$ denotes the space of measurable functions on $R$ which are square integrable, with the norm

$$\|f\| = \Big(\int |f(x)|^2 dx\Big)^{1/2}.$$

$H^l (l \geq 0)$ denotes the Sobolev space of $L^2$-functions $f$ on $R$ whose derivatives $\partial_x^j f, j = 1, \cdots, l$, are also $L^2$-functions, with the norm

$$\|f\|_l = \Big(\sum_{j=0}^{l} \|\partial_x^j f\|^2\Big)^{1/2}.$$

$L_w^2$ denotes the space of measurable functions on $R$ which satisfy $w(x)^{1/2} f \in L^2$, where $w(x) > 0$ is a called weight function, with the norm

$$|f|_w = \Big(\int w(x)|f(x)|^2 dx\Big)^{1/2}.$$

$H_w^l$ $(l \geq 0)$ denotes the weighted Sobolev space of $L_w^2$-functions $f$ on $R$ whose derivatives $\partial_x^j f, j = 1, \cdots, l$, are also $L_w^2$-functions, where $w(x) > 0$ is a called weight function, with the norm

$$|f|_{l,w} = \Big(\sum_{j=0}^{l} |\partial_x^j f|_w^2\Big)^{1/2}.$$

$C_w^l$ $(l \geq 0)$ denotes the weighted $l$-times continuously differentiable space whose functions $f(x)$ satisfy $w(x)\partial_x^j f \in C^0, j = 0, 1, \cdots, l$, with the norm

$$\|f\|_{C_w^l} = \sup_{x \in R} \sum_{j=0}^{l} w(x)|\partial_x^j f|.$$

Denoting

$$\langle x\rangle_+ = \begin{cases} \sqrt{1+x^2}, & \text{if} \quad x \geq 0 \\ 1, & \text{if} \quad x < 0, \end{cases} \tag{1.2.1}$$

we will make use of the space $L^2_{\langle x\rangle_+}$ and $H^l_{\langle x\rangle_+}(l = 1,2)$ later. We also denote $f(x) \sim g(x)$ as $x \to a$ when $C^{-1}g \leq f \leq Cg$ in a neighborhood of $a$. Here and after here, we denote several constants by $C_i$, or $c_i$, $i = 1, 2, \cdots$, or by $C$ without confusion. When $C^{-1} \leq w(x) \leq C$ for $x \in R$, we note that $L^2 = H^0 = L^2_w = H^0_w$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$. Especially, when $w(x) = \langle x\rangle_+$, we have the following embedding theorem which will be used in Chapter 3.

**Embedding Theorem.** *There exists the embedding relation* $H^l_{\langle x\rangle_+} \hookrightarrow H^l$ *i.e., if* $f \in H^l_{\langle x\rangle_+}$, *then* $f \in H^l$ *and it holds the following inequality*

$$\|f\|_l \leq C|f|_{l,\langle x\rangle_+}. \tag{1.2.2}$$

*Moreover, if* $f \in H^l_{\langle x\rangle_+}$ *for* $l \leq 2$, *then* $\langle x\rangle_+^{1/2} f \in H^l$ *and it holds the inequality*

$$\|\langle x\rangle_+^{1/2} f\|_l \leq C|f|_{l,\langle x\rangle_+}. \tag{1.2.3}$$

*When* $l \geq 1$, *then* $H^l_{\langle x\rangle_+} \hookrightarrow C^0_{\langle x\rangle_+^{1/2}} \hookrightarrow C^0$, *i.e., if* $f \in H^l_{\langle x\rangle_+}$, *then* $\langle x\rangle_+^{1/2} f \in C^0$ *and it holds the inequality*

$$\sup_{x\in R} |f(x)| \leq C \sup_{x\in R} \langle x\rangle_+^{1/2} |f(x)| \leq C|f|_{2,\langle x\rangle_+}. \tag{1.2.4}$$

*Furthermore, the embedding relation* $H^l \cap L^2_{\langle x\rangle_+} \hookrightarrow C^0_{\langle x\rangle_+^{1/4}} \hookrightarrow C^0$, *and the inequality*

$$\sup_{x\in R} |f(x)| \leq C \sup_{x\in R} \langle x\rangle_+^{1/4} |f(x)| \leq C(\|f\|_l + |f|_{\langle x\rangle_+}) \tag{1.2.5}$$

*hold for any* $l \geq 1$.

**Proof.** Noting the facts $\langle x\rangle_+ \geq C$ for all $x \in R$, and $|\frac{d^k}{dx^k}\langle x\rangle_+^{1/2}| \leq C$ for $x \in (-\infty, -0]$ and $x \in [+0, +\infty)$, and using Sobolev's embedding theorem $H^l \hookrightarrow C^0 (l \geq 1)$, we can prove (1.2.2)-(1.2.4) by simple but tedious calculations. By Hölder inequality, we have

$$\begin{aligned}\langle x\rangle_+^{1/2} f(x)^2 &= \int_{-\infty}^{x} \frac{d}{dy}(\langle y\rangle_+^{1/2} f(y)^2)dy \\ &= \int_{-\infty}^{x} \frac{1}{2}(1+y^2)^{-\frac{3}{4}} y f(y)^2 dy + \int_{-\infty}^{x} \langle y\rangle_+^{1/2} 2 f(y) f'(y) dy \\ &\leq C(\|f\|^2 + |f|^2_{\langle x\rangle_+} + \|f_x\|^2),\end{aligned}$$

which proves (1.2.5).

Let $T$ and $B$ be a positive constant and a Banach space, respectively. We denote $C^k(0,T;B)$ $(k \geq 0)$ as the space of $B$- valued $k$- times continuously differentiable functions on $[0,T]$, and $L^2$ $(0,T;B)$ as the space of $B$-valued $L^2$-functions on $[0,T]$. The corresponding spaces of $B$-valued function on $[0,\infty)$ are defined similarly.

# Chapter 2. Scalar Viscous Conservation Laws

In this chapter, we study the asymptotic stability of shock profile for general non-convex scalar conservation laws (1.1.1),(1.1.2). In the convex case $f'' > 0$, the stability and time decay rates have been studied by many authers [4,6,9,21,22]. In the non-convex case, when $f(u)$ has one inflection point, the stability are studied by Kawashima and Matsumura [7] and Mei [15]. Mei also proved the polynomial and exponential time decay rates, and studied the stability of degenerate shock profile at the first time. When $f(u)$ is generally non-convex, the stability and the polynomial time decay rate are studied by Matsumura and Nishihara [14] and Jones et al [5]. But the time decay rate of the later is less optimal than the former. Matsumura and Nishihara [14] also showed the asymptotic stability of degenerate shock profile for the first time. Other interesting results can be found in Osher and Ralstin [20] and Weinberger [23]. However, the exponential time decay rate in the generally non-convex case is unsolved yet. To investigate it is our main purpose in this chapter. We shall prove that the solution of (1.1.1),(1.1.2) converges exponentially to the non-degenerat viscous shock profile of (1.1.1) for the initial perturbation with some exponential decay orders as $x \to \pm\infty$.

Our plan in this chapter is as follows. We first study the properties of non-degenerate viscous shock profile in Section 2.1, then state our main stability theorem and transfer our original probelm into new one by a reformulation in Section 2.2. Section 2.3 is to prove *a priori* estimates which is the key step for our stablity proof. In the last section, as a remark, we disuss the relation between time decay rates and spatial decay orders.

## 2.1. Viscous Shock Profiles

This section is to summarize properties of viscous shock profile of (1.1.1). If $u(t,x) = U(x-st)$ is a smooth solution of (1.1.1) satisfying $U(\pm\infty) = u_\pm$, then we call $U(x-st)$ a vicous shock profile of (1.1.1) which connects $u_+$ and $u_-$, and call $s$ a shock speed. Here, $u_\pm$ and $s$ satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + (f(u_+) - f(u_-)) = 0 \qquad (2.1.1)$$

and the generalized shock condition, or say Oleinik's shock condition

$$h(u) \equiv -s(u-u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0, & \text{if} \quad u_+ < u < u_- \\ > 0, & \text{if} \quad u_- < u < u_+. \end{cases} \qquad (2.1.2)$$

If Lax's shock condition

$$f'(u_+) < s < f'(u_-) \qquad (2.1.3)$$

is satisfied, we say the viscous shock profile is non-degenerate. Correspondingly, the degenerate shock profile means that the degenerate shock condition $s = f'(u_+)$, or $s = f'(u_-)$, or $s = f'(u_\pm)$ holds.

By the Rankine-Hugoniot condition (2.1.1), as a smooth solution of (1.1.1), $U(\xi)$ ($\xi = x - st$) must satisfy the following ordinary differential equation

$$\mu U_\xi = -su + f(u) - a_1 \equiv h(U), \tag{2.1.4}$$

where $a_1 = -su_\pm + f(u_\pm)$ is an integral constant.

Throught this chapter, we focus on the case of non-degenerate shock, *i.e.*, Lax's shock condition (2.13) holds. With a similar proof as in Kawashima and Matsumura [7], we find (2.1.4) admits a smooth solution if and only if (2.1.1) and (2.1.2) hold. This result is stated as follows without proof.

**Proposition 2.1.1.**

(i) *If (1.1.1),(1.1.2) admits a traveling wave solution with shock profile $U(x - st)$ connecting $u_\pm$, then $u_\pm$ and $s$ must satisfy the Rankine-Hugoniot condition (2.1.1) and the generalized shock condition (2.1.2).*

(ii) *Conversely, suppose that (2.1.1) and (2.1.2) hold, then there exists a shock profile $U(x - st)$ of (1.1.1),(1.1.2) which connects $(v_\pm, u_\pm)$. The $U(\xi)(\xi = x - st)$ is unique up to a shift in $\xi$ and is a monotone function of $\xi$. In particular, under Lax's shock condition (2.1.3), then it holds*

$$|h(U)| \sim |U - u_\pm| \sim exp(-c_\pm|\xi|), \quad as \quad \xi \to \pm\infty, \tag{2.1.5}$$

*where $c_\pm = |f'(u_\pm) - s|/\mu$.*

Let us make some preparation for the following sections. Without loss of generality, we assume that $u_+ < u_-$. Define

$$w_\alpha(u) = \frac{(u - u_+)^{1-\alpha}(u_- - u)^{1-\alpha}}{-h(u)}, \quad 0 < \alpha < 1, \quad u_+ < u < u_-, \tag{2.1.7}$$

$$g_\alpha(u) = (1-\alpha)[\alpha(u_+ + u_- - 2u)^2 + 2(u-u_+)(u_- - u)], \quad 0<\alpha<1, \tag{2.1.8}$$

$$k_\alpha(u) = -\frac{h(u)(w_\alpha h)''(u)}{2\mu w_\alpha(u)}, \quad 0<\alpha<1, \tag{2.1.9}$$

in which, $w_\alpha(u)$ is called as a weight function, and will play a key role in the proof of asymptotic stability below. When $U(\xi)$ is non-degenerate, we have by (2.1.5)

$$w_\alpha(U) \sim |U-u_\pm|^{-\alpha} \sim exp(\alpha c_\pm|\xi|), \quad as \quad \xi \to \pm\infty, \tag{2.1.10}$$

$$\left|\frac{h(U)}{\sqrt{2\mu}(U-u_+)(u_- - U)}\right|^2 \geq c_1, \quad \xi \in (-\infty,\infty), \tag{2.1.11}$$

for some constant $c_1 > 0$. Put

$$\theta_\alpha := c_1 \min_{u_+<u<u_-} g_\alpha(u), \tag{2.1.12}$$

then a straight computation leads to

$$\theta_\alpha = \begin{cases} c_1\alpha(1-\alpha)(u_- - u_+)^2, & 0<\alpha<\frac{1}{2}, \\ \frac{c_1}{4}(u_- - u_+)^2, & \alpha = \frac{1}{2}, \\ \frac{c_1(1-\alpha)}{2}(u_- - u_+)^2, & \frac{1}{2}<\alpha<1, \end{cases} \tag{2.1.13}$$

which implies that

$$\max_{0<\alpha<1} \theta_\alpha = \theta_{\frac{1}{2}}. \tag{2.1.14}$$

By (2.1.9),(2.1.11) and (2.1.13) we have

$$k_\alpha(U) = \left|\frac{h(U)}{\sqrt{2\mu}(U-u_+)(u_- - U)}\right|^2 g_\alpha(U) \geq \theta_\alpha. \tag{2.1.15}$$

All these conclusions will be used in Section 2.3.

## 2.2. Main Theorem and Reformulation of Problem

Let $U(x-st)$ be a non-degenerate viscous shock profile connecting $u_\pm$, and let us define $x_0$ by

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x))dx = x_0(u_+ - u_-). \tag{2.2.1}$$

We note that $x_0$ is uniquely determined by (2.2.1), provided that $u_0 - U$ is integrable over $R$. Then the shifted function $U(x - st + x_0)$ is also a shock profile connecting $u_\pm$ such that

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))dx = 0. \tag{2.2.2}$$

Without loss of generality, we assume $x_0 = 0$ for simplicity. We also define

$$\phi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y))dy. \tag{2.2.3}$$

Our main theorem is as follows.

**Theorem 2.2.1** (Decay Rates). *Suppose that (2.2.1) and (2.1.1)-(2.1.3) hold. If $\phi_0 \in H^2_{w_\alpha(U(x))}$ for $0 < \alpha \le \frac{1}{2}$, then there exists a positive constant $\delta_{2-1}$ such that if $|\phi_0|_{2,w_\alpha} \le \delta_{2-1}$, the Cauchy problem (1.1.1),(1.1.2) has a unique global solution $u(t,x)$ satisfying*

$$u - U \in C^0(0,\infty; H^1_{w_\alpha}) \cap L^2(0,\infty; H^2_{w_\alpha}).$$

*Moreover, the solution verifies the following decay rate estimate*

$$\sup_{x\in R} |u(t,x) - U(x - st)| \le Ce^{-\theta_\alpha t}|\phi_0|_{2,w_\alpha}, \tag{2.2.4}$$

*where $\theta_\alpha$ is defined as (2.1.13).*

*Remark 2.2.1.* 1. In Theorem 2.2.1, by (2.1.10), we see that the initial perturbation $\phi_0(x)$ has such exponential decay order $exp(-\frac{\alpha c_\pm}{2}|x|)$ for $0 < \alpha < 1$ as $x \to \pm\infty$.

Hence, our exponential decay result is much better than that in the previous works [4,15,18,21], because, when $\alpha$ is very closed to 1, $\phi_0(x)$ can have much slower spatial decay order than ones in [4,15,18,21], to get the exponential time decay rate.

2. The fact (2.1.14) shows us that when the initial perturbation $\phi_0(x)$ has a stronger spatial decay rate with $\alpha > \frac{1}{2}$, we cannot always have better time decay rate by our present method. This is quite different from the polynomial decay case.

3. Same as in Matsumura and Nishihara [14], we get the stability for *any shock profile* (weak or not).

In order to prove Theorem 2.2.1, like the previous works, we make a reformulation of our problem by changing unknown variable as the form

$$u(t,x) = U(\xi) + \phi_\xi(t,\xi), \ \xi = x - st. \tag{2.2.5}$$

Then the problem (1.1.1),(1.1.2) is reduced to the "integrated" equation

$$\phi_t + h'(U)\phi_\xi - \mu\phi_{\xi\xi} = F(U,\phi_\xi), \tag{2.2.6}$$

$$\phi(0,\xi) = \phi_0(\xi), \tag{2.2.7}$$

where

$$F = -\{f(U+\phi_\xi) - f(U) - f'(U)\phi_\xi\}. \tag{2.2.8}$$

The problem (2.2.6),(2.2.7) can be solved globally in time as follows.

**Theorem 2.2.2.** *Suppose that $\phi_0 \in H^2_{w_\alpha}$ for $0 < \alpha \le \frac{1}{2}$, and the conditions in Theorem 2.2.1 hold. Then there exists a positive constant $\delta_{2-2}$ such that if*

$|\phi_0|_{2,w_\alpha} \le \delta_{2-2}$, *the problem (2.2.6),(2.2.7) has a unique global solution* $\phi(t,\xi)$ *satisfying*

$$\phi \in C^0(0,\infty; H^2_{w_\alpha}) \cap C^1(0,\infty; L^2_{w_\alpha}), \ \ \phi_\xi \in L^2(0,\infty; H^2_{w_\alpha}), \tag{2.2.9}$$

*and the decay estimate*

$$e^{2\theta_\alpha t}|\phi(t)|^2_{2,w_\alpha} + \int_0^t e^{2\theta_\alpha \tau}|\phi_\xi(\tau)|^2_{2,w_\alpha} d\tau \le C|\phi_0|^2_{2,w_\alpha} \tag{2.2.10}$$

*holds for* $t \ge 0$.

Since we can easily prove Theorem 2.2.1 from Theorem 2.2.2, it is sufficient to prove Theorem 2.2.2 for our purpose. To do that, we shall combine a local existence result together with *a priori* estimates.

