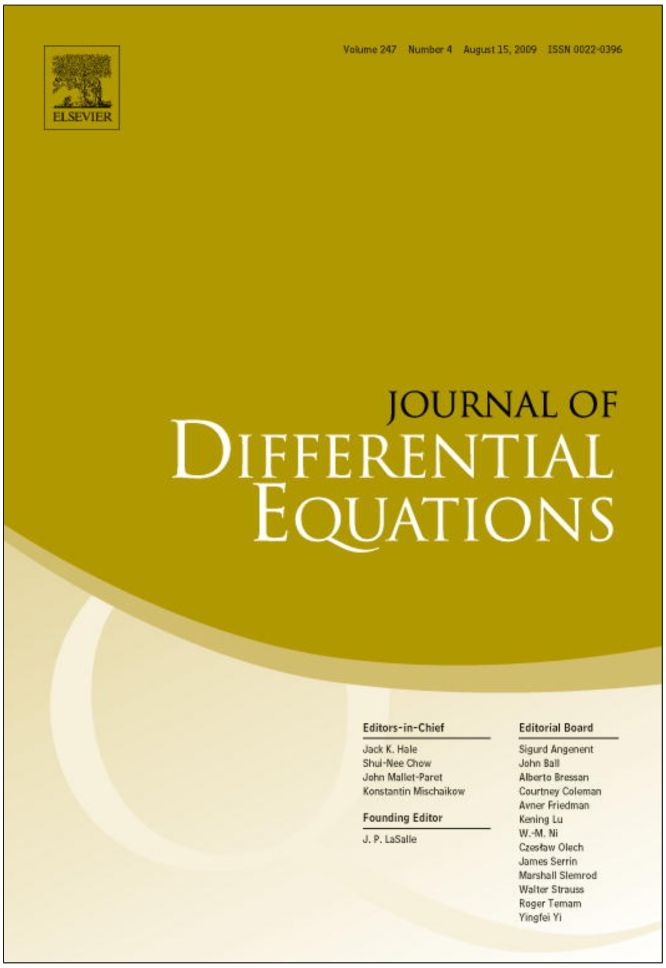
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Nonlinear diffusion waves for hyperbolic p-system with nonlinear damping

Ming Mei [a,b,*,1]

[a] Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, QC, J4P 3P2, Canada
[b] Department of Mathematics and Statistics, McGill University, Montreal, QC, H3A 2K6, Canada

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ABSTRACT

This paper is concerned with the p-system of hyperbolic conservation laws with nonlinear damping. When the constant states are small, the solutions of the Cauchy problem for the damped p-system globally exist and converge to their corresponding nonlinear diffusion waves, which are the solutions of the corresponding nonlinear parabolic equation given by the Darcy's law. The optimal convergence rates are also obtained. In order to overcome the difficulty caused by the nonlinear damping, a couple of correction functions have been technically constructed. The approach adopted is the elementary energy method together with the technique of approximating Green function. On the other hand, when the constant states are large, the solutions of the Cauchy problem for the p-system will blow up at a finite time.

* Address for correspondence: Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, QC, J4P 3P2, Canada.

E-mail addresses: mmei@champlaincollege.qc.ca, mei@math.mcgill.ca.

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1. Introduction and main results

We study the p-system of hyperbolic conservation laws with nonlinear damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta|u|^{q-1}u, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with the initial value condition

$$(v,u)(x,t)|_{t=0} = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty, \tag{1.2}$$

which models the compressible flow through porous media with nonlinear dissipative external force field in Lagrangian coordinates. Here, $v = v(x,t) > 0$ is the specific volume, $u = u(x,t)$ is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$. A typical example, in the case of a polytropic gas, is $p(v) = v^{-\gamma}$ with $\gamma \geqslant 1$. The external term $-\alpha u - \beta|u|^{q-1}u$, called the nonlinear damping, appears in the momentum equation, where $\alpha > 0$ is a constant, $\beta \neq 0$ is another constant but can be negative or positive, $q > 1$ is a given number, and $-\beta|u|^{q-1}u$ is regarded as a nonlinear perturbation to the linear damping $-\alpha u$. $v_\pm > 0$ and $u_\pm$ are constant states.

According to Darcy's law, the solutions $(v,u)(x,t)$ of (1.1) and (1.2) are expected to behave time-asymptotically as the self-similar solutions $(\bar v, \bar u)(x,t) = (\bar v, \bar u)(x/\sqrt{1+t})$ of the following (parabolic) porous media equation

$$\begin{cases} \bar v_t - \bar u_x = 0, \\ p(\bar v)_x = -\alpha \bar u, \end{cases} \quad \text{or} \quad \begin{cases} \bar v_t = -\dfrac{1}{\alpha} p(\bar v)_{xx}, \\ p(\bar v)_x = -\alpha \bar u, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.3}$$

with

$$(\bar v, \bar u)(x,t) \to (v_\pm, 0) \quad \text{as } x \to \pm\infty. \tag{1.4}$$

Such self-similar solutions $(\bar v, \bar u)(x/\sqrt{1+t})$ are usually called nonlinear diffusion waves of the p-system (1.1). The existence of the self-similar solutions was proved by C.T. Duyn and L.A. van Peletier in [2]. To prove the convergence of the solutions (v,u) to the diffusion waves $(\bar v, \bar u)$ with small $|u_\pm|$ is one of our main purposes in this present paper. The other target is to show that the solutions $(v,u)(x,t)$ will blow up when $|u_\pm|$ is large.

When $\beta = 0$, the system (1.1) is linear damping. In this case, Hsiao and Liu [3] first showed the convergence to the diffusion waves in the form of $\|(v - \bar v, u - \bar u)(t)\|_{L^\infty} = O(1)(t^{-1/2}, t^{-1/2})$, then Nishihara [15] succeeded in improving the convergence rates as $\|(v - \bar v, u - \bar u)(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$. Furthermore, by constructing an approximating Green function, Nishihara, Wang and Yang [19] and Wang and Yang [21] improved the rates as $\|(v - \bar v, u - \bar u)(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$, which is optimal in the sense comparing with the heat equation. For other studies related to this topic, we refer to [1,4–6,8–10,16–19,22–25] and the references therein. In [8], Li and Saxton considered a more general (quasilinear) system

$$\begin{cases} v_t - \big(h(v)u\big)_x = 0, \\ u_t + \sigma(v)_x = f(v)u, \end{cases}$$

where $\sigma'(v) < 0$, $h(v) > 0$ and $f(v) < 0$, and obtained the convergence of the solutions to the corresponding diffusion waves. Although the damping term $f(v)u$ is nonlinear, the nonlinearity $f(v)$ is only for v but not for u. Regarding the damping effect which is essentially governed by u for the second equation, the damping term $f(v)u$ is still linear with respect to u. So, the convergence to the diffusion waves obtained in [8] is still regarded as the case of linear damping.

When $\beta \neq 0$, the damping effect is nonlinear. The study in this case is very limited as we know. The convergence to the nonlinear diffusion waves has been recently investigated by Zhu and Jiang in [26] under the stiff condition $u_+ = u_- = 0$ for $q \geqslant 3$ with the less sufficient convergence rates as $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$, see also the boundary case by them in [7]. The main difficulty as they mentioned is hard to construct a pair of correction functions to eliminate a gap between $(v, u)(x, t)$ and $(\bar{v}, \bar{u})(x, t)$ at $x = \pm\infty$, so then they needed to assume $u_+ = u_- = 0$. However, by a deep observation, we succeed in this paper in constructing such a pair of correction functions such that we can obtain the convergence to the nonlinear diffusion waves without the restriction $u_+ = u_- = 0$, and also we can release $q \geqslant 5/2$. Furthermore, as showed in [21], by the technique of constructing a minimizing Green function, we can improve the convergence rate to be optimal: $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$. In what follows, we introduce how to construct the correction functions and how to set up properly the working equations.

