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GLOBAL STABILITY OF MONOSTABLE TRAVELING WAVES FOR NONLOCAL TIME-DELAYED REACTION-DIFFUSION EQUATIONS*

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Abstract. For a class of nonlocal time-delayed reaction-diffusion equations, we prove that all noncritical wavefronts are globally exponentially stable, and critical wavefronts are globally algebraically stable when the initial perturbations around the wavefront decay to zero exponentially near the negative infinity regardless of the magnitude of time delay. This work also improves and develops the existing stability results for local and nonlocal reaction-diffusion equations with delays. Our approach is based on the combination of the weighted energy method and the Green function technique.

Key words. nonlocal reaction-diffusion equations, time delays, traveling waves, global stability, the Fisher–KPP equation, L^1-weighted energy, Green functions

AMS subject classifications. 35K57, 34K20, 92D25

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1. Introduction.

Regarding the spatial dynamics of a single-species population with age-structure and spatial diffusion such as the Australian blowflies population distribution, there is a class of time-delayed reaction-diffusion equations with nonlocal nonlinearity (see, e.g., [6, 12, 32, 40, 41])

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + d(u(t,x)) = \varepsilon \int_{-\infty}^{\infty} f_\alpha(y) b(u(t-\tau, x-y))dy, \quad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

with the initial data

$$u(s,x) = u_0(s,x), \quad s\in[-\tau,0],\ x\in\mathbb{R}. \tag{1.2}$$

Here $u(t,x)$ denotes the total mature population of the species (with age greater than the maturation age $\tau>0$) at time t and position x, $D>0$ is the spatial diffusion rate for the mature population, $\alpha>0$ is the total amount of diffusion for the immature species and satisfies $\alpha\le\tau D$, $\varepsilon>0$ is the survival rate of the species in time τ period and represents the impact of the death rate of the immature population, and $f_\alpha(y)$ is the heat kernel in the form of

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha} \quad \text{with} \quad \int_{-\infty}^{\infty} f_\alpha(y)dy = 1.$$

The nonlinear functions $d(u)$ and $b(u)$ denote the death and birth rates of the mature population, respectively, and satisfy the following hypotheses:

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(H1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = \varepsilon b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;
(H2) $\varepsilon b'(0) > d'(0) \geq 0$ and $0 \leq \varepsilon b'(u_+) < d'(u_+)$, and $d'(u_+)^2 > \varepsilon^2 b'(0)b'(u_+)$;
(H3) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$, but either $d''(u) > 0$ or $|b''(u)| > 0$.

The equation (1.1) includes a lot of evolution equations for the single species population with an age structure. For example, by taking the death rate function as $d(u) = \delta u$ with a positive coefficient $\delta > 0$, (1.1) reduces to the following *nonlocal Nicholson's blowflies population model* (see, e.g., [12, 19, 20, 27, 28, 32, 40, 41, 45]):

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \delta u = \varepsilon \int_{-\infty}^{\infty} f_\alpha(y) b(u(t-\tau, x-y))dy, \quad t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

where the birth rate function $b(u)$ is usually taken as

$$b_1(u) = pue^{-au^q}, \quad b_2(u) = \frac{pu}{1 + au^q}, \quad p > 0,\ a > 0,\ q > 0.$$

In particular, when $q = 1$, $b_1(u)$ is just the so-called Nicholson's birth rate function.

If we further assume that the immature species is almost nonmobile, i.e., the impact factor α of spatial diffusion for the immature population is sufficiently close to zero, by using the property of the heat kernel $f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha}$,

$$b(u(t-\tau, x)) = \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} f_\alpha(y) b(u(t-\tau, x-y))dy,$$

we then obtain the following *local Nicholson's blowflies equation* (see, e.g., [12, 13, 23, 26, 29, 39]):

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \delta u = \varepsilon b(u(t-\tau, x)), \quad t > 0,\ x \in \mathbb{R}. \tag{1.4}$$

On the other hand, if we take

$$d(u) = \delta u^2,\ \delta > 0 \ \text{ and } \ \varepsilon b(u) = pe^{-\gamma\tau} u,\ p > 0,\ \gamma > 0,$$

then (1.1) reduces to the following *nonlocal age-structured population model* (see, e.g., [1, 2, 3, 6, 11, 12, 32, 41, 44])

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \delta u^2 = pe^{-\gamma\tau} \int_{-\infty}^{\infty} f_\alpha(y) u(t-\tau, x-y)dy,\ t > 0,\ x \in \mathbb{R}, \tag{1.5}$$

and by taking $\alpha \to 0^+$ in (1.5) for consideration of the nonmobile immature population, we further derive the *local age-structured population model* (see, e.g., [2, 10, 11, 12, 18, 25, 30])

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \delta u^2 = pe^{-\gamma\tau} u(t-\tau, x), \quad t > 0,\ x \in \mathbb{R}. \tag{1.6}$$

In particular, if we consider the case without time delay, i.e., $\tau = 0$, and simply take $D = \delta = p = 1$ in (1.6), we get the well-known Fisher-KPP equation (c.f. [8, 9, 15, 16, 31, 35, 43, 47])

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(1-u), \quad t > 0,\ x \in \mathbb{R}. \tag{1.7}$$

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From (H1), it can be verified that both $u_- = 0$ and $u_+ > 0$ are constant equilibria of (1.1), and from (H2) we see that $u_- = 0$ is unstable and u_+ is stable for the spatially homogeneous equation associated with (1.1). (H3) implies that, in $[u_-, u_+]$ under consideration, both the birth rate function $b(u)$ and the death rate function $d(u)$ are nondecreasing, and $b(u)$ is concave downward and $d(u)$ is concave upward. These characters are summarized from those typical examples in (1.3)–(1.6).

A *traveling wavefront* of (1.1) is a special solution in the form of $u(t,x) = \phi(x+ct)$ with $\phi(\pm\infty) = u_\pm$, where c is the wave speed. The main purpose of this paper is to study the global asymptotic stability of traveling wavefronts $\phi(x+ct)$ of (1.1), including the case of the *critical wave* $\phi(x+c_*t)$. Here the number c_* is called the *critical speed* (or the *minimum speed*) in the sense that a traveling wave $\phi(x+ct)$ exists if $c \geq c_*$, while no traveling wave $\phi(x+ct)$ exists if $c < c_*$.

There have been extensive investigations on the stability of traveling waves for reaction-diffusion equations without time delay; see, e.g., [5, 7, 9, 14, 15, 24, 31, 35, 37, 42, 46], the monographs [4, 43], the survey paper [47], and the references therein. Regarding time-delayed reaction-diffusion equations such as those in [6, 12, 21, 22, 29, 32, 33, 36, 40, 41], the study of stability of traveling waves is quite limited. The first study on the linearized stability was given by Schaaf [36] via a spectral analysis. When the spatially homogeneous equation possesses two stable constant equilibria (i.e., the bistable case), the stability of bistable traveling waves for local equations was obtained by Smith and Zhao [38] via the upper-lower solutions method coupled with the squeezing technique; see also the recent contribution by Wang, Li, and Ruan [44] for nonlocal equations. In the monostable case (i.e., one equilibrium is stable, but the other is unstable), the study of the stability of traveling waves is much harder, due to the difficulty caused by the unstable equilibrium. The first work related to this case for the local Nicholson's blowflies equation was given by Mei et al. [29] via the weighted L^2-energy method, where the fast waves (i.e., the wave speeds are large) were proved to be locally stable (i.e., the initial perturbation around the waves must be small enough). Later on, a similar result for the nonlocal Nicholson's blowflies equation was obtained by Mei and So [28]. Furthermore, the global stability for all waves, including those slow waves (but except for the critical one), was proved by Mei et al. in [26] for the local equation and in [27] for the nonlocal equation via a development of the ideas in [23]. Note that the nonlocal Nicholson's blowflies equation was considered in [27] under the condition that the total diffusion for the immature population, α, is sufficiently small when the wave speed c is sufficiently close to the critical wave speed c_*. This condition is acceptable but still a bit stiff because when $\alpha \ll 1$, the nonlocal birth-rate term can be regarded as a small perturbation of the corresponding local birth-rate term, which implies that the nonlocal equation is just a small perturbation of the local equation for $\alpha \ll 1$. For the monostable equation with age-structure, the linear stability for all slow waves (except for the critical one) was studied by Gourley [10] when the time delay $\tau \ll 1$. Further, these waves were proved to be nonlinearly stable, also globally stable by Li, Mei, and Wong [18] but still with a small τ. Recently, such a smallness on time delay τ was removed by Mei and Wong [30].

The stability of the critical traveling wave solutions to either local or nonlocal time-delayed equations has been a challenging open problem. It is well known that the stability of the critical waves is very important in the study of biological invasions. This is because the critical wave speed is also the spreading speed for all solutions with initial data having compact supports (see, e.g., [41, 21, 22] and the references therein). Since the traditional methods, including the weighted L^2-energy method, the upper-

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lower solution method, as well as the spectral analysis approach, may not be used to prove the stability of the critical traveling wavefronts for these generalized nonlocal time-delayed reaction-diffusion equations, we need to look for a new strategy to attack the problem. By a profound observation on the standing equation and using the concavity of the nonlinear birth and death rate functions, we first establish a weighted L^1-energy estimate of solutions and then obtain the desired L^2-energy estimate as well as the exponential convergence rate to the noncritical wave by the ordinary weighted energy method. When the wave is critical, the convergence rate to the wave is proved to be algebraic by the Green function method. These stability results improve and develop the existing works on monostable waves. As the applications of our main result, we obtain the global and exponential stability of all noncritical traveling waves and the algebraic stability of the critical wave for the local/nonlocal Nicholson's blowflies equations and the local/nonlocal population equations with age-structure. In particular, the classical stability results for the Fisher–KPP equation, for example, the exponential stability of all noncritical waves given by Sattinger [35] and the algebraic stability of the critical waves shown by Moet [31], Kirchgassner [15], and Gallay [9] are consequences of our main theorem.