**Proposition 2.2.3** (Local Existence). *Suppose that* $\phi_0 \in H^2$ *and the conditions in Theorem 2.2.1 hold. Then there is a positive constant* $T_0$ *such that the problem (2.2.6),(2.2.7) has a unique solution* $\phi(t,\xi)$ *satisfying*

$$\phi \in C^0(0,T_0; H^2) \cap C^1(0,T_0; L^2), \ \ \phi_\xi \in L^2(0,T_0; H^2),$$

$$\sup_{t\in[0,T_0]} \|\phi(t)\|_2 \le 2\|\phi_0\|_2. \tag{2.2.11}$$

*Moreover, if* $\phi_0 \in H^2_{w_\alpha}$ *for some* $0 < \alpha < 1$, *then* $\phi \in C^0(0,T_0; H^2_{w_\alpha})$ *and* $\phi_\xi \in L^2(0,T_0; H^2_{w_\alpha})$.

**Proposition 2.2.4** (*A Priori* Estimate). *Let* $T$ *be a positive constant, and* $\phi(t,\xi)$ *be a solution of the problem (2.2.6),(2.2.7) satisfying*

$$\phi \in C^0(0,T; H^2_{w_\alpha}) \cap C^1(0,T; L^2_{w_\alpha}),$$

$$\phi_\xi \in L^2(0,T;H^2_{w_\alpha}). \tag{2.2.12}$$

*Then there exist positive constants $\delta_{2-3}$ and $C$ which are independent of $T$ such that if $\sup_{0\le t\le T}|\phi(t)|^2_{2,w_\alpha} \le \delta_{2-3}$, then the estimate (2.2.10) holds for $t\in[0,T]$.*

Since Proposition 2.2.3 can be proved in the standard way, we omit its proof. Once Proposition 2.2.4 is proved, using the continuation arguments based on Propositions 2.2.3 and 2.2.4, we can show Theorem 2.2.2. This scheme is the same as in the previous papers [6,15], so we also omit its proof. To prove Proposition 2.2.4 is our main goal which will be showed in the following section.

## 2.3. The Proof of A Priori Estimate

We now define the solution space of (2.2.6) and (2.2.7)

$$X(0,T) = \{\phi \in C^0(0,T;H^2_{w_\alpha}), \phi_\xi \in L^2(0,T;H^2_{w_\alpha})\},$$

with $0 < T \le +\infty$. Put

$$N(t) = \sup_{0\le\tau\le t}|\phi(\tau)|_{2,w_\alpha},$$

then we first prove a basic energy estimate as follows

**Lemma 2.3.1** (Basic Energy Estimate). *Let $\phi(t,\xi)\in X(0,T)$ be a solution of (2.2.6),(2.2.7) for some $T>0$. Then it holds*

$$e^{2\theta_\alpha t}|\phi(t)|^2_{w_\alpha} + \int_0^t e^{2\theta_\alpha\tau}|\phi_\xi(\tau)|^2_{w_\alpha}\,d\tau \le C|\phi_0|^2_{w_\alpha} \tag{2.3.1}$$

*provided $N(T)$ is suitably small.*

**Proof.** Multiplying (2.2.6) by $e^{2\theta_\alpha t}w_\alpha(U)\phi(t,\xi)$, we have

$$(\frac{1}{2}e^{2\theta_\alpha t}w_\alpha(U)\phi^2)_t + e^{2\theta_\alpha t}(\frac{1}{2}(w_\alpha h)'(U)\phi^2 - \mu w_\alpha(U)\phi\phi_\xi)_\xi$$

$$+\mu e^{2\theta_\alpha t}w_\alpha(U)\phi_\xi^2 + [k_\alpha(U)-\theta_\alpha]e^{2\theta_\alpha t}w_\alpha(U)\phi^2 = e^{2\theta_\alpha t}w_\alpha(U)\phi F. \qquad (2.3.2)$$

Here we used $\mu U_\xi = h(U)$, $k_\alpha(U)$ is defined as in (2.1.9).

Integrating (2.3.2) over $[0,t]\times R$, and noting (2.1.14) and the facts $|F| \le C\phi_\xi^2$, $\sup_{0\le\tau\le t}|\phi(t,\xi)| \le CN(t)$ and $k_\alpha(U)\ge\theta_\alpha$ (see (2.1.15)), we obtain

$$e^{2\theta_\alpha t}|\phi(t)|_{w_\alpha}^2 + (1-CN(t))\int_0^t e^{2\theta_\alpha\tau}|\phi_\xi(\tau)|_{w_\alpha}^2 d\tau \le C|\phi_0|_{w_\alpha}^2. \qquad (2.3.3)$$

Letting $N(t)$ be suitably small, we then complete the proof of Lemma 2.3.1.

Based on the basic energy estimate (2.3.1), we can derive the following energy estimates for higher order derivatives of $\phi(t,\xi)$.

**Lemma 2.3.2.** *There hold for suitably small $N(T)$,*

$$e^{2\theta_\alpha t}|\phi_\xi(t)|_{w_\alpha}^2 + \int_0^t e^{2\theta_\alpha\tau}|\phi_{\xi\xi}(\tau)|_{w_\alpha}^2 d\tau \le C|\phi_0|_{1,w_\alpha}^2, \qquad (2.3.4)$$

$$e^{2\theta_\alpha t}|\phi_{\xi\xi}(t)|_{w_\alpha}^2 + \int_0^t e^{2\theta_\alpha\tau}|\phi_{\xi\xi\xi}(\tau)|_{w_\alpha}^2 d\tau \le C|\phi_0|_{2,w_\alpha}^2. \qquad (2.3.5)$$

**Proof.** Similar to Lemma 2.3.1, differentiating equation (2.2.6) respect with $\xi$, and multiplying the resulant equality by $e^{2\theta_\alpha t}w_\alpha(U)\phi_\xi$ and integrating it over $[0,t]\times R$, using $|F_\xi| \le O(1)(|\phi_\xi|^2 + |\phi_\xi||\phi_{\xi\xi}|)$ and (2.3.1), we can get the estimate (2.3.4) for some small $N(T)$. The estimate (2.3.5) can be proved by a scheme somehow same as above. We here omit the details.

By Lemmas 2.3.1 and 2.3.2, we have proved Proposition 2.2.4.

## 2.4. Remarks

By some computation like above, we can also choose the weight function as

$$w_{\alpha,\beta}(u) = \frac{(u-u_+)^{1-\alpha}(u_- - u)^{1-\beta}}{-h(u)},$$

for $u_+ \le u \le u_-$. Here $\alpha$ and $\beta$ must statisfy $0 < \alpha < 1$, $1-\alpha \le \beta \le \alpha$ for $\alpha \ge \frac{1}{2}$, or $\alpha \le \beta \le 1-\alpha$ for $\alpha \le \frac{1}{2}$.

When $U(\xi)$ is non-degenerate shock profile, we have

$$w_{\alpha,\beta}(U) \sim \begin{cases} |U-u_+|^{-\alpha} \\ |U-u_-|^{-\beta} \end{cases} \sim \begin{cases} exp(-\alpha c_+|\xi|, & \text{as} \quad \xi \to +\infty, \\ exp(-\beta c_-|\xi|), & \text{as} \quad \xi \to -\infty. \end{cases}$$

If we apply this weight function to make an energy estimate likes (2.2.10), we can also show the exponential time decay rate $e^{-\theta_{\alpha,\beta}t}$ for some $\theta_{\alpha,\beta} > 0$. Which means $\phi_0(x)$ allows some different spatial decay order for $x$ near $-\infty$ or $+\infty$, to get the asymptotics of time. Especially, when $\alpha = \beta = \frac{1}{2}$, the time decay rate $exp(-\theta_{\frac{1}{2},\frac{1}{2}}t) = exp(-\theta_{\frac{1}{2}}t)$ is the largest in the sense of the present method.

# Chapter 3. System of Viscoelasticity (I)

In this chapter, we are concerned with the asymptotic stability of shock profiles for Cauchy problem to a non-convex system of viscoelasticity (1.1.3)-(1.1.5) under the non-convexity conditions (1.1.6),(1.1.7). With weaker conditions on nonlinearity, initial perturbations and weight, we shall improve the stability results in Kawashima and Matsumura [7] and Nishihara [17].

We find that the system (1.1.3),(1.1.4) with $\mu = 0$ is strictly hyperbolic, with the characteristic roots

$$\lambda = \pm\lambda(v), \qquad \text{where} \quad \lambda(v) = \sqrt{\sigma'(v)}$$

and with the corresponding right eigenvectors

$$r_{\pm}(v) = \begin{pmatrix} 1 \\ \mp\lambda(v) \end{pmatrix}.$$

Moreover, we see that both characteristic fields are neither genuinely nonlinear nor linearly degenerate in the neighborhood of $v = 0$. In fact, the quantity

$$\nabla\lambda(v) \cdot r_{\pm}(v) = \lambda'(v) = \sigma''(v)/2\sqrt{\sigma'(v)},$$

changes its sign at $v = 0$, where $\nabla$ denotes the gradient with respect to $(v, u)$.

Throughout this chapter, without loss of generality, let us suppose $\sigma(0) = 0$. In fact, if $\sigma(0) \neq 0$, setting $\sigma_1(v) = \sigma(v) - \sigma(0)$, then $\sigma_1(0) = 0$ and $\sigma_1(v)$ satisfies equations (1.1.3), (1.1.4) and (1.1.6), (1.1.7) corresponding to $\sigma(v)$. Thus, we may denote $\sigma_1(v)$ by $\sigma(v)$ again.

Our plan in this chapter is as follows. After analysing the properties of the viscous shock profile of (1.1.3),(1.1.4) in Section 3.1, we give our stability theorems in Section 3.2, whose proofs are also given by admitting *a priori* estimates in this section. In the final section, we complete the proofs of *a priori* estimates, where to introduce some suitable weight functions plays a key role.

## 3.1. Properties of Traveling Wave Solution

In this section, we state the properties of traveling wave solution with shock profile. The traveling wave solutions are solutions of the form

$$(v,u)(t,x) = (V,U)(\xi), \quad \xi = x - st, \tag{3.1.1}$$

$$(V,U)(\xi) \to (v_\pm, u_\pm), \qquad \xi \to \pm\infty, \tag{3.1.2}$$

where $s$ is the shock speed and $(v_\pm, u_\pm)$ are constant stats at $\pm\infty$. Let the system (1.1.3),(1.1.4) admit the traveling wave solutions, then both $(v_\pm, u_\pm)$ and $s$ satisfy the Rankine-Hugoniot condition

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) - (\sigma(v_+) - \sigma(v_-)) = 0, \end{cases} \tag{3.1.3}$$

and the generalized shock condition

$$\frac{1}{s}h(v) \equiv \frac{1}{s}[-s^2(v - v_\pm) + \sigma(v) - \sigma(v_\pm)] \begin{cases} < 0, & \text{if} \quad v_+ < v < v_- \\ > 0, & \text{if} \quad v_- < v < v_+. \end{cases} \tag{3.1.4}$$

We note that the condition (3.1.4) with (1.1.6) and (1.1.7) implies

$$\lambda(v_+) \le s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_+) \le s < -\lambda(v_-), \tag{3.1.5}$$

and that, especially when $\sigma''(v) > 0$, the condition (3.1.4) is equivalent to

$$\lambda(v_+) < s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_-) < s < -\lambda(v_+), \tag{3.1.6}$$

which is well-known as Lax's shock condition(Lax[8]). We call the condition (3.1.5) with $s = \lambda(v_+)$ (or $s = -\lambda(v_+)$) and the condition (3.1.6) as the degenerate or non-degenerate shock condition respectively.

If $(v, u)(t, x) = (V, U)(\xi)$ ($\xi = x - st$) is the traveling wave solution with shock profile of (1.1.3),(1.1.4), then $(V, U)(\xi)$ must satisfy

$$\begin{cases} -sV' - U' = 0, \\ -sU' - \sigma(V)' = \mu U'' \end{cases} \tag{3.1.7}$$

Integrating (3.1.7) over $(-\infty, \infty)$, we have Rankine-Hugoniot condition (3.1.3). We integrate (3.1.7) over $(-\infty, \xi]$ and eliminate $U$, then we obtain a single ordinary differential equation for $V(\xi)$

$$\mu s V' = -s^2 V + \sigma(V) - a_2 \equiv h(V), \tag{3.1.8}$$

where

$$a_2 = -s^2 v_\pm + \sigma(v_\pm). \tag{3.1.9}$$

Letting $(v_+, u_+) \neq (v_-, u_-)$ and $s > 0$, we are now ready to summarize a characterization of the generalized shock condition (3.1.4) and the results on the existence of shock profiles studied in [7]:

**Proposition 3.1.1.** *Suppose that (1.1.6) and (1.1.7) hold. Then the following statements are equivalent to each other.*

(i) *The generalized shock condition (3.1.4) holds.*

(ii) $\sigma'(v_+) \le s^2$, *i.e.*, $\lambda(v_+) \le s$.

(iii) $\sigma'(v_+) \le s^2 < \sigma'(v_-)$, *i.e.*, $\lambda(v_+) \le s < \lambda(v_-)$.

(iv) *There exists uniquely a* $v_* \in (v_+, v_-)$ *such that* $\sigma'(v_*) = s^2$ *and it holds*

$$\sigma'(v) < s^2 \text{ for } v \in (v_+, v_*), \quad s^2 < \sigma'(v) \text{ for } v \in (v_*, v_-). \tag{3.1.10}$$

*i.e.*,

$$h'(v_*) = 0, \quad h'(v) < 0 \text{ for } v \in (v_+, v_*), \quad h'(v) > 0 \text{ for } v \in (v_*, v_-). \tag{3.1.11}$$

*Moreover, if one of the above four conditions holds, then we must have* $v_- \neq 0$. *In addition,* $v_+ \lessgtr v_-$ *and* $v_* \gtrless 0$ *hold when* $v_- \gtrless 0$.

**Proposition 3.1.2.** *Suppose that (1.1.6) and (1.1.7) hold.*

(i) *If (1.1.3),(1.1.4) admits a traveling wave solution with shock profile* $(V(x - st), U(x - st))$ *connecting* $(v_\pm, u_\pm)$, *then* $(v_\pm, u_\pm)$ *and* $s$ *must satisfy the Rankine-Hugoniot condition (3.1.3) and the generalized shock condition (3.1.4).*

(ii) *Conversely, suppose that (3.1.3) and (3.1.4) hold, then there exists a shock profile* $(V, U)(x - st)$ *of (1.1.3),(1.1.4) which connects* $(v_\pm, u_\pm)$. *The* $(V, U)(\xi)(\xi = x - st)$ *is unique up to a shift in* $\xi$ *and is a monotone function of* $\xi$. *In particular, when* $v_+ \lessgtr v_-$ *(and hence* $u_+ \gtrless u_-$*) we have*

$$u_+ \gtrless U(\xi) \gtrless u_-, \quad U_\xi(\xi) \gtrless 0, \tag{3.1.12}$$

$$v_+ \lessgtr V(\xi) \lessgtr v_-, \quad V_\xi(\xi) \lessgtr 0, \tag{3.1.13}$$

*for all $\xi \in R$. Moreover, $(V,U)(\xi) \to (v_\pm, u_\pm)$ exponentially as $\xi \to \pm\infty$, with the following exceptional case: when $\lambda(v_+) = s$, $(V,U)(\xi) \to (v_+, u_+)$ at the rate $|\xi|^{-1}$ as $\xi \to +\infty$, and $|h(V)| = |\mu s V_\xi| = O(|\xi|^{-2})$ as $\xi \to \infty$.*

For the graphes of $\sigma(v)$ and $h(v)$, may see Figures 3.1 and 3.2.