For a given diffusion wave $(\bar{v}, \bar{u})(x, t)$, we consider its shifted diffusion wave $(\bar{v}, \bar{u})(x + x_0, t)$ with some shift constant x_0, which will be determined later. First of all, we are interested in the perturbation of $(v, u)(x, t) - (\bar{v}, \bar{u})(x + x_0, t)$. From the first equation of (1.1) and the first equation of (1.3), we have

$$(v - \bar{v})_t - (u - \bar{u})_x = 0. \tag{1.5}$$

Integrating (1.5) over $(-\infty, \infty)$ with respect to x, and noting (1.4), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} \big[v(x, t) - \bar{v}(x + x_0, t)\big]\, dx &= \big[u(+\infty, t) - \bar{u}(+\infty, t)\big] - \big[u(-\infty, t) - \bar{u}(-\infty, t)\big] \\ &= u(+\infty, t) - u(-\infty, t). \end{aligned} \tag{1.6}$$

In order to get $u(+\infty, t) - u(-\infty, t) = 0$, Zhu and Jiang [26] had to assume $u_- = u_+ = 0$. Because this yields $u(+\infty, t) = u(-\infty, t) = 0$. However, for any given constant state $u_\pm$, it usually holds $u(+\infty, t) - u(-\infty, t) \neq 0$. To overcome this difficulty, we need technically to construct a couple of correction functions $(\hat{v}, \hat{u})(x, t)$ so that we can eliminate the gap of $u(+\infty, t) - u(-\infty, t)$.

First of all, we investigate $u(\pm\infty, t)$. Let

$$u^\pm(t) := u(\pm\infty, t) = \lim_{x \to \pm\infty} u(x, t). \tag{1.7}$$

Taking the limits to the second equation of (1.1) as $x \to \pm\infty$, and noting that $p(v)_x$ will be vanishing, then we find that $u^\pm(t)$ satisfy formally the following modified Bernoulli's ODEs:

$$\begin{cases} \dfrac{d}{dt} u^\pm(t) = -\alpha u^\pm(t) - \beta \big|u^\pm(t)\big|^{q-1} u^\pm(t), \quad t > 0, \\ u^\pm(0) = u(\pm\infty, 0) = u_0(\pm\infty) = u_\pm. \end{cases} \tag{1.8}$$

Using the method of separation of variables, by a straightforward but tedious calculation, we can exactly solve (1.8) as (for the details, see Appendix A in the last section)

$$u^\pm(t) = \frac{C_\pm e^{-\alpha t}}{(1 - \frac{\beta}{\alpha}(|C_\pm| e^{-\alpha t})^{q-1})^{\frac{1}{q-1}}}, \tag{1.9}$$

where $C_\pm$ are the integration constants. In order to avoid the solution to blow up at a finite time, we need to restrict $C_\pm$ in (1.9) to be $1 > \frac{|\beta|}{\alpha} |C_\pm|^{q-1}$, namely,

$$|C_\pm| < \left(\frac{\alpha}{|\beta|}\right)^{\frac{1}{q-1}}. \tag{1.10}$$

Determining by the initial conditions in (1.8),

$$u_\pm = \frac{C_\pm}{(1 - \frac{\beta}{\alpha}|C_\pm|^{q-1})^{\frac{1}{q-1}}},$$

which implies that $C_\pm$ and $u_\pm$ both have the same signs, i.e., $\mathrm{sign}(C_\pm) = \mathrm{sign}(u_\pm)$, then we further specify

$$C_\pm = \frac{u_\pm}{(1 + \frac{\beta}{\alpha}|u_\pm|^{q-1})^{\frac{1}{q-1}}}. \tag{1.11}$$

From (1.11), the restriction (1.10) is equivalent to

$$\frac{|\beta|}{\alpha}|u_\pm|^{q-1} < \left|1 + \frac{\beta}{\alpha}|u_\pm|^{q-1}\right|. \tag{1.12}$$

Note that, when $\beta > 0$, the condition (1.12) automatically holds. While, when $\beta < 0$, (1.12) is also true if we ask

$$|u_\pm| < \left(\frac{\alpha}{2|\beta|}\right)^{1/(q-1)},$$

which implies that $|u_\pm|$ needs to be suitably small. Thus, if $|u_\pm| \ll 1$, then (1.12) is always true, and there is no blowing-up for $u^\pm(t)$.

Substituting (1.11) to (1.9), we obtain

$$u^\pm(t) = \frac{u_\pm e^{-\alpha t}}{(1 + \frac{\beta}{\alpha}|u_\pm|^{q-1}[1 - e^{-\alpha(q-1)t}])^{\frac{1}{q-1}}}. \tag{1.13}$$

Obviously, it holds

$$\left|u(\pm\infty, t)\right| = \left|u^\pm(t)\right| \sim O(1)|u_\pm|e^{-\alpha t} \quad \text{as } t \to \infty. \tag{1.14}$$

Next is to construct the correction functions such that we can eliminate the gap of $u(+\infty, t) - u(-\infty, t)$ in (1.6). Let us consider the function $\hat{u}(x,t)$ such that

$$\begin{cases} \dfrac{d\hat{u}}{dt} = -\alpha\hat{u} - \beta|\hat{u}|^{q-1}\hat{u}, & x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ \hat{u}(x,t) \to u^\pm(t) & \text{as } x \to \pm\infty. \end{cases} \tag{1.15}$$

As shown in (1.9), we can similarly solve (1.15) as

$$\hat{u}(x,t) = \frac{m(x)e^{-\alpha t}}{(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}}, \tag{1.16}$$

where $m(x)$ is an integration constant (with respect to t). Note that $\hat{u}(x,t) \to u^\pm(t)$ as $x \to \pm\infty$, we further confirm

$$m(x) \to C_\pm \quad \text{as } x \to \pm\infty. \tag{1.17}$$

Let $m_0(x) > 0$, $m_0(x) \in C_0^\infty(\mathbb{R})$ and $\int_{-\infty}^{\infty} m_0(x)\,dx = 1$, then we construct the desired function $m(x)$ as

$$m(x) := C_- + (C_+ - C_-) \int_{-\infty}^{x} m_0(y)\,dy. \tag{1.18}$$

It can be verified that $m(x)$ is sufficiently smooth and satisfies (1.17) as well as

$$\left|m(x)\right| \leqslant \min\left\{|C_+|, |C_-|\right\} < \left(\frac{\alpha}{|\beta|}\right)^{\frac{1}{q-1}}, \tag{1.19}$$

which ensures no blowing-up for $\hat{u}(x,t)$.

Now we are going to construct another correction function $\hat{v}(x,t)$. Let

$$\begin{aligned}
\hat{w}(x,t) :&= \int_0^t \hat{u}_x(x,\tau)\,d\tau = \left\{ \int_0^t \hat{u}(x,\tau)\,d\tau \right\}_x \\
&= \left\{ \int_0^t \frac{m(x)e^{-\alpha\tau}}{(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha\tau}]^{q-1})^{\frac{1}{q-1}}}\,d\tau \right\}_x \\
&= \left\{ \int_0^t \frac{m(x)e^{-\alpha\tau}}{(1 - \frac{\beta}{\alpha}[\sqrt{[m(x)]^2}\ e^{-\alpha\tau}]^{q-1})^{\frac{1}{q-1}}}\,d\tau \right\}_x \\
&= \left\{ \int_0^t \frac{m(x)\ e^{-\alpha\tau}}{(1 - \frac{\beta}{\alpha}[\sqrt{[m(x)e^{-\alpha\tau}]^2}]^{q-1})^{\frac{1}{q-1}}}\,d\tau \right\}_x \quad \left[\text{change of variables: } s = m(x)e^{-\alpha\tau}\right] \\
&= \left\{ -\frac{1}{\alpha} \int_{m(x)}^{m(x)e^{-\alpha t}} \frac{1}{(1 - \frac{\beta}{\alpha}|s|^{q-1})^{\frac{1}{q-1}}}\,ds \right\}_x \\
&= \frac{m'(x)}{\alpha(1 - \frac{\beta}{\alpha}|m(x)|^{q-1})^{\frac{1}{q-1}}} - \frac{m'(x)e^{-\alpha t}}{\alpha(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}}.
\end{aligned} \tag{1.20}$$

Define

$$\hat{v}(x,t) := -\frac{m'(x)e^{-\alpha t}}{\alpha(1 - \frac{\beta}{\alpha}[|m(x)|e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}}, \qquad n(x) := \frac{m'(x)}{\alpha(1 - \frac{\beta}{\alpha}|m(x)|^{q-1})^{\frac{1}{q-1}}}, \tag{1.21}$$

we then have $\hat{w}(x,t) = n(x) + \hat{v}(x,t)$ and $\hat{v}_t = \hat{w}_t$. From (1.20), i.e., $\hat{w}_t = \hat{u}_x$, we further obtain $\hat{v}_t = \hat{u}_x$. Thus, the constructed correction functions $(\hat{v}, \hat{u})(x,t)$ satisfy

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha\hat{u} - \beta\hat{u}^q. \end{cases} \tag{1.22}$$

Therefore, from (1.1), (1.3) and (1.22), we get

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + \left[p(v) - p(\bar{v})\right]_x = -\alpha(u - \bar{u} - \hat{u}) - \beta\left(|u|^{q-1}u - |\hat{u}|^{q-1}\hat{u}\right) - \bar{u}_t, \end{cases} \tag{1.23}$$

where $(\bar{v}, \bar{u})$ is the shifted diffusion wave $(\bar{v}, \bar{u})(x + x_0, t)$ with the shift x_0, which will be specified below.