The rest of this paper is organized as follows. In section 2, we introduce some necessary notations and present the main results on the exsitence and nonlinear stability of traveling wavefronts. In section 3, we build up some energy estimates in the weighted L^1 space, then establish the energy estimates in H^1, and further prove the global asymptotic stability result with a time-exponential decay for the noncritical traveling waves and a time-algebraic decay for the critical traveling wave, respectively. Section 4 is devoted to the application of our main result to the aforementioned evolution equations, including the Fisher–KPP equation. In section 5, we present a generalization of our stability result to a larger class of nonlocal time-delayed reaction-diffusion equations and give a remark on a time-delayed integro-differential vector disease model.

2. Main results. Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \dots)$ represents a specific constant. Let I be an interval, typically $I = \mathbb{R}$. $L^p(I)$ $(p \ge 1)$ is the Lebesque space of the integrable functions defined on I, $W^{k,p}(I)$ $(k \ge 0, p \ge 1)$ is the Sobolev space of the L^p-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$ $(i = 1, \dots, k)$ also belong to $L^p(I)$, and in particular we denote $W^{k,2}(I)$ as $H^k(I)$. Further, $L^p_w(I)$ denotes the weighted L^p-space for a weight function $w(x) > 0$ with the norm defined as

$$\|f\|_{L^p_w} = \left(\int_I w(x)|f(x)|^p dx\right)^{1/p},$$

$W^{k,p}_w(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{W^{k,p}_w} = \left(\sum_{i=0}^k \int_I w(x)\left|\frac{d^i}{dx^i}f(x)\right|^p dx\right)^{1/p},$$

and $H^k_w(I)$ is defined with the norm

$$\|f\|_{H^k_w} = \left(\sum_{i=0}^k \int_I w(x)\left|\frac{d^i}{dx^i}f(x)\right|^2 dx\right)^{1/2}.$$

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Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C^0([0,T],\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, $L^2([0,T],\mathcal{B})$ as the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

Recall that a traveling wavefront to (1.1) connecting $u_\pm$ is a solution in the form of $\phi(x+ct)$ with a speed c. Namely, the function ϕ satisfies the following differential equation:

$$\begin{cases} c\phi' - D\phi'' + d(\phi) = \varepsilon \int_{-\infty}^{\infty} f_\alpha(y) b(\phi(\xi - y - c\tau))dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{2.1}$$

where $' = \frac{d}{d\xi}$, $\xi = x + ct$.

Note that the existence of monotone traveling wavefronts of (1.1) can be proved by the method of upper-lower solutions in a similar way as in [36, 40, 39, 41, 20]. However, the nonexistence of traveling wavefronts may not be obtained by the linearization of (2.1) at its zero solution since (2.1) is a mixed-type functional differential equation. It is easy to see that (1.1) generates a monotone semiflow on $C(\mathbb{R},[0,u_+])$ equipped with the compact open topology. Consequently, the abstract results in [21, 22] imply the following result on the existence of the minimum (critical) wave speed.

Lemma 2.1 (existence of traveling waves). *Under the conditions* (H1)–(H3), *there exist a minimum wave speed (also called the critical wave speed) $c_* > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying*

$$F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \qquad F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),$$

where

$$F_c(\lambda) = \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau}, \qquad G_c(\lambda) = c\lambda - D\lambda^2 + d'(0),$$

and (c_*, λ_*) is the tangent point of $F_c(\lambda)$ and $G_c(\lambda)$, namely,

$$\begin{aligned} \varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_* \tau} &= c_*\lambda_* - D\lambda_*^2 + d'(0), \\ \varepsilon b'(0)(2\alpha\lambda_* - c_*\tau) e^{\alpha\lambda_*^2 - \lambda_* c_* \tau} &= c_* - 2D\lambda_*, \end{aligned}$$

such that for any $c \ge c_*$, a monotone traveling wavefront $\phi(x+ct)$ of (2.1) *connecting $u_\pm$ exists, and for any $c < c_*$, no traveling wave $\phi(x+ct)$ exists. When $c > c_*$, there exist two numbers depending on c: $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $F_c(\lambda_i) = G_c(\lambda_i)$, i.e.,*

$$\varepsilon b'(0) e^{\alpha\lambda_i^2 - \lambda_i c\tau} = c\lambda_i - D\lambda_i^2 + d'(0), \quad i = 1, 2,$$

such that

$$F_c(\lambda) < G_c(\lambda) \quad \text{for} \ \ \lambda_1 < \lambda < \lambda_2,$$

and particularly

$$F_c(\lambda_*) < G_c(\lambda_*) \ \ \text{with} \ \ \lambda_1 < \lambda_* < \lambda_2.$$

When $c = c_*$, it holds that

$$F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \ \ \text{with} \ \ \lambda_1 = \lambda_* = \lambda_2.$$

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Now we are going to define a weight function. Let x_0 be sufficiently large so that

$$2d'(\phi(x_0)) - \frac{\varepsilon}{\eta} b'(\phi(x_0)) - \varepsilon\eta b'(0) > 0,$$

where $\eta > 0$ is taken as

$$0 < \frac{d'(u_+) - \sqrt{d'(u_+)^2 - \varepsilon^2 b'(0)b'(u_+)}}{\varepsilon b'(0)} < \eta < \frac{d'(u_+) + \sqrt{d'(u_+)^2 - \varepsilon^2 b'(0)b'(u_+)}}{\varepsilon b'(0)}.$$

For the choices of x_0 and η, we refer the details to section 3 (see (3.29)–(3.32) below). For any given $c \geq c_*$, we define

$$w(x) = \begin{cases} e^{-\lambda(x-x_0)} & \text{for } x \leq x_0, \\ 1 & \text{for } x > x_0, \end{cases} \tag{2.2}$$

where λ is any fixed number in $(\lambda_1, \lambda_*]$ when $c > c_*$, but $\lambda = \lambda_*$ when $c = c_*$. It is easy to see that $w(\xi) \geq 1$ for all $\xi \in \mathbb{R}$ and $w(-\infty) = \infty$.

THEOREM 2.2 (stability of traveling waves). *Let $d(u)$ and $b(u)$ satisfy* (H1)–(H3). *For a given traveling wave $\phi(x+ct)$ of* (1.1) *with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfies*

$$0 = u_- \leq u_0(s,x) \leq u_+ \ \forall (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x+cs)$ is in $C([-\tau,0], L^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (1.1) *and* (1.2) *uniquely exists and satisfies*

$$0 = u_- \leq u(t,x) \leq u_+ \ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$
$$u(t,x) - \phi(x+ct) \in C([0,\infty), L^1_w(\mathbb{R}) \cap H^1(\mathbb{R})).$$

When $c > c_*$, the solution $u(t,x)$ converges to the noncritical traveling wave $\phi(x+ct)$ exponentially,

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ce^{-\mu t}, \quad t > 0,$$

for a positive constant $\mu = \mu_1/3$, where $\mu_1 = \mu_1(c,\lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_*]$ satisfies

$$G_c(\lambda) - F_c(\lambda) - \mu_1 - F_c(\lambda)(e^{\mu_1\tau} - 1) \geq 0 \tag{2.3}$$

and

$$d'(u_+) - \varepsilon e^{\mu_1\tau} b'(u_+) - \mu_1 > 0. \tag{2.4}$$

When $c = c_*$, the solution $u(t,x)$ converges to the critical traveling wave $\phi(x+c_*t)$ algebraically,

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c_*t)| \leq Ct^{-\frac{1}{2}}, \quad t > 0.$$

Remark 1. Theorem 2.2, as applied to monostable evolution equations (1.3)–(1.6), implies the global stability of the critical wave $\phi(x+c_*t)$, which was left open in the earlier works [10, 18, 23, 28, 29, 26, 27, 30].

Remark 2. For the Fisher–KPP equation (1.7), Theorem 2.2 also provides the stability of all traveling wavefronts including the critical one, with a time-exponential decay to the noncritical waves and a time-algebraic decay to the critical wave, which are the same as those in [9, 15, 31, 35]. For more details, we refer to section 4.

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3. Proof of the global stability. The existence and uniqueness of the solution to (1.1) and (1.2) can been proved via the standard energy method and continuity-extension method (cf., [29, 28]) or the theory of abstract functional differential equations [17], and we omit the details here. The main target in this section is to prove the stability for all noncritical traveling waves to (1.1) with an exponential convergence rate and, in particular, the stability for the critical traveling wave with an algebraic convergence rate. As in [18, 23, 26, 27], we will use the comparison principle and the weighted energy method to prove the exponential stability for the noncritical waves in Theorem 2.2, and use the Green function method to prove the algebraic stability for the critical waves in Theorem 2.2. As usual, the crucial step is to establish the L^2-energy estimate for the solution in a suitable weighted Sobolev space H^1_w. However, such a weighted L^2-energy method cannot be directly applied to the case of the critical wavefront. Here, we develop a new strategy. Instead of the weighted L^2-energy estimate, we first establish a weighted L^1-energy estimate by selecting a suitable weight function and carefully treating each term. Then using this crucial L^1-estimate, we further obtain the desired L^2-energy estimate.