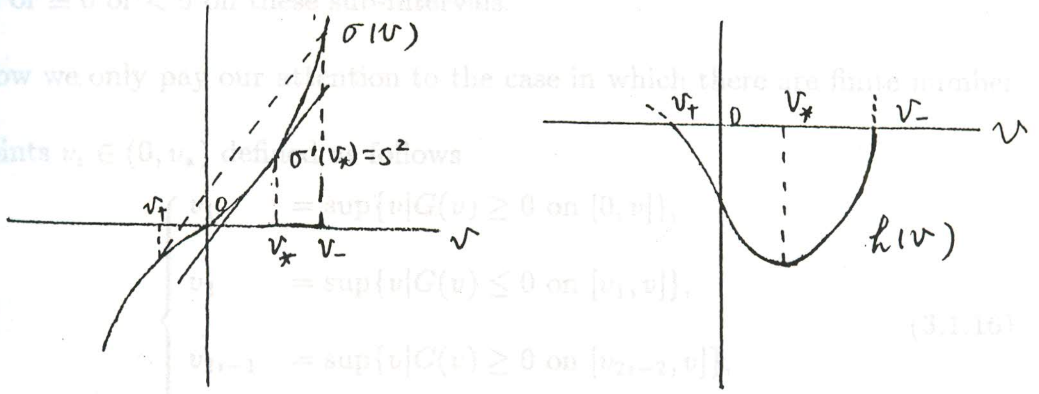


Figure 3.1. Non-degenerate case

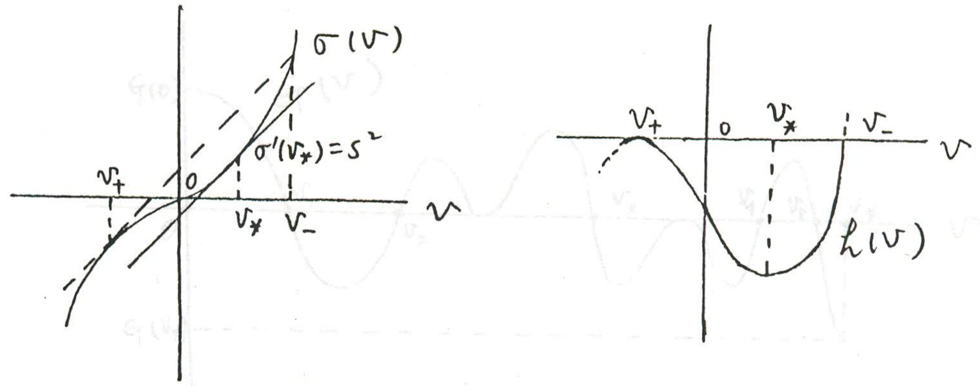


Figure 3.2. Degenerate case

Now we give a function of the form

$$G(v) \equiv h(v)\sigma''(v) - h'(v)\sigma'(v) = h(v)^2\Big(\frac{\sigma'(v)}{h(v)}\Big)', \qquad v \in [0, v_*] \tag{3.1.14}$$

which plays an important role in our proof. We know that $G(v)$ is continuous, and

$G(v)$ satisfies, by virtue of (3.1.11),

$$G(0) = -\sigma'(0)h'(0) > 0, \qquad G(v_*) = h(v_*)\sigma''(v_*) < 0. \tag{3.1.15}$$

According to these facts, we know that there exist some finite or infinite points in $(0, v_*)$ such that $G(v) = 0$. These points divide $[0, v_*]$ into sub-intervals such that $G(v) > 0$ or $\equiv 0$ or $< 0$ on these sub-intervals.

Now we only pay our attention to the case in which there are finite number of the points $v_i \in (0, v_*)$ defined as follows

$$\begin{cases} v_1 & = \sup\{v | G(v) \geq 0 \text{ on } [0, v]\}, \\ v_2 & = \sup\{v | G(v) \leq 0 \text{ on } [v_1, v]\}, \\ v_{2i-1} & = \sup\{v | G(v) \geq 0 \text{ on } [v_{2i-2}, v]\}, \\ v_{2i} & = \sup\{v | G(v) \leq 0 \text{ on } [v_{2i-1}, v]\}. \end{cases} \tag{3.1.16}$$

In this case, the graph of $G(v)$ looks like the following Figure 3.3.

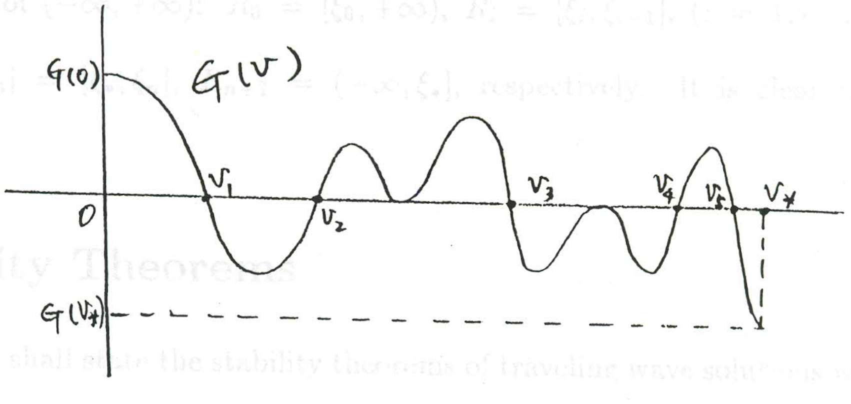


Figure 3.3

The function $G(v)$ may have infinitely many $v_i$'s, whose case will be remarked later (see Remark 3.3.6). By our choice (3.1.16), we get the properties of $G(v)$ in the followings

**Proposition 3.1.3.** *There exist the odd number points, without loss of generality,* *say n points (n is an odd number),* $v_i \in (0, v_*), i = 1, \cdots, n$, *such that* $G(v_i) = 0$ *and*

$$\begin{cases} G(v) \geq 0 \text{ on } I_{2j-1} \equiv [v_{2j-2}, v_{2j-1}], j = 1, 2, \cdots, \frac{n+1}{2}, \\ G(v) \leq 0 \text{ on } I_{2j} \equiv [v_{2j-1}, v_{2j}], j = 1, 2, \cdots, \frac{n+1}{2}, \end{cases} \tag{3.1.17}$$

*where* $v_0$ *and* $v_{n+1}$ *denote 0 and* $v_*$, *respectively. Especially,* $G(v) < 0$ *on* $(v_n, v_*]$.

**Proof.** By the continuity of $G(v)$ and (3.1.15), and our choice (3.1.16), then we see easily that Proposition 3.1.3 is true.

We also denote $I_0$ and $I_{n+2}$ as the following intervals

$$I_0 \equiv (v_+, 0], \quad I_{n+2} \equiv [v_*, v_-]. \tag{3.1.18}$$

Since $V(\xi)$ is monotonic on $[v_+, v_-]$(see (3.1.13)), then there exist the unique numbers $\xi_0$, $\xi_i (i = 1, 2, \cdots, n)$, and $\xi_*$ such that $V(\xi_0) = v_0 \equiv 0$, $V(\xi_i) = v_i$, $i = 1, \cdots, n$ and $V(\xi_*) = v_*$. Here, we also denote $R_i$ $(i = 0, 1, \cdots, n+1, n+2)$ as the following sub-intervals of $(-\infty, +\infty)$: $R_0 = [\xi_0, +\infty)$, $R_i = [\xi_i, \xi_{i-1}]$, $(i = 1, \cdots, n)$, $R_{n+1} = [\xi_{n+1}, \xi_n] = [\xi_*, \xi_n]$, $R_{n+2} = (-\infty, \xi_*]$, respectively. It is clear that $R = \cup_{i=0}^{n+2} R_i$.

## 3.2. Stability Theorems

In this section, we shall state the stability theorems of traveling wave solutions with shock profiles for (1.1.3)-(1.1.7) without the condition $\sigma'''(v) > 0$. To state our result in the degenerate case, we set

$$\overline{\sigma}(v) \equiv \sigma(v) - \sigma'(0)v. \tag{3.2.1}$$

Then we have $\overline{\sigma}(0) = \overline{\sigma}'(0) = \overline{\sigma}''(0)$, $\overline{\sigma}''(v) \lessgtr 0$ for $v \lessgtr 0$ and

$$0 < -\overline{\sigma}(v_+) < -v_+\overline{\sigma}'(v_+) \quad \text{for} \quad v_+ < 0 < v_- \tag{3.2.2}$$

by an observation of the graph of the function $\overline{\sigma}'(v)$, $v_+ \leq v \leq 0$ (see Figure 3.4).

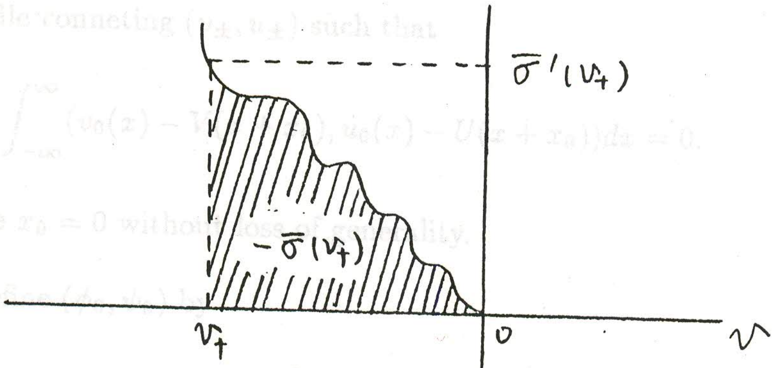


Figure 3.4

Now, we assume that there is a constant $\overline{\delta}(0 < \overline{\delta} < 1)$ such that

$$-\overline{\sigma}(v_+) < -\overline{\delta} v_+ \overline{\sigma}'(v_+) \quad \text{as} \quad v_+ \to 0_-. \tag{3.2.3}$$

We note that $\overline{\delta}$ can be taken as $\overline{\delta} = \frac{1}{3} + \epsilon$ if $\overline{\sigma}'''(0)(= \sigma'''(0)) > 0$, where $\epsilon > 0$ is any given constant. In fact, we have

$$-\overline{\sigma}(v_+) = -\frac{\sigma'''(0)}{3!} v_+^3 + o(v_+^3),$$

$$-v_+ \overline{\sigma}'(v_+) = -\frac{\sigma'''(0)}{2!} v_+^3 + o(v_+^3),$$

which means (3.2.3) holds for $\overline{\delta} > \frac{1}{3}$ as $v_+ \to 0$.

Now, without loss of generality, we restrict our attention to the case

$$s > 0 \quad \text{and} \quad v_+ < 0 < v_-, \quad i.e., \quad \mu s V_\xi = h(V) < 0. \tag{3.2.4}$$

Let $(V, U)(x - st)$ be a pair of traveling wave solutions connecting $(v_\pm, u_\pm)$, we assume the integrability of $(v_0 - V, u_0 - U)(x)$ over $R$ and express that integral in the form

$$\int_{-\infty}^{\infty} (v_0 - V, u_0 - U)(x) dx = x_0 (v_+ - v_-, u_+ - u_-). \tag{3.2.5}$$

Then the shifted function $(V,U)(x-st+x_0)$ is also a pair traveling wave solution with shock profile conneting $(v_\pm, u_\pm)$ such that

$$\int_{-\infty}^{\infty} (v_0(x) - V(x+x_0), u_0(x) - U(x+x_0))dx = 0. \tag{3.2.6}$$

We also suppose $x_0 = 0$ without loss of generality.

Let us define $(\phi_0, \psi_0)$ by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (v_0 - V, u_0 - U)(y)dy. \tag{3.2.7}$$

Our main theorems are the followings.

**Theorem 3.2.1.**( Non-degenerate case: $\lambda(v_+) < s < \lambda(v_-)$). *Suppose (1.1.6), (1.1.7), (3.1.3), (3.1.4), (3.2.6), and* $(\phi_0, \psi_0) \in H^2$. *Then there exists a positive constant* $\delta_{3-1}$ *such that if* $|(v_+ - v_-, u_+ - u_-)| + \|(\phi_0, \psi_0)\|_2 < \delta_{3-1}$, *then (1.1.3)-(1.1.5) has a unique global solution* $(v,u)(t,x)$ *satisfying*

$$v - V \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^1),$$

$$u - U \in C^0([0,\infty); H^1) \cap L^2([0,\infty); H^2).$$

*Furthermore, the solution verifies*

$$\sup_{x\in R} |(v,u)(t,x) - (V,U)(x-st)| \to 0 \quad as \quad t \to \infty. \tag{3.2.8}$$

**Theorem 3.2.2.**( Degenerate case: $\lambda(v_+) = s < \lambda(v_-)$). *Suppose (1.1.6), (1.1.7), (3.1.3), (3.1.4) and (3.2.6). Assume* $|(v_+ - v_-, u_+ - u_-)| \ll 1$ *and (3.2.3), then the followings hold:*

*(i) Suppose that* $(\phi_0,\psi_0)\in H^2_{\langle x\rangle_+}$*, then there exists a positive constant* $\delta_{3-2}$ *such that if* $|(\phi_0,\psi_0)|_{2,\langle x\rangle_+}<\delta_{3-2}$*, then (1.1.3)-(1.1.5) has a unique global solution* $(v,u)(t,x)$ *satisfying*

$$v-V\in C^0([0,\infty);H^1_{\langle x\rangle_+})\cap L^2([0,\infty);H^1_{\langle x\rangle_+^{\frac{1}{2}}})$$

$$u-U\in C^0([0,\infty);H^1_{\langle x\rangle_+})\cap L^2([0,\infty);H^2_{\langle x\rangle_+}).$$

*Furthermore, the solution verifies the asymptotic stability (3.2.8).*

*(ii) Suppose that* $(\phi_0,\psi_0)\in H^2\cap L^2_{\langle x\rangle_+}$ *and* $\phi_{0,x}\in L^2_{\langle x\rangle_+^{\frac{3}{4}}}$*. Then there exists a positive constant* $\delta_{3-3}$ *such that if* $\|(\phi_0,\psi_0)\|_2+|(\phi_0,\psi_0)|_{\langle x\rangle_+}+|\phi_{0,x}|_{\langle x\rangle_+^{\frac{3}{4}}}<\delta_{3-3}$*, then (1.1.3)-(1.1.5) has a unique global solution* $(v,u)(t,x)$ *satisfying*

$$v-V\in C^0([0,\infty);H^1\cap L^2_{\langle x\rangle_+})\cap L^2([0,\infty);H^1\cap L^2_{\langle x\rangle_+^{\frac{3}{4}}})$$

$$u-U\in C^0([0,\infty);H^1\cap L^2_{\langle x\rangle_+})\cap L^2([0,\infty);H^2\cap L^2_{\langle x\rangle_+}).$$

*Furthermore, the solution verifies the asymptotic stability (3.2.8).*

*Remark 3.2.3.* In the stability results in [7,17] both $\sigma'''(v)>0$ and smallness of shock strength $|(v_+-v_-,u_+-u_-)|$ are assumed as sufficient conditions. In the non-degenerate shock case, Theorem 3.2.1 deletes the condition $\sigma'''(v)>0$. In the degenerate shock condition, $\lambda(v_+)=s<\lambda(v_-)$, the condition (3.2.3) in Theorem 3.2.2 seems to be much weaker than the condition $\sigma'''(v)>0$, and also the weight is improved compared to that in Nishihara [17]. As an example of $\sigma(v)$, we have

$$\sigma(v)=bv+\int_0^v\int_0^x y^k(sin\frac{1}{y}+2)dydx,\quad k=1,3,5,\cdots,$$

where $b$ is a constant satisfying

$$b > \max_{v_+ \le v \le 0} \left| \int_0^v x^k (sin\frac{1}{x} + 2) dx \right|.$$

Then, note that $\sigma'''(v)$ does not exist for $k = 1$ and $\sigma'''(v)$ changes the sign on $[v_+, v_-]$ for $k \ge 3$.

In order to solve the stability, we make a reformulation for the problem (1.1.3)-(1.1.5) by changing the unknown variables as

$$(v, u)(t, x) = (V, U)(\xi) + (\phi_\xi, \psi_\xi)(t, \xi), \quad \xi = x - st. \tag{3.2.9}$$

Then the problem (1.1.3)-(1.1.5) is reduced to the following "integrated" system

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - \sigma'(V)\phi_\xi - \mu\psi_{\xi\xi} = F \\ (\phi, \psi)(0, \xi) = (\phi_0, \psi_0)(\xi) \end{cases} \tag{3.2.10}$$

with

$$F = \sigma(V + \phi_\xi) - \sigma(V) - \sigma'(V)\phi_\xi.$$

For any fixed $T \in (0, \infty)$, let us define the solution spaces of (3.2.10) by

$$X_0(0, T) = \{(\phi, \psi) \in C^0([0, T]; H^2), \phi_\xi \in L^2([0, T]; H^1),$$

$$\psi_\xi \in L^2([0, T]; H^2)\},$$

$$X_1(0, T) = \{(\phi, \psi) \in C^0([0, T]; H^2_{\langle\xi\rangle_+}), \phi_\xi \in L^2([0, T]; H^1_{\langle\xi\rangle_+^{\frac{1}{2}}}),$$

$$\psi_\xi \in L^2([0, \infty); H^2_{\langle\xi\rangle_+})\},$$

$$X_2(0,T) = \{(\phi,\psi) \in C^0([0,T]; H^2 \cap L^2_{\langle\xi\rangle_+}), \phi_\xi \in L^2([0,T]; H^1 \cap L^2_{\langle\xi\rangle_+^{\frac{3}{4}}}),$$
$$\psi_\xi \in L^2([0,T]; H^2 \cap L^2_{\langle\xi\rangle_+})\}.$$

Setting

$$N_0(t) = \sup_{0\le\tau\le t} \|(\phi,\psi)(\tau)\|_2,$$

$$N_1(t) = \sup_{0\le\tau\le t} |(\phi,\psi)(\tau)|_{2,\langle\xi\rangle_+},$$

$$N_2(t) = \sup_{0\le\tau\le t} (\|(\phi,\psi)(\tau)\|_2 + |(\phi,\psi)(\tau)|_{\langle\xi\rangle_+} + |\phi_\xi(\tau)|_{\langle\xi\rangle_+^{\frac{3}{4}}}),$$

we have by the embedding theorem in Section 1.2, Chapter 1

$$\begin{cases} \sup_{\xi\in R} |(\phi,\psi)(t,\xi)| \le CN_0(t), \\ \sup_{\xi\in R} |(\phi,\psi)(t,\xi)| \le C\sup_{\xi\in R} |\langle\xi\rangle_+^{1/2}(\phi,\psi)(t,\xi)| \le CN_1(t), \\ \sup_{\xi\in R} |\psi(t,\xi)| \le C\sup_{\xi\in R} |\langle\xi\rangle_+^{3/4}\psi(t,\xi)| \le CN_2(t), \\ \sup_{\xi\in R} |(\phi,\psi)(t,\xi)| \le CN_2(t). \end{cases} \tag{3.2.11}$$

Theorem 3.2.1 and Theorem 3.2.2 can be regarded as the direct consequences from the following theorem.

