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ON NONLINEAR COUPLED REACTION-DIFFUSION SYSTEMS*

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Abstract

In this paper, the problem of initial boundary value for nonlinear coupled reaction-diffusion systems arising in biochemistry, engineering and combustion theory is considered.

§ 1. Introduction

We consider the problem of initial boundary value for nonilinear coupled reaction-diffusion systems which arising in biochemistry, engineering and combustion theory

$$u_t-\Delta u=f_1(u,\ v,\ D_xu,\ D_xv) \quad (x,\ t)\in \boldsymbol{R}_+^n\times \boldsymbol{R}_+ \tag{1}$$
$$v_t-\Delta v=f_2(u,\ v,\ D_xu,\ D_xv) \tag{2}$$
$$u|_{t=0}=\varphi_1(x) \quad x\in \boldsymbol{R}_+^n \tag{3}$$
$$v|_{t=0}=\varphi_2(x) \tag{4}$$
$$u|_{x_j=0}=\psi_{1j}(\hat{x}_j,\ t) \quad (j=1,\ 2,\ \cdots,\ n) \quad (\hat{x}_j,\ t)\in \boldsymbol{R}_+^{n-1}\times \boldsymbol{R}_+ \tag{5}$$
$$v|_{x_j=0}=\psi_{2j}(\hat{x}_j,\ t) \tag{6}$$

where Δ is Laplacian, i.e.

$$\Delta=\sum_{i=1}^{n}\partial^2/\pi x_i^2,$$
$$\boldsymbol{R}_+^n=\{(x_1,\ \cdots,\ x_n)\in \boldsymbol{R}^n|x_i>0,\ i=1,\ 2,\ \cdots,\ n\},$$
$$D_xu=(u_{x_1},\ u_{x_2},\ \cdots,\ u_{x_n}),$$
$$D_xv=(v_{x_1},\ v_{x_2},\ \cdots,\ v_{x_n})$$

$\hat{x}_j$ denotes the vector $\hat{x}_j=(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)\in \boldsymbol{R}_+^{n-1}$, $j=2,\ 3,\ \cdots,\ n$, $\hat{x}_1=(x_2,x_3,\cdots,x_n)$, $\hat{x}_n=(x_1,\ x_2,\ \cdots,\ x_{n-1})$.

For convenience's sake, the following signs are given $(\hat{x}_j,\ x_j)$ denotes the vector $x=(x_1,\ \cdots,\ x_{j-1},\ x_j,\ x_{j+1},\ \cdots,\ x_n)\in \boldsymbol{R}_+^n$ and $(\widehat{x_{jl},\ 0})$ denotes the vector which let $x_l=0$ in $\hat{x}_j$, i.e.

$$(\widehat{x_{jl},\ 0})=(x_1,\ \cdots,\ x_{l-1},\ 0,\ x_{l+1},\ \cdots,\ x_{j-1},\ x_{j+1},\ \cdots,\ x_n)\in \boldsymbol{R}_+^{n-1} \quad \text{as } l<j$$
$$(\widehat{x_{j1},\ 0})=(0,\ x_2,\ \cdots,\ x_{j-1},\ x_{j+1},\ \cdots,\ x_n)$$

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or

$$\widehat{(x_{jl}, 0)} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{l-1}, 0, x_{l+1}, \cdots, x_n) \in R_+^{n-1} \quad \text{as } l>j$$
$$\widehat{(x_{jn}, 0)} = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{n-1}, 0)$$
$$j=1, 2, \cdots, n;\ l=1, 2, \cdots, n;\ j\neq l.$$

We give some elementary hypotheses now

(H_1) f_1, f_2, φ_1, φ_2 are real continuous functions

(H_2)
$$\psi_{1j}, \psi_{2j} \in C^{2,1}(R_+^{n-1}, R_+)$$
$$\psi_{1j}(\widehat{x_{jl}, 0}, t) = \frac{\partial}{\partial x_l}\psi_{1j}(\widehat{x_{jl}, 0}, t) = 0$$
$$\psi_{2j}(\widehat{x_{jl}, 0}, t) = \frac{\partial}{\partial x_l}\psi_{2j}(\widehat{x_{jl}, 0}, t) = 0$$
$$j=1, 2, \cdots, n;\ l=1, 2, \cdots, n;\ j\neq l$$

(H_3)
$$\varphi_1(\hat{x}_j, x_j)\,|_{x_j=0} = \psi_{1j}(\hat{x}_j, t)\,|_{t=0} = 0$$
$$\varphi_2(\hat{x}_j, x_j)_{x_j=0} = \psi_{2j}(\hat{x}_j, t)\,|_{t=0} = 0$$
$$j=1, 2, \cdots, n.$$

There are many studies for reaction–diffusion equations (see[1—5]), the main method is usually the scheme of upper–lower solutions. We know that the method is often based on the comparison theorem of the nonlinear systems which are investigated, so it must depend on the extremal principle, but as for now the method of upper–lower solutions cannot be used for the non–weak coupled situation because the extremal principle of the systems does not hold. B. G. Pachpatte[6] discussed coupled nonlinear reaction–diffusion equations, under very strict conditions, he proved the existence of the solutions. In this paper, the problems (1)—(6) which we investigate are nonlinear non–weak coupled systems, they are wider than the problems studied by B. G. Pachpatte under certain means (our systems contain the coupled terms of $D_x u$ and $D_x v$). We prove the existence and uniqueness of classical solutions for the systems under more weakly condition than in[6].In section 2, our approach to the problems is based on converting the systems into the nonlinear coupled Volterra–Fredholm type integral equations, then we give some sharp estimates on the solutions and use Banach theory to prove the existence and uniqueness of the classical solutions for the equations. This scheme is similar to H. Fujita[7],[9]. Here we can avoid the difficulty of using the method of upper–lower solutions. Finally, we will point out similar results held still for multiple coupled equations.