To determine the shift x_0, let us integrate the first equation of (1.23) over $(-\infty, \infty)$ with respect to x,

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}\big[v(x,t) - \bar{v}(x+x_0,t) - \hat{v}(x,t)\big]\,dx \\
&= \big[u(+\infty,t) - \bar{u}(+\infty,t) - \hat{u}(+\infty,t)\big] - \big[u(-\infty,t) - \bar{u}(-\infty,t) - \hat{u}(-\infty,t)\big] \\
&= \big[u^+(t) - 0 - u^+(t)\big] - \big[u^-(t) - 0 - u^-(t)\big] \\
&= 0,
\end{aligned}$$

and integrate the above equation with respect to t to have

$$\int_{-\infty}^{\infty}\big[v(x,t) - \bar{v}(x+x_0,t) - \hat{v}(x,t)\big]\,dx = \int_{-\infty}^{\infty}\big[v_0(x) - \bar{v}(x+x_0,0) - \hat{v}(x,0)\big]\,dx =: I(x_0). \tag{1.24}$$

Now we are going to determine x_0 such that $I(x_0) = 0$. Since

$$\begin{aligned}
I'(x_0) &= \frac{\partial}{\partial x_0}\left(\int_{-\infty}^{\infty}\big[v_0(x) - \bar{v}(x+x_0,0) - \hat{v}(x,0)\big]\,dx\right) \\
&= -\int_{-\infty}^{\infty}\bar{v}'(x+x_0,0)\,dx = -\big[\bar{v}(\infty,0) - \bar{v}(-\infty,0)\big] \\
&= -(v_+ - v_-),
\end{aligned} \tag{1.25}$$

we have

$$I(x_0) - I(0) = \int_0^{x_0} I'(y)\,dy = -(v_+ - v_-)x_0,$$

which gives, with $I(x_0) = 0$, that

$$x_0 := \frac{1}{v_+ - v_-}I(0) = \frac{1}{v_+ - v_-}\int_{-\infty}^{\infty}\big[v_0(x) - \bar{v}(x,0) - \hat{v}(x,0)\big]\,dx. \tag{1.26}$$

Define

$$\begin{cases}
V(x,t) := \displaystyle\int_{-\infty}^{x}\big[v(y,t) - \bar{v}(y+x_0,t) - \hat{v}(y,t)\big]\,dy, \\
z(x,t) := u(x,t) - \bar{u}(x+x_0,t) - \hat{u}(x,t),
\end{cases} \tag{1.27}$$

and

$$\begin{cases} V_0(x) := \displaystyle\int_{-\infty}^{x} \big[v_0(y) - \bar{v}(y + x_0, 0) - \hat{v}(y, 0)\big]\,dy, \\ z_0(x) := u_0(x) - \bar{u}(x + x_0, 0) - \hat{u}(x, 0), \end{cases} \tag{1.28}$$

we deduce (1.23) into

$$\begin{cases} V_t - z = 0, \\ z_t + \big(p'(\bar{v})V_x\big)_x = -\alpha z - F_1 - F_2, \\ (V, z)|_{t=0} = (V_0, z_0)(x), \end{cases} \tag{1.29}$$

where

$$F_1 := -\frac{1}{\alpha} p(\bar{v})_{xt} + \big\{p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x\big\}_x, \tag{1.30}$$

$$F_2 := g(z + \bar{u} + \hat{u}) - g(\hat{u}) = g(V_t + \bar{u} + \hat{u}) - g(\hat{u}), \tag{1.31}$$

$$g(u) := \beta|u|^{q-1}u. \tag{1.32}$$

Instead of (1.1) and (1.2), we study the initial value problem (1.29).

Notations. Before stating our main results, we give some notations as follows. Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \ldots)$ represents a specific constant. $L^2(\mathbb{R})$ is the space of square integrable functions, and $H^k(\mathbb{R})$ $(k \geqslant 0)$ is the Sobolev space of L^2-functions $f(x)$ whose derivatives $\frac{d^i}{dx^i} f$, $i = 1, \ldots, k$, also belong to $L^2(\mathbb{R})$. The norms of $L^2(\mathbb{R})$ and $H^k(\mathbb{R})$ are denoted as $\|f\|_{L^2(\mathbb{R})}$ and $\|f\|_{H^k(\mathbb{R})}$, respectively. For the sake of simplicity, we also denote $\|(f, g, h)\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})} + \|h\|^2_{L^2(\mathbb{R})}$. Let $T > 0$ and let $\mathcal{B}$ be a Banach space. We denote by $C^0([0, T]; \mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0, T]$, and $L^2([0, T]; \mathcal{B})$ as the space of $\mathcal{B}$-valued L^2-functions on $[0, T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0, \infty)$ are defined similarly.

Our first result is as follows.

Theorem 1.1 *(Convergence). Let $q > \frac{5}{2}$, $(V_0, z_0)(x)$ be in $H^3(\mathbb{R}) \times H^2(\mathbb{R})$, and $u_\pm$ satisfy*

$$|u_\pm| < \left(\frac{\alpha}{2|\beta|}\right)^{1/(q-1)}. \tag{1.33}$$

There exists a number $\varepsilon_1 > 0$, when the initial perturbation and

$$\delta := |v_+ - v_-| + |u_+| + |u_-|$$

are suitably small such that

$$\delta + \|V_0\|_{H^3(\mathbb{R})} + \|z_0\|_{H^2(\mathbb{R})} \leqslant \varepsilon_1, \tag{1.34}$$

then the global solution $(V, z)(x, t)$ of (1.29) uniquely exists and satisfies

$$V(x,t) \in C^k\big(0, \infty; H^{3-k}(\mathbb{R})\big), \quad k = 0, 1, 2, 3, \qquad z(x,t) \in C^k\big(0, \infty; H^{2-k}(\mathbb{R})\big), \quad k = 0, 1, 2,$$

and

$$\sum_{k=0}^{3}(1+t)^k\left\|\partial_x^k V(t)\right\|_{L^2(\mathbb{R})}^2+\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_x^k z(t)\right\|_{L^2(\mathbb{R})}^2$$
$$+\int_0^t\left[\sum_{k=0}^{3}(1+s)^{k-1}\left\|\partial_x^k V(s)\right\|_{L^2(\mathbb{R})}^2+\sum_{k=0}^{2}(1+s)^{k+1}\left\|\partial_x^k z(s)\right\|_{L^2(\mathbb{R})}^2\right]ds$$
$$\leqslant C\left(\|V_0\|_{H^3(\mathbb{R})}^2+\|z_0\|_{H^2(\mathbb{R})}^2+\delta\right). \tag{1.35}$$

Furthermore, (1.35) *can be improved as the following optimal convergence rates*

$$\left\|\partial_x^k V(t)\right\|_{L^2(\mathbb{R})}\leqslant C\left(\|V_0\|_3^2+\|z_0\|_2^2+\delta\right)(1+t)^{-\frac{1}{4}-\frac{k}{2}},\quad k=0,1,2,3, \tag{1.36}$$
$$\left\|\partial_x^k z(t)\right\|_{L^2(\mathbb{R})}\leqslant C\left(\|V_0\|_3^2+\|z_0\|_2^2+\delta\right)(1+t)^{-\frac{5}{4}-\frac{k}{2}},\quad k=0,1,2. \tag{1.37}$$

Notice that $V_x=v-\bar v-\hat v$, $z=u-\bar u-\hat u$, and use (1.16) and (1.21), i.e., $|\hat v(x,t)|$, $|\hat u(x,t)|\sim O(1)e^{-\alpha t}$, and Sobolev's embedding inequalities $\|f\|_{L^\infty(\mathbb{R})}\leqslant\sqrt{2}\|f\|_{L^2(\mathbb{R})}^{1/2}\|f_x\|_{L^2(\mathbb{R})}^{1/2}$, we immediately obtain the following decay rates.