Let $c \geq c_*$ and the initial data $u_0(s,x)$ be such that $0 = u_- \leq u_0(s,x) \leq u_+$ for $(s,x) \in [-\tau, 0] \times \mathbb{R}$, and define

$$\begin{cases} U_0^+(s,x) = \max\{u_0(s,x), \quad \phi(x+cs)\}, \\ U_0^-(s,x) = \min\{u_0(s,x), \quad \phi(x+cs)\}, \end{cases} \quad \forall (s,x) \in [-\tau,0]\times\mathbb{R},$$

which implies

$$0 = u_- \leq U_0^-(s,x) \leq u_0(s,x) \leq U_0^+(s,x) \leq u_+ \; \forall (s,x) \in [-\tau,0]\times\mathbb{R},$$
$$0 = u_- \leq U_0^-(s,x) \leq \phi(x+cs) \leq U_0^+(s,x) \leq u_+ \; \forall (s,x) \in [-\tau,0]\times\mathbb{R}.$$

Let $U^+(t,x)$ and $U^-(t,x)$ be the corresponding solutions of (1.1) with the initial data $U_0^+(s,x)$ and $U_0^-(s,x)$, respectively, that is,

$$\frac{\partial U^\pm}{\partial t} - D\frac{\partial^2 U^\pm}{\partial x^2} + d(U^\pm) = \varepsilon \int_{\mathbb{R}} f_\alpha(y) b(U^\pm(t-\tau, x-y))dy,$$
$$U^\pm(s,x) = U_0^\pm(s,x), \quad x \in \mathbb{R}, \; s \in [-\tau, 0].$$

By similar arguments as in [23, 26, 27, 18] or the abstract results in [17], it easily follows that (1.1) admits the comparison principle. Thus, we have

$$u_- \leq U^-(t,x) \leq u(t,x) \leq U^+(t,x) \leq u_+ \; \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.1}$$
$$u_- \leq U^-(t,x) \leq \phi(x+ct) \leq U^+(t,x) \leq u_+ \; \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.2}$$

In what follows, we are going to complete the proof for the stability in three steps.

Step 1 (the convergence of $U^+(t,x)$ to $\phi(x+ct)$). For any given $c \geq c_*$, let $\xi := x + ct$ and

$$v(t,\xi) := U^+(t,x) - \phi(x+ct), \qquad v_0(s,\xi) := U_0^+(s,x) - \phi(x+cs).$$

It follows from (3.1) and (3.2) that

$$v(t,\xi) \geq 0, \qquad v_0(s,\xi) \geq 0.$$

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We see from (1.1) that $v(t,\xi)$ satisfies (by linearizing it at 0)

$$\begin{aligned}
&\frac{\partial v}{\partial t}+c\frac{\partial v}{\partial \xi}-D\frac{\partial^2 v}{\partial \xi^2}+d'(0)v\\
&\quad-\varepsilon b'(0)\int_{\mathbb{R}} f_\alpha(y)v(t-\tau,\xi-y-c\tau)dy\\
&=-Q_1(t,\xi)+\varepsilon\int_{\mathbb{R}} f_\alpha(y)Q_2(t-\tau,\xi-y-c\tau)dy+[d'(0)-d'(\phi(\xi))]v\\
&\quad+\varepsilon\int_{\mathbb{R}} f_\alpha(y)[b'(\phi(\xi-y-c\tau)-b'(0)]v(t-\tau,\xi-y-c\tau)dy\\
&=: I_1(t,\xi)+I_2(t,\xi)+I_3(t,\xi)+I_4(t,\xi),
\end{aligned}\tag{3.3}$$

with the initial data

$$v(s,\xi)=v_0(s,\xi),\ \ s\in[-\tau,0],\tag{3.4}$$

where

$$Q_1(t,\xi)=d(\phi+v)-d(\phi)-d'(\phi)v\tag{3.5}$$

with $\phi=\phi(\xi)$ and $v=v(t,\xi)$, and

$$Q_2(t-\tau,\xi-y-c\tau)=b(\phi+v)-b(\phi)-b'(\phi)v\tag{3.6}$$

with $\phi=\phi(\xi-y-c\tau)$ and $v=v(t-\tau,\xi-y-c\tau)$. Here $I_i(t,\xi), i=1,2,3,4$, denotes the ith term in the right-side of line above (3.3).

LEMMA 3.1. *It holds that*

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})}+\int_0^{t-\tau} e^{-\mu_1(t-s)}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds\le Ce^{-\mu_1 t}\qquad \text{for } c>c_*,\tag{3.7}$$

and

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})}+\int_0^{t-\tau}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds\le C\qquad \text{for } c=c_*,\tag{3.8}$$

where $w_1(\xi)=e^{-\lambda(\xi-x_0)}$, λ *is chosen as in* (2.2), *and* $\mu_1>0$ *is the small constant given in* (2.3) *and* (2.4) *for* $c>c_*$.

Proof. In order to establish the energy estimate (3.7), technically we need the good enough regularity for the solution of (3.3) and (3.4). To do it, the usual approach is via the mollification. Now let us mollify the initial data as

$$v_{0\bar\epsilon}(s,\xi)=(J_{\bar\epsilon}*v_0)(s,\xi)=\int_{\mathbb{R}} J_{\bar\epsilon}(\xi-y)v_0(s,y)dy\in C^0([-\tau,0],W^{2,1}_w(\mathbb{R})\cap H^2(\mathbb{R})),$$

where $J_{\bar\epsilon}(\xi)$ is the mollifier. Let $v_{\bar\epsilon}(t,\xi)$ be the solution to (3.3) with the above mollified initial data. We then have

$$v_{\bar\epsilon}(t,\xi)\in C^0([0,\infty),W^{2,1}_w(\mathbb{R})\cap H^2(\mathbb{R})).\tag{3.9}$$

To show (3.9), we first prove the local existence for the solution in the designed solution space within $[0,t_0]$ for some $t_0>0$. Then, by Zorn's lemma (for example, see [50]), the solution either globally exists in the given solution space or blows up

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at a finite time in the norm of the given space. We further show that, by using the energy method, for any time $T > 0$, the solution within the designed space (3.9) is bounded by a constant depending on T and doesn't blow up. Consequently, we obtain the global existence in the given solution space. Here we omit the detail of the proof since it is rather standard.

Next we are going to derive (3.7) and (3.8) for all $t > 0$. Multiplying (3.3) by $w_1(\xi)e^{\mu_1 t}$, where $\mu_1 > 0$ is given in (2.3) (we will show how to determine it later), we have

$$
\begin{aligned}
(3.10)\qquad & \frac{\partial}{\partial t}(e^{\mu_1 t}w_1 v_{\bar\epsilon}) + e^{\mu_1 t}\frac{\partial}{\partial \xi}\Big\{cw_1 v_{\bar\epsilon} - Dw_1 v_{\bar\epsilon\xi} + Dw_1' v_{\bar\epsilon}\Big\} - \mu_1 e^{\mu_1 t}w_1 v_{\bar\epsilon} \\
&\quad + e^{\mu_1 t}\Big\{ - cw_1' - Dw_1'' + d'(0)w_1\Big\}v_{\bar\epsilon} \\
&\quad - \varepsilon b'(0)w_1 e^{\mu_1 t}(\xi)\int_{\mathbb{R}} f_\alpha(y)v_{\bar\epsilon}(t-\tau,\xi-y-c\tau)dy \\
&= e^{\mu_1 t}w_1(\xi)[I_1(t,\xi)+I_2(t,\xi)+I_3(t,\xi)+I_4(t,\xi)].
\end{aligned}
$$

Integrating the above equation over $\mathbb{R}\times[0,t]$ with respect to ξ and t gives

$$
\begin{aligned}
& e^{\mu_1 t}\int_{\mathbb{R}} w_1(\xi)v_{\bar\epsilon}(t,\xi)d\xi \\
&\quad + \int_0^t\int_{\mathbb{R}} e^{\mu_1 s}\Big\{ - cw_1'(\xi) - Dw_1''(\xi) + d'(0)w_1(\xi) - \mu_1 w_1(\xi)\Big\}v_{\bar\epsilon}(s,\xi)d\xi ds \\
&\quad - \varepsilon b'(0)\int_0^t\int_{\mathbb{R}} e^{\mu_1 s}w_1(\xi)\Big[\int_{\mathbb{R}} f_\alpha(y)v_{\bar\epsilon}(s-\tau,\xi-y-c\tau)dy\Big]d\xi ds \\
(3.11)\qquad & = \|v_{0\bar\epsilon}(0)\|_{L^1_{w_1}} + \int_0^t\int_{\mathbb{R}} e^{\mu_1 s}w_1(\xi)(I_1+I_2+I_3+I_4)d\xi ds.
\end{aligned}
$$

Here we have used (3.9) to ensure that the integral of the second term in (3.10) is zero. By applying Taylor's expansion to (3.5) and (3.6) and noting (H3), we have

$$
\begin{aligned}
Q_1(t,\xi) &= d(\phi+v_{\bar\epsilon}) - d(\phi) - d'(\phi)v_{\bar\epsilon} = d''(\bar\phi_1)v_{\bar\epsilon}^2 \ge C_1 v_{\bar\epsilon}^2, \\
Q_2(t-\tau,\xi-c\tau) &= b(\phi+v_{\bar\epsilon}) - b(\phi) - b'(\phi)v_{\bar\epsilon} = b''(\bar\phi_2)v_{\bar\epsilon}^2 \le -C_2 v_{\bar\epsilon}^2,
\end{aligned}
$$

for some $\bar\phi_1,\ \bar\phi_2 \in [0,\phi+v_{\bar\epsilon}]$ and nonnegative constants $C_i \ge 0$ $(i=1,2)$ with $C_1+C_2>0$, namely, at least one of C_1 and C_2 is positive (see (H3)), which implies

$$
\begin{aligned}
(3.12)\qquad & \int_0^t\int_{\mathbb{R}} e^{\mu_1 s}w_1(\xi)I_1(s,\xi)d\xi ds \le -C_1\int_0^t e^{\mu_1 s}\int_{\mathbb{R}} w_1(\xi)v_{\bar\epsilon}^2(s,\xi)d\xi ds, \\
& \int_0^t\int_{\mathbb{R}} e^{\mu_1 s}w_1(\xi)I_2(s,\xi)d\xi ds \le -\varepsilon C_2\int_0^t e^{\mu_1 s}\int_{\mathbb{R}} w_1(\xi) \\
(3.13)\qquad & \qquad\qquad \left[\int_{\mathbb{R}} f_\alpha(y)\times v_{\bar\epsilon}^2(s-\tau,\xi-y-c\tau)dy\right]d\xi ds.
\end{aligned}
$$