§ 2. Main Result and Proof

In the section, we turn the nonlinear coupled systems into the nonlinear coupled Volterra–Fredholm type integral equations, then define a Banach space, and give some sharp estimates on the solutions (u, v) in this space, moreover, use the Banach fixed point theorem to prove that the classical global solutions exists.

Noticing the elementary hypothese (H_1)—(H_3), we known that the systems (1)

—(6) are equal to the following integral equations (see [4], [6], [8])

$$u(x, t)=u_0(x, t)+\int_0^t d\tau\int_{R_+^n} G(x, t; \xi, \tau)f_1(u, v, D_x u, D_x v)(\xi, \tau)d\xi, \tag{7}$$

$$v(x, t)=v_0(x, t)+\int_0^t d\tau\int_{R_+^n} G(x, t; \xi, \tau)f_2(u, v, D_x u, D_x v)(\xi, \tau)d\xi, \tag{8}$$

for $(x, t)\in \boldsymbol{R}_+^n\times \boldsymbol{R}_+$, where

$$u_0(x, t)=\int_{R_+^n} G(x, t; \xi, 0)\varphi_1(\xi)d\xi$$
$$+\sum_{j=1}^n\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial}{\partial\xi_j}G(x, t; (\hat{\xi}_j, 0), \tau)\cdot\psi_{1j}(\hat{\xi}_j, \tau)d\hat{\xi}_j, \tag{9}$$

$$v_0(x, t)=\int_{R_+^n} G(x, t; \xi, 0)\varphi_2(\xi)d\xi$$
$$+\sum_{j=1}^n\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial}{\partial\xi_j}G(x, t; (\hat{\xi}_j, 0), \tau)\psi_{2j}(\hat{\xi}_j, \tau)d\hat{\xi}_j, \tag{10}$$

$G(x, t; \xi, \tau)$ is the Green function of the systems (1)—(6), and

$$G(x, t; \xi, \tau)=\sum_{l_1,\cdots,l_n=0}^{1}(-1)^{l_1+\cdots+l_n}\Gamma((-1)^{l_1}x_1, \cdots, (-1)^{l_n}x_n, t; \xi, \tau), \tag{11}$$

$$\Gamma((-1)^{l_1}x_1, \cdots, (-1)^{l_n}x_n, t; \xi, \tau)=\frac{1}{[4\pi(t-\tau)]^{n/2}}\exp\left\{-\frac{\sum_{i=1}^n[(-1)^{l_i}x_i+\xi_i]^2}{4(t-\tau)}\right\}. \tag{12}$$

In order to look for the classical solutions of the problems (7)—(10), a fit function space and several lemmas must be given.

Definition. Assume B denotes a real value functional set

$$\{(u, v)\in \boldsymbol{R}\times\boldsymbol{R}\,|\,(x, t)\in \boldsymbol{R}_+^n\times\boldsymbol{R}_+\},$$

and (u, v) satisfies the conditions

(i) $u(x, t)$, $v(x, t)$ exist continuous first partial derivatives.

(ii) $|u(x, t)|$, $|v(x, t)|$, $|D_x u(x, t)|$, $|D_x v(x, t)|\leqslant M\cdot E(x, t+r)$ for $(x, t)\in \boldsymbol{R}_+^n\times\boldsymbol{R}_+$, where

$$E(x, t+r)=\frac{1}{[4\pi(t+r)]^{n/2}}\exp\left[-\frac{|x|^2}{4(t+r)}\right],$$

r is a positive constant, $M>0$ and may depend on u and v.

(iii) Introduce a norm in B

$$\|(u, v)\|_B=\sup_{(x,t)\in R_+^n\times R_+}\left\{\frac{|u(x, t)|+|v(x, t)|+\sum_{i=1}^n(|D_{x_i}u(x, t)|+|D_{x_i}v(x, t)|)}{E(x, t+r)}\right\}.$$

It is known easily that B is a Banach space.

We must point out that the condition (ii) is important and necessary for the uniqueness of the solutions, because some papers gave the result of the non-uniqueness of solutions with the condition (ii) unsatisfied (see [10] and its references).

Lemma 1.

$$\int_{R_+^n} |G(x, t; \xi, \tau)| \cdot E(\xi, r) d\xi \leqslant 2^{n^2-n} E(x, t-\tau+r).$$

Proof. We have the following by (11)

$$\int_{R_+^n} |G(x, t; \xi, \tau)| \cdot E(\xi, r) d\xi$$

$$\leqslant \int_{R_+^n} \sum_{l_1, \cdots, l_n=0}^{1} [4\pi(t-\tau)]^{-n/2} \cdot (4\pi r)^{-n/2}$$

$$\cdot \exp\left\{ -\frac{\sum_{i=1}^{n} [(-1)^{l_i} x_i - \xi_i]^2}{4(t-\tau)} - \frac{\sum_{i=1}^{n} \xi_i^2}{4r} \right\} d\xi$$

$$\leqslant (4\pi)^{-n} [(t-\tau) r]^{-n/2} \prod_{i=1}^{n} \left\{ \exp\left[-\frac{x_i^2}{4(t-\tau)} + \frac{x_i^2 r}{4(t-\tau)(t-\tau+r)} \right] \right.$$

$$\left. \cdot \int_0^\infty \sum_{l_1, \cdots, l_n=0}^{1} \exp\left[-\left(\sqrt{\frac{t+r-\tau}{4r(t-\tau)}}\, \xi_i - (-1)^{l_i} \sqrt{\frac{r}{4(t-\tau)(t-\tau+r)}}\, x_i \right)^2 \right] d\xi_i \right\}$$