**Theorem 3.2.4.** (A) (Non-degenerate Case): *Suppose the assumptions in Theorem 3.2.1. Then there exists a positive constant* $\delta_{3-4}$ *such that if* $\|(\phi_0,\psi_0)\|_2 < \delta_{3-4}$, *then (3.2.10) has a unique global solution* $(\phi,\psi) \in X_0(0,\infty)$ *satisfying*

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\} d\tau \le C\|(\phi_0,\psi_0)\|_2^2 \tag{3.2.12$_0$}$$

*for any* $t \ge 0$. *Moreover, the stability holds in the following sense:*

$$\sup_{\xi\in R} |(\phi_\xi,\psi_\xi)(t,\xi)| \to 0 \quad as \quad t\to\infty. \tag{3.2.13}$$

(B) (Degenerate Case): *Suppose the assumptions in Theorem 3.2.2, then we have the followings.*

*(i) There exists a positive constant* $\delta_{3-5}$ *such that if* $|(\phi_0,\psi_0)|_{2,\langle\xi\rangle_+} < \delta_{3-5}$, *then (3.2.10) has a unique global solution* $(\phi,\psi) \in X_1(0,\infty)$ *satisfying*

$$|(\phi,\psi)(t)|^2_{2,\langle\xi\rangle_+} + \int_0^t \{|\phi_\xi(\tau)|^2_{1,\langle\xi\rangle_+^{1/2}} + |\psi_\xi(\tau)|^2_{2,\langle\xi\rangle_+}\}d\tau$$
$$\leq C|(\phi_0,\psi_0)|^2_{2,\langle\xi\rangle_+} \qquad (3.2.12)_1$$

*for any* $t \geq 0$. *Moreover, the stability (3.2.13) holds.*

*(ii) There exists a positive constant* $\delta_{3-6}$ *such that if* $\|(\phi_0,\psi_0)\|_2 + |(\phi_0,\psi_0)|_{\langle\xi\rangle_+}$ $+|\phi_{0,\xi}|_{\langle\xi\rangle_+^{\frac{3}{4}}} < \delta_{3-6}$, *then (3.2.10) has a unique global solution* $(\phi,\psi) \in X_2(0,\infty)$ *satisfying*

$$\|(\phi,\psi)(t)\|_2^2 + |(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + |\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}$$
$$+ \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{3/4}} + \|\psi_\xi(\tau)\|_2^2 + |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}\}d\tau$$
$$\leq C(\|(\phi_0,\psi_0)\|_2^2 + |(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}), \qquad (3.2.12)_2$$

*for any* $t \geq 0$. *Moreover, the stability (3.2.13) also holds.*

Theorem 3.2.4 is proved by a weighted energy method combining the local existence with *a priori* estimates.

**Proposition 3.2.5.**(Local existence) *For any* $\delta_0 > 0$, *there exists a positive constant* $T_0$ *depending on* $\delta_0$ *which satisfies the followings.*

*(A) (Non-degenerate Case): If* $(\phi_0, \psi_0) \in H^2$ *and* $\|(\phi_0, \psi_0)\|_2 \le \delta_0$*, then the problem (3.2.10) has a unique solution* $(\phi, \psi) \in X_0(0, T_0)$ *satisfying* $\|(\phi, \psi)(t)\|_2 \le 2\delta_0$ *for* $0 \le t \le T_0$.

*(B) (Degenerate Case): (i) If* $(\phi_0, \psi_0) \in H^2_{\langle\xi\rangle_+}$ *and* $|(\phi_0, \psi_0)|_{2,\langle\xi\rangle_+} \le \delta_0$*, then the problem (3.2.10) has a unique solution* $(\phi, \psi) \in X_1(0, T_0)$ *satisfying* $|(\phi, \psi)(t)|_{2,\langle\xi\rangle_+} \le 2\delta_0$ *for* $0 \le t \le T_0$.

*(ii) If* $(\phi_0, \psi_0) \in H^2 \cap L^2_{\langle\xi\rangle_+}$*, and* $\phi_{0,\xi} \in L^2_{\langle\xi\rangle_+^{\frac{3}{4}}}$*,* $\|(\phi_0, \psi_0)\|_2 + |(\phi_0, \psi_0)|_{\langle\xi\rangle_+} + |\phi_{0,\xi}|_{\langle\xi\rangle_+^{\frac{3}{4}}} \le \delta_0$*, then the problem (3.2.10) has a unique solution* $(\phi, \psi) \in X_2(0, T_0)$ *satisfying* $\|(\phi, \psi)(t)\| + |(\phi, \psi)(t)|_{\langle\xi\rangle_+} + |\phi_\xi|_{\langle\xi\rangle_+^{\frac{3}{4}}} \le 2\delta_0$ *for* $0 \le t \le T_0$.

**Proposition 3.2.6.** (A priori estimate) *(A) (Non-degenerate Case): Let* $(\phi, \psi) \in X_0(0, T)$ *be a solution for a positive* $T$*. Then there exists a positive constant* $\delta_{3-7}$ *independent of* $T$ *such that if* $N_0(T) < \delta_{3-7}$*, then* $(\phi, \psi)$ *satisfies the a priori estimate* $(3.2.12)_0$ *for* $0 \le t \le T$.

*(B) (Degenerate Case): (i) Let* $(\phi, \psi) \in X_1(0, T)$ *be a solution for a positive* $T$*. Then there exists a positive constant* $\delta_{3-8}$ *independent of* $T$ *such that if* $N_1(T) < \delta_{3-8}$*, then* $(\phi, \psi)$ *satisfies the a priori estimate* $(3.2.12)_1$ *for* $0 \le t \le T$.

*(ii) Let* $(\phi, \psi) \in X_2(0, T)$ *be a solution for a positive* $T$*. Then there exists a positive constant* $\delta_{3-9}$ *independent of* $T$ *such that if* $N_2(T) < \delta_{3-9}$*, then* $(\phi, \psi)$ *satisfies the a priori estimate* $(3.2.12)_2$ *for* $0 \le t \le T$.

Proposition 3.2.5 can be proved in the standard way. So we omit the proof.

We shall prove Proposition 3.2.6 in the next section.

## 3.3. The Proofs of A Priori Estimates

This section is a key step to complete the proofs of the stability theorems. At first, let's introduce our desired weight functions which pay a key role for our *a priori* estimates. Let a weight function be

$$w(v) = \begin{cases} w_0(v) & = \frac{v^2 - v_+^2}{h(v)}, \quad v \in I_0, \\ w_{2j-1}(v) & = k_{2j-1} \cdot \frac{-1}{h(v)}, \quad v \in I_{2j-1}, \\ w_{2j}(v) & = k_{2j} \cdot \frac{1}{\sigma'(v)}, \quad v \in I_{2j}, \\ w_{n+2}(v) & = k_{n+1} \cdot \frac{1}{\sigma'(v)}, \quad v \in I_{n+2}, \end{cases} \tag{3.3.1}$$

where $j = 1, \cdots, \frac{n+1}{2}$, $I_i$ $(i = 0, 1, \cdots, n+1, n+2)$ are mentioned in (3.1.16)-(3.1.18) and $k_1 = v_+^2$, $k_2 = -k_1\sigma'(v_1)/h(v_1)$, $k_{2j-1} = -k_{2j-2}h(v_{2j-2})/\sigma'(v_{2j-2})$, $k_{2j} = -k_{2j-1}\sigma'(v_{2j-1})/h(v_{2j-1})$, $j = 2, \cdots, \frac{n+1}{2}$. So $k_i > 0$ $(i = 1, 2, \cdots, n+1)$.

We also denote $r(\xi)$ as another weight function in the form

$$r(\xi) = \begin{cases} 1 + \xi - \xi_0, & \text{as } \xi \geq \xi_0, \\ 1, & \text{as } \xi \leq \xi_0, \end{cases} \tag{3.3.2}$$

where $\xi_0$ is defined as such number that $V(\xi_0) = 0$ in Section 3.1. Then we know that $w(V) \in C^0(v_+, v_-]$, $w(V) \notin C^1(v_+, v_-]$, but $w_i(V) \in C^2(I_i)$, $i = 0, 1, \cdots, n+1, n+2$. $r(\xi)$ has the same property of $w(V)$. Moreover, we find

$$\text{non-degenerate case:} \quad w(V(\xi)) \sim Const., \quad L_w^2 = L^2, \tag{3.3.3$_1$}$$

$$\text{degenerate case:} \quad w(V(\xi)) \sim r(\xi) \sim \langle\xi\rangle_+, \quad L_w^2 = L_r^2 = L_{\langle\xi\rangle_+}^2. \tag{3.3.3$_2$}$$

Now we are going to prove part (B) of Proposition 3.2.6 by the following two sub-sections. Since part (A) of Proposition 3.2.6 can be proved in the same procedure as (B), we omit its detail and only give a remark in the following sub-section.

## 3.3.1. The Proof of Proposition 3.2.6 B(i)

Let $(\phi, \psi) \in X_1(0,T)$ be a solution of (3.2.10). On the every interval $R_i$ ($i = 0, 1, \cdots, n+2$), multiplying the first equation of (3.2.10) by $(w_i\sigma')(V)\phi$ and the second equation of (3.2.10) by $w_i(V)\psi$ and adding those equations, we have

$$\begin{aligned}&\frac{1}{2}\{(w_i\sigma')(V)\phi^2 + w_i(V)\psi^2\}_t - \{(w_i\sigma')(V)\phi\psi + \mu w_i(V)\psi\psi_\xi\}_\xi \\ &- \frac{s}{2}\{(w_i\sigma')(V)\phi^2 + w_i(V)\psi^2\}_\xi + \mu w_i(V)\psi_\xi^2 + A_i(t,\xi) \\ &= F w_i(V)\psi, \qquad (3.3.4)\end{aligned}$$

where

$$\begin{aligned}A_i(t,\xi) &= \frac{s}{2}(w_i\sigma')'(V)V_\xi\phi^2 + \mu w_i'(V)V_\xi\psi\psi_\xi \\ &+ (w_i\sigma')'(V)V_\xi\phi\psi + \frac{s}{2}w_i'(V)V_\xi\psi^2, \quad i = 0, 1, \cdots, n+2. \qquad (3.3.5)\end{aligned}$$

Integrating (3.3.4) over $R_i$ and adding thses integrated equations, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_R ((w\sigma')(V)\phi^2 + w(V)\psi^2)d\xi + \mu\sum_{i=0}^{n+2}\int_{R_i} w(V)\psi_\xi^2 d\xi \\ &+ \sum_{i=0}^{n+2}\int_{R_i} A_i(t,\xi)d\xi = \int_R Fw(V)\psi d\xi. \qquad (3.3.6)\end{aligned}$$

*Step 1.* When $\xi \in R_0$, *i.e.*, $v \in I_0 = (v_+, 0]$, we can check the facts $w_0'(v) < 0$, $(w_0\sigma')'(v) \le 0$ and $(w_0 h)''(v) = 2$, similar to Nishihara[17]. By (3.3.5), (3.2.4) and

Cauchy's inequality, and noting

$$-\frac{1}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 = -(\frac{1-\alpha}{2s}+\frac{\alpha}{2s})w_0'(0)h(0)\psi(t,\xi_0)^2,$$

where $\alpha$ is a constant which will be suitably chosen as $0<\alpha<1$, we obtain

$$\begin{aligned}
&\int_{R_0} A_0(t,\xi)d\xi \\
&= -\frac{1}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 \\
&\quad + \int_{R_0}\frac{sV_\xi}{2}[(w_0\sigma')'(V)(\phi+\frac{1}{s}\psi)^2 - \frac{(w_0h)''(V)}{s^2}\psi^2]d\xi \\
&\geq -\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 \\
&\quad + \frac{\alpha}{2s}w_0'(0)\int_{R_0}\frac{\partial}{\partial\xi}(h(V)\psi(t,\xi)^2)d\xi + \int_{R_0} -\frac{V_\xi}{s}\psi^2 d\xi \\
&\geq -\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 \\
&\quad - \frac{\mu}{2}\int_{R_0} w_0(V)\psi_\xi^2 d\xi + \int_{R_0} -\frac{V_\xi}{2s}p_\alpha(V)\psi^2 d\xi, \qquad (3.3.7)
\end{aligned}$$

where

$$p_\alpha(V) = 2 - \alpha w_0'(0)h'(V) + \frac{\alpha^2 w_0'(0)^2 h(V)}{w_0(V)}. \qquad (3.3.8)$$

**Lemma 3.3.1.** *Suppose that (3.2.3) holds. Let* $\alpha=(1-\bar{\delta})^2$, *then* $p_\alpha(V)\geq\bar{\delta}(2-\bar{\delta})$.

**Proof.** Since $\sigma''(V)<0$ $(V<0)$, and $\sigma'(V)>0$, *i.e.*, $\sigma'(V)$ is decrease on $I_0$ and $\sigma(V)$ is increase on $[v_+,v_-]$, we have $0<(s^2-\sigma'(V))/(s^2-\sigma'(0))\leq 1$, $0<-(V-v_+)/(V+v_+)<1$, and $0\leq\left(s^2-\frac{\sigma(V)-\sigma(v_+)}{V-v_+}\right)/\left(s^2-\frac{\sigma(0)-\sigma(v_+)}{-v_+}\right)\leq 1$. Noting $w_0'(0)=\frac{v_+^2 h'(0)}{h(0)^2}$ and (3.3.8), both above-mentioned facts, we obtain

$$p_\alpha(V) = 2-\alpha\Big(\frac{v_+h'(0)}{h(0)}\Big)^2\Big\{\frac{s^2-\sigma'(V)}{s^2-\sigma'(0)}$$

$$-\alpha\frac{V-v_+}{V+v_+}\Big(s^2-\frac{\sigma(V)-\sigma(v_+)}{V-v_+}\Big)^2/\Big(s^2-\frac{\sigma(0)-\sigma(v_+)}{-v_+}\Big)^2\Big\}$$

$$\geq 2-\alpha(1+\alpha)\Big(\frac{v_+h'(0)}{h(0)}\Big)^2. \qquad (3.3.9)$$

By (3.2.3), we find

$$\Big|\frac{v_+h'(0)}{h(0)}\Big|=\frac{-v_+\overline{\sigma}'(v_+)}{\overline{\sigma}(v_+)-v_+\overline{\sigma}'(v_+)}=\frac{1}{1-\overline{\delta}}.$$

Due to (3.3.9) and $\alpha=(1-\overline{\delta})^2$, we obtain $p_\alpha\geq\overline{\delta}(2-\overline{\delta})$. This completes the proof of Lemma 3.3.1.

**Lemma 3.3.2.** *Consider the non-degenerate case. For any fixed $\alpha$ $(0<\alpha<1)$, if $|v_+-v_-|$ is suitably small, then there exits a positive constant $C_\alpha$ depending on $\alpha$, such that $p_\alpha(V(\xi))\geq C_\alpha$ for any $\xi\in R$.*

**Proof.** Let $H(v_+)\equiv-\frac{v_+h'(0)}{h(0)}>0$ be a function on $v_+$. Due to (3.1.4),(3.3.10) and $s^2=\frac{\sigma(v_+)-\sigma(v_-)}{v_+-v_-}$, here, $v_+$ and $v_-$ are independent, we know $lim_{v_+\to0}H(v_+)$ exists with the type of "$\frac{0}{0}$" as the following

$$\lim_{v_+\to0}H(v_+)=\lim_{v_+\to0}\frac{-v_+(\sigma'(0)-s^2)}{\sigma(0)-\sigma(v_+)+s^2v_+}=1. \qquad (3.3.11)$$

For an arbitrary given constant $\varepsilon_1>0$, by (3.3.11), there exists a $\delta=\delta(\varepsilon_1)>0$ such that $|v_+|<\delta$, then $0<H(v_+)<1+\varepsilon_1$ holds. Thus, by (3.3.9), we get

$$p_\alpha(V(\xi))\geq2-\alpha(1+\alpha)(1+\varepsilon_1)^2\equiv C_\alpha,\ \text{for}\ \xi\in R$$

by choosing such positive constant $\varepsilon_1$ that $\varepsilon_1 < \sqrt{\frac{2}{\alpha(1+\alpha)}} - 1$ for any $\alpha \in (0,1)$.