Notice from (H3) that $d'(u)$ is increasing and $b'(u)$ is decreasing, which implies

$$
d'(0) - d'(\phi) \le 0 \quad\text{and}\quad b'(\phi) - b'(0) \le 0 \quad\text{for } \phi \ge 0,
$$

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namely,

$$I_3(t,\xi) \le 0 \quad \text{and} \quad I_4(t,\xi) \le 0.$$

Thus, we have

$$\int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi)[I_3(s,\xi) + I_4(s,\xi)] d\xi ds \le 0. \tag{3.14}$$

Applying (3.12), (3.13), and (3.14) to (3.11) gives

$$\begin{aligned}
& e^{\mu_1 t}\|v_{\bar{\epsilon}}(s)\|_{L^1_{w_1}(\mathbb{R})} \\
& + \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} \left\{-cw_1'(\xi) - Dw_1''(\xi) + d'(0)w_1(\xi) - \mu_1 w_1(\xi)\right\} v_{\bar{\epsilon}}(s,\xi) d\xi ds \\
& - \varepsilon b'(0) \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) \left[\int_{\mathbb{R}} f_\alpha(y) v_{\bar{\epsilon}}(s-\tau, \xi - y - c\tau) dy\right] d\xi ds \\
& + C_1 \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) v_{\bar{\epsilon}}^2(s,\xi) d\xi ds \\
& + \varepsilon C_2 \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) \left[\int_{\mathbb{R}} f_\alpha(y) v_{\bar{\epsilon}}^2(s-\tau, \xi - y - c\tau) dy\right] d\xi ds \\
& \le \|v_{0\bar{\epsilon}}(0)\|_{L^1_{w_1}(\mathbb{R})}.
\end{aligned} \tag{3.15}$$

By changing variables $y \to y,\ \xi - y - c\tau \to \xi,\ s - \tau \to s$, and using the fact

$$\int_{\mathbb{R}} f_\alpha(y) \frac{w_1(\xi + y + c\tau)}{w_1(\xi)} dy = e^{\alpha\lambda^2 - \lambda c\tau},$$

we obtain

$$\begin{aligned}
& \varepsilon b'(0) \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) \left[\int_{\mathbb{R}} f_\alpha(y) v_{\bar{\epsilon}}(s-\tau, \xi - y - c\tau) dy\right] d\xi ds \\
& = \varepsilon b'(0) \int_{-\tau}^{t-\tau} \int_{\mathbb{R}} e^{\mu_1 (s+\tau)} \left[\int_{\mathbb{R}} w_1(\xi + y + c\tau) f_\alpha(y) dy\right] v_{\bar{\epsilon}}(s,\xi) d\xi ds \\
& = \varepsilon b'(0) \int_0^{t-\tau} \int_{\mathbb{R}} e^{\mu_1 (s+\tau)} \left[\int_{\mathbb{R}} \frac{w_1(\xi + y + c\tau)}{w_1(\xi)} f_\alpha(y) dy\right] w_1(\xi) v_{\bar{\epsilon}}(s,\xi) d\xi ds \\
& \quad + \varepsilon b'(0) \int_{-\tau}^{0} \int_{\mathbb{R}} e^{\mu_1 (s+\tau)} \left[\int_{\mathbb{R}} \frac{w_1(\xi + y + c\tau)}{w_1(\xi)} f_\alpha(y) dy\right] w_1(\xi) v_{0\bar{\epsilon}}(s,\xi) d\xi ds \\
& \le \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau} e^{\mu_1 \tau} \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) v_{\bar{\epsilon}}(s,\xi) d\xi ds \\
& \quad + \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau} e^{\mu_1 \tau} \int_{-\tau}^0 e^{\mu_1 s} \|v_{0\bar{\epsilon}}(s)\|_{L^1_{w_1}(\mathbb{R})} ds
\end{aligned} \tag{3.16}$$

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and

$$
\begin{aligned}
&\varepsilon C_2 \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} w_1(\xi) \left[\int_{\mathbb{R}} f_\alpha(y) v_{\bar\epsilon}^2(s-\tau, \xi - y - c\tau) dy \right] d\xi ds \\
&= \varepsilon C_2 \int_{-\tau}^{t-\tau} \int_{\mathbb{R}} e^{\mu_1(s+\tau)} \int_{\mathbb{R}} w_1(\xi + y + c\tau) f_\alpha(y) v_{\bar\epsilon}^2(s,\xi) dy d\xi ds \\
&= \varepsilon C_2 \int_0^{t-\tau} \int_{\mathbb{R}} e^{\mu_1(s+\tau)} \left[\int_{\mathbb{R}} \frac{w_1(\xi+y+c\tau)}{w_1(\xi)} f_\alpha(y) dy \right] w_1(\xi) v_{\bar\epsilon}^2(s,\xi) dy d\xi ds \\
&\quad + \varepsilon C_2 \int_{-\tau}^{0} \int_{\mathbb{R}} e^{\mu_1(s+\tau)} \left[\int_{\mathbb{R}} \frac{w_1(\xi+y+c\tau)}{w_1(\xi)} f_\alpha(y) dy \right] w_1(\xi) v_{0\bar\epsilon}^2(s,\xi) dy d\xi ds \\
&\geq \varepsilon C_2 e^{\alpha\lambda^2 - \lambda c\tau} e^{\mu_1 \tau} \int_0^{t-\tau} e^{\mu_1 s} \|v_{\bar\epsilon}(s)\|^2_{L^2_{w_1}(\mathbb{R})} ds. \qquad (3.17)
\end{aligned}
$$

Substituting (3.17) and (3.16) to (3.15), we then get

$$
\begin{aligned}
&e^{\mu_1 t} \|v_{\bar\epsilon}(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} e^{\mu_1 s} \bar{A}(c,\mu_1,\xi) w_1(\xi) v_{\bar\epsilon}(s,\xi) d\xi ds \\
&\quad + C_1 \int_0^t e^{\mu_1 s} \|v_{\bar\epsilon}(s)\|^2_{L^2_{w_1}(\mathbb{R})} ds + \varepsilon C_2 e^{\alpha\lambda^2 - \lambda c\tau} e^{\mu_1\tau} \int_0^{t-\tau} e^{\mu_1 s} \|v_{\bar\epsilon}(s)\|^2_{L^2_{w_1}(\mathbb{R})} ds \\
&\leq C \left(\|v_{0\bar\epsilon}(0)\|_{L^1_{w_1}(\mathbb{R})} + \int_{-\tau}^0 \|v_{0\bar\epsilon}(s)\|_{L^1_{w_1}(\mathbb{R})} ds \right), \qquad (3.18)
\end{aligned}
$$

where

$$
\bar{A}(c,\mu_1,\xi) := A(c,\xi) - \mu_1 - \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau} [e^{\mu_1\tau} - 1]
$$

and

$$
A(c,\xi) := -c \frac{w_1'(\xi)}{w_1(\xi)} - D \frac{w_1''(\xi)}{w_1(\xi)} + d'(0) - \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau}.
$$

Using the facts that $c\lambda - D\lambda^2 + d'(0) > (\text{ or } =) \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau}$ for $c > (\text{ or } =) c_*$ and that $w_1(\xi) = e^{-\lambda(\xi - x_0)}$, we further obtain

$$
\begin{aligned}
A(c,\xi) &= c\lambda - D\lambda^2 + d'(0) - \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau} \\
&= \begin{cases} G_c(\lambda) - F_c(\lambda) > 0 & \text{for } c > c_*,\ \lambda_1 < \lambda \leq \lambda_*, \\ G_{c_*}(\lambda_*) - F_{c_*}(\lambda_*) = 0 & \text{for } c = c_*,\ \lambda_1 = \lambda = \lambda_*. \end{cases}
\end{aligned}
$$

Thus, when $c > c_*$, we choose a small $\mu_1 > 0$ such that

$$
\begin{aligned}
\bar{A}(c,\mu_1,\xi) &= A(c,\xi) - \mu_1 - \varepsilon b'(0) e^{\alpha\lambda^2 - \lambda c\tau} [e^{\mu_1\tau} - 1] \\
&= G_c(\lambda) - F_c(\lambda) - \mu_1 - F_c(\lambda)(e^{\mu_1\tau} - 1) \\
&\geq 0, \qquad (3.19)
\end{aligned}
$$

and when $c = c_*$, we can take only $\mu_1 = 0$ such that

$$
\bar{A}(c_*, 0, \xi) = A(c_*, \xi) \geq 0. \qquad (3.20)
$$

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Applying (3.19) for $c > c_*$ and (3.20) for $c = c_*$ to (3.18) and noting that $C_1 + C_2 > 0$, namely, at least one of them is positive, we then establish the following key energy estimate:

$$e^{\mu_1 t}\|v_{\bar\epsilon}(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^{t-\tau} e^{\mu_1 s}\|v_{\bar\epsilon}(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds \le C \quad \text{for } c > c_* \tag{3.21}$$

and

$$\|v_{\bar\epsilon}(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^{t-\tau} \|v_{\bar\epsilon}(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds \le C \quad \text{for } c = c_*. \tag{3.22}$$

Letting $\bar\epsilon \to 0$ in (3.21) and (3.22), we finally arrive at

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^{t-\tau} e^{-\mu_1(t-s)}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds \le Ce^{-\mu_1 t} \quad \text{for } c > c_*$$

and

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^{t-\tau} \|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}ds \le C \quad \text{for } c = c_*.$$

This proves (3.7) and (3.8). ∎

LEMMA 3.2. *For any $c \ge c_*$, it holds that*

$$\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v(s)\|^2_{H^1(\mathbb{R})}ds \le C, \quad t \ge 0. \tag{3.23}$$

Proof. Since $w_1(\xi) = e^{-\lambda(\xi - x_0)} \ge 1$ for $\xi \in (-\infty, x_0]$, (3.7) and (3.8) guarantee that for any $c \ge c_*$,

$$\int_{-\infty}^{x_0} v(t,\xi)d\xi + \int_0^{t-\tau}\int_{-\infty}^{x_0} v^2(s,\xi)d\xi ds \le C \quad \text{for all } t \ge 0,$$

and in particular by taking $t = \infty$, we have

$$\int_0^{\infty}\int_{-\infty}^{x_0} v^2(s,\xi)d\xi ds \le C. \tag{3.24}$$

Although we cannot directly work on the original equations (3.3) and (3.4) due to the lack of regularity for the solution as illustrated in the proof of Lemma 3.1, we can get a mollified solution first and then take the limit to get the corresponding energy estimate for the original solution $v(t,\xi)$. Therefore, for the sake of simplicity, we formally use $v(t,\xi)$ to establish the desired energy estimates in what follows.