$$\leqslant (4\pi)^{-n} [(t-\tau) r]^{-n/2} \prod_{i=1}^{n} \left\{ \exp\left[-\frac{x_i^2}{4(t-\tau+r)} \right] \right.$$

$$\left. \cdot \left(\frac{t+r-\tau}{4r(t-\tau)} \right)^{-\frac{1}{2}} \cdot \sum_{l_1, \cdots, l_n=0}^{1} \frac{\sqrt{\pi}}{2} \right\}$$

$$\leqslant 2^{n^2-n} \cdot E(x, t-\tau+r).$$

Let

$$\Psi_1(x, t) = \sum_{j=1}^{n} \psi_{1j}(\hat{x}_j, t) \quad x \in R_+^n,\ t \in R_+. \tag{13}$$

$$\Psi_2(x, t) = \sum_{j=1}^{n} \psi_{2j}(\hat{x}_j, t) \quad x \in R_+^n,\ t \in R_+. \tag{14}$$

Since

$$G(x, t; \xi, \tau)|_{\xi_j=\infty} = 0, \quad j = 1, 2, \cdots, n \tag{15}$$

$$\frac{\partial}{\partial \tau} G + \sum_{i=1}^{n} \frac{\partial^2}{\partial \xi_i^2} G = 0, \tag{16}$$

$$G(x, t; \xi, t)|_{\tau \to t} \to 0. \tag{17}$$

Noticing (H_3) so we obtain

$$\Psi_i(x, t)_{t=0} = 0, \quad i = 1, 2, \quad x \in R_+^n. \tag{18}$$

According to (15)—(18) and (H_1)—(H_3), using the method of integration by prats, we get

$$\int_0^t d\tau \int_{R^n} G(x, t; \xi, \tau) \left(\frac{\partial \Psi_1}{\partial \tau} - \Delta \Psi_1 \right)(\xi, \tau) d\xi$$

$$= \int_0^t d\tau \int_{R_+^n} G \cdot \frac{\partial \Psi_1}{\partial \tau}(\xi, \tau) d\xi$$

$$- \int_0^t d\tau \int_{R_+^n} G \cdot \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_j^2} \Psi_1(\xi, \tau) d\xi$$

$$= \int_{R_+^n} G(x, t; \xi, \tau) \Psi_1(\xi, \tau) |_{\tau=0}^{\tau=t} d\xi$$

$$- \int_0^t d\tau \int_{R_+^n} \frac{\partial G(x, t; \xi, \tau)}{\partial \tau} \Psi_1(\xi, \tau) d\xi$$

$$-\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^{n-1}} G\cdot\frac{\partial\Psi_1}{\partial\xi_j}(\xi,\tau)\,\Big|_{\xi_j=0}^{\xi_j=+\infty} d\hat{\xi}_j$$

$$+\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^n}\frac{\partial G(x,t;\xi,\tau)}{\partial\xi_i}\cdot\frac{\partial\Psi_1}{\partial\xi_j}(\xi,\tau)d\xi$$

$$=-\int_0^t d\tau\int_{R_+^n}\frac{\partial}{\partial\tau}G(x,t;\xi,\tau)\Psi_1(\xi,\tau)d\xi$$

$$+\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^n}\frac{\partial G(x,t;\xi,\tau)}{\partial\xi_j}\cdot\frac{\partial\Psi_1}{\partial\xi_j}(\xi,\tau)d\xi$$

$$=-\int_0^t d\tau\int_{R_+^n}\frac{\partial}{\partial\tau}G(x,t;\xi,\tau)\Psi_1(\xi,\tau)d\xi$$

$$+\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial G}{\partial\xi_j}(x,t;\xi,\tau)\Psi_1(\xi,\tau)\,\Big|_{\xi_j=0}^{\xi_j=+\infty}d\hat{\xi}_j$$

$$-\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^n}\frac{\partial^2}{\partial\xi_j^2}G(x,t;\xi,\tau)\Psi_1(\xi,\tau)d\xi$$

$$=-\int_0^t\Big(\frac{\partial G}{\partial\tau}+\sum_{j=1}^{n}\frac{\partial^2 G}{\partial\xi_j^2}\Big)(x,t;\xi,\tau)\Psi_1(\xi,\tau)d\xi$$

$$-\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial}{\partial\xi_j}G(x,t;(\hat{\xi}_j,0),\tau)\cdot\Psi_{1j}(\hat{\xi}_j,\tau)d\hat{\xi}_j$$

$$=-\sum_{j=1}^{n}\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial}{\partial\xi_j}G(x,t;(\hat{\xi}_j,0),\tau)\cdot\Psi_{1j}(\hat{\xi}_j,\tau)d\hat{\xi}_j. \tag{19}$$

Similarly for $\psi_{2j}(\hat{\xi}_j, t)$, it holds

$$\sum_{j=0}^{n}\int_0^t d\tau\int_{R_+^{n-1}}\frac{\partial}{\partial\xi_j}G(x,t;(\hat{\xi}_j,0),\tau)\cdot\psi_{2j}(\hat{\xi}_j,\tau)d\hat{\xi}_j$$

$$=-\int_0^t d\tau\int_{R_+^n}G(x,t;\xi,\tau)\Big(\frac{\partial\Psi_2}{\partial\tau}-\sum_{j=1}^{n}\frac{\partial^2}{\partial\xi_j^2}\Psi_2\Big)(\xi,\tau)d\xi. \tag{20}$$