Corollary 1.2 *(Convergence to diffusion waves). Under the conditions in Theorem* 1.1*, the system* (1.1) *and* (1.2) *possesses a uniquely global solution* $(v,u)(x,t)$*, which converges to its nonlinear diffusion wave* $(\bar v,\bar u)(x+x_0,t)$ *in the form of*

$$\left\|(v-\bar v)(t)\right\|_{L^\infty(\mathbb{R})}=O(1)(1+t)^{-1}, \tag{1.38}$$
$$\left\|(u-\bar u)(t)\right\|_{L^\infty(\mathbb{R})}=O(1)(1+t)^{-3/2}. \tag{1.39}$$

The rates showed in (1.38) *and* (1.39) *are optimal.*

If $\beta<0$ and $|u_\pm|>(\frac{\alpha}{|\beta|})^{1/(q-1)}$, then, from (1.13), $u(\pm\infty,t)$ will blow up at the finite time $t_{**}:=\frac{1}{\alpha(q-1)}\ln\frac{|\beta||u_\pm|^{q-1}}{|\beta||u_\pm|^{q-1}-\alpha}$. Since $\|u(t)\|_{L^\infty(\mathbb{R})}\geqslant|u(\pm\infty,t)|$, we immediately recognize that $(v,u)(x,t)$ will blow up at a finite time.

Remark 1.3. When $\beta<0$ and

$$|u_\pm|>\left(\frac{\alpha}{|\beta|}\right)^{1/(q-1)}, \tag{1.40}$$

then the solution $(v,u)(x,t)$ of (1.1) and (1.2) does not globally exist, and

$$\lim_{t\to t_*^-}\left\|(v,u)(t)\right\|_{L^\infty}=+\infty, \tag{1.41}$$

where

$$0<t_*\leqslant\frac{1}{\alpha(q-1)}\ln\frac{|\beta||u_\pm|^{q-1}}{|\beta||u_\pm|^{q-1}-\alpha}. \tag{1.42}$$

The paper is organized as follows. In Section 2, we prepare some preliminaries, which are useful in the proof of theorems. Section 3 is devoted to the proof of the convergence of the solution $(v,u)(x,t)$ to the nonlinear diffusion waves $(\bar{v},\bar{u})((x+x_0)/\sqrt{1+t})$ (i.e., Theorem 1.1). The adopted approach is the elementary energy method together with the technique of approximating Green function.

2. Preliminaries

In this section, we are going to introduce some well-known results, i.e., the decay rates of the nonlinear diffusion waves $(\hat{v},\hat{u})(x,t)$, the decay rates of the correction functions $(\hat{v},\hat{u})(x,t)$, and the fundamental properties of the linear damped wave equation.

Let $(\bar{v},\bar{u})(x,t)=(\bar{v},\bar{u})(x/\sqrt{1+t})$ be the self-similar solution of (1.3) satisfying the "boundary" condition (1.4). It has been proved in [2] (see also, for example, [3,26], etc.) that the so-called nonlinear diffusion wave $(\bar{v},\bar{u})(x,t)$ exists and behaves as follows.

Lemma 2.1. *For each $p\in[1,\infty]$, it holds*

$$\left\|\partial_x^k\bar{v}(t)\right\|_{L^p(\mathbb{R})}=O(1)|v_+-v_-|(1+t)^{-(\frac{k}{2}-\frac{1}{2p})},\quad k=1,2,3,\ldots,\tag{2.1}$$

$$\left\|\partial_x^k\bar{u}(t)\right\|_{L^p(\mathbb{R})}=O(1)|v_+-v_-|(1+t)^{-(\frac{k+1}{2}-\frac{1}{2p})},\quad k=0,1,2,3,\ldots,\tag{2.2}$$

$$\left\|\partial_x^k\bar{v}_t(t)\right\|_{L^p(\mathbb{R})}=O(1)|v_+-v_-|(1+t)^{-(\frac{k+2}{2}-\frac{1}{2p})},\quad k=0,1,2,3,\ldots,\tag{2.3}$$

$$\left\|\partial_x^k\bar{u}_t(t)\right\|_{L^p(\mathbb{R})}=O(1)|v_+-v_-|(1+t)^{-(\frac{k+3}{2}-\frac{1}{2p})},\quad k=0,1,2,3,\ldots.\tag{2.4}$$

As shown in (1.16) and (1.21), where $m(x)$ and $C_\pm$, defined in (1.11) and (1.18), respectively, are bounded, the correction function $(\hat{v},\hat{u})(x,t)$ can be immediately confirmed to satisfy

Lemma 2.2. *It holds*

$$\left\|\hat{v}(t)\right\|_{L^\infty(\mathbb{R})}=O(1)|v_+-v_-|e^{-\alpha t},\tag{2.5}$$

$$\left\|\hat{u}(t)\right\|_{L^\infty(\mathbb{R})}=O(1)\max\bigl\{|u_+|,|u_-|\bigr\}e^{-\alpha t}.\tag{2.6}$$

Furthermore, we introduce the following famous lemma, which can be founded, for example, in [10,11,13,20].

Lemma 2.3. *Let $a>0$, $b>0$. If $\max(a,b)>1$, then*

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}\,ds=O(1)(1+t)^{-\min(a,b)}.\tag{2.7}$$

If $\max(a,b)=1$, then

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}\,ds=O(1)(1+t)^{-\min(a,b)}\ln(2+t).\tag{2.8}$$

If $\max(a,b)<1$, then

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}\,ds=O(1)(1+t)^{1-a-b}.\tag{2.9}$$

3. Proof of Theorem 1.1

Substituting the first equation of (1.29) into the second equation of (1.29), we obtain

$$\begin{cases} V_{tt} + \alpha V_t + \big(p'(\bar{v})V_x\big)_x = -F_1 - F_2, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ (V, V_t)|_{t=0} = (V_0, z_0)(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

It is known that Theorem 1.1 can be proved by the classical continuation method based on the local existence and the *a priori* estimates. The local existence of the solution for (3.1) can be obtained by the standard iteration method (cf. [12,14]), so we will omit its details. To establish the *a priori* estimates for the solution usually is technical, which will be the main effort in this section.

Let $T \in [0, \infty]$, we define

$$N(T)^2 := \sup_{0 \leqslant t \leqslant T} \left\{ \sum_{k=0}^{3} (1+t)^k \big\| \partial_x^k V(t) \big\|_{L^2(\mathbb{R})}^2 + \sum_{k=0}^{2} (1+t)^{k+2} \big\| \partial_x^k V_t(t) \big\|_{L^2(\mathbb{R})}^2 \right\}. \tag{3.2}$$

We first establish the following basic energy estimate.

Lemma 3.1 *(Basic energy estimates). It follows that*

$$\big\| (V, V_x, V_t)(t) \big\|_{L^2(\mathbb{R})}^2 + \int_0^t \big\| (V_x, V_t)(s) \big\|_{L^2(\mathbb{R})}^2 \, ds \leqslant C\big(\|V_0\|_{H^1(\mathbb{R})}^2 + \|z_0\|_{L^2(\mathbb{R})}^2 + \delta \big) \tag{3.3}$$

provided $N(T) + \delta \ll 1$.