Let us multiply (3.3) by $v(t,\xi)$ and integrate it over $\mathbb{R} \times [0,t]$ with respect to ξ and t. Then we have

$$\begin{aligned}
&\|v(t)\|^2_{L^2(\mathbb{R})} + 2D\int_0^t \|v_\xi(s)\|^2_{L^2(\mathbb{R})}ds + 2\int_0^t\int_{\mathbb{R}} d'(\phi(\xi))v^2(s,\xi)d\xi ds \\
&\quad - 2\varepsilon\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(y)b'(\phi(\xi-y-c\tau))v(s-\tau,\xi-y-c\tau)v(s,\xi)dyd\xi ds \\
&= \|v_0(0)\|^2_{L^2(\mathbb{R})} - 2\int_0^t\int_{\mathbb{R}} v(s,\xi)Q_1(s,\xi)d\xi ds \\
&\quad + 2\varepsilon\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(y)v(s,\xi)Q_2(s-\tau,\xi-y-c\tau)dyd\xi ds.
\end{aligned} \tag{3.25}$$

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Using the Cauchy inequality $|ab| \le \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$ for $\eta > 0$, which will be specified later, we obtain

$$\begin{aligned}
&2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v(s-\tau, \xi - y - c\tau) v(s,\xi) dy d\xi ds \\
&\le \frac{\varepsilon}{\eta} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v^2(s-\tau, \xi - y - c\tau) dy d\xi ds \\
&\quad + \varepsilon\eta \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v^2(s,\xi) dy d\xi ds. \qquad (3.26)
\end{aligned}$$

By changing variables $y \to y,\ \xi - y - c\tau \to \xi,\ s - \tau \to s$, we have

$$\begin{aligned}
&\frac{\varepsilon}{\eta} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v^2(s-\tau, \xi - y - c\tau) dy d\xi ds \\
&= \frac{\varepsilon}{\eta} \int_{-\tau}^{t-\tau} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} f_\alpha(y) dy \Big] b'(\phi(\xi)) v^2(s,\xi) d\xi ds \\
&= \frac{\varepsilon}{\eta} \int_0^{t-\tau} \int_{\mathbb{R}} b'(\phi(\xi)) v^2(s,\xi) d\xi ds + \frac{\varepsilon}{\eta} \int_{-\tau}^0 \int_{\mathbb{R}} b'(\phi(\xi)) v_0^2(s,\xi) d\xi ds \\
&\le C + \frac{\varepsilon}{\eta} \int_0^{t-\tau} \int_{-\infty}^{x_0} b'(\phi(\xi)) v^2(s,\xi) d\xi ds + \frac{\varepsilon}{\eta} \int_0^{t-\tau} \int_{x_0}^{\infty} b'(\phi(\xi)) v^2(s,\xi) d\xi ds \\
&\le C + \frac{\varepsilon}{\eta} b'(\phi(x_0)) \int_0^t \int_{x_0}^{\infty} v^2(s,\xi) d\xi ds. \qquad (3.27)
\end{aligned}$$

Here we have used (3.24) and the fact that $b'(\phi(\xi))$ is decreasing. Similarly, we can obtain

$$\begin{aligned}
&\varepsilon\eta \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v^2(s,\xi) dy d\xi ds \\
&\le \varepsilon\eta b'(0) \int_0^t \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f_\alpha(y) dy \right] v^2(s,\xi) d\xi ds \\
&= \varepsilon\eta b'(0) \int_0^t \int_{-\infty}^{x_0} v^2(s,\xi) d\xi ds + \varepsilon\eta b'(0) \int_0^t \int_{x_0}^{\infty} v^2(s,\xi) d\xi ds \\
&\le C + \varepsilon\eta b'(0) \int_0^t \int_{x_0}^{\infty} v^2(s,\xi) d\xi ds. \qquad (3.28)
\end{aligned}$$

Substituting (3.27) and (3.28) to (3.26) yields

$$\begin{aligned}
&2\varepsilon \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_\alpha(y) b'(\phi(\xi - y - c\tau)) v(s-\tau, \xi - y - c\tau) v(s,\xi) dy d\xi ds \\
&\le C + \frac{\varepsilon}{\eta} b'(\phi(x_0)) \int_0^t \int_{x_0}^{\infty} v^2(s,\xi) d\xi ds + \varepsilon\eta b'(0) \int_0^t \int_{x_0}^{\infty} v^2(s,\xi) d\xi ds. \qquad (3.29)
\end{aligned}$$

By applying (3.29) to (3.25) and noting that $v(t,\xi) \ge 0$, $Q_1(t,\xi) \ge 0$, and $Q_2(t - \tau, \xi - y - c\tau) \le 0$, we then obtain

$$\begin{aligned}
&\|v(t)\|^2_{L^2(\mathbb{R})} + 2D \int_0^t \|v_\xi(s)\|^2_{L^2(\mathbb{R})} ds + 2 \int_0^t \int_{-\infty}^{x_0} d'(\phi(\xi)) v^2(s,\xi) d\xi ds \\
&\quad + \int_0^t \int_{x_0}^{\infty} \left[2d'(\phi(\xi)) - \frac{\varepsilon}{\eta} b'(\phi(x_0)) - \varepsilon\eta b'(0) \right] v^2(s,\xi) d\xi ds \\
&\le C. \qquad (3.30)
\end{aligned}$$

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Since $d'(u_+)^2 > \varepsilon^2 b'(0)b'(u_+)$ (see (H2)), we can choose $\eta > 0$ such that

$$0 < \frac{d'(u_+) - \sqrt{d'(u_+)^2 - \varepsilon^2 b'(0)b'(u_+)}}{\varepsilon b'(0)} < \eta < \frac{d'(u_+) + \sqrt{d'(u_+)^2 - \varepsilon^2 b'(0)b'(u_+)}}{\varepsilon b'(0)}.$$

It then follows that

$$2d'(u_+) - \frac{\varepsilon}{\eta} b'(u_+) - \varepsilon\eta b'(0) > 0. \tag{3.31}$$

Thus, choosing x_0 sufficiently large such that $|\phi(x_0) - u_+| \ll 1$, we reach

$$C_3 := 2d'(\phi(x_0)) - \frac{\varepsilon}{\eta} b'(\phi(x_0)) - \varepsilon\eta b'(0) > 0. \tag{3.32}$$

Since $d'(u_+) \geq d'(\phi(\xi) \geq d'(\phi(x_0))$ for $\xi \in [x_0, \infty)$ (from (H3), $d'(\phi(\xi))$ is increasing), we see that (3.32) implies

$$2d'(\phi(\xi)) - \frac{\varepsilon}{\eta} b'(\phi(x_0)) - \varepsilon\eta b'(0) \geq C_3 > 0, \quad \xi \in [x_0, \infty). \tag{3.33}$$

Applying (3.33) to (3.30) and adding it with (3.24), that is, $C_3 \int_0^t \int_{-\infty}^{x_0} v^2(s, \xi) d\xi ds \leq C$, we further obtain

$$\|v(t)\|^2_{L^2(\mathbb{R})} + 2D \int_0^t \|v_\xi(s)\|^2_{L^2(\mathbb{R})} ds + C_3 \int_0^t \|v(s)\|^2_{L^2(\mathbb{R})} ds \leq C.$$

This proves (3.23). ∎

Next we derive the L^2-energy estimate for $v_\xi(t, \xi)$. Let us differentiate (3.3) with respect to ξ and multiply the resulting equation by $v_\xi(t, \xi)$ and then integrate it over $\mathbb{R} \times [0, t]$ with respect to ξ and t. By using the key estimates (3.23), we can similarly obtain the following high order estimate. The detail of proof is omitted.

Lemma 3.3. *For any $c \geq c_*$, it holds that*

$$\|v_\xi(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v_\xi(s)\|^2_{H^1(\mathbb{R})} ds \leq C, \quad t \geq 0. \tag{3.34}$$

Based on the above lemmas, we can prove the following two convergence results. One is the exponential stability for the noncritical traveling waves with $c > c_*$, and the other one is the algebraic stability for the critical traveling wave with $c = c_*$. We first prove the exponential stability.

Lemma 3.4. *For any $c > c_*$, there holds*

$$\|v(t)\|_{L^\infty(-\infty, x_0]} \leq Ce^{-\mu_1 t/3}, \quad t \geq 0.$$

Proof. Let $I = (-\infty, x_0]$. Then we have

$$\|v(t)\|^2_{L^2(I)} = \int_{-\infty}^{x_0} |v(\xi, t)|^2 d\xi \leq \|v(t)\|_{L^\infty(I)} \|v(t)\|_{L^1(I)}, \tag{3.35}$$

and

$$v^2(\xi, t) = \int_{-\infty}^{\xi} \partial_\xi(v^2) d\xi = 2\int_{-\infty}^{\xi} v(\xi, t) v_\xi(\xi, t) d\xi,$$

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which, by the Hölder inequality, implies the following Sobolev inequality:

$$\|v(t)\|^2_{L^\infty(I)} \le 2\|v(t)\|_{L^2(I)}\|v_\xi(t)\|_{L^2(I)}. \tag{3.36}$$

Combining (3.35) and (3.36), we obtain

$$\|v(t)\|_{L^\infty(I)} \le \sqrt[3]{4}\|v(t)\|^{\frac{1}{3}}_{L^1(I)}\|v_\xi(t)\|^{\frac{2}{3}}_{L^2(I)}. \tag{3.37}$$

In view of $\|v_\xi(t)\|_{L^2(I)} \le C$ from (3.34), $w_1(\xi) = e^{-\lambda(\xi - x_0)} \ge 1$ for $\xi \in I = (-\infty, x_0]$, and (3.7), it follows that

$$\|v(t)\|_{L^1(I)} \le \|v(t)\|_{L^1_{w_1}(I)} \le Ce^{-\mu_1 t}. \tag{3.38}$$

Thus, (3.37) and (3.38) immediately yield

$$\|v(t)\|_{L^\infty(I)} \le Ce^{-\mu_1 t/3}, \quad t \ge 0,$$

and

$$v(t, x_0) \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

This completes the proof. ∎

Now we are going to prove the exponential stability for noncritical traveling waves in $[x_0, \infty)$.