By Lemma 3.3.1, substituting (3.3.7) back into (3.3.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \{(w\sigma')(V)\phi^2 + w(V)\psi^2\}d\xi + \frac{\mu}{2}\int_{R_0} w_0(V)\psi_\xi^2 d\xi \\
&+\mu\sum_{i=1}^{n+2}\int_{R_i} w_i(V)\psi_\xi^2 d\xi - \frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 \\
&+\int_{R_0} -\frac{V_\xi}{2s}\overline{\delta}(2-\overline{\delta})\psi(t,\xi)^2 d\xi + \sum_{i=1}^{n+2}\int_{R_i} A_i(t,\xi)d\xi \\
&\le \int_R Fw(V)\psi d\xi. \qquad (3.3.12)
\end{aligned}$$

*Step 2.* Due to the continunity of $w(V)$, *i.e.*, $w_i(v_i) = w_{i+1}(v_i)$, and $w_0'(0) = -w_1(0)h'(0)/h(0)$, $h'(v_*) = 0$ (see(3.1.5)), we have

$$\begin{aligned}
&-\frac{1-\alpha}{2s}w_0'(0)h(0)\psi(t,\xi_0)^2 = \frac{1-\alpha}{2s}w_1(0)h'(0)\psi(t,\xi_0)^2 \\
&= \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\{w_i(v_{i-1})h'(v_{i-1})\psi(t,\xi_{i-1})^2 - w_i(v_i)h'(v_i)\psi(t,\xi_i)^2\} \\
&= \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\int_{R_i}\frac{\partial}{\partial\xi}(w_i(V)h'(V)\psi(t,\xi)^2)d\xi \\
&= \frac{1-\alpha}{2s}\sum_{i=1}^{n+1}\int_{R_i} B_i(t,\xi)d\xi, \qquad (3.3.13)
\end{aligned}$$

where

$$\begin{aligned}
B_i(t,\xi) = &[w_i'(V)h'(V) + w_i(V)h''(V)]V_\xi\psi^2 \\
&+ 2w_i(V)h'(V)\psi\psi_\xi, \quad i = 1,\cdots,n+1. \qquad (3.3.14)
\end{aligned}$$

Substituting (3.3.13) into (3.3.12), we have

$$\frac{1}{2}\frac{d}{dt}\int_R \{(w\sigma')(V)\phi^2 + w(V)\psi^2\}d\xi + \frac{\mu}{2}\int_{R_0} w_0(V)\psi_\xi^2 d\xi$$

$$+\mu\sum_{i=1}^{n+2}\int_{R_i} w_i(V)\psi_\xi^2 d\xi + \int_{R_0} -\frac{V_\xi}{2s}\bar{\delta}(2-\bar{\delta})\psi(t,\xi)^2 d\xi$$

$$+\sum_{i=1}^{n+1}\int_{R_i}(A_i(t,\xi)+\frac{1-\alpha}{2s}B_i(t,\xi))d\xi + \int_{R_i} A_{n+2}(t,\xi)d\xi$$

$$\leq \int_R Fw(V)\psi d\xi. \tag{3.3.15}$$

Using Cauchy's inequality and $\mu s V_\xi = h(V)$, we obtain

$$\sum_{i=1}^{n+1}\int_{R_i}(A_i(t,\xi)+\frac{1-\alpha}{2s}B_i(t,\xi))d\xi$$

$$=\int_{R_i}\{\frac{1}{2s}(w_i\sigma')'(V)V_\xi(s\phi+\psi)^2$$

$$-\frac{1-\alpha}{2s}w_i(V)V_\xi[\frac{w_i'(V)}{w_i(V)}h'(V)+h''(V)]\psi^2$$

$$+\mu w_i(V)[\frac{w_i'(V)}{w_i(V)}V_\xi+\frac{1-\alpha}{s\mu}h'(V)]\psi\psi_\xi\}d\xi$$

$$\geq \int_{R_i} -\frac{V_\xi}{2s}\{\alpha w_i(V)z_i(V)\psi^2 - y_i(V)(s\phi+\psi)^2\}d\xi$$

$$-\frac{\mu}{2}\int_{R_i} w_i(V)\psi_\xi^2 d\xi, \qquad i=1,\cdots,n+1, \tag{3.3.16}$$

where

$$z_i(V) = h''(V) + h'(V)\frac{w_i'(V)}{w_i(V)} + \frac{h(V)}{\alpha}\Big[\frac{w_i'(V)}{w_i(V)} + (1-\alpha)\frac{h'(V)}{h(V)}\Big]^2, \tag{3.3.17}$$

$$y_i(V) = (w_i\sigma')'(V). \tag{3.3.18}$$

We can prove the followings

**Lemma 3.3.3.** *It holds*

$$z_i(V)\geq 0, \quad y_i(V)\leq 0 \quad \textit{for} \quad V\in I_i, \quad i=1,\cdots,n+1, \tag{3.3.19}$$

*provided the shock strength is suitably small.*

**Proof.** Since the weight functions $w_{2j}(V)$ on $I_{2j}$ are different from the $w_{2j-1}(V)$ on $I_{2j-1}$, $j = 1, \cdots, \frac{n+1}{2}$, we have to divide the arguments into two parts to discuss (3.3.19) as follows.

*Part 1.* When $V \in I_{2j}$, *i.e.*, $\xi \in R_{2j}$, $j = 1, \cdots, \frac{n+1}{2}$, noting $w_{2j} = k_{2j}\sigma'(V)^{-1}$, $\sigma''(V) = h''(V) \geq 0$ and $G(V) \leq 0$, *i.e.*, $0 < \frac{h'(V)}{h(V)} \leq \frac{\sigma''(V)}{\sigma'(V)}$, we have $y_{2j}(V) \equiv 0$ and

$$z_{2j}(V) = s^2\frac{\sigma''(V)}{\sigma'(V)} + \frac{h(V)}{\alpha}\Big[-\frac{\sigma''(V)}{\sigma'(V)} + (1-\alpha)\frac{h'(V)}{h(V)}\Big]^2$$

$$\geq s^2\frac{\sigma''(V)}{\sigma'(V)} + \frac{h(V)}{\alpha}\Big(\frac{\sigma''(V)}{\sigma'(V)}\Big)^2$$

$$= s^2\frac{\sigma''(V)}{\sigma'(V)}(1 - q_{2j,\alpha}(V)) \geq 0,$$

where

$$q_{2j,\alpha}(V) = -\frac{h(V)\sigma''(V)}{s^2\alpha\sigma'(V)} \geq 0,$$

and $max_{V\in I_{2j}} q_{2j,\alpha}(V) < 1$ as $|v_+ - v_-| \ll 1$.

*Part 2.* When $V \in I_{2j-1}$, *i.e.*, $\xi \in R_{2j-1}$, $j = 1, \cdots, \frac{n+1}{2}$, since $w_{2j-1} = -k_{2j-1}/h(V)$, $\sigma''(V) = h''(V) \geq 0$, $h(V) < 0$ and $G(V) \geq 0$, we have

$$y_{2j-1}(V) = (w_{2j-1}\sigma')'(V)$$

$$= w'_{2j-1}(V)\sigma'(V) + w_{2j-1}(V)\sigma''(V)$$

$$= k_{2j-1}[\frac{h'(V)\sigma'(V)}{h(V)^2} - \frac{\sigma''(V)}{h(V)}]$$

$$= -k_{2j-1}G(V)/h(V)^2 \leq 0,$$

and

$$z_{2j-1}(V) = h''(V) + h'(V)\frac{w'_{2j-1}(V)}{w_{2j-1}(V)} + \frac{h(V)}{\alpha}[\frac{w'_{2j-1}(V)}{w_{2j-1}(V)} + (1-\alpha)\frac{h'(V)}{h(V)}]^2$$

$$= h''(V) - (1-\alpha)\frac{h'(V)^2}{h(V)} \geq 0.$$

Thus, we have proved Lemma 3.3.3.

By (3.3.15), (3.3.16) and (3.3.19), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R ((w\sigma')(V)\phi^2 + w(V)\psi^2)d\xi + \frac{\mu}{2}\int_{R_0} w_0(V)\psi_\xi^2 d\xi \\
&+ \frac{\mu}{2}\sum_{i=1}^{n+1}\int_{R_i} w_i(V)\psi_\xi^2 d\xi + \mu\int_{R_{n+2}} w_{n+2}(V)\psi_\xi^2 d\xi \\
&+ \int_{R_0} -\frac{V_\xi}{2s}\bar{\delta}(2-\bar{\delta})\psi(t,\xi)^2 d\xi + \int_{R_{n+2}} A_{n+2}(t,\xi)d\xi \\
&\leq \int_R Fw(V)\psi d\xi. \qquad (3.3.20)
\end{aligned}$$

*Step 3.* Now we consider the last term in the left hand side of (3.3.20). When $V \in I_{n+2}$, *i.e.*, $\xi \in R_{n+2} = (-\infty, \xi_*]$, due to Kawashima and Matsumura [7], we have

$$\begin{aligned}
&\int_{R_{n+2}} A_{n+2}(t,\xi)d\xi \\
&= \int_{R_{n+2}} -\frac{w_{n+2}(V)}{2s}V_\xi[\frac{w'_{n+2}(V)}{w_{n+2}(V)}h'(V) + h''(V)]\psi^2 d\xi
\end{aligned}$$

$$+\int_{R_{n+2}}\frac{V_\xi}{2s}(w_{n+2}\sigma')'(V)(s\phi+\psi)^2d\xi+\mu\int_{R_{n+2}}w'_{n+2}(V)V_\xi\psi\psi_\xi d\xi$$

$$\geq -\frac{\mu}{2}\int_{R_{n+2}}w_{n+2}(V)\psi_\xi^2d\xi+\int_{R_{n+2}}\frac{V_\xi}{2s}y_{n+2}(V)(s\phi+\psi)^2d\xi$$

$$-\int_{R_{n+2}}\frac{V_\xi}{2s}w_{n+2}(V)z_{n+2}(V)\psi^2d\xi, \qquad (3.3.21)$$

where $y_{n+2}(V)=(w_{n+2}\sigma')'(V)$ and

$$z_{n+2}(V)=h''(V)+h'(V)\frac{w'_{n+2}(V)}{w_{n+2}(V)}+2h(V)[\frac{w'_{n+2}(V)}{w_{n+2}(V)}]^2.$$

Corresponding [7], it is easily checked that

$$y_{n+2}(V)\equiv 0 \quad \text{and} \quad z_{n+2}(V)\geq 0 \qquad (3.3.22)$$

provided $|v_+-v_-|\ll 1$.

Substituting (3.3.21)-(3.3.22) into (3.3.20) and integrating the resultant inequality over $[o,t]$, we have the following first Key Lemma.

**Key Lemma 3.3.4.** *It holds*

$$|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+}+\int_0^t|\psi_\xi(\tau)|^2_{\langle\xi\rangle_+}d\tau+\int_0^t\int_{R_0}|V_\xi|\psi(\tau,\xi)^2d\xi d\tau$$

$$\leq C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+N_1(t)\int_0^t|\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}d\tau). \qquad (3.3.23)$$

*Remark 3.3.5.* In the non-degenerate case, noting Lemma 3.3.2 and $(3.3.3)_1$, both step 1-step 3, we get

$$\|(\phi,\psi)(t)\|^2+\int_0^t\|\psi_\xi(\tau)\|^2d\tau+\int_0^t\int_{R_0}|V_\xi|\psi(\tau,\xi)^2d\xi d\tau$$

$$\leq C(\|(\phi_0,\psi_0)\|^2+N_0(t)\int_0^t\|\phi_\xi(\tau)\|^2d\tau).$$

Thus, we can prove part (A) of Proposition 3.2.6 corresponding the procedure in [7,17].

*Remark 3.3.6.* $G(v)$ may have infinitely many $v_i$'s defined in (3.1.16), so that there are cluster points. But, both endpoints 0 and $v_*$ are not cluster points by (3.1.15). Denote a cluster point by $\overline{v}^i$ with $v_j^i \to \overline{v}^i$ as $j \to \infty$. Then, we have $\lim_{j\to\infty} k_j^i \equiv k^i < +\infty$ and also $\lim_{j\to\infty} \frac{k_{2j-1}^i}{k_{2j}^i} = -\frac{h(\overline{v}^i)}{\sigma'(\overline{v}^i)}$. Due to this, at each cluster point, by changing $\sum_{j=0}^{n+2}$ to $\sum_{ij}$, the procedures in steps 2-3 are still available.

The next Key Lemma is to estimate the last term in (3.3.23) provided small $N_1(t)$.

**Key Lemma 3.3.7.** *It holds*

$$
\begin{aligned}
&|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} + (1 - CN_1(t)) \int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} d\tau \\
&\leq C(|(\phi_0, \psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}). \qquad (3.3.24)
\end{aligned}
$$

**Proof.** From equations (3.2.10), we have

$$
\mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + \sigma'(V)\phi_\xi + s\psi_\xi - \psi_t = -F. \qquad (3.3.25)
$$

Since $L^2_{w(V)} = L^2_{r(\xi)} = L^2_{\langle\xi\rangle_+}$, firstly, let's consider our problem in the weighted space $L^2_{r(\xi)}$.

*(i).* On the interval $[\xi_0, +\infty) = R_0$, *i.e.*, $v \in (v_+, 0]$, multiplying (3.3.25) by $r(\xi)^{\frac{1}{2}}\phi_\xi$, here $r(\xi) = 1 + \xi - \xi_0$ (see (3.3.2)), we get

$$
\frac{\mu}{2}\{r(\xi)^{\frac{1}{2}}\phi_\xi^2\}_t - \frac{s\mu}{2}\{r(\xi)^{\frac{1}{2}}\phi_\xi^2\}_\xi + \frac{s\mu}{4}r(\xi)^{-\frac{1}{2}}\phi_\xi^2
$$

$$+ r(\xi)^{\frac{1}{2}}\sigma'(V)\phi_\xi^2 + sr(\xi)^{\frac{1}{2}}\phi_\xi\psi_\xi - r(\xi)^{\frac{1}{2}}\psi_t\phi_\xi = -Fr(\xi)^{\frac{1}{2}}\phi_\xi. \qquad (3.3.26)$$

By the first equation in (3.2.10), we note that

$$\begin{aligned}
-r(\xi)^{\frac{1}{2}}\psi_t\phi_\xi &= -\{r(\xi)^{\frac{1}{2}}\psi\phi_\xi\}_t + r(\xi)^{\frac{1}{2}}\psi\phi_{\xi t} \\
&= -\{r(\xi)^{\frac{1}{2}}\psi\phi_\xi\}_t + r(\xi)^{\frac{1}{2}}\psi(s\phi_\xi + \psi_\xi)_\xi \\
&= -\{r(\xi)^{\frac{1}{2}}\psi\phi_\xi\}_t + \{r(\xi)^{\frac{1}{2}}\psi(s\phi_\xi + \psi_\xi)\}_\xi \\
&\quad - sr(\xi)^{\frac{1}{2}}\psi_\xi\phi_\xi - r(\xi)^{\frac{1}{2}}\psi_\xi^2 \\
&\quad - \frac{s}{2}r(\xi)^{-\frac{1}{2}}\psi\phi_\xi - \{\frac{1}{4}r(\xi)^{-\frac{1}{2}}\psi^2\}_\xi - \frac{1}{8}r(\xi)^{-\frac{3}{2}}\psi^2, \qquad (3.3.27)
\end{aligned}$$

and

$$|-\frac{s}{2}r(\xi)^{-\frac{1}{2}}\psi\phi_\xi| \le \varepsilon_2 r(\xi)^{\frac{1}{2}}\sigma'(V)\phi_\xi^2 + (16\sigma'(0)\varepsilon_2)^{-1}s^2 r(\xi)^{-\frac{3}{2}}\psi^2, \qquad (3.3.28)$$

where $0 < \varepsilon_2 < 1$.

Substituting (3.3.28), (3.3.27) into (3.3.26), and integrating it over $R_0$, we have

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_0} - \frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi\psi d\xi \\
&+ \frac{s\mu}{4}\int_{\xi_0}^{+\infty} r(\xi)^{-\frac{1}{2}}\phi_\xi^2 + (1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\sigma'(V)\phi_\xi^2 d\xi \\
&- (\frac{s^2}{16\sigma'(0)\varepsilon_2} + \frac{1}{8})\int_{\xi_0}^{+\infty} r(\xi)^{-\frac{3}{2}}\psi^2 d\xi \\
&- \int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\psi_\xi^2 d\xi + \frac{1}{4}r(\xi_0)^{-\frac{1}{2}}\psi(t,\xi_0)^2 \\
&\le -\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi F d\xi, \qquad (3.3.29)
\end{aligned}$$

where $\{\cdots\} = -\frac{s\mu}{2}r(\xi)^{\frac{1}{2}}\phi_\xi^2 + r(\xi)^{\frac{1}{2}}\psi(s\phi_\xi + \psi_\xi)$. Moreover, by Cauchy's inequality, we find the fact

$$
\begin{aligned}
&\frac{1}{4}r(\xi_0)^{-\frac{1}{2}}\psi(t,\xi_0)^2 \\
&= -\frac{1}{4}r(\xi_0)^{-\frac{1}{4}}\int_{\xi_0}^{+\infty}\frac{\partial}{\partial\xi}(r(\xi)^{-\frac{1}{4}}\psi(t,\xi)^2)d\xi \\
&= -\frac{1}{4}r(\xi_0)^{-\frac{1}{4}}\int_{\xi_0}^{+\infty}\{-\frac{1}{4}r(\xi)^{-\frac{5}{4}}\psi^2 + r(\xi)^{-\frac{1}{4}}2\psi\psi_\xi\}d\xi \\
&\geq \frac{1}{4}\int_{\xi_0}^{+\infty}\{\frac{1}{4}r(\xi)^{-\frac{5}{4}}\psi^2 - r(\xi)^{\frac{-3}{2}}\psi^2 - r(\xi)\psi_\xi^2\}d\xi,
\end{aligned}
\tag{3.3.30}
$$

where $r(\xi_0) = 1$.