So

$$u_0(x,t)=\int_{R_+^n}G(x,t;\xi,0)\varphi_1(\xi)d\xi$$

$$-\int_0^t d\tau\int_{R_+^n}G(x,t;\xi,\tau)\Big(\frac{\partial\Psi_1}{\partial\tau}-\sum_{j=1}^{n}\frac{\partial^2\Psi_1}{\partial\xi_j^2}\Big)(\xi,\tau)d\xi, \tag{21}$$

$$v_0(x,t)=\int_{R_+^n}G(x,t;\xi,0)\varphi_2(\xi)d\xi$$

$$-\int_0^t d\tau\int_{R_+^n}G(x,t;\xi,\tau)\Big(\frac{\partial\Psi_2}{\partial\tau}-\sum_{j=1}^{n}\frac{\partial^2\Psi_2}{\partial\xi_j^2}\Big)(\xi,\tau)d\xi. \tag{22}$$

Lemma 2. Suppose $n\alpha>2$, $0<\alpha<1$. If the initial boundary values satisfy respectively

$$|\varphi_1(x)|\cdot|\varphi_2(x)|\cdot|D_{x_j}\varphi_1(x)|\cdot|D_{x_j}\varphi_2(x)|\leqslant\delta\cdot E(x,t+r),\quad j=1,2,\cdots,n, \tag{23}$$

$$\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)D_{x_j}\Psi_i(x,t)\Big|\leqslant\frac{\delta}{(t+r)^{n\alpha/2}}\cdot E(x,t+r),$$

$$\Big|\Big(\frac{\partial}{\partial t}-\Delta\Big)\Psi_i(x,t)\Big|\leqslant\frac{\delta}{(t+r)^{n\alpha/2}}\cdot E(x,t+r),$$

$$i=1,2;\ j=1,2,\cdots,n. \tag{24}$$

$$\left|\left(\frac{\partial}{\partial t}-\Delta\right)\psi_{1j}(\hat{x}_j,\ t)\right|\leqslant\frac{\delta}{(t+r)^{n\alpha/2}}\cdot E(\hat{x}_j,\ t+r),$$

$$\left|\left(\frac{\partial}{\partial t}-\Delta\right)\psi_{2j}(\hat{x}_j,\ t)\right|\leqslant\frac{\delta}{(t+r)^{n\alpha/2}}\cdot E(\hat{x}_j,\ t+r),\tag{25}$$

$$i=1,\ 2;\ j=1,\ 2,\ \cdots,\ n.\ x\in\boldsymbol{R}^n_+,\ t\in\boldsymbol{R}_+,\ \hat{x}_j\in\boldsymbol{R}^{n-1}_+,$$

where
$$E(\hat{x}_j,\ t)=(4\pi t)^{-(n-1)/2}\cdot\exp\left[-\sum_{i=1,i\neq j}^{n}x_i^2/4t\right],\tag{26}$$

then $(u_0,\ v_0)\in B$, and

$$\|(u_0,\ v_0)\|_B\leqslant C_1\delta.\tag{27}$$

Here and after here, C denotes different constant, and is not relative to u, v, x and t. δ is a positive constant.

Proof. According to (21), (22), thanks to Lemma 1 and (23)—(25), we have firstly

$$\begin{aligned}|u_0(x,\ t)|&\leqslant\int_{R^n_+}|G(x,\ t;\ \xi,\ 0)|\cdot|\varphi_1(\xi)|d\xi\\&\quad+\int_0^t d\tau\int_{R^n_+}|G(x,\ t;\ \xi,\ \tau)|\cdot\left|\frac{\partial\Psi_1}{\partial\tau}-\sum_{j=1}^{n}\frac{\partial^2}{\partial\xi_j^2}\Psi_1(\xi,\ \tau)\right|d\xi\\&\leqslant2^{n^2-n}\delta\cdot E(x,\ t+r)\\&\quad+\int_0^t d\tau\int_{R^n_+}|G(x,\ t;\ \xi,\ \tau)|\cdot\frac{\delta}{(t+r)^{n\alpha/2}}\cdot E(\xi,\ \tau+r)d\xi\\&\leqslant2^{n^2-n}\delta\cdot E(x,\ t+r)+2^{n^2-n}\delta E(x,\ t+r)\int_0^t(\tau+r)^{-n\alpha/2}d\tau\\&\leqslant2^{n^2-n}(C_0+1)\delta E(x,\ t+r)\\&\leqslant C_1'\delta E(x,\ t+r),\end{aligned}\tag{28}$$

where
$$C_0=\int_0^\infty(\tau+r)^{-n\alpha/2}d\tau$$

converges to a positive constant as $n\alpha>2$.

Reason by analogy of (28), we gain

$$|v_0(x,\ t)|\leqslant C_1'\delta E(x,\ t+r).\tag{29}$$

Moreover, it is easily known

$$\begin{aligned}&\frac{\partial}{\partial x_i}\Gamma((-1)^{l_1}x_1,\ \cdots,\ (-1)^{l_n}x_n,\ t;\ \xi,\ \tau)\\&=(-1)^{l_i+1}\frac{\partial}{\partial\xi_i}\Gamma((-1)^{l_1}x_1,\cdots,\ (-1)^{l_n}x_n,\ t;\ \xi.\ \tau),\end{aligned}\tag{30}$$

as $0\leqslant t\leqslant\varepsilon$, ε is an arbitrary selected constant, let $\eta=t-\tau$

$$\begin{aligned}\int_0^t\sqrt{\frac{t+r}{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}&=\int_0^t\frac{\sqrt{t+r}}{\sqrt{\eta}}\cdot\frac{d\eta}{(t-\eta+r)^{(n\alpha+1)/2}}\\&\leqslant r^{-(n\alpha+1)/2}\sqrt{t+r}\int_0^t\frac{d\eta}{\sqrt{\eta}}\\&=r^{-(n\alpha+1)/2}\sqrt{t+r}\cdot2\sqrt{t}\end{aligned}$$