Proof. Multiplying (3.1) by $\lambda V + V_t$ with small constant $0 < \lambda \ll 1$, we have

$$\big\{E_1(V, V_x, V_t)\big\}_t + E_2(V_x, V_t) + \big\{E_3(x,t)\big\}_x = -(F_1 + F_2)(\lambda V + V_t), \tag{3.4}$$

where

$$E_1(V, V_x, V_t) := \frac{1}{2} V_t^2 + \lambda V V_t + \frac{\alpha\lambda}{2} V^2 - \frac{1}{2} p'(\bar{v}) V_x^2, \tag{3.5}$$

$$E_2(V_x, V_t) := (\alpha - \lambda) V_t^2 + \left[-\lambda p'(\bar{v}) + \frac{1}{2} p''(\bar{v}) \bar{v}_t \right] V_x^2, \tag{3.6}$$

$$E_3(x,t) := p'(\bar{v}) V_x (\lambda V + V_t). \tag{3.7}$$

Notice that $p'(\bar{v}) < 0$. When $\lambda \ll 1$, $|v_+ - v_-| \ll 1$, the following estimates hold

$$C_1\big(V^2 + V_x^2 + V_t^2\big) \leqslant E_1(V, V_x, V_t) \leqslant C_2\big(V^2 + V_x^2 + V_t^2\big), \tag{3.8}$$

$$C_3\big(V_x^2 + V_t^2\big) \leqslant E_2(V_x, V_t), \tag{3.9}$$

for some positive constants C_i $(i = 1, 2, 3)$. Integrating (3.4) over $\mathbb{R} \times [0, t]$ with respect to x and t, we have

$$
\begin{aligned}
&\big\|(V, V_x, V_t)(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big\|(V_x, V_t)(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^1(\mathbb{R})} + \|z_0\|^2_{L^2(\mathbb{R})} + \delta\big) + C\int_0^t\int_{\mathbb{R}} \big(|F_1| + |F_2|\big)\big(\lambda|V| + |V_t|\big)\,dx\,ds.
\end{aligned} \tag{3.10}
$$

As shown in [9,15,26], we can estimate

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} |F_1|\big(\lambda|V| + |V_t|\big)\,dx\,ds &\leqslant CN(t)\big\|V_x(t)\big\|^2 + C\big[\delta + N(t)\big]\int_0^t \big\|V_x(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad + C\|V_0\|^2_{H^1(\mathbb{R})} + C\big[1 + N(t)\big]\delta.
\end{aligned} \tag{3.11}
$$

Now we are going to estimate the second part of the last term in (3.10). Notice that

$$
\begin{aligned}
|F_2| &= \big|g(V_t + \bar{u} + \hat{u}) - g(\hat{u})\big| \\
&\leqslant C|V_t + \bar{u}|\big(|V_t + \bar{u}|^{q-1} + |\hat{u}|^{q-1}\big) \\
&\leqslant C\big(|V_t|^q + |\bar{u}|^q + |\hat{u}|^{q-1}|V_t| + |\hat{u}|^{q-1}|\bar{u}|\big),
\end{aligned} \tag{3.12}
$$

we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} |F_2|\big(\lambda|V| + |V_t|\big)\,dx\,ds \\
&\quad\leqslant C\int_0^t\int_{\mathbb{R}} \big(|V_t|^q + |\bar{u}|^q + |\hat{u}|^{q-1}|V_t| + |\hat{u}|^{q-1}|\bar{u}|\big)\big(\lambda|V| + |V_t|\big)\,dx\,ds.
\end{aligned} \tag{3.13}
$$

From (3.2), we can first estimate

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}} |V_t|^q\big(\lambda|V| + |V_t|\big)\,dx\,ds &\leqslant \lambda \sup_{0\leqslant s\leqslant t}\Big[\big\|V_t(s)\big\|^{q-2}_{L^\infty(\mathbb{R})}\big\|V(s)\big\|_{L^\infty(\mathbb{R})}\Big]\int_0^t \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad + \sup_{0\leqslant s\leqslant t}\big\|V_t(s)\big\|^{q-1}_{L^\infty(\mathbb{R})}\int_0^t \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\leqslant CN(t)^{q-1}\int_0^t \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds.
\end{aligned} \tag{3.14}
$$

Notice from (3.2) that

$$
\big\|V(s)\big\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\big\|V(s)\big\|^{\frac12}_{L^2(\mathbb{R})}\big\|V_x(s)\big\|^{\frac12}_{L^2(\mathbb{R})} \leqslant CN(t)(1+s)^{-\frac14}, \tag{3.15}
$$

$$
\big\|V_t(s)\big\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\big\|V_t(s)\big\|^{\frac12}_{L^2(\mathbb{R})}\big\|V_{xt}(s)\big\|^{\frac12}_{L^2(\mathbb{R})} \leqslant CN(t)(1+s)^{-\frac54}, \tag{3.16}
$$

and (2.2), we then obtain

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} |\bar{u}|^q \big(\lambda|V| + |V_t|\big)\,dx\,ds &\leqslant \int_0^t \big[\lambda \big\|V(s)\big\|_{L^\infty(\mathbb{R})} + \big\|V_t(s)\big\|_{L^\infty(\mathbb{R})}\big]\big\|\bar{u}(s)\big\|^q_{L^q(\mathbb{R})}\,ds \\
&\leqslant CN(t)|v_+ - v_-|^q \int_0^t \big[(1+s)^{-\frac{1}{4}} + (1+s)^{-\frac{3}{4}}\big](1+s)^{-\frac{q-1}{2}}\,ds \\
&\leqslant C|v_+ - v_-|^q N(t), \quad \text{for } q > \frac{5}{2}. \qquad (3.17)
\end{aligned}$$

Furthermore, notice (2.6), we can similarly prove

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} |\hat{u}|^{q-1}|V_t|\big(\lambda|V| + |V_t|\big)\,dx\,ds &\leqslant \int_0^t \big\|\hat{u}(s)\big\|^{q-1}_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |V_t|\big(\lambda|V| + |V_t|\big)\,dx\,ds \\
&\leqslant C\big(|u_+| + |u_-|\big)^{q-1} \int_0^t e^{-\alpha(q-1)s}\big[\big\|V(s)\big\|^2_{L^2(\mathbb{R})} + \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\big]\,ds \\
&\leqslant C\big(|u_+| + |u_-|\big)^{q-1} N(t)^2, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |\hat{u}|^{q-1}|\bar{u}|\big(\lambda|V| + |V_t|\big)\,dx\,ds \\
&\quad\leqslant \int_0^t \big\|\hat{u}(s)\big\|^{q-1}_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\bar{u}|\big(\lambda|V| + |V_t|\big)\,dx\,ds \\
&\quad\leqslant C\big(|u_+| + |u_-|\big)^{q-1} \int_0^t e^{-\alpha(q-1)s}\big\|\bar{u}(s)\big\|_{L^2(\mathbb{R})}\big[\big\|V(s)\big\|_{L^2(\mathbb{R})} + \big\|V_t(s)\big\|_{L^2(\mathbb{R})}\big]\,ds \\
&\quad\leqslant C\big(|u_+| + |u_-|\big)^{q-1}|v_+ - v_-|N(t). \qquad (3.19)
\end{aligned}$$

Substituting (3.14), (3.17), (3.18) and (3.19) into (3.13), we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |F_2|\big(\lambda|V| + |V_t|\big)\,dx\,ds \\
&\quad\leqslant CN(t)^{q-1} \int_0^t \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})} + C|v_+ - v_-|^q N(t) \\
&\qquad + C\big(|u_+| + |u_-|\big)^{q-1} N(t)^2 + C\big(|u_+| + |u_-|\big)^{q-1}|v_+ - v_-|N(t). \qquad (3.20)
\end{aligned}$$

Substituting (3.11) and (3.20) into (3.10), and taking $N(t) + \delta \ll 1$, where $\delta = |v_+ - v_-| + |u_+| + |u_-|$, we finally prove (3.3). The proof is completed. □

Lemma 3.2 *(Higher order energy estimates). It holds that*

$$\big\|(V_x, V_{xx}, V_{xt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big\|(V_{xx}, V_{xt})(s)\big\|^2_{L^2(\mathbb{R})}\,ds \leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \tag{3.21}$$

provided $N(T) + \delta \ll 1$.