LEMMA 3.5. *For any $c > c_*$, there holds*

$$\|v(t)\|_{L^\infty[x_0,\infty)} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

Proof. Multiplying (3.3) by $e^{\mu_1 t}$ and integrating it with respect to (ξ, t) over $\mathbb{R} \times [0, t]$, and noting that $-Q_1 \le 0$ and $Q_2 \le 0$, we have

$$\begin{aligned} &e^{\mu_1 t}\|v(t)\|_{L^1(\mathbb{R})} + \int_0^t e^{\mu_1 s}\int_{\mathbb{R}} d'(\phi(\xi))v(t,\xi)d\xi ds - \mu_1\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1(\mathbb{R})}ds \\ &\quad - \varepsilon\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mu_1 s}f_\alpha(y)b'(\phi(\xi - y - c\tau))v(s-\tau, \xi - y - c\tau)dyd\xi ds \\ &\quad \le \|v_0(0)\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.39}$$

As shown before, by the change of variables $\xi - y - c\tau \to \xi$ and $s - \tau \to s$, and using the fact $\int_{\mathbb{R}} f_\alpha(y)dy = 1$, we have

$$\begin{aligned} &\varepsilon\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mu_1 s}f_\alpha(y)b'(\phi(\xi - y - c\tau))v(t-\tau, \xi - y - c\tau)dyd\xi ds \\ &= \varepsilon\int_{-\tau}^{t-\tau}\int_{\mathbb{R}} e^{\mu_1\tau}f_\alpha(y)dy\int_{\mathbb{R}} e^{\mu_1 s}b'(\phi(\xi))v(s,\xi)d\xi ds \\ &= \varepsilon e^{\mu_1\tau}\int_{-\tau}^{t-\tau}\int_{\mathbb{R}} e^{\mu_1 s}b'(\phi(\xi))v(s,\xi)d\xi ds. \end{aligned} \tag{3.40}$$

Substituting (3.40) into (3.39), we obtain

$$\begin{aligned} &e^{\mu_1 t}\|v(t)\|_{L^1(\mathbb{R})} + \int_0^t e^{\mu_1 s}\int_{\mathbb{R}} d'(\phi(\xi))v(t,\xi)d\xi ds \\ &\quad - \mu_1\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1(\mathbb{R})}ds - \varepsilon e^{\mu_1\tau}\int_{-\tau}^{t-\tau}\int_{\mathbb{R}} e^{\mu_1 s}b'(\phi(\xi))v(s,\xi)d\xi ds \\ &\le \|v_0(0)\|_{L^1(\mathbb{R})}. \end{aligned}$$

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Splitting each integral on the above inequality into two parts due to $\mathbb{R} = (-\infty, x_0] \cup [x_0, \infty)$, we get

$$e^{\mu_1 t} \int_{x_0}^{\infty} v(t,\xi) d\xi + \int_0^t e^{\mu_1 s} \int_{x_0}^{\infty} d'(\phi(\xi)) v(t,\xi) d\xi ds$$
$$- \mu_1 \int_0^t e^{\mu_1 s} \int_{x_0}^{\infty} v(s,\xi) d\xi ds - \varepsilon e^{\mu_1 \tau} \int_0^{t-\tau} \int_{x_0}^{\infty} e^{\mu_1 s} b'(\phi(\xi)) v(s,\xi) d\xi ds$$
$$\le \|v_0(0)\|_{L^1(\mathbb{R})} - J(t), \tag{3.41}$$

where

$$J(t) := e^{\mu_1 t} \int_{-\infty}^{x_0} v(t,\xi) d\xi + \int_0^t e^{\mu_1 s} \int_{-\infty}^{x_0} d'(\phi(\xi)) v(t,\xi) d\xi ds$$
$$-\mu_1 \int_0^t e^{\mu_1 s} \int_{-\infty}^{x_0} v(s,\xi) d\xi ds - \varepsilon e^{\mu_1 \tau} \int_{-\tau}^{t-\tau} \int_{-\infty}^{x_0} e^{\mu_1 s} b'(\phi(\xi)) v(s,\xi) d\xi ds.$$

In view of Lemma 3.1 and $w_1(\xi) \ge 1$ in $(-\infty, x_0]$, we obtain

$$e^{\mu_1 t} \int_{-\infty}^{x_0} v(t,\xi) d\xi \le C \quad \text{and} \quad \int_0^t e^{\mu_1 s} \int_{-\infty}^{x_0} v(t,\xi) d\xi ds \le C,$$

which, together with the boundedness of $d'(\phi(\xi))$ and $b'(\phi(\xi))$, implies that

$$|J(t)| \le C \quad \text{for } t \ge 0. \tag{3.42}$$

Applying (3.42) to (3.41), we then get

$$e^{\mu_1 t} \int_{x_0}^{\infty} v(t,\xi) d\xi + \int_0^t e^{\mu_1 s} \int_{x_0}^{\infty} [d'(\phi(\xi)) - \varepsilon e^{\mu_1 \tau} b'(\phi(\xi)) - \mu_1] v(s,\xi) d\xi ds \le C. \tag{3.43}$$

Since u_+ is stable (see (H2)), i.e., $d'(u_+) - \varepsilon b'(u_+) > 0$, there exists a small $\mu_1 > 0$ so that

$$d'(u_+) - \varepsilon e^{\mu_1 \tau} b'(u_+) - \mu_1 > 0.$$

By the continuity of $\lim_{\xi \to +\infty} \phi(\xi) = v_+$, it then follows that for all $x_0 \gg 1$, we have

$$d'(\phi(\xi)) - \varepsilon b'(\phi(\xi)) - \mu_1 \ge 0, \quad \xi \in [x_0, \infty). \tag{3.44}$$

Applying (3.44) to (3.43), we see that

$$\int_{x_0}^{\infty} v(t,\xi) d\xi \le C e^{-\mu_1 t}.$$

Based on this inequality, as shown in Lemma 3.4, we can similarly prove the following convergence in $[x_0, \infty)$:

$$\|v(t)\|_{L^\infty[x_0,\infty)} \le \sqrt[3]{4} \|v(t)\|_{L^1[x_0,\infty)}^{\frac{1}{3}} \|v_\xi(t)\|_{L^2[x_0,\infty)}^{\frac{2}{3}} \le C e^{-\mu_1 \frac{t}{3}}.$$

This completes the proof. ∎

Combining Lemmas 3.4 and 3.5, we have the following result.

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LEMMA 3.6. *For any $c > c_*$, there holds*

$$\sup_{x\in\mathbb{R}} |U^+(t,x) - \phi(x+ct)| = \|v(t)\|_{L^\infty(\mathbb{R})} \leq Ce^{-\mu_1 t/3}, \quad t \geq 0.$$

Next, we are going to prove the algebraic stability for the critical traveling wave with $c = c_*$. In this case, we have $\lambda = \lambda_1 = \lambda_*$. Using the linearization of (3.3) at 0, we can rewrite (3.3) as

$$\begin{aligned}
&\frac{\partial v}{\partial t} + c_* \frac{\partial v}{\partial \xi} - D\frac{\partial^2 v}{\partial \xi^2} + d'(0)v \\
&\quad - \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) v(t-\tau, \xi - y - c_*\tau) dy \\
&= -Q_1(t,\xi) + \varepsilon \int_{\mathbb{R}} f_\alpha(y) Q_2(t-\tau, \xi - y - c_*\tau) dy + [d'(0) - d'(\phi(\xi))]v \\
&\quad + \varepsilon \int_{\mathbb{R}} f_\alpha(y)[b'(\phi(\xi - y - c_*\tau)) - b'(0)] v(t-\tau, \xi - y - c_*\tau) dy.
\end{aligned}$$

From (H2) and (H3), we see that

$$-Q_1(t,\xi) \leq 0, \quad Q_2(t-\tau, \xi - y - c\tau) \leq 0,$$

and

$$d'(0) - d'(\phi(\xi)) \leq 0, \quad b'(\phi(\xi - y - c_*\tau) - b'(0) \leq 0.$$

It then follows that

$$\frac{\partial v}{\partial t} + c_* \frac{\partial v}{\partial \xi} - D\frac{\partial^2 v}{\partial \xi^2} + d'(0)v - \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) v(t-\tau, \xi - y - c_*\tau) dy \leq 0.$$

Let $\bar{v}(t,\xi)$ be the solution of the following equation with the same initial data $v_0(s,\xi)$:

(3.45)
$$\begin{cases}
\dfrac{\partial \bar{v}}{\partial t} + c_* \dfrac{\partial \bar{v}}{\partial \xi} - D\dfrac{\partial^2 \bar{v}}{\partial \xi^2} + d'(0)\bar{v} - \varepsilon b'(0) \displaystyle\int_{\mathbb{R}} f_\alpha(y) \bar{v}(t-\tau, \xi - y - c_*\tau) dy = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\
\bar{v}(s,\xi) = v_0(s,\xi), \quad s \in [-\tau, 0], x \in \mathbb{R}.
\end{cases}$$