$$\leqslant 2r^{-(n\alpha+1)/2}\sqrt{s+r}\cdot\sqrt{s}$$
$$\leqslant C, \tag{31}$$

as $t>s$, we have

$$\int_0^t \frac{\sqrt{t+r}}{\sqrt{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}$$
$$=\int_0^{t/2} \frac{\sqrt{t+r}}{\sqrt{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}$$
$$+\int_{\frac{t}{2}}^t \frac{\sqrt{t+r}}{\sqrt{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}, \tag{32}$$

but the right-hand first term of (32) holds

$$\int_0^{t/2}\sqrt{\frac{t+r}{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}\leqslant\sqrt{\frac{t+r}{t/2}}\cdot\int_0^\infty(\tau+r)^{-(n\alpha+1)/2}d\tau$$
$$\leqslant\sqrt{2+2r/s}\cdot C_0'\leqslant C, \tag{33}$$

where
$$C_0'=\int_0^\infty(\tau+r)^{-(n\alpha+1)/2}d\tau,$$

it converges as $n\alpha>2$.

Let $\eta=t-\tau$, the right-hand second term of (32) holds

$$\int_{t/2}^t\sqrt{\frac{t+r}{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}=\int_0^{t/2}\frac{\sqrt{t+r}}{\sqrt{\eta}}\cdot\frac{d\eta}{(t-\eta+r)^{(n\alpha+1)/2}}$$
$$\leqslant\frac{\sqrt{t+r}}{\left(\frac{1}{2}t+r\right)^{(n\alpha+1)/2}}\cdot\int_0^{t/2}\frac{d\eta}{\sqrt{\eta}}$$
$$\leqslant\frac{\sqrt{2}}{4}\sqrt{\frac{(t+r)t}{\left(\frac{1}{2}t+r\right)^{n\alpha+1}}}\leqslant C. \tag{34}$$

So, as $t>s$, by (33)—(34), (32) may express

$$\int_0^t\sqrt{\frac{t+r}{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}\leqslant C, \tag{35}$$

therefore

$$\int_0^t\sqrt{\frac{t+r}{t-\tau}}\cdot\frac{d\tau}{(\tau+r)^{(n\alpha+1)/2}}\leqslant C, \quad \text{for } t\in[0,+\infty). \tag{36}$$

Paying attention to (23)—(25), (36) and (35) and condition (H_3), and thanking to Lemma 1, we get the following inequality by means of integration by parts.

$$|D_{x_i}u_0|\leqslant\int_{R_+^n}\sum_{l_1,\cdots,l_n=0}^{1}|D_{x_i}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,0)|\cdot|\varphi_1(\xi)|d\xi$$
$$+\int_0^t d\tau\int_{R_+^n}\sum_{l_1,\cdots,l_n=0}^{1}|D_{x_i}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,\tau)|$$
$$\cdot\left|\left(\frac{\partial\Psi_1}{\partial\tau}-\Delta\Psi_1\right)(\xi,\tau)\right|d\xi$$
$$\leqslant\int_{R_+^n}\sum_{l_1,\cdots,l_n=0}^{1}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,0)|D_{\xi_i}\varphi_1(\xi)|d\xi$$

$$
\begin{aligned}
&+\int_0^t d\tau\int_{R_+^{n-1}}\sum_{l_1,\cdots,l_n=0}^{1}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;(\hat{\xi}_i,0),\tau)\\
&\cdot\left|\left(\frac{\partial}{\partial\tau}\Psi_1-\Delta\Psi_1\right)((\hat{\xi}_i,0),\tau)\right|d\hat{\xi}_i\\
&+\int_0^t d\tau\int_{R_+^n}\sum_{l_1,\cdots,l_n=0}^{1}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,\tau)\\
&\cdot\left|\left(\frac{\partial}{\partial\tau}-\Delta\right)D_{\xi_i}\Psi_1(\xi,\tau)\right|d\xi\\
\leqslant&\,\delta\int_{R_+^n}\sum_{l_1,\cdots,l_n=0}^{1}\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,0)\cdot E(\xi,r)d\xi\\
&+\int_0^t d\tau\int_{R_+^{n-1}}\sum\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;(\hat{\xi}_i,0),\tau)\\
&\cdot\left|\left(\frac{\partial}{\partial\tau}\psi_{1i}-\Delta\psi_{1i}\right)(\hat{\xi}_i,\tau)\right|d\hat{\xi}_i\\
&+\delta\int_0^t d\tau\int_{R_+^n}\sum\Gamma((-1)^{l_1}x_1,\cdots,(-1)^{l_n}x_n,t;\xi,\tau)\\
&\cdot(\tau+r)^{-(n\alpha+1)/2}E(\xi,\tau+r)d\xi\\
\leqslant&\,2^{n^2-n}\delta E(x,t+r)+\delta\exp\left[-\frac{x_i^2}{4(t+r)}\right]\cdot\frac{1}{\sqrt{4\pi(t+r)}}\\
&\cdot\int_0^t\sqrt{\frac{t+r}{t-\tau}}\,(\tau+r)^{-(n\alpha+1)/2}\exp\left[-\frac{x_i^2(\tau+r)}{4(t-\tau)(t+r)}\right]d\tau\\
&\cdot\int_{R_+^{n-1}}\sum\Gamma(\widehat{(-1)^{l_i}x_i},t;\hat{\xi}_i,\tau)E(\hat{\xi}_i,\tau+r)d\tau\\
&+2^{n^2-n}C_0\delta E(x,t+r)\\
\leqslant&\,2^{n^2-n}\delta E(x,t+r)+2^{(n-1)(n-2)}C\delta E(x,t+r)\\
&+2^{n^2-n}C_0\delta E(x,t+r)\\
\leqslant&\,C\delta E(x,t+r),\quad j=1,2,\cdots,n.
\end{aligned}\tag{37}
$$

Similarly, we have same estimates for $|v_0|$, $|D_{x_j}v_0|$. So $\|(u_0,v_0)\|_B\leqslant C$. The lemma has been proved completedly.