Substituting (3.3.30) into (3.3.29), we have

$$
\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_0} - \frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi\psi d\xi \\
&+\frac{s\mu}{4}\int_{\xi_0}^{+\infty} r(\xi)^{-\frac{1}{2}}\phi_\xi^2 + (1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\sigma'(V)\phi_\xi^2 d\xi + \int_{\xi_0}^{+\infty} C_{\varepsilon_2}(\xi)\psi^2 d\xi \\
&\leq CN_1(t)\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi^2 d\xi + C\int_{\xi_0}^{+\infty} r(\xi)\psi_\xi^2 d\xi,
\end{aligned}
\tag{3.3.31}
$$

where

$$
C_{\varepsilon_2}(\xi) = \frac{r(\xi)^{-\frac{3}{2}}}{16}\{r(\xi)^{\frac{1}{4}} - \frac{s^2}{\varepsilon_4\sigma'(0)} - 6\}. \tag{3.3.32}
$$

Since $r(\xi) = O(|\xi|)$ as $\xi \to \infty$, thus we know there exists a larger number $\xi_{**}(> \xi_0)$ such that

$$
C_{\varepsilon_2}(\xi) \geq 0 \text{ on } [\xi_{**}, +\infty), \quad |C_{\varepsilon_2}(\xi)| \leq Const. \text{ on } [\xi_0, \xi_{**}] \tag{3.3.33}
$$

Due to (3.3.23) in Key Lemma 3.3.4 and the boundness of $|V_\xi|$ on $[\xi_0, \xi_{**}]$, noting (3.3.33) and $w(V) \sim \langle\xi\rangle_+ \sim r(\xi)$, we obtain

$$
\int_0^t\int_{\xi_0}^{\xi_{**}} |C_{\varepsilon_2}(\xi)|\psi(\tau,\xi)^2 d\xi d\tau
$$

$$\leq C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + N_1(t)\int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} d\tau), \tag{3.3.34}$$

$$\int_{\xi_{**}}^{+\infty} C_{\varepsilon_2}(\xi)\psi(\tau,\xi)^2 d\xi \geq 0. \tag{3.3.35}$$

Then we can rewrite (3.3.31) as follows

$$\frac{\mu}{2}\frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi^2 d\xi + \{\cdots\}|_{\xi=\xi_0} - \frac{d}{dt}\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi\psi d\xi$$
$$+ (1-\varepsilon_2)\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\sigma'(V)\phi_\xi^2 d\xi + \int_{\xi_{**}}^{+\infty} C_{\varepsilon_2}(\xi)\psi^2 d\xi$$
$$\leq \int_{\xi_0}^{\xi_{**}} |C_{\varepsilon_2}(\xi)|\psi^2 d\xi + CN_1(t)\int_{\xi_0}^{+\infty} r(\xi)^{\frac{1}{2}}\phi_\xi^2 d\xi$$
$$+ C\int_{\xi_0}^{+\infty} r(\xi)\psi_\xi^2 d\xi. \tag{3.3.36}$$

*(ii).* On the another interval $(-\infty,\xi_0]$, *i.e.*, $v \in [0, v_-]$, mutilplying (4.3.25) by $\phi_\xi$, and integrating it over $(-\infty,\xi_0]$, here $r(\xi)=1$, we have

$$\frac{\mu}{2}\frac{d}{dt}\int_{-\infty}^{\xi_0} \phi_\xi^2 d\xi - \{\cdots\}|_{\xi=\xi_0} - \frac{d}{dt}\int_{-\infty}^{\xi_0} \phi_\xi\psi d\xi + \int_{-\infty}^{\xi_0} \sigma'(V)\phi_\xi^2 d\xi$$
$$\leq CN_1(t)\int_{-\infty}^{\xi_0} \phi_\xi^2 d\xi + C\int_{-\infty}^{\xi_0} \psi_\xi^2 d\xi, \tag{3.3.37}$$

The continuity of $r(\xi)$ at $\xi_0$ admits the addition of (3.3.36) and (3.3.37). Noting

$$\int_{-\infty}^{+\infty} r(\xi)^{\frac{1}{2}}|\phi_\xi\psi| d\xi \leq \frac{\mu}{4}|\phi_\xi|^2_{r(\xi)^{\frac{1}{2}}} + \frac{1}{\mu}|\psi|^2_{r(\xi)},$$

and (3.3.34)–(3.3.35), Key Lemma 3.3.4, and $\langle\xi\rangle_+ \sim r(\xi)$, we have proved (3.3.24).

By Key Lemma 3.3.4 and Key Lemma 3.3.7, we have the following

**Lemma 3.3.8.** *It holds*

$$|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+} d\tau \leq C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}), \tag{3.3.39}$$

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} + \int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{1}{2}}} d\tau \le C(\|(\phi_0,\psi_0)\|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}), \tag{3.3.40}$$

*provided $N_1(T)$ is suitably small.*

Next, we shall derive the higher order estimates on the solution $(\phi, \psi)$. Let us differentiate equations (3.2.10) on $\xi$, and mutiply the frist equation by $w(V)\sigma(V)\phi_\xi$ and the second equation by $w(V)\psi_\xi$, respectively. We add them and integrate the resulted equation over $[0,t]\times R$, similar to the procedures in Lemmas 3.3.1 – 3.3.8. Then by the fact $w(V) \sim \langle\xi\rangle_+ \sim r(\xi)$ and $L^2_{w(V)} = L^2_{\langle\xi\rangle_+} = L^2_{r(\xi)}$, we have

**Lemma 3.3.9.** *It holds*

$$|(\phi_\xi,\psi_\xi)(t)|^2_{w(V)} + \int_0^t |\psi_{\xi\xi}(\tau)|^2_{w(V)} d\tau \le C(\|(\phi_0,\psi_0)\|^2_{1,w(V)} + |\phi_{0,\xi\xi}|^2_{w(V)^{\frac{1}{2}}}), \tag{3.3.41}$$

$$|\phi_{\xi\xi}(t)|^2_{w(V)^{\frac{1}{2}}} + \int_0^t |\phi_{\xi\xi}(\tau)|^2_{w(V)^{\frac{1}{2}}} d\tau \le C(\|(\phi_0,\psi_0)\|^2_{1,w(V)} + |\phi_{0,\xi\xi}|^2_{w(V)^{\frac{1}{2}}}), \tag{3.3.42}$$

*for suitably small $N_1(T)$.*

Similarly, the second order estimate of the solutions can be proved as follows.

**Lemma 3.3.10.** *It holds*

$$|(\phi_{\xi\xi},\psi_{\xi\xi})(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_{\xi\xi\xi}(\tau)|^2_{\langle\xi\rangle_+} d\tau \le C\|(\phi_0,\psi_0)\|^2_{2,\langle\xi\rangle_+} \tag{3.3.43}$$

*for suitably small $N_1(T)$.*

**The Proof of Proposition 4.2.6 B(i).** Combining Lemmas 3.3.8–3.3.10, we have

$$\|(\phi,\psi)(t)\|^2_{2,\langle\xi\rangle_+} + \int_0^t \{\|\phi_\xi(\tau)\|^2_{1,\langle\xi\rangle_+^{\frac{1}{2}}} + \|\psi_\xi(\tau)\|^2_{2,\langle\xi\rangle_+}\} d\tau \le C\|(\phi_0,\psi_0)\|^2_{2,\langle\xi\rangle_+}$$

for suitably small $N_1(T)$.

Thus, we have completed the proof of (i) of part (B) in Proposition 3.2.6.

### 3.3.2. The proof of Proposition 3.2.6 B(ii)

Let $(\phi, \psi) \in X_2(0,T)$ be a solution of (3.2.10). By the same procedures as in the last sub-section, we estabilish the key estimates correspoding to Lemmas 3.3.4−3.3.7 by the weight functions $w(V)$ and $r(\xi)$. Noting $w(V) \sim \langle\xi\rangle_+ \sim r(\xi)$,and $L^2_{w(V)} = L^2_{\langle\xi\rangle_+} = L^2_{r(\xi)}$, we obtain the following lemma.

**Lemma 3.3.11.** *It holds*

$$|(\phi,\psi)(t)|^2_{\langle\xi\rangle_+} + \int_0^t |\psi_\xi(\tau)|^2_{\langle\xi\rangle_+} d\tau \le C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}), \tag{3.3.44}$$

$$|\phi_\xi(t)|^2_{\langle\xi\rangle_+^{\frac{3}{4}}} + \int_0^t |\phi_\xi(\tau)|^2_{\langle\xi\rangle_+^{\frac{3}{4}}} d\tau \le C(|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}). \tag{3.3.45}$$

Next, we shall derive the higher order estimates on the solution $(\phi, \psi)$ without weight function. This procedure is simpler than previous one. According to Lemma 3.3.11, we can prove the following Lemmas same way as in [7,17]. So, here we also only give the sketch of the proofs.

Multiplying the second equation of (3.2.10) by $-\psi_{\xi\xi}$, and integrating it over $[0,t] \times R$, we have by Lemma 3.3.11

**Lemma 3.3.12.** *It holds*

$$\|\psi_\xi(t)\|^2 + \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau \le C(\|(\phi_{0,\xi},\psi_{0,\xi})\|^2 + |(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+} + |\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}) \tag{3.3.46}$$

*for suitably small* $N_2(T)$.

When we differentiate (3.3.25) in $\xi$ and multiply it by $\phi_{\xi\xi}$ and integrate the resultant equation over $[0,t]\times R$, then we get

**Lemma 3.3.13.** *It holds*

$$\|\phi_{\xi\xi}(t)\|^2+\int_0^t\|\phi_{\xi\xi}(\tau)\|^2d\tau$$

$$\leq C(\|\phi_{0,\xi}\|_1^2+\|\psi_{0,\xi}\|^2+|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{3}{4}}}) \tag{3.3.47}$$

*forsuitably small* $N_2(T)$.

Differentiating the second equation of (3.2.10) in $\xi$ and multiplying it by $-\psi_{\xi\xi\xi}$, then integrating the resultant equation over $[0,t]\times R$, we obtain

**Lemma 3.3.14.** *It holds*

$$\|\psi_{\xi\xi}(t)\|^2+\int_0^t\|\psi_{\xi\xi\xi}(\tau)\|^2d\tau$$

$$\leq C(\|(\phi_{0,\xi},\psi_{0,\xi}\|_1^2+|(\phi_0,\psi_0)|^2_{\langle\xi\rangle_+}+|\phi_{0,\xi}|^2_{\langle\xi\rangle_+^{\frac{1}{2}}}) \tag{3.3.48}$$

*for suitably small* $N_2(T)$.

Finally, combining Lemmas 3.3.11–3.3.14, we complete the proof of Proposition 3.2.6 B(ii).

# Chapter 4. System of Viscoelasticity (II)

The main aim in this chapter is to study the stability of viscous shock profile for Cauchy problem (1.1.3)-(1.1.5) with the non-convexity (1.1.8), which is an open problem proposed by Kawashima and Matsumura [7]. Quite different from the single equation case, the procedures in the previous works [6,7,12-19] are invalid for this problem. Therefore, we expect to develop a new recipe to solve this open problem which is the transform-weighted energy method as we call, and the details will be showed in Section 4.3.

The arrangement in this chapter is as follows. After sating some preliminaries and the main asymptotic stability theorem in Section 4.1, we prove this main theorem based on a basic energy estimate in Section 4.2. Finally, we comlete the proof of the basic estimate in Section 4.3.

## 4.1. Preliminaries and Main Theorem

In this section, before stating our main theorem, we now recall the properties of traveling wave solution with shock profile as stated in Chapter 3.

We know the traveling wave solution with shock profile for (1.1.6) and (1.1.7) has the form

$$(v,u)(t,x) = (V,U)(\xi), \quad \xi = x - st, \tag{4.1.1}$$

$$(V,U)(\xi) \to (v_\pm, u_\pm), \qquad \xi \to \pm\infty, \tag{4.1.2}$$

where $s$ is the shock speed and $(v_\pm, u_\pm)$ are constant states at $\pm\infty$ satisfying the Rankine-Hugoniot condition

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) - (\sigma(v_+) - \sigma(v_-)) = 0, \end{cases} \tag{4.1.3}$$

and the generalized shock condition

$$\frac{1}{s}h(v) \equiv \frac{1}{s}[-s^2(v - v_\pm) + \sigma(v) - \sigma(v_\pm)] \begin{cases} < 0, & \text{if} \quad v_+ < v < v_- \\ > 0, & \text{if} \quad v_- < v < v_+. \end{cases} \tag{4.1.4}$$

We note that the condition (4.1.4) with (1.1.6) and (1.1.8) implies

$$\lambda(v_+) < s \le \lambda(v_-) \quad \text{or} \quad -\lambda(v_+) < s \le -\lambda(v_-), \tag{4.1.5}$$

where $\lambda(v) = \sqrt{\sigma'(v)}$ is the positive characteristic root, and that, especially when $\sigma''(v) > 0$, the condition (4.1.4) is equivalent to

$$\lambda(v_+) < s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_+) < s < -\lambda(v_-), \tag{4.1.6}$$

which is well-known as Lax's shock condition [8]. We also call the condition (4.1.5) with $s = \lambda(v_-)$ or $= -\lambda(v_-)$ (resp. the condition (4.1.6)) the degenerate (resp. non-degenerate) shock condition.

If $(v,u)(t,x) = (V,U)(\xi)$ $(\xi = x - st)$ is the viscous shock profile, then $(V,U)(\xi)$ must satisfy

$$\begin{cases} -sV' - U' = 0, \\ -sU' - \sigma(V)' = \mu U'' \end{cases} \tag{4.1.7}$$

Integrating (4.1.7) and eliminating $U$, then we obtain a single ordinary differential equation for $V(\xi)$:

$$\mu s V' = -s^2 V + \sigma(V) - a_3 \equiv h(V), \tag{4.1.8}$$

where

$$a_3 = -s^2 v_\pm + \sigma(v_\pm). \tag{4.1.9}$$

Let $(v_+, u_+) \neq (v_-, u_-)$ and $s > 0$ (the case $s < 0$ can be treated similarly). We are now ready to summarize a characterization of the generalized shock condition (4.1.4) and the results on the existence of viscous shock profile corresponding to one in the above chapters:

**Proposition 4.1.1.** *Suppose that (1.1.6) and (1.1.8) hold. Then the following statements are equivalent to each other.*

(i) *The generalized shock condition (4.1.4) holds.*

(ii) $\sigma'(v_-) \geq s^2$, *i.e.*, $\lambda(v_-) \geq s$.

(iii) $\sigma'(v_+) < s^2 \leq \sigma'(v_-)$, *i.e.*, $\lambda(v_+) < s \leq \lambda(v_-)$.

(iv) *There exists uniquely a* $v_* \in (v_+, v_-)$ *such that* $\sigma'(v_*) = s^2$ *and it holds*

$$\sigma'(v) < s^2 \text{ for } v \in (v_+, v_*), \quad s^2 < \sigma'(v) \text{ for } v \in (v_*, v_-). \tag{4.1.10}$$

*i.e.*,

$$h'(v_*) = 0, \quad h'(v) < 0 \text{ for } v \in (v_+, v_*), \quad h'(v) > 0 \text{ for } v \in (v_*, v_-). \tag{4.1.11}$$

*Moreover, if one of the above four conditions holds, then we must have* $v_+ \neq 0$. *In addition,* $v_+ \lesseqgtr v_-$ *and* $v_* \gtreqless 0$ *hold when* $v_+ \gtreqless 0$, *i.e.*, $v_* v_+ > 0$.