Proof. Similarly, let us differentiate (3.1) with respect to x and multiply it by V_{xt}, then integrate the resultant equation over $\mathbb{R} \times [0,t]$ with respect to x and t. Applying the basic energy estimate (3.3), we can further prove the higher order energy estimate (3.21). Here we omit the details of the proof. □

Lemma 3.3 *(Decay rate for V_x). It holds that*

$$(1+t)\big\|(V_x, V_t)(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \leqslant C\big(\|V_0\|^2_{H^1(\mathbb{R})} + \|z_0\|^2_{L^2(\mathbb{R})} + \delta\big) \tag{3.22}$$

provided $N(T) + \delta \ll 1$.

Proof. Multiplying (3.1) by $(1+t)V_t$ and integrating it over $\mathbb{R} \times [0,t]$ with respect to x ant t, we have

$$\begin{aligned}
&(1+t)\int_{\mathbb{R}} \big[V_t^2 - p'(\bar v)V_x^2\big]\,dx + 2\alpha \int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad = \int_{\mathbb{R}} \big[z_0^2 - p'\big(\bar v(x,0)\big)V_{0,x}^2\big]\,dx + \int_0^t \int_{\mathbb{R}} \big[V_t^2 - p'(\bar v)V_x^2\big]\,dx\,ds \\
&\qquad - \int_0^t (1+s)\int_{\mathbb{R}} p''(\bar v)\bar v_t V_x^2\,dx\,ds - 2\int_0^t (1+s)\int_{\mathbb{R}} (F_1 + F_2)V_t\,dx\,ds.
\end{aligned} \tag{3.23}$$

From (3.3), we can estimate

$$\int_0^t \int_{\mathbb{R}} \big[V_t^2 - p'(\bar v)V_x^2\big]\,dx\,ds \leqslant C\int_0^t \big\|(V_x, V_t)(s)\big\|^2_{L^2(\mathbb{R})}\,ds \leqslant C\big(\|V_0\|^2_{H^1(\mathbb{R})} + \|z_0\|^2_{L^2(\mathbb{R})} + \delta\big), \tag{3.24}$$

and notice $|\bar v_t| \sim O(1)(1+t)^{-3/2}$, we further have

$$\begin{aligned}
\int_0^t (1+s)\int_{\mathbb{R}} p''(\bar v)\bar v_t V_x^2\,dx\,ds &\leqslant C\int_0^t (1+s)^{-1/2}\big\|V_x(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\leqslant C\int_0^t \big\|V_x(s)\big\|^2_{L^2(\mathbb{R})}\,ds \leqslant C\big(\|V_0\|^2_{H^1(\mathbb{R})} + \|z_0\|^2_{L^2(\mathbb{R})} + \delta\big).
\end{aligned} \tag{3.25}$$

For the last term of (3.23), as exactly shown in [15], the first part on F_1 can be estimated as

$$\begin{aligned}
&\int_0^t (1+s)\int_{\mathbb{R}} F_1 V_t\,dx\,ds \\
&\quad \leqslant CN(t)(1+t)\big\|V_x(t)\big\|^2_{L^2(\mathbb{R})}+\frac{\alpha}{2}\int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\qquad + C\delta N(t)\int_0^t (1+s)\big\|V_x(s)\big\|^2_{L^2(\mathbb{R})}\,ds + C\big(\|V_0\|^2_{H^1(\mathbb{R})}+\delta\big).
\end{aligned}\tag{3.26}$$

While, the second part on F_2 in the last term of (3.23) is estimated as

$$\begin{aligned}
\int_0^t (1+s)\int_{\mathbb{R}} F_2 V_t\,dx\,ds &\leqslant C\int_0^t (1+s)\int_{\mathbb{R}}\big[|V_t|^q+|\bar u|^q+|\hat u|^{q-1}|V_t|+|\hat u|^{q-1}|\bar u|\big]|V_t|\,dx\,ds \\
&\leqslant C\int_0^t (1+s)\big\|V_t(s)\big\|^{q-1}_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|V_t|^2\,dx\,ds + C\int_0^t (1+s)\big\|V_t(s)\big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\bar u|^q\,dx\,ds \\
&\quad + C\int_0^t (1+s)\big\|\hat u(s)\big\|^{q-1}_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|V_t|^2\,dx\,ds + C\int_0^t (1+s)\big\|\hat u(s)\big\|^{q-1}_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\hat u V_t|\,dx\,ds \\
&=: I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.27}$$

Thanks to (3.16), we first have

$$\begin{aligned}
I_1 &\leqslant CN(t)^{q-1}\int_0^t (1+s)(1+s)^{-\frac{5(q-1)}{4}}\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\leqslant CN(t)^{q-1}\int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds.
\end{aligned}\tag{3.28}$$

Notice also from Lemma 2.1 that $\|\bar u(s)\|_{L^q(\mathbb{R})}\leqslant C|v_+-v_-|(1+s)^{-(\frac12-\frac{1}{2q})}$, and notice $q>5/2$, we then estimate

$$\begin{aligned}
I_2 &\leqslant CN(t)\int_0^t (1+s)(1+s)^{-\frac54}\big\|\bar u(s)\big\|^q_{L^q(\mathbb{R})}\,ds \\
&\leqslant C|v_+-v_-|^q N(t)\int_0^t (1+s)^{-\frac{2q-1}{4}}\,ds \\
&\leqslant C|v_+-v_-|^q N(t) \\
&\leqslant C\delta^q, \quad \text{for } N(t)\ll 1.
\end{aligned}\tag{3.29}$$

Applying Lemma 2.2, we further estimate I_3 and I_4 as

$$I_3 \leqslant C\big[\max\big\{|u_+|, |u_-|\big\}\big]^{q-1} \int_0^t e^{-\alpha(q-1)s}(1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds$$

$$\leqslant C\delta^{q-1} \int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds, \tag{3.30}$$

and

$$I_4 \leqslant C\big[\max\big\{|u_+|, |u_-|\big\}\big]^{q-1} \int_0^t e^{-\alpha(q-1)s}(1+s)\big[\big\|\bar{u}(s)\big\|^2_{L^2(\mathbb{R})} + \big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\big]\,ds$$

$$\leqslant C\delta^{q-1} + C\delta^{q-1} \int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds. \tag{3.31}$$

Substituting (3.28)–(3.31) into (3.27), we obtain

$$\int_0^t (1+s) \int_{\mathbb{R}} F_2 V_t\,dx\,ds \leqslant C\big(\delta^q + \delta^{q-1}\big) + C\big[N(t)^{q-1} + \delta^{q-1}\big] \int_0^t (1+s)\big\|V_t(s)\big\|^2_{L^2(\mathbb{R})}\,ds. \tag{3.32}$$

Applying (3.24)–(3.26) and (3.32) into (3.23), we finally prove (3.22) by providing $N(T) + \delta \ll 1$. □

For the following estimates, since the proofs are similar to the previous three lemmas, we give only the outline of the proofs, and omit their details.

Lemma 3.4 *(Decay rate for V_{xx}). It holds that*

$$(1+t)^2\big\|(V_{xx}, V_{xt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big[(1+s)\big\|V_{xx}(s)\big\|^2_{L^2(\mathbb{R})} + (1+s)^2\big\|V_{xt}(s)\big\|^2_{L^2(\mathbb{R})}\big]\,ds$$

$$\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \tag{3.33}$$

provided $N(T) + \delta \ll 1$.