Lemma 3. Suppose $n\alpha>2$, $0<\alpha<1$, then

$$\int_0^t d\tau\int_{R_+^n}|G(x,t;\xi,\tau)|\cdot E^{1+\alpha}(\xi,\tau<r)d\xi\leqslant C\cdot E(x,t+r),\tag{38}$$

$$\int_0^t d\tau\int_{R_+^n}|D_{x_i}G(x,t;\xi,\tau)|\cdot E^{1+\alpha}(\xi,\tau+r)d\xi\leqslant CE(x,t+r).\tag{39}$$

Proof. We can prove (38) easily, as

$$\zeta^N\cdot\exp(-\zeta^2)\leqslant C,\quad N\geqslant 0\tag{40}$$

so

$$
\begin{aligned}
&\int_0^t d\tau\int_{R_+^n}|G(x,t;\xi,\tau)|E^{1+\alpha}(\xi,\tau+r)d\xi\\
&=\int_0^t E^\alpha(\xi,\tau+r)d\tau\int_{R_+^n}|G(x,t;\xi,\tau)|\cdot E(\xi,\tau+r)d\xi\\
&\leqslant\int_0^t[4\pi(\tau+r)]^{-n\alpha/2}d\tau\cdot 2^{n^2-n}E(x,t+r)
\end{aligned}
$$

$$\leqslant 2^{n^2-n}C_0E(x,\ t+r)\leqslant C\cdot E(x,\ t+r), \tag{41}$$

where
$$C=2^{n^2-n}\int_0^\infty \frac{d\tau}{[4\pi(t+r)]^{n\alpha/2}},$$

and converges to a positive constant as $n\alpha>2$.

Now, we prove (39). According to Lamma 1, and (36), (38) as $n\alpha>2$, we get by the method of integration by parts

$$\begin{aligned}
&\int_0^t d\tau\int_{R_+^n}|D_{x_i}G(x,\ t;\ \xi,\ \tau)|\cdot E^{1+\alpha}(\xi,\ \tau+r)d\xi\\
&\leqslant\int_0^t d\tau\int_{R_+^{n-1}}\sum\Gamma((-1)^{l_1}x_1,\ \cdots,\ (-1)^{l_n}x_n,\ t;\ (\hat{\xi}_i,\ 0),\ \tau)\cdot E^{1+\alpha}((\hat{\xi}_i,\ 0),\ \tau+r)d\hat{\xi}_i\\
&\quad+\int_0^t d\tau\int_{R_+^n}\sum\Gamma((-1)^{l_1}x_1,\ \cdots,\ (-1)^{l_n}x_n,\ t;\ \xi,\ \tau)\cdot|D_{\xi_i}E^{1+\alpha}(\xi,\ \tau+r)|d\xi\\
&\leqslant\int_0^t E^\alpha((\hat{\xi}_i,\ 0),\ \tau+r)d\tau\int_{R_+^{n-1}}\sum\Gamma((-1)^{l_1}x_1,\ \cdots,\ (-1)^{l_n}x_n,\ t;\ (\hat{\xi}_i,\ 0),\ \tau)\\
&\quad\cdot E((\hat{\xi}_i,\ 0),\ \tau+r)d\hat{\xi}_i\\
&\quad+\int_0^t\frac{1+\alpha}{\sqrt{\alpha}\,(4\pi)^{n\alpha/2}}\cdot\frac{1}{(\tau+r)^{(n\alpha+1)/2}}\cdot\exp\Big[-\frac{\alpha|\xi|^2}{4(\tau+r)}\Big]\\
&\quad\cdot\frac{\sqrt{\alpha}\,|\hat{\xi}|}{2\sqrt{\tau+r}}\,d\tau\int_{R_+^n}\sum\Gamma((-1)^{l_1}x_1,\ \cdots,\ (-1)^{l_n}x_n,\ t;\ \xi,\ \tau)\cdot E(\xi,\ \tau+r)d\xi\\
&\leqslant\frac{1}{\sqrt{4\pi(t+r)}}\cdot\exp\Big[-\frac{((-1)^{l_i}x_i)^2}{4(t+r)}\Big]\cdot\int_0^t\frac{\sqrt{t+r}}{\sqrt{t-\tau}}\cdot(\tau+r)^{-(n\alpha+1)/2}\\
&\quad\cdot\exp\Big[-\frac{\alpha|\xi|^2}{4(\tau+r)}\Big]\cdot\exp\Big[-\frac{x_i^2(\tau+r)}{4(t-\tau)(t+r)}\Big]\cdot\int_{R_+^{n-1}}\sum\Gamma(\widehat{(-1^{l_i}x_i},\ t;\ \hat{\xi}_i,\ \tau)\\
&\quad\cdot E(\hat{\xi}_i,\ \tau+r)d\hat{\xi}_i+CE(x,\ t+r)\\
&\leqslant C\cdot\frac{1}{\sqrt{4\pi(t+r)}}\cdot\exp\Big[-\frac{x_i^2}{4(t+r)}\Big]\cdot E(\hat{x}_i,\ t+r\ )+CE(x,\ t+r)\\
&\leqslant CE(x,\ t+r).
\end{aligned}\tag{42}$$

So the proof has been completed.