By the comparison principle, we have

$$v(t,\xi) \leq \bar{v}(t,\xi) \quad \text{for } (t,\xi) \in R_+ \times \mathbb{R}.$$

Let

$$\tilde{v}(t,\xi) := w_1(\xi)\bar{v}(t,\xi). \tag{3.46}$$

From (3.45), we see that $\tilde{v}(t,\xi)$ satisfies

$$\frac{\partial \tilde{v}}{\partial t} + k_1 \frac{\partial \tilde{v}}{\partial \xi} - D\frac{\partial^2 \tilde{v}}{\partial \xi^2} + k_2 \tilde{v} = \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y + c_*\tau)} \tilde{v}(t-\tau, \xi - y - c_*\tau) dy,$$

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where

$$k_1 := c_* - 2D\lambda_* \quad \text{and} \quad k_2 := c_*\lambda_* - D\lambda_*^2 + d'(0) > 0.$$

Furthermore, letting

$$\hat{v}(t,\xi) := e^{k_2 t}\tilde{v}(t,\xi), \tag{3.47}$$

we have

$$\frac{\partial \hat{v}}{\partial t} + k_1 \frac{\partial \hat{v}}{\partial \xi} - D\frac{\partial^2 \hat{v}}{\partial \xi^2} = \varepsilon b'(0) e^{k_2\tau} \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y+c_*\tau)} \hat{v}(t-\tau, \xi - y - c_*\tau)dy,$$

which is equivalent to

$$\begin{aligned} \hat{v}(t,\xi) = &\int_{\mathbb{R}} G(t,\xi-\zeta)\hat{v}_0(0,\zeta)d\zeta \\ &+ \varepsilon b'(0)e^{k_2\tau}\int_0^t \int_{\mathbb{R}} G(t-s,\xi-\zeta) \\ &\qquad \times \int_{\mathbb{R}} f_\alpha(y)e^{-\lambda_*(y+c_*\tau)}\hat{v}(s-\tau,\zeta-y-c_*\tau)dyd\zeta ds, \end{aligned} \tag{3.48}$$

where the Green function $G(t,\xi-\zeta)$ is defined as

$$G(t,\xi-\zeta) = \frac{1}{\sqrt{4\pi D t}}\; e^{-\frac{(\xi-\zeta+k_1 t)^2}{4Dt}}.$$

Lemma 3.7. *It holds that*

$$\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}} e^{k_2 t}, \quad t > 0. \tag{3.49}$$

Proof. Note that

$$\varepsilon b'(0)\int_{\mathbb{R}} f_\alpha(y)e^{-\lambda_*(y+c_*\tau)}dy = \varepsilon b'(0)e^{\alpha\lambda_*^2 - c_*\lambda_*\tau} = k_2$$

and

$$0 < G(t,\xi) \le (4\pi D)^{-\frac{1}{2}} t^{-\frac{1}{2}} \quad \text{and} \quad \int_{\mathbb{R}} G(t,\xi)d\xi = 1. \tag{3.50}$$

If

$$\|\hat{v}(t-\tau)\|_{L^\infty} \le C(\theta + t - \tau)^{-\frac{1}{2}} e^{k_2(t-\tau)}, \quad t \ge 0 \tag{3.51}$$

where we take $\theta > 2\tau$ in order to avoid the singularity, then (3.48) together with

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(3.50) implies that for all $t > 0$,

$$
\begin{aligned}
\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} &\le t^{-\frac{1}{2}}\|\hat{v}_0\|_{L^1(\mathbb{R})} \\
&\quad + \varepsilon b'(0)e^{k_2\tau}\int_0^t \|\hat{v}(s-\tau)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} G(t-s,\xi-\zeta) \\
&\qquad \times \int_{\mathbb{R}} f_\alpha(y)e^{-\lambda_*(y+c_*\tau)}dyd\zeta ds \\
&\le C\left[t^{-\frac{1}{2}} + k_2e^{k_2\tau}\int_0^t(\theta+s-\tau)^{-\frac{1}{2}}e^{k_2(s-\tau)}ds\right] \\
&= C\left[t^{-\frac{1}{2}} + k_2\int_0^{\frac{t}{2}}(\theta+s-\tau)^{-\frac{1}{2}}e^{k_2s}ds + k_2\int_{\frac{t}{2}}^t(\theta+s-\tau)^{-\frac{1}{2}}e^{k_2s}ds\right] \\
&\le C\left[t^{-\frac{1}{2}} + k_2e^{\frac{k_2t}{2}}\int_0^{\frac{t}{2}}(\theta+s-\tau)^{-\frac{1}{2}}ds + k_2\left(\theta+\frac{t}{2}-\tau\right)^{-\frac{1}{2}}\int_{\frac{t}{2}}^t e^{k_2s}ds\right] \\
&= C\left[t^{-\frac{1}{2}} + 2k_2e^{\frac{k_2t}{2}}\left[\left(\theta+\frac{t}{2}-\tau\right)^{\frac{1}{2}} - \left(\theta-\tau\right)^{\frac{1}{2}}\right]\right. \\
&\qquad \left. + \left(\theta+\frac{t}{2}-\tau\right)^{-\frac{1}{2}}[e^{k_2t}-e^{k_2\frac{t}{2}}]\right] \\
&\le C\left(t^{-\frac{1}{2}} + e^{\frac{k_2t}{2}}(\theta+t-\tau)^{\frac{1}{2}} + (\theta+t-\tau)^{-\frac{1}{2}}e^{k_2t}\right).
\end{aligned}
\tag{3.52}
$$

Since $\|\hat{v}(0)\|_{L^\infty(\mathbb{R})} \le C$, $\hat{v}(t,\xi)$ has no singularity for t around 0, and hence the first term $t^{-\frac{1}{2}}$ on the last line of (3.52) could be replaced by $(\theta+t)^{-\frac{1}{2}}$ (this is the standard way in the heat equation). It then follows from (3.52) that

$$
\begin{aligned}
\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} &\le C\Big((\theta+t)^{-\frac{1}{2}} + e^{\frac{k_2t}{2}}(\theta+t-\tau)^{\frac{1}{2}} + (\theta+t-\tau)^{-\frac{1}{2}}e^{k_2t}\Big) \\
&\le C(\theta+t)^{-\frac{1}{2}}e^{k_2t}\Big[e^{-k_2t} + e^{-\frac{k_2t}{2}}(\theta+t-\tau)+1\Big] \\
&\le C(\theta+t)^{-\frac{1}{2}}e^{k_2t}.
\end{aligned}
\tag{3.53}
$$

Here we have used the fact that

$$e^{-\frac{k_2t}{2}}(\theta+t-\tau) \le C,\ \forall t \ge 0.$$

It is easy to check that

$$\|\hat{v}(t-\tau)\|_{L^\infty(\mathbb{R})} = \|\hat{v}_0(t-\tau)\|_{L^\infty(\mathbb{R})} \le C(\theta+t)^{-\frac{1}{2}}e^{k_2(t-\tau)},\ \forall t\in[0,\tau],$$

namely, (3.51) holds for all $t \in [0,\tau]$. It then follows from (3.53) that

$$\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}e^{k_2t},\ \forall t\in[0,\tau]. \tag{3.54}$$

Next, for any $t\in[\tau,2\tau]$, i.e., $t-\tau\in[0,\tau]$, (3.54) immediately implies that

$$\|\hat{v}(t-\tau)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}e^{k_2(t-\tau)},\ \forall t\in[\tau,2\tau],$$

and hence (3.51) holds for all $t\in[\tau,2\tau]$. Thus, we can apply (3.53) to get

$$\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}e^{k_2t},\ \forall t\in[\tau,2\tau]. \tag{3.55}$$

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For $t \in [n\tau, (n+1)\tau]$, by repeating this procedure, we then obtain

$$\|\hat{v}(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}} e^{k_2 t}, \ \forall t \in [n\tau, (n+1)\tau]. \tag{3.56}$$

Combining (3.54), (3.55), and (3.56), we see that (3.49) holds for all $t \ge 0$. ∎

As a consequence of (3.46), (3.47), and (3.49), we have the following result on the algebraic decay for $\bar{v}(t,\xi)$.

LEMMA 3.8. *For $c = c_*$, it holds that*

$$\|\bar{v}(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}, \quad \forall t > 0.$$

Since $v(t,\xi) \le \bar{v}(t,\xi)$, we immediately obtain the following decay for v in the case of $c = c_*$.

LEMMA 3.9. *For $c = c_*$, it holds that*

$$\|v(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}, \ \forall t > 0$$

and

$$\|v(t)\|_{L^\infty(-\infty,x_0]} \le C(1+t)^{-\frac{1}{2}}, \ \forall t > 0, \tag{3.57}$$

due to $w_1(\xi) \ge 1$ for $\xi \in (-\infty, x_0]$.

On the other hand, v possesses the exponential decay for $\xi \in [x_0, \infty)$ (because $v = u_+$ is stable):

$$\lim_{\xi \to \infty} v(t,\xi) \le Ce^{-\mu_2 t}, \quad \forall t > 0 \tag{3.58}$$

for some μ_2 satisfying

$$0 < \mu_2 < d'(u_+) - \varepsilon b'(u_+).$$

Thus, (3.57) and (3.58) lead to the following algebraic decay property.

LEMMA 3.10. *For $c = c_*$, there holds*

$$\sup_{x\in\mathbb{R}} |U^+(t,x) - \phi(x + c_* t)| = \|\bar{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}, \quad \forall t > 0.$$

Combining Lemmas 3.6 and 3.10, we have the following result.

LEMMA 3.11. *There hold the exponential decay*

$$\sup_{x\in\mathbb{R}} |U^+(t,x) - \phi(x + ct)| \le Ce^{-\mu t}, \qquad \forall t \ge 0, \quad \textit{for } c > c_*,$$

and the algebraic decay

$$\sup_{x\in\mathbb{R}} |U^+(t,x) - \phi(x + c_* t)| \le C(1+t)^{-\frac{1}{2}}, \qquad \forall t \ge 0, \quad \textit{for } c = c_*.$$

Step 2 (the convergence of $U^-(t,x)$ to $\phi(x+ct)$). For any given $c \ge c_*$, let $\xi = x + ct$ and

$$v(t,\xi) = \phi(x+ct) - U^-(t,x), \quad v_0(s,\xi) = \phi(x+cs) - U_0^-(s,x).$$

As in Step 1, we can similarly prove that $U^-(t,x)$ converges to $\phi(x+ct)$ as follows.

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LEMMA 3.12. *There hold the exponential decay*

$$\sup_{x\in\mathbb{R}} |U^-(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \qquad \forall t \ge 0, \quad \textit{for } c > c_*,$$

and the algebraic decay

$$\sup_{x\in\mathbb{R}} |U^-(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \qquad \forall t \ge 0, \quad \textit{for } c = c_*.$$

Step 3 (the convergence of $u(t,x)$ to $\phi(x+ct)$). Finally, we prove that $u(t,x)$ converges to $\phi(x+ct)$ as follows.

LEMMA 3.13. *There hold the exponential decay*

$$\sup_{x\in\mathbb{R}} |U^-(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \textit{for } c > c_*,$$

and the algebraic decay

$$\sup_{x\in\mathbb{R}} |U^-(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \textit{for } c = c_*.$$

Proof. Since the initial data satisfy $U_0^-(s,x) \le u_0(s,x) \le U_0^+(s,x)$, the comparison principle implies that

$$U^-(t,x) \le u(t,x) \le U^+(t,x), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Thanks to Lemmas 3.11 and 3.12, we have the following convergence results:

$$\sup_{x\in\mathbb{R}} |U^\pm(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \text{for } c > c_*,$$

and

$$\sup_{x\in\mathbb{R}} |U^\pm(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \text{for } c = c_*.$$

By these inequalities and the squeezing argument, it then follows that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \text{for } c > c_*,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \text{for } c = c_*.$$

This completes the proof. ∎

4. Applications.

In this section, we apply Theorem 2.2 to the monostable evolution equations mentioned in (1.3)–(1.7) to obtain the global stability of all traveling waves, including the critical one.

4.1. Nonlocal Nicholson's blowflies equation.

Consider the nonlocal Nicholson's blowflies equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + \delta u = \varepsilon p \displaystyle\int_{\mathbb{R}} f_\alpha(y) u(t-\tau, x-y) e^{-au(t-\tau,x-y)} dy, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}. \end{cases} \tag{4.1}$$

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Here the death rate function is $d(u) = \delta u$ and the birth rate is $b(u) = pue^{-au}$ with $\delta > 0$, $p > 0$, and $a > 0$. The constant equilibria of (4.1) is

$$u_- = 0 \quad \text{and} \quad u_+ = \frac{1}{a} \ln \frac{\varepsilon p}{\delta}.$$

For $1 < \frac{\varepsilon p}{\delta} \le e$, it can be easily checked that the conditions (H1)–(H3) are satisfied, where the condition $d'(u_+)^2 > \varepsilon^2 b'(0) b'(u_+)$ is equivalent to $\frac{\delta}{\varepsilon p} > 1 - \ln \frac{\varepsilon p}{\delta}$, which automatically holds when $\frac{\varepsilon p}{\delta} \approx e$. From Theorem 2.2, we immediately obtain the following result.

THEOREM 4.1. *Let $1 < \frac{\varepsilon p}{\delta} \le e$ and $\frac{\delta}{\varepsilon p} > 1 - \ln \frac{\varepsilon p}{\delta}$. For a given traveling wave $\phi(x + ct)$ of (4.1) with $c \ge c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

$$0 = u_- \le u_0(s,x) \le u_+, \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x + cs)$ is in $C([-\tau,0]; L^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (4.1) converges to the traveling wave $\phi(x + ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \text{for } c > c_* \quad \text{and} \quad \mu = \mu(c) > 0,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \text{for } c = c_*.$$

Remark 3. The stability of traveling wavefronts of (4.1) was studied earlier in [28] for fast waves and then improved in [26] for all slow waves under the condition that $\alpha \ll 1$ if $c \approx c_*$. Here we obtain the stability for all waves, including the critical wave, without any restriction on the delay time τ and α (the total diffusion of immature population).

4.2. Local Nicholson's blowflies equation. Let $\alpha \to 0$. We then reduce the nonlocal Nicholson's blowflies equation (4.1) to the following local Nicholson's blowflies equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + \delta u = \varepsilon p u(t-\tau,x) e^{-au(t-\tau,x)}, & t > 0, \, x \in \mathbb{R}, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau,0], \, x \in \mathbb{R}. \end{cases} \tag{4.2}$$

By a similar calculation as in section 3, we find that the condition $\frac{\delta}{\varepsilon p} > 1 - \ln \frac{\varepsilon p}{\delta}$, which is needed in Theorem 4.1, can be removed. Our new stability is as follows.

THEOREM 4.2. *Let $1 < \frac{\varepsilon p}{\delta} \le e$. For a given traveling wave $\phi(x + ct)$ of (4.2) with $c \ge c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

$$0 = u_- \le u_0(s,x) \le u_+, \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x + cs)$ is in $C([-\tau,0]; L^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (4.2) converges to the traveling wave $\phi(x + ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \text{for } c > c_* \quad \text{and} \quad \mu = \mu(c) > 0,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c_*t)| \le C(1+t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \text{for } c = c_*.$$

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Remark 4. For the local time-delayed reaction-diffusion equation (4.2), it was showed that fast waves are locally stable with an exponential decay in [29] and globally stable in [23]. But the stability for those slow waves (except for the critical wave) holds only when the delay time $\tau \ll 1$. Recently, the global stability of all noncritical traveling waves was proved in [26] regardless of the magnitude of time delay. The stability result presented in Theorem 4.2 for the critical wave improves these earlier works in [23, 26, 29].

4.3. A nonlocal population model with age structure. Letting $d(u) = \delta u^2$ with $\delta > 0$ and $\varepsilon = 1, b(u) = pe^{-\gamma\tau}u$ with $p > 0$ and $\gamma > 0$, we then reduce (1.1) to the following age-structured population model which was first derived in [3]:

$$(4.3) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + \delta u^2 = pe^{-\gamma\tau}\displaystyle\int_{-\infty}^{\infty} f_\alpha(y)u(t-\tau, x-y)dy, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}. \end{cases}$$

It is clear that the constant equilibria of (4.3) are $u_- = 0$ and $u_+ = \frac{p}{\delta}e^{-\gamma\tau}$, and the conditions (H1)–(H3) are satisfied automatically. The following result is a straightforward consequence of Theorem 2.2.

THEOREM 4.3. *For a given traveling wave $\phi(x+ct)$ of (4.3) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

$$0 = u_- \leq u_0(s,x) \leq u_+, \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x+cs)$ is in $C([-\tau,0]; L^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (4.3) converges to the traveling wave $\phi(x+ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ce^{-\mu t}, \ \forall t \geq 0, \quad \text{for } c > c_* \quad \text{and} \quad \mu = \mu(c) > 0,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c_*t)| \leq C(1+t)^{-\frac{1}{2}}, \ \forall t \geq 0, \quad \text{for } c = c_*.$$

4.4. A local population model with age-structure. Letting $\alpha \to 0$ in (4.3), we obtain the following local population model with age-structure:

$$(4.4) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + \delta u^2 = pe^{-\gamma\tau}u(t-\tau, x), \quad t > 0, x \in \mathbb{R}, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}. \end{cases}$$

Here $u_- = 0$ and $u_+ = \frac{p}{\delta}e^{-\gamma\tau}$. Now we give a complete answer to the stability for all traveling waves, including the critical wave.

THEOREM 4.4. *For a given traveling wave $\phi(x+ct)$ of (4.4) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy*

$$0 = u_- \leq u_0(s,x) \leq u_+, \quad \text{for } (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x+cs)$ is in $C([-\tau,0]; L^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then the solution of (4.4) converges to the traveling wave $\phi(x+ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ce^{-\mu t}, \ \forall t \geq 0, \quad \text{for } c > c_* \quad \text{and} \quad \mu = \mu(c) > 0,$$

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and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x + c_*t)| \le C(1 + t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \textit{for } c = c_*.$$

Remark 5. The linear stability of all slow waves (except for the critical wave) was given in [10] in the case where the time delay τ is sufficiently small. The global nonlinear stability of noncritical waves was proved in [18] still with $\tau \ll 1$. Such a restriction of smallness for the time delay was further removed in [30]. However, the stability of the critical wave was not addressed in [10, 18, 30] because the weighted L^2-energy method cannot apply to this case. The stability of the critical wave in Theorem 4.4 complements the existing results in [10, 18, 30].

4.5. Fisher–KPP equation. Taking $\tau = 0$, $D = \delta = p = 1$ in (4.4), we then get the following well-known Fisher–KPP equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = u(1-u), & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4.5}$$

Here $u_- = 0$, $u_+ = 1$, $c_* = 2$, $\lambda_* = \frac{c_*}{2} = 1$, $\lambda_1 = \frac{c-\sqrt{c^2-4}}{2}$, and $\lambda_2 = \frac{c+\sqrt{c^2-4}}{2}$ for $c > c_* = 2$. Let the weight function $w(x)$ be defined as in (2.2). We further choose a large number x_0 such that

$$4\phi(x_0) > \eta + \frac{1}{\eta}.$$

Here $\phi(x + ct)$ is the given traveling wave, and η is a positive constant satisfying

$$2 - \sqrt{3} < \eta < 2 + \sqrt{3}.$$

Thus, Theorem 2.2 implies the following result.

THEOREM 4.5. *For a given traveling wave solution $\phi(x + ct)$ of the Fisher–KPP equation* (4.5) *with $c \ge c_* = 2$ and $\