**Proposition 4.1.2.** *Suppose that (1.1.6) and (1.1.8) hold.*

(i) *If (1.1.3), (1.1.4) admits a viscous shock profile* $(V(x-st), U(x-st))$ *connecting* $(v_\pm, u_\pm)$, *then* $(v_\pm, u_\pm)$ *and* $s$ *must satisfy the Rankine-Hugoniot condition (4.1.3) and the generalized shock condition (4.1.4).*

(ii) *Conversely, suppose that (4.1.3) and (4.1.4) hold, then there exists a viscous shock profile* $(V, U)(x - st)$ *of (1.1.3),(1.1.4) which connects* $(v_\pm, u_\pm)$. *The* $(V, U)(\xi)(\xi = x - st)$ *is unique up to a shift in* $\xi$ *and is a monotone function of* $\xi$. *In particular, when* $v_+ \lesseqgtr v_-$ *(and hence* $u_+ \gtreqless u_-$*) we have*

$$u_+ \gtreqless U(\xi) \gtreqless u_-, \quad U_\xi(\xi) \gtreqless 0, \tag{4.1.12}$$

$$v_+ \lesseqgtr V(\xi) \lesseqgtr v_-, \quad V_\xi(\xi) \lesseqgtr 0, \tag{4.1.13}$$

*for all* $\xi \in R$. *Moreover,* $(V, U)(\xi) \to (v_\pm, u_\pm)$ *exponentially as* $\xi \to \pm\infty$, *with the following exceptional case: when* $\lambda(v_-) = s$, $(V, U)(\xi) \to (v_-, u_-)$ *at the rate* $|\xi|^{-1}$ *as* $\xi \to -\infty$, *and* $|h(V)| = |\mu s V_\xi| = O(|\xi|^{-2})$ *as* $\xi \to -\infty$.

We see the following graphes of $\sigma(v)$ and $h(v)$.

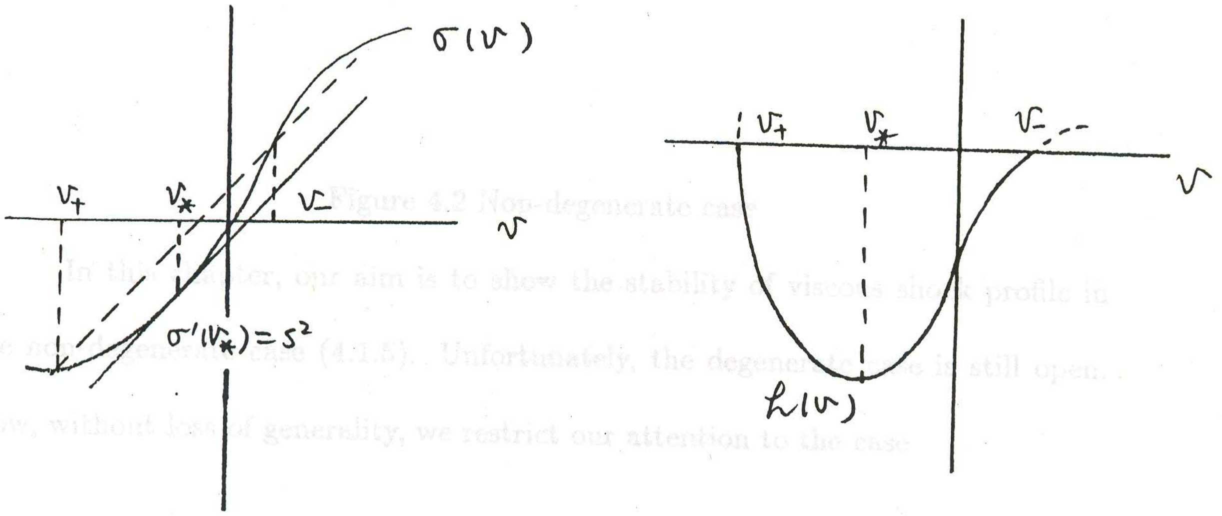


Figure 4.1 Non-degenerate case

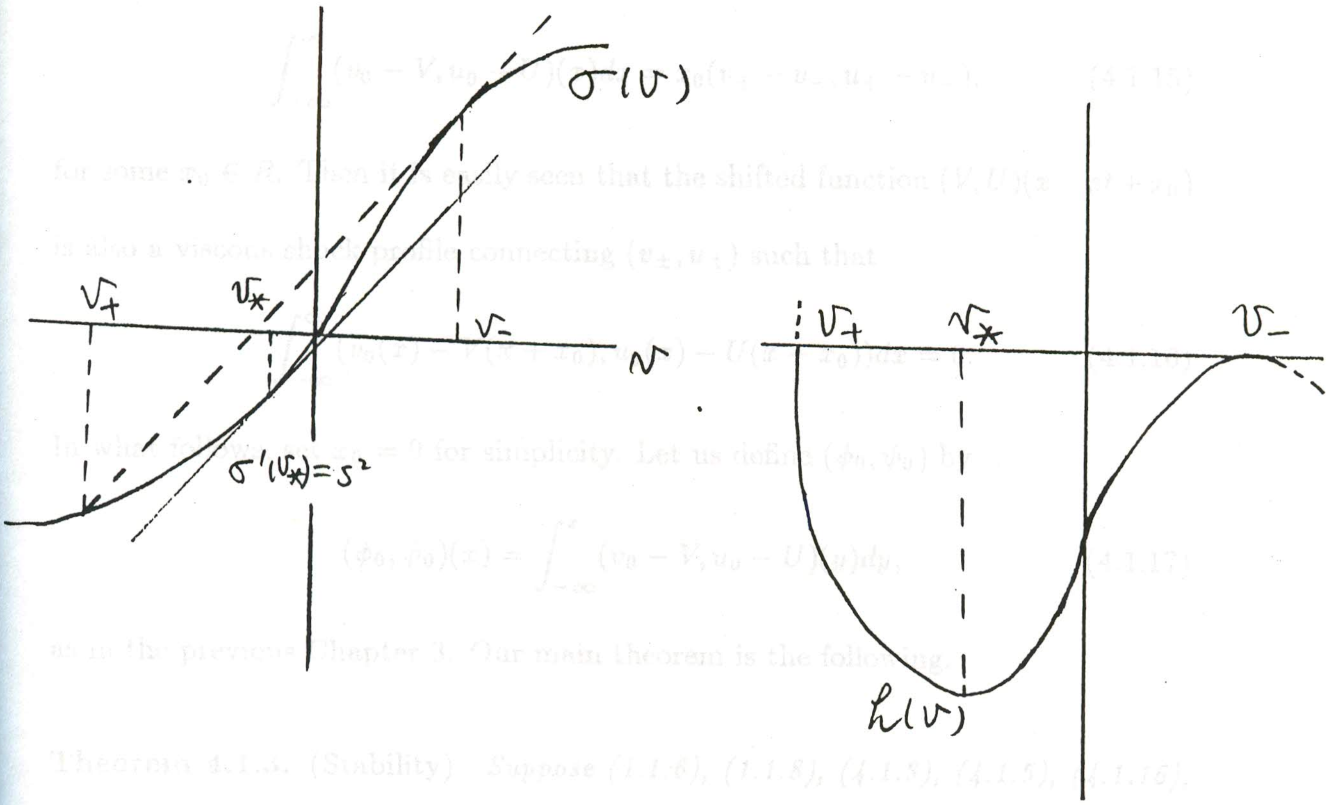


Figure 4.2 Non-degenerate case

In this chapter, our aim is to show the stability of viscous shock profile in the non-degenerate case (4.1.5). Unfortunately, the degenerate case is still open. Now, without loss of generality, we restrict our attention to the case

$$s > 0 \quad \text{and} \quad v_+ < 0 < v_-, \quad i.e., \quad \mu s V_\xi = h(V) < 0. \tag{4.1.14}$$

Let $(V, U)(x - st)$ be a viscous shock profile connecting $(v_\pm, u_\pm)$, we assume the integrability of $(v_0 - V, u_0 - U)(x)$ over $R$ and

$$\int_{-\infty}^{\infty} (v_0 - V, u_0 - U)(x)dx = x_0(v_+ - v_-, u_+ - u_-), \tag{4.1.15}$$

for some $x_0 \in R$. Then it is easily seen that the shifted function $(V, U)(x - st + x_0)$ is also a viscous shock profile connecting $(v_\pm, u_\pm)$ such that

$$\int_{-\infty}^{\infty} (v_0(x) - V(x + x_0), u_0(x) - U(x + x_0))dx = 0. \tag{4.1.16}$$

In what follows, set $x_0 = 0$ for simplicity. Let us define $(\phi_0, \psi_0)$ by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (v_0 - V, u_0 - U)(y)dy, \tag{4.1.17}$$

as in the previous Chapter 3. Our main theorem is the following.

**Theorem 4.1.3.** (Stability) *Suppose (1.1.6), (1.1.8), (4.1.3), (4.1.5), (4.1.16), and* $(\phi_0, \psi_0) \in H^2$. *Further, assume that*

$$s^2 < \sigma'(v_-) + \frac{1}{2}\sigma''(v_-)[2(v_* - v_+) + v_- - v_+], \tag{4.1.18}$$

$$\sigma'''(v) < 0, \quad v \in (v_+, 0) \cup (0, v_-). \tag{4.1.19}$$

*Then there exists a positive constant* $\delta_{4-1}$ *such that if* $\|(\phi_0, \psi_0)\|_2 < \delta_{4-1}$, *then (1.1.3)-(1.1.5) has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v - V \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^1),$$

$$u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2)$$

*and the asymptotic behavior*

$$\sup_{x \in R} |(v, u)(t, x) - (V, U)(x - st)| \to 0 \quad as \quad t \to \infty. \tag{4.1.20}$$

*Remark 4.1.4.* 1. First note that our condition (4.1.18) is, as easily seen, much stronger than Lax's shock condition. We get the stability of *any viscous shock* (weak shock or not) as long as the condition (4.1.18) is satisfied. This means that we don't necessarily assume that the viscous shock profile is weak, i.e., $|v_+ - v_-| \ll 1$, which is a sufficient condition in the previous works.

2. An important example is $\sigma(v) = \alpha v - \beta v^3$ for $v \in [v_+, v_-]$, where $\alpha$, $\beta$ are any given positive constants. It is easy to see that $\sigma(v)$ satisfies (1.1.6), (1.1.8) for some $v_+$ and $v_-$. In this case, Lax's condition is equivalent to $v_+ < -2v_-$, and our condition (4.1.18) to $v_+ < -a_* v_-$, where $a_* = 7.418190\cdots$ is a unique positive root of

$$-x^2 + 10x + 5 = 2\sqrt{3}\sqrt{x^2 - x + 1}.$$

3. For the general stress $\sigma(v)$, if viscous shock is weak, i.e., $|v_+ - v_-| \ll 1$, and suppose $\sigma'''(0) \neq 0$, then the condition (4.1.18) is equivalent to the condition $v_+ < -a_* v_-$. A significant example is $\sigma(v) = v/\sqrt{1+v^2}$.

## 4.2. Proof of Stability Theorem

In this section, we shall prove the Stability Theorem 4.1.3 by means of a key estimate which will be proved in the next section. In order to show the stability, we first make a reformulation for the problem (1.1.3)-(1.1.5) by changing unknown variables

$$(v, u)(t, x) = (V, U)(\xi) + (\phi_\xi, \psi_\xi)(t, \xi), \quad \xi = x - st \tag{4.2.1}$$

Then the problem (1.1.3)-(1.1.5) is reduced to the following "integrated" system

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - \sigma'(V)\phi_\xi - \mu\psi_{\xi\xi} = F \\ (\phi,\psi)(0,\xi) = (\phi_0,\psi_0)(\xi) \end{cases} \tag{4.2.2}$$

where

$$F = \sigma(V+\phi_\xi) - \sigma(V) - \sigma'(V)\phi_\xi.$$

For any interal $I \subset [0,\infty)$, we define the solution space of (4.2..2) as

$$X(I) = \{(\phi,\psi) \in C^0(I;H^2), \phi_\xi \in L^2(I;H^1), \psi_\xi \in L^2(I;H^2)\},$$

and set

$$N(t) = \sup_{0\le\tau\le t} \|(\phi,\psi)(\tau)\|_2.$$

It is well-known as in the previous papers that Theorem 4.1.3 can be proved by the following theorem to the problem (4.2.2).

**Theorem 4.2.1.** *Suppose the assumptions in Theorem 4.1.3. Then there exist positive constants $\delta_{4-2}$ and $C$ such that if $\|(\phi_0,\psi_0)\|_2 < \delta_{4-2}$, then (4.2.2) has a unique global solution $(\phi,\psi) \in X([0,\infty))$ satisfying*

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\} d\tau \le C\|(\phi_0,\psi_0)\|_2^2 \tag{4.2.3}$$

*for any $t \ge 0$. Moreover, the stability holds in the following sense:*

$$\sup_{\xi\in R} |(\phi_\xi,\psi_\xi)(t,\xi)| \to 0 \quad as \quad t \to \infty. \tag{4.2.4}$$

By the same continuation procedure as in the last chapters, we can prove Theorem 4.2.1 combining the following local existence and *a priori* estimate.

**Proposition 4.2.2.** (Local existence) *For any $\delta_0 > 0$, there exists a positive constant $T_0$ depending on $\delta_0$ such that, if $(\phi_0, \psi_0) \in H^2$ and $\|(\phi_0, \psi_0)\|_2 \le \delta_0$, then the problem (4.2.2) has a unique solution $(\phi, \psi) \in X([0, T_0])$ satisfying $\|(\phi, \psi)(t)\|_2 \le 2\delta_0$ for $0 \le t \le T_0$.*

**Proposition 4.2.3.** (A priori estimate) *Under the assumptions in Theorem 4.1.3, let $(\phi, \psi) \in X([0, T])$ be a solution of (4.2.2) for a positive $T$. Then there exist positive constants $\delta_{4-3}$ and $C$ which are independent of $T$ such that if $N(T) < \delta_{4-3}$, then $(\phi, \psi)$ satisfies the a priori estimate* (4.2.3) *for $0 \le t \le T$.*

The proof of Proposition 4.2.2 is standard, so we here omit it. In the rest of this paragraph, our purpose is to prove Proposition 4.2.3 by using the following key estimate. In what follows, we assume that $(\phi, \psi) \in X([0, T])$ is a solution of (4.2.2) for a positive $T$ and $N(T) \le 1$.

**Key Lemma 4.2.4.** (Basic Estimate) *Suppose the assumptions in Theorem 4.1.3. Then it holds*

$$\|(\phi, \psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \le C(\|(\phi_0, \psi_0)\|^2 + N(t) \int_0^t \|\phi_\xi(\tau)\|^2 d\tau) \quad (4.2.5)$$

*for* $t \in [0, T]$.

**Proof of Proposition 4.2.3.** Since the proof is given exactly in the same way as in [7], we only show its rough sketch. From the equations (4.2.2), we have

$$\mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + \sigma'(V)\phi_\xi + s\psi_\xi - \psi_t = -F. \tag{4.2.6}$$

Multiplying (4.2.6) by $\phi_\xi$ and integrating the resultant equality over $[0,t] \times R$, using the basic estimate of Key Lemma 4.2.4, we obtain

$$\|\phi_\xi(t)\|^2 + (1 - CN(t)) \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \leq C(\|(\phi_0, \psi_0)\|^2 + \|\phi_{0,\xi}\|^2). \tag{4.2.7}$$

For the estimates of $\psi_\xi$, we may differentiate the equations (4.2.2) in $\xi$, and mutiply the frist equation by $\sigma'(V)\phi_\xi$ and the second one by $\psi_\xi$ respectively, then may add them up and integrate the resultant equality over $[0,t] \times R$. Then, combined with (4.2.5) and (4.2.7), it consequently gives us

$$\|(\phi_\xi, \psi_\xi)(t)\|^2 + \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau \leq C\|(\phi_0, \psi_0)\|_1^2 \tag{4.2.8}$$

provided $N(T)$ is suitably small. Similarly, for the estimates of $\phi_{\xi\xi}$, differentiating equation (4.2.6) in $\xi$, multiplying it by $\phi_{\xi\xi}$, and integrating the resultant equality over $[0,t] \times R$, we then obtain, combined with (4.2.5), (4.2.7) and (4.2.8),

$$\|\phi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi}(\tau)\|^2 d\tau \leq C(\|(\phi_0, \psi_0)\|_1^2 + \|\phi_{0,\xi\xi}\|^2) \tag{4.2.9}$$

provided $N(T)$ is suitably small. Furthermore, we differentiate the second equation of (4.2.2) in $\xi$ twice, and mutiply it by $\psi_{\xi\xi\xi}$. Then, for suitably small $N(T)$, we can similarly show

$$\|(\phi_{\xi\xi}, \psi_{\xi\xi})(t)\|^2 + \int_0^t \|\psi_{\xi\xi\xi}(\tau)\|^2 d\tau \leq C\|(\phi_0, \psi_0)\|_2^2. \tag{4.2.10}$$

Combining (4.2.5)–(4.2.10) yields

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\}d\tau \le C\|(\phi_0,\psi_0)\|_2^2$$

for suitably small $N(T)$, say $N(T) < \delta_{4-3}$. Thus, we have completed the proof of Proposition 4.2.3.

## 4.3. Proof of Basic Estimate

To prove the stability by energy method, the key step is to establish the basic estimate (5.2.5) in Key Lemma 4.2.4. Since the previous procedures in [7,12-19] are invalid for the non-convexity condition (1.1.8), so we have to find another way to arrive at our goal. Here, our idea is that after transforming the system (4.2.2) into a new one by selecting a suitable transform function, we prove the basic estimate (4.2.5) by the weighted energy method with a suitable weight.