We set an integral operator

$$S_i:\ \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R},\quad i=1,\ 2$$

$$S_1(u,\ v)=\int_0^t d\tau\int_{R_+^n}G(x,\ t;\ \xi,\ \tau)f_1(u,\ v,\ D_xu,\ D_xv)(\xi,\ \tau)d\xi, \tag{43}$$

$$S_2(u,\ v)=\int_0^t d\tau\int_{R_+^n}G(x,\ t;\ \xi,\ \tau)f_2(u,\ v,\ D_xu,\ D_xv)(\xi,\ \tau)d\xi. \tag{43'}$$

Lemma 4. Suppose $n\alpha>2$, $0<\alpha<1$, f satisfies the following assumption

f_i: $\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n\to\boldsymbol{R}$ is a function which belongs to $C^{1+\alpha}(K_{N'})$, the set $K_{N'}=\{(\xi,\ \eta,\ \zeta,\ s)\in\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ $|\xi|,\ |\eta|,\ |\zeta|,\ |s|<N',\ N'>0\}$, and f_i satisfies $f_i(0,\ 0,\ 0,\ 0)=f'_{i\xi}(0,\ 0,\ 0,\ 0)=f'_{i\eta}(0,\ 0,\ 0,\ 0)=D_\zeta f_i(0,\ 0,\ 0,\ 0)=D_sf_i(0,\ 0,\ 0,\ 0)$, $i=1,\ 2$. (44)

If (u, v), $(u', v')\in B$ and $\|(u, v)\|_B$, $\|(u', v')\|_B\leqslant N$, as $N<N'$ then

$$\|(S_1(u, v)-S_1(u', v'), S_2(u, v)-S_2(u', v'))\|_B \leqslant CN^\alpha\|(u-u', v-v')\|_B, \tag{45}$$

$$\|(S_1(u, v), S_2(u, v))\|_B\leqslant CN^{1+\alpha}, \tag{46}$$

where C is independent of (u, v), (u', v') and N.

Proof. Since $f_i(\xi, \eta, \zeta, \varepsilon)\in C^{1+\alpha}(K_{N'})$ according to the Hölder continuity of f_i and (44), then the following estimates hold

$$\begin{cases}|f'_{i\xi}(\xi, \eta, \zeta, \varepsilon)|,\ |f'_{i\eta}(\xi, \eta, \zeta, \varepsilon)|\leqslant C(|\xi|+|\eta|+|\zeta|+|\varepsilon|)^\alpha,\\ |D_\zeta f_i(\xi, \eta, \zeta, \varepsilon)|,\ |D_\varepsilon f_i(\xi, \eta, \zeta, \varepsilon)|\leqslant C(|\xi|+|\eta|+|\zeta|+|\varepsilon|)^\alpha.\end{cases} \tag{47}$$

Due to the mid-value theorem of multiple function and (47), we may obtain

$$\begin{aligned}&|f_i(u, v, D_xu, D_xv)-f_i(u', v', D_xu', D_xv')|\\ &\quad\leqslant|f'_{i\xi}(\xi, \eta, D_x\xi, D_x\eta)|\cdot|u-u'|+|f'_{i\eta}f_i(\xi, \eta, \zeta, \varepsilon)|\cdot|v-v'|\\ &\qquad+|D_\zeta f_i(\xi, \eta, D_x\xi, D_x\eta)|\cdot|D_xu-D_xu'|\\ &\qquad+|D_\varepsilon f_i(\xi, \eta, D_x\xi, D_x\eta)|\cdot|D_xv-D_xv'|\\ &\quad\leqslant C(|\xi|+|\eta|+|D_x\xi|+|D_x\eta|)^\alpha(|u-u'|+|D_xu-D_xu'|\\ &\qquad+|v-v'|+|D_xv-D_xv'|)\\ &\quad\leqslant CN^\alpha\|(u-u', v-v')\|_BE^{1+\alpha}(x, t+\tau),\end{aligned} \tag{48}$$

where $(\xi, \eta, D_x\xi, D_x\eta)$ is a point which situates on the line made up of (u, v, D_xu, D_xv), (u', v', D_xu', D_xv'). Since $\|(u, v)\|_B$, $\|(u', v')\|_B\leqslant N$, $(\xi, \eta, D_x\xi, D_x\eta)$ satisfies the following (49)

$$\begin{aligned}&(|\xi|+|\eta|+|D_x\xi|+|D_x\eta|)^\alpha\\ &\quad\leqslant\max\{(|u|+|v|+|D_xu|+|D_xv|)^\alpha, (|u'|+|v'|+|D_xu'|+|D_xv'|)^\alpha\}\\ &\quad\leqslant N^\alpha E^\alpha(x, t+\tau).\end{aligned} \tag{49}$$

For (48), thanks to Lemma 3, then next two estimates hold

$$\begin{aligned}&|S_i(u, v)-S_i(u', v')|\\ &\quad\leqslant\int_0^t d\tau\int_{R_+^n}|G(x, t; \xi, \tau)|\cdot|(f_i(u, v, D_xu, D_xv)\\ &\qquad-f_i(u', v', D_xu', D_xv'))(\xi, \tau)|d\xi\\ &\quad\leqslant CN^\alpha\|(u-u', v-v')\|_B\\ &\qquad\cdot\int_0^t d\tau\int_{R_+^n}|G(x, t; \xi, \tau)|\cdot E^{1+\alpha}(\xi, \tau+\tau)d\xi\\ &\quad\leqslant CN^\alpha\|(u-u', v-v')\|_BE(x, t+\tau),\end{aligned} \tag{50}$$

and

$$\begin{aligned}&|D_{x_j}S_i(u, v)-D_{x_j}S_i(u', v')|\\ &\quad\leqslant\int_0^t d\tau\int_{R_+^n}|D_{x_j}G(x, t; \xi, \tau)|\cdot|(f_i(u, v, D_xu, D_xv)\\ &\qquad-f_i(u', v', D_xu', D_xv'))(\xi, \tau)|d\xi\\ &\quad\leqslant CN^\alpha\|(u-u', v-v')\|_BE(x, t+\tau).\end{aligned} \tag{51}$$

Therefore, the estimates of the operator S_i on (u, v) and (u', v') in the space B are gained

$$\|(S_1(u, v)-S_1(u', v'), S_2(u, v)-S_2(u', v'))\|_B \leqslant C\cdot N^{\alpha}\cdot\|(u-u', v-v')\|_B. \tag{52}$$

So, (45) is proved completely.