Proof. Differentiating (3.1) with respect to x, and multiplying it by $(1+t)V_{xt}$, then integrating the resultant equation over $\mathbb{R} \times [0, t]$ with respect to x and t, as shown in Lemma 3.3, we can similarly prove that

$$(1+t)\big\|(V_x, V_{xx}, V_{xt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)\big\|(V_{xt}, V_{xx})(s)\big\|^2_{L^2(\mathbb{R})}\,ds$$

$$\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \tag{3.34}$$

provided $N(T) + \delta \ll 1$. Furthermore, by taking $\partial_x(3.1) \times (1+t)^2 V_{xt}$ and integrating it over $\mathbb{R} \times [0, t]$ with respect to x and t, and applying (3.34), we then obtain

$$
\begin{aligned}
&(1+t)^2\big\|(V_{xx}, V_{xt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^2\big\|V_{xt}(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \qquad (3.35)
\end{aligned}
$$

provided $N(T)+\delta \ll 1$. Thus, combining (3.34) and (3.35) gives (3.33). The proof is completed. $\square$

Lemma 3.5 *(Decay rate for V_{xxx}). It holds that*

$$
\begin{aligned}
&(1+t)^3\big\|(V_{xxx}, V_{xxt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big[(1+s)^2\big\|V_{xxx}(s)\big\|^2_{L^2(\mathbb{R})} + (1+s)^3\big\|V_{xxt}(s)\big\|^2_{L^2(\mathbb{R})}\big]\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^3(\mathbb{R})} + \|z_0\|^2_{H^2(\mathbb{R})} + \delta\big) \qquad (3.36)
\end{aligned}
$$

provided $N(T)+\delta \ll 1$.

Proof. In the same manner of Lemma 3.4, by calculating $\int_0^t\int_{\mathbb{R}} \partial_x^2(3.1)\times(1+t)^2 V_{xxt}\,dx\,dt$, we first have

$$
\begin{aligned}
&(1+t)^2\big\|(V_{xx}, V_{xxx}, V_{xxt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^2\big\|(V_{xxt}, V_{xxx})(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^3(\mathbb{R})} + \|z_0\|^2_{H^2(\mathbb{R})} + \delta\big) \qquad (3.37)
\end{aligned}
$$

provided $N(T)+\delta \ll 1$, and by calculating $\int_0^t\int_{\mathbb{R}} \partial_x^2(3.1)\times(1+t)^3 V_{xxt}\,dx\,dt$, we further have

$$
\begin{aligned}
&(1+t)^3\big\|(V_{xxx}, V_{xxt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^3\big\|V_{xxt}(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \qquad (3.38)
\end{aligned}
$$

provided $N(T)+\delta \ll 1$. Combining (3.37) and (3.38) deduces (3.36). $\square$

Lemma 3.6 *(Decay rate for V_t). It holds that*

$$
\begin{aligned}
&(1+t)^2\big\|(V_t, V_{xt}, V_{tt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^2\big\|(V_{xt}, V_{tt})(s)\big\|^2_{L^2(\mathbb{R})}\,ds \\
&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big) \qquad (3.39)
\end{aligned}
$$

provided $N(T)+\delta \ll 1$.

Proof. Differentiating (3.1) with respect to t to have

$$
V_{ttt} + \alpha V_{tt} + \big(p'(\bar{v})V_x\big)_{xt} = -F_{1t} - F_{2t}, \qquad (3.40)
$$

and multiplying (3.40) by $\lambda V_t + V_{tt}$ ($\lambda \ll 1$), then integrating it over $\mathbb{R}\times[0,t]$ with respect to x and t, and using Lemma 3.2, we first have

$$\begin{aligned}&\big\|(V_t, V_{xt}, V_{tt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big\|(V_{xt}, V_{tt})(s)\big\|^2_{L^2(\mathbb{R})}\,ds\\&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big)\end{aligned}\tag{3.41}$$

with small $N(T)+\delta$.

Secondly, multiplying (3.40) by $(1+t)(\lambda V_t + V_{tt})$ ($\lambda \ll 1$), and integrating it over $\mathbb{R}\times[0,t]$, and using the above estimate (3.41), we obtain

$$\begin{aligned}&(1+t)\big\|(V_t, V_{xt}, V_{tt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)\big\|(V_{xt}, V_{tt})(s)\big\|^2_{L^2(\mathbb{R})}\,ds\\&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big)\end{aligned}\tag{3.42}$$

with small $N(T)+\delta$.

Finally, multiplying (3.40) by $(1+t)^2(\lambda V_t + V_{tt})$ ($0<\lambda \ll 1$), and integrating it over $\mathbb{R}\times[0,t]$, and using the above two estimates (3.41) and (3.5), we obtain

$$\begin{aligned}&(1+t)^2\big\|(V_t, V_{xt}, V_{tt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^2\big\|(V_{xt}, V_{tt})(s)\big\|^2_{L^2(\mathbb{R})}\,ds\\&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big)\end{aligned}\tag{3.43}$$

with small $N(T)+\delta$. The proof is completed. □

Lemma 3.7 *(Decay rate for V_{xt}). It holds that*

$$\begin{aligned}&(1+t)^3\big\|(V_{xt}, V_{tt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t (1+s)^3\big\|V_{tt}(s)\big\|^2_{L^2(\mathbb{R})}\,ds\\&\quad\leqslant C\big(\|V_0\|^2_{H^2(\mathbb{R})} + \|z_0\|^2_{H^1(\mathbb{R})} + \delta\big)\end{aligned}\tag{3.44}$$

provided $N(T)+\delta \ll 1$.

Proof. Multiplying (3.40) by $(1+t)^3 V_{tt}$ and integrating it over $\mathbb{R}\times[0,t]$, then using Lemma 3.6, we can obtain (3.44). □

Lemma 3.8 *(Decay rate for V_{xxt}). It holds that*

$$\begin{aligned}&(1+t)^4\big\|(V_{xxt}, V_{xtt})(t)\big\|^2_{L^2(\mathbb{R})} + \int_0^t \big[(1+s)^3\big\|V_{xxt}(s)\big\|^2_{L^2(\mathbb{R})} + (1+s)^4\big\|V_{xtt}(s)\big\|^2_{L^2(\mathbb{R})}\big]\,ds\\&\quad\leqslant C\big(\|V_0\|^2_{H^3(\mathbb{R})} + \|z_0\|^2_{H^2(\mathbb{R})} + \delta\big)\end{aligned}\tag{3.45}$$

provided $N(T)+\delta \ll 1$.

Proof. In a similar way as before, differentiating (3.40) with respect to x, and multiplying it by $(1+t)^4 V_{xtt}$, and then integrating the resultant equation over $\mathbb{R}\times[0,t]$ with respect to x and t, also using Lemmas 3.2–3.7, we can prove (3.45) provided $N(T)+\delta \ll 1$. $\square$

Based on Lemmas 3.1–3.8, we have proved (1.35) in Theorem 1.1. Next, we are going to adopt the technique of approximating Green function to obtain the optimal decay rates (1.36) and (1.37).

As in [19], we rewrite Eq. (3.1) as

$$\alpha V_t - \big(a(x,t)V_x\big)_x = -F_1 - F_2 - V_{tt}, \tag{3.46}$$

where $a(x,t) = -p'(\bar v(x,t)) > C_0 > 0$, and construct a minimizing Green function as

$$G(x,t;y,s) = \left(\frac{\alpha}{4\pi a(x,t)(t-s)}\right)^{1/2} \exp\left(\frac{-\alpha(x-y)^2}{4A(y,s,t)(t-s)}\right), \tag{3.47}$$

where $A(y,s,t) = -p'(\bar v(\eta))$, $\bar v = \bar v(\frac{y}{\sqrt{1+t}})$ is the diffusion wave in the form of self-similar solution, and η is defined as

$$\eta = \begin{cases} y/\sqrt{1+s}, & s > t/2, \\ y/\sqrt{1+t/2}, & s \leqslant t/2. \end{cases}$$