Let us introduce a transform function $T(v)$ and a weight function $w(v)$ as follows:

$$T(v) = \begin{cases} C_0(v+b), & v \in [0, v_-], \\ \sqrt{\sigma'(v)}(v+b), & v \in [v_+, 0], \end{cases} \tag{4.3.1}$$

$$w(v) = (v+b)^2, \quad v \in [v_+, v_-], \tag{4.3.2}$$

where $C_0 = \sqrt{\sigma'(0)}$ and $b$ is a positive constant choosen as

$$0 < 2v_* - 3v_+ < b < 2(s^2 - \sigma'(v_-))\sigma''(v_-)^{-1} - v_- \tag{4.3.3}$$

corresponding to the assumption (4.1.18) and Proposition 4.1.1. It is noted that $T(v)$ and $w(v)$ are bounded and positive on $[v_+, v_-]$, and are in $C^1[v_+, v_-]$, but the continunity of $T''(v)$ at the point $v = 0$ cannot be ensured, for example, in the case of $\sigma'''(0) \neq 0$. We shall show how to choose $T(v)$ and $w(v)$ in the following procedure.

Let $(\phi, \psi)(t, \xi)$ be the solution of equations (4.2.2). We define a transformation in the form

$$(\phi, \psi)(t, \xi) = T(V(\xi))(\Phi(t, \xi), \Psi(t, \xi)), \tag{4.3.4}$$

where $V(\xi)$ is the viscous shock profile. We denote $\xi_0$ as a number in R such that $V(\xi_0) = 0$. It is easily seen that $\xi_0$ is unique because of the monotonicity of $V(\xi)$, i.e., $V_\xi(\xi) < 0$. Then the equations (4.2.2) can be transformed into

$$\begin{cases} \Phi_t - s\Phi_\xi - \Psi_\xi - s\frac{T_\xi}{T}\Phi - \frac{T_\xi}{T}\Psi = 0, \\ \Psi_t - (s + 2\mu\frac{T_\xi}{T})\Psi_\xi - \sigma'(V)\Phi_\xi - \mu\Psi_{\xi\xi} \\ \quad -(s\frac{T_\xi}{T} + \mu\frac{T_{\xi\xi}}{T})\Psi - \sigma'(V)\frac{T_\xi}{T}\Phi = F/T(V), \\ (\Phi, \Psi)(0, \xi) = (\phi_0, \psi_0)(\xi)/T(V), \end{cases} \tag{4.3.5}$$

in respect of two spatial parts $\xi \in (-\infty, \xi_0]$ and $\xi \in [\xi_0, \infty)$ due to the discontinuity of $T_{\xi\xi}$ at the point $\xi_0$, where $T$ denotes the transform function $T(V)$, $T_\xi = \frac{\partial T(V)}{\partial \xi}$ and $T_{\xi\xi} = \frac{\partial^2 T(V)}{\partial \xi^2}$.

Multiplying the first equation of (4.3.5) by $\sigma'(V)w(V)\Phi$ and the second one by $w(V)\Psi$ respectively, noting $\mu V_\xi = h(V)$, we have

$$\begin{aligned} &\frac{1}{2}\{(w\sigma')(V)\Phi^2 + w(V)\Psi^2\}_t - \{\cdots\}_\xi + \mu w(V)\Psi_\xi^2 \\ &+ \frac{|V_\xi|}{2s}w(V)Y(V)(s\Phi + \Psi)^2 + \frac{|V_\xi|}{s}w(V)Z(V)\Psi^2 \\ &= Fw(V)\Psi/T(V), \end{aligned} \tag{4.3.6}$$

in respect of the two spatial parts $(-\infty, \xi_0]$ and $[\xi_0, \infty)$, where

$$\begin{aligned}\{\cdots\} =& \frac{s}{2}\sigma'(V)w(V)\Phi^2 + \sigma'(V)w(V)\Phi\Psi \\ &+ \Big[(\frac{s}{2} + \mu\frac{T_\xi}{T})w(V) + \frac{\mu}{2}w(V)_\xi\Big]\Psi^2,\end{aligned} \tag{4.3.7}$$

$$Y(V) = -\sigma''(V) - \sigma'(V)\Big(\frac{w'(V)}{w(V)} - 2\frac{T'(V)}{T(V)}\Big), \tag{4.3.8}$$

$$\begin{aligned}Z(V) =& \frac{\sigma''(V)}{2} + h'(V)\frac{w'(V)}{w(V)} - h(V)\frac{w'(V)}{w(V)}\frac{T'(V)}{T(V)} - h'(V)\frac{T'(V)}{T(V)} \\ &+ h(V)\Big(\frac{T'(V)}{T(V)}\Big)^2 + \frac{h(V)}{2}\Big(\frac{w'}{w}\Big)'(V) + \frac{h(V)}{2}\Big(\frac{w'(V)}{w(V)}\Big)^2.\end{aligned} \tag{4.3.9}$$

We see that the coefficient functions in (4.3.7) are continuous in $R$ since $w(V(\xi))$ and $T(V(\xi))$ are in $C^1(-\infty, \infty)$, so $\{\cdots\}_\xi$ will disappear after integration over $(-\infty, \infty)$. The most essential point of this paper is to choose $w(V)$ and $T(V)$ properly so that both $Y(V)$ and $Z(V)$ are non-negative in (4.3.6).

**Lemma 4.3.1.** *Under the sufficient conditions (4.1.18) and (4.1.19), let $T(v)$ and $w(v)$ be chosen as in (4.3.1) and (4.3.2). Then it holds*

$$Y(v) \geq 0, \qquad Z(v) \geq C_1|\sigma''(v)| \tag{4.3.10}$$

*for all $v \in [v_+, v_-]$, where $C_1 > 0$ is a constant.*

**Proof.** Since $\sigma''(v)$ changes its sign depending on the sign of $v$, we have to divide the region of $v$ into two parts as follows.

*Part 1.* When $v \in [0, v_-]$, *i.e.*, $\sigma''(v) < 0$ and $h'(v) = \sigma'(v) - s^2 > 0$, we find $T(v)$ and $w(v)$ satisfy

$$\frac{w'(v)}{w(v)} = 2\frac{T'(v)}{T(v)},$$

which yields $Y(v) = -\sigma''(v) > 0$, and

$$\begin{aligned} Z(v) &= \frac{\sigma''(v)}{2} + \frac{h'(v)}{2}\frac{w'(v)}{w(v)} + \frac{h(v)}{2}\Big(\frac{w'}{w}\Big)'(v) + \frac{h(v)}{4}\Big(\frac{w'(v)}{w(v)}\Big)^2 \\ &= \frac{q_1(v)}{v+b}, \end{aligned} \tag{4.3.11}$$

where

$$q_1(v) \equiv h'(v) + \frac{1}{2}\sigma''(v)(v+b). \tag{4.3.12}$$

Therefore, in order to see (4.3.10), we should show $q_1(v)$ is positive on $[0, v_-]$. We first note that $q_1(v)$ is monotonicaly decreasing since $q_1'(v) = \frac{3}{2}\sigma''(v) + \frac{1}{2}\sigma'''(v)(v + b) < 0$, $\sigma''(v) < 0$, $\sigma'''(v) < 0$ and $v + b > 0$(see (4.1.18) and (4.3.3)). Then we have $q_1(v) \geq q_1(v_-) = h'(v_-) + \frac{1}{2}\sigma''(v_-)(v_- + b) > 0$ by (4.3.3). Thus, we observe that

$$Z(v) \geq \frac{q_1(v_-)}{v_- + b} \geq \frac{q_1(v_-)|\sigma''(v)|}{(v_- + b)|\sigma''(v_-)|}, \quad v \in [0, v_-].$$

*Part 2.* When $v \in [v_+, 0]$, *i.e.*, $\sigma''(v) > 0$ and $h'(v) = \sigma'(v) - s^2 \lessgtr 0$ for $v \lessgtr v_*$, see (5.1.11) in Sect.5.1, we find $T(v)$ and $w(v)$ satisfy

$$\frac{w'(v)}{w(v)} = 2\frac{T'(v)}{T(v)} - \frac{\sigma''(v)}{\sigma'(v)},$$

which yields $Y(v) \equiv 0$, and

$$Z(v) = \frac{s^2}{2}\frac{\sigma''(v)}{\sigma'(v)} + \frac{h(v)}{4}\Big(\frac{\sigma''(v)}{\sigma'(v)}\Big)^2 + \frac{\sigma'(v) - s^2}{v+b}. \tag{5.3.13}$$

To show $Z(v) > C\sigma''(v)$, we further devide the region $[v_+, 0]$ into $[v_*, 0]$ and $[v_+, v_*]$.

When $v \in [v_*, 0]$, since $\sigma'(v) - s^2 > 0$ and $b + v > 0$ for $v \in [v_*, 0]$, we have

$$Z(v) > \frac{s^2}{4}\frac{\sigma''(v)}{\sigma'(v)}\Big(2 + \frac{h(v)\sigma''(v)}{s^2\sigma'(v)}\Big) > C\sigma''(v). \tag{5.3.14}$$

Here we used the fact

$$0 < q_2(v) \equiv -\frac{h(v)\sigma''(v)}{s^2\sigma'(v)} < 1, \quad \text{for} \quad v \in [v_+, 0]. \tag{5.3.15}$$

To see (5.3.15), making use of $h(v_+) = \sigma''(0) = 0$, and $\sigma'''(v) < 0$, we observe that $q_2(v_+) = q_2(0) = 0$, and $q_2(v) > 0$ for $v \in (v_+, 0)$. Consequently, $q_2(v)$ attains its maximum over $[v_+, 0]$ at a point $v = \bar{v}$ in $(v_+, 0)$, and hence $\bar{q}_2 = \max_{v\in[v_+,0]} q_2(v) = q_2(\bar{v}) > 0$ and $q_2'(\bar{v}) = 0$. Rewriting $q_2(v)$ as $-h(v)\sigma''(v) = s^2\sigma'(v)q_2(v)$ and differentiating it with respect to $v$ at $v = \bar{v}$, we have

$$\bar{q}_2 = 1 - \frac{\sigma'(\bar{v})}{s^2} - \frac{h(\bar{v})\sigma'''(\bar{v})}{s^2\sigma''(\bar{v})} < 1.$$

When $v \in [v_+, v_*]$, i.e., $\sigma'(v) - s^2 < 0$, there exists a point $\tilde{v} \in (v, v_*)$ such that

$$\sigma'(v) - s^2 = \sigma''(\tilde{v})(v - v_*) > \sigma''(v)(v - v_*), \tag{5.3.16}$$

because of $\sigma'''(v) < 0$. Substituting (5.3.16) back into (5.3.13), we have

$$\begin{aligned} Z(v) &\geq \frac{\sigma''(v)}{4\sigma'(v)^2}\{2s^2\sigma'(v) + h(v)\sigma''(v) + 4\sigma'(v)^2\frac{v - v_*}{v + b}\} \\ &\equiv \frac{\sigma''(v)}{4\sigma'(v)^2}q_3(v). \end{aligned} \tag{5.3.17}$$

Differentiating $q_3(v)$ with repect to $v$, and making use of $h(v) < 0$ and $\sigma'''(v) < 0$, we have

$$q_3'(v) = h(v)\sigma'''(v) + \sigma''(v)q_4(v) + 4\frac{\sigma'(v)}{(v + b)^2}q_5(v)$$

$$\geq \sigma''(v)q_4(v) + 4\frac{\sigma'(v)}{(v+b)^2}q_5(v), \tag{5.3.18}$$

where

$$q_4(v) = s^2 + \sigma'(v)\frac{5v - 4v_* + b}{v+b}, \quad q_5(v) = \sigma'(v)(v_* + b) + (v+b)(v - v_*)\sigma''(v).$$

Making use of $s^2 \geq \sigma'(v)$ and $v + b > 0$ on $[v_+, v_*]$, and (4.1.18) and (4.3.3), we know $q_4(v) > 0$ for $v \in [v_+, v_*]$. On the other hand, we can see that

$$q_5(v) \geq q_5(-b) = \sigma'(-b)(v_* + b) > 0. \tag{4.3.19}$$

In fact, by $\sigma''(v) > 0$, $v + b > 0$, (4.1.18)(see also (4.3.3)) and (4.1.19), we have

$$q_5'(v) = 2\sigma''(v)(v+b) + (v+b)(v - v_*)\sigma'''(v) \geq 0$$

for $v_* \geq v \geq -b$, which implies (4.3.19). Consequently, we have proved $q_3'(v) > 0$ for $v \in [v_+, v_*]$ in (4.3.18). Thus using $s^2 > \sigma'(v_+)$, (4.1.18) and (4.3.3), we obtain

$$q_3(v) \geq q_3(v_+) \geq 2\sigma'(v_+)^2\{1 + 2\frac{v_+ - v_*}{v_+ + b}\} > 0, \quad v \in [v_+, v_*].$$

Therefore, by (4.3.17), we can see that $Z(v) >$ Const.$\sigma''(v) > 0$ for $v \in [v_+, v_*]$.

Combining Parts 1 and 2 together, we have completed the proof of (4.3.10).

Integrating (4.3.6) over $[0,t] \times (-\infty, \xi_0]$ and $[0,t] \times [\xi_0, \infty)$ respectively, and adding them, we obtain the followings by Lemma 4.3.1.

**Lemma 4.3.2.** *Suppose the assumptions in Theorem 4.1.3. Then it holds*

$$\|(\Phi, \Psi)(t)\|^2 + \int_0^t \|\Psi_\xi(\tau)\|^2 d\tau + \int_0^t \int_{-\infty}^{\infty} |V_\xi| w(V) Z(V) \Psi^2 d\xi d\tau$$

$$\leq C(\|(\Phi_0, \Psi_0)\|^2 + \int_0^t \int_{-\infty}^{\infty} \frac{w(V)}{T(V)} |F||\Psi| d\xi d\tau) \tag{4.3.20}$$

*for* $t \in [0, T]$.

**The Proof of Key Lemma 4.3.4:** Since $\|(\Phi, \Psi)\| \sim \|(\phi, \psi)\|$ by the boundedness of $T(V)$, and $|F| = O(|\phi_\xi|^2)$, we have due to Lemma 4.3.2

$$\|(\phi, \psi)(t)\|^2 + \int_0^t \|\Psi_\xi(\tau)\|^2 d\tau + \int_0^t \int_{-\infty}^{\infty} |V_\xi| w(V) Z(V) \Psi^2 d\xi d\tau$$
$$\leq C(\|(\phi_0, \psi_0)\|^2 + N(t) \int_0^t \|\phi_\xi(\tau)\|^2 d\tau). \tag{4.3.21}$$

Furthermore, multiplying the first equation of (4.2.2) by $\phi$ and the second one by $\psi \sigma'(V)^{-1}$ respectively, and adding them, we have

$$\{\frac{\phi^2}{2} + \frac{\psi^2}{2\sigma'(V)}\}_t - \{\frac{s\phi^2}{2} + \frac{s\psi^2}{2\sigma'(V)} + \phi\psi + \frac{\mu}{\sigma'(V)}\psi\psi_\xi\}_\xi$$
$$+ \frac{\mu}{\sigma'(V)}\psi_\xi^2 + \frac{s\sigma''(V)V_\xi}{\sigma'(V)}\psi^2 + \frac{\mu\sigma''(V)V_\xi}{\sigma'(V)}\psi\psi_\xi = \frac{F\psi}{\sigma'(V)}. \tag{4.3.22}$$

We note that

$$\left|\frac{\mu\sigma''(V)V_\xi}{\sigma'(V)}\psi\psi_\xi\right| \leq \varepsilon \frac{\mu\psi_\xi^2}{\sigma'(V)} + \frac{\mu\sigma''(V)^2 V_\xi^2 \psi^2}{4\varepsilon\sigma'(V)^3} \tag{4.3.23}$$

for $0 < \varepsilon < 1$. Substituting (4.3.23) into (4.3.22), and integrating the resultant inequality over $[0, t] \times R$, we have

$$\|(\phi, \psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau$$
$$\leq C(\|(\phi_0, \psi_0)\|^2 + \int_0^t \int_{-\infty}^{\infty} |\sigma''(V)V_\xi|\psi^2 d\xi d\tau + N(t) \int_0^t \|\phi_\xi(\tau)\|^2 d\tau) \tag{4.3.24}$$

Making use of Lemma 4.3.1 and $w(V) \sim T(V) \sim Const.$, we obtain

$$\int_0^t \int_{-\infty}^{\infty} |\sigma''(V)V_\xi|\psi^2 d\xi d\tau \leq C \int_0^t \int_{-\infty}^{\infty} |V_\xi| Z(V) w(V) \Psi^2 d\xi d\tau. \tag{4.3.25}$$

Applying (4.3.25) and (4.3.21) to (4.3.24), we finally have (4.2.5). This completes the proof of Key Lemma 4.2.4.

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