Let $(u', v')=(0, 0)$, then (46) can be proved easily.

Theorem 1. Suppose $n\alpha>2$, elementary hypothese (H_1)—(H_3) hold, initial-boundary values and f_i satisfies respectively the assumption (23)—(25) in Lemma 2 and the assumptions (44) in Lemma 4, then as δ is small suitably, there exists a unique classical solution for problems (1)—(6).

Proof. Define an integral operator $T: B\to B$

$$T(u, v)=(u_0, v_0)+(S_1(u, v), S_2(u, v)). \tag{53}$$

Now let B_1 be a subspace of B

$$B_1=\{(u, v)\in B\mid \|(u, v)\|_B\leqslant \bar{N}, \bar{N}<N'\},$$

$\bar{N}$ is a positive constant that may be chosen arbitrarily.

According to Lemma 2 and Lemma 4, we have

$$\|T(u, v)\|_B\leqslant\|(u_0, v_0)\|_B+\|(S_1(u, v), S_2(u, v))\|_B\leqslant C_1\delta+C\bar{N}^{1+\alpha}. \tag{54}$$

Choosing $\bar{N}<(1/C)^{1/\alpha}$, so $1-C\bar{N}^{\alpha}>0$, and let $\delta<(1/C_1)(\bar{N}-C\bar{N}^{1+\alpha})$, it is got that $\|T(u, v)\|_B\leqslant\bar{N}$. Therefore, $T(u, v)\in B_1$, i.e. $T: B_1\to B_1$.

For (u, v), $(u', v')\in B_1$, by Lemma 4, the following estimate of T about (u, v) and (u', v') hold $(1-C\bar{N}^{\alpha}>0)$

$$\begin{aligned}&\|T(u, v)-T(u', v')\|_B\\&=\|(S_1(u, v)-S_1(u', v'), S_2(u, v)-S_2(u', v'))\|_B\\&\leqslant C\bar{N}^{\alpha}\|(u-u', v-v')\|_B\\&<\|(u-u', v-v')\|_B.\end{aligned} \tag{55}$$

So we point out that the operator T has a unique fixed point in B_1 by means of Banach fixed point theorem. This point of T is however a solution of BIVP (1)—(6).

This completes the proof of the theorem.

Theorem 2. Assume that the hypotheses hold in Theorem 1. Then for any $(u^{(0)}, v^{(0)})\in B_1$, the sequence $\{(u^{(m)}, v^{(m)})\}=\{T(u^{(m-1)}, v^{(m-1)})\}$ converges in B_1 to a unique classical solution (u, v) of BIVP (1)—(6). Moreover

$$\|(u^{(m)}, v^{(m)})\|_B<\frac{(C\bar{N}^{\alpha})^m}{1-C\bar{N}^{\alpha}}\|(u^{(1)}, v^{(1)})-(u^{(0)}, v^{(0)})\|_B. \tag{56}$$

It is easily to prove the theorem.

According to (55) in Theorem 1, we know

$$\begin{aligned}\|(u^{(m)}, v^{(m)})-(u, v)\|_B&=\|T(u^{(m-1)}, v^{(m-1)})-T(u, v)\|_B\\&\leqslant\|T(u^{(m-1)}, v^{(m-1)})-T(u^{(m)}, v^{(m)})\|_B\\&\quad+\|T(u^{(m)}, v^{(m)})-T(u, v)\|_B\\&\leqslant C\bar{N}^{\alpha}\|(u^{(m)}, v^{(m)})-(u^{(m-1)}, v^{(m-1)})\|_B\\&\quad+C\bar{N}^{\alpha}\|(u^{(m)}, v^{(m)})-(u, v)\|_B.\end{aligned} \tag{57}$$

So we can prove (56) by (57)

$$\|(u^{(m)}, v^{(m)})-(u, v)\|_B \leqslant \frac{C\bar{N}^\alpha}{1-C\bar{N}^\alpha}\|(u^{(m)}, v^{(m)})-(u^{(m-1)}, v^{(m-1)})\|_B$$
$$\leqslant \frac{(C\bar{N}^\alpha)^m}{1-C\bar{N}^\alpha}\|(u^{(1)}, v^{(1)})-(u^{(0)}, v^{(0)})\|_B.$$

(56) is an error estimate of the convergemential sequence.

Notation. Theorem 1 and Theorem 2 may be extended similarly to the situation of multiple reaction-diffusion equations

$$\frac{\partial u_i}{\partial t}-\Delta u_i=f_i(u_1, \cdots, u_m, D_x u_1, \cdots, D_x u_m) \quad (x, t)\in \boldsymbol{R}_+^n\times \boldsymbol{R}_+ \tag{58}$$

$$u_i|_{t=0}=\varphi_i(x) \quad x\in \boldsymbol{R}_+^n \tag{59}$$

$$u_i|_{x_j=0}=\psi_{ij}(\hat{x}_j, t) \quad (\hat{x}_j, t)\in \boldsymbol{R}_+^{n-1}\times \boldsymbol{R}_+ \tag{60}$$

$$j=1, 2, \cdots, n;\ i=1, 2, \cdots, m.$$

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