Then the solution of (3.46) can be written in the integral form

$$\begin{aligned} V(x,t) = {} & \int_{-\infty}^{\infty} G(x,t;y,0)V_0(y)\,dy \\ & + \alpha^{-1}\int_0^t\int_{-\infty}^{\infty} G(x,t;y,s)\big[-F_1(y,s) - F_2(y,s) - V_{ss}(y,s)\big]\,dy\,ds \\ & + \int_0^t\int_{-\infty}^{\infty} R_G(x,t;y,s)V(y,s)\,dy\,ds, \end{aligned} \tag{3.48}$$

where

$$R_G(x,t;y,s) := G_s(x,t;y,s) + \alpha^{-1}\big\{a(y,s)G_y(x,t;y,s)\big\}_y.$$

Differentiating (3.48) with respect to x and t, we have, for $l \leqslant 1$, $k+l \leqslant 3$,

$$\begin{aligned} \partial_t^l\partial_x^k V(x,t) = {} & \partial_t^l\partial_x^k \int_{-\infty}^{\infty} G(x,t;y,0)V_0(y)\,dy \\ & - \alpha^{-1}\partial_t^l\partial_x^k \int_0^t\int_{-\infty}^{\infty} G(x,t;y,s)F_1(y,s)\,dy\,ds \\ & - \alpha^{-1}\partial_t^l\partial_x^k \int_0^t\int_{-\infty}^{\infty} G(x,t;y,s)F_2(y,s)\,dy\,ds \end{aligned}$$

$$
\begin{aligned}
&- \alpha^{-1}\partial_t^l\partial_x^k \int_0^t \int_{-\infty}^{\infty} G(x,t;y,s)V_{ss}(y,s)\,dy\,ds \\
&+ \partial_t^l\partial_x^k \int_0^t \int_{-\infty}^{\infty} R_G(x,t;y,s)V(y,s)\,dy\,ds \\
=:{}& I_1^{l,k} + I_2^{l,k} + I_3^{l,k} + I_4^{l,k} + I_5^{l,k}.
\end{aligned}
\tag{3.49}
$$

Based on the estimates obtained in (1.35), and by applying the decay of the approximating Green function $G(x,t;y,s)$, as exactly showed in [19], we can further prove, for $l \leqslant 1$, $k+l \leqslant 3$,

$$\left\| I_1^{l,k} \right\|_{L^2(\mathbb{R})} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.50}$$

$$\left\| I_2^{l,k} \right\|_{L^2(\mathbb{R})} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.51}$$

$$\left\| I_3^{l,k} \right\|_{L^2(\mathbb{R})} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.52}$$

$$\left\| I_4^{l,k} \right\|_{L^2(\mathbb{R})} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}, \tag{3.53}$$

$$\left\| I_5^{l,k} \right\|_{L^2(\mathbb{R})} = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}}. \tag{3.54}$$

The details are omitted.

Combing (3.49)–(3.54), we then obtain the optimal rates (1.36) and (1.37), namely,

Lemma 3.9 *(Optimal decay rates). It holds that*

$$
\begin{aligned}
\left\| \partial_x^k V(t) \right\|_{L^2(\mathbb{R})} &= O(1)(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 0,1,2,3, \\
\left\| \partial_x^k V_t(t) \right\|_{L^2(\mathbb{R})} &= O(1)(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad k = 0,1,2.
\end{aligned}
$$

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Appendix A

In this appendix, we give the detailed proof for finding the solution (1.9) to the ODE (1.8). The approach adopted is the standard method of separation of variables.

Let us consider the following modified Bernoulli's differential equation

$$\frac{d}{dt} f(t) = -\alpha f(t) - \beta \left| f(t) \right|^{q-1} f(t). \tag{A.1}$$

It can be separated as

$$\frac{df}{f(1+\frac{\beta}{\alpha}|f|^{q-1})} = -\alpha\,dt,$$

and can be reduced to the form of partial fractions

$$\left(\frac{1}{f}-\frac{\frac{\beta}{\alpha}|f|^{q-2}\frac{|f|}{f}}{1+\frac{\beta}{\alpha}|f|^{q-1}}\right)df=-\alpha\,dt.$$

After integration, it yields

$$\ln|f|-\int\frac{\frac{\beta}{\alpha}|f|^{q-2}\frac{|f|}{f}}{1+\frac{\beta}{\alpha}|f|^{q-1}}\,df=-\alpha t+C_1, \tag{A.2}$$

where C_1 is an integration constant. Let us define

$$\operatorname{sign}(f)=\begin{cases}1, & \text{if } f(t)\geqslant 0,\\ -1, & \text{if } f(t)<0.\end{cases} \tag{A.3}$$

Then it holds

$$|f|=\operatorname{sign}(f)f \quad\text{and}\quad f=\operatorname{sign}(f)|f| \tag{A.4}$$

and formally

$$\frac{d|f|}{dt}=\operatorname{sign}(f)\frac{df}{dt}. \tag{A.5}$$

We may also treat $|f|$ by taking

$$|f|=\sqrt{f^2} \quad\text{and}\quad \frac{d|f|}{dt}=\frac{f}{\sqrt{f^2}}\frac{df}{dt}=\operatorname{sign}(f)\frac{df}{dt}. \tag{A.6}$$

Now let

$$h=1+\frac{\beta}{\alpha}|f|^{q-1},$$

then, from (A.6), it can be verified that

$$\begin{aligned}
dh&=\frac{d}{df}\left(1+\frac{\beta}{\alpha}|f|^{q-1}\right)df=\frac{d}{df}\left(1+\frac{\beta}{\alpha}\big[\sqrt{f^2}\big]^{q-1}\right)df\\
&=\frac{\beta}{\alpha}(q-1)\big[\sqrt{f^2}\big]^{q-2}\frac{d}{df}\big(\sqrt{f^2}\big)\,df=\frac{\beta}{\alpha}(q-1)\big[\sqrt{f^2}\big]^{q-2}\frac{1}{2\sqrt{f^2}}\frac{d}{df}\big(f^2\big)\,df\\
&=\frac{\beta}{\alpha}(q-1)\big[\sqrt{f^2}\big]^{q-2}\frac{f}{\sqrt{f^2}}\,df=\frac{\beta}{\alpha}(q-1)|f|^{q-2}\frac{f}{|f|}\,df\\
&=\frac{\beta}{\alpha}(q-1)|f|^{q-2}\frac{|f|}{f}\,df,
\end{aligned}$$

where the last step we used $\frac{f}{|f|}=\frac{|f|}{f}=\operatorname{sign}(f)$. So, we can integrate

$$\int\frac{\frac{\beta}{\alpha}|f|^{q-2}\frac{|f|}{f}}{1+\frac{\beta}{\alpha}|f|^{q-1}}\,df=\frac{1}{q-1}\int\frac{dh}{h}=\frac{1}{q-1}\ln|h|=\ln|h|^{\frac{1}{q-1}}=\ln\left|1+\frac{\beta}{\alpha}|f|^{q-1}\right|^{\frac{1}{q-1}}. \tag{A.7}$$

Substituting (A.7) into (A.2), we obtain

$$\ln|f| - \ln\left|1 + \frac{\beta}{\alpha}|f|^{q-1}\right|^{\frac{1}{q-1}} = -\alpha t + C_1,$$

which gives

$$\frac{|f|}{|1 + \frac{\beta}{\alpha}|f|^{q-1}|^{\frac{1}{q-1}}} = e^{-\alpha t} e^{C_1},$$

namely,

$$\frac{|f|^{q-1}}{|1 + \frac{\beta}{\alpha}|f|^{q-1}|} = e^{-(q-1)\alpha t} e^{(q-1)C_1}. \tag{A.8}$$

In order to avoid the solution to blow up at a finite time t, we need to look for a small solution $f(t)$ such that

$$1 + \frac{\beta}{\alpha}|f|^{q-1} > 0.$$

Thus, (A.8) is reduced to

$$|f|^{q-1} = \left(1 + \frac{\beta}{\alpha}|f|^{q-1}\right) e^{-(q-1)\alpha t} e^{(q-1)C_1},$$

which can be solved as

$$|f| = \frac{e^{C_1} e^{-\alpha t}}{(1 - \frac{\beta}{\alpha}[e^{C_1} e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}},$$

namely,

$$f = \frac{\pm e^{C_1} e^{-\alpha t}}{(1 - \frac{\beta}{\alpha}[e^{C_1} e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}}.$$

Let

$$C_2 = \pm e^{C_1},$$

and note that $|C_2| = e^{C_1}$, we finally obtain

$$f(t) = \frac{C_2 e^{-\alpha t}}{(1 - \frac{\beta}{\alpha}[|C_2| e^{-\alpha t}]^{q-1})^{\frac{1}{q-1}}}.$$

This proves (1.9).

Furthermore, when $|C_2| \ll 1$, we can easily confirm $|f(t)| \ll 1$, which ensures $1 + \frac{\beta}{\alpha}|f|^{q-1} > 0$ as we required before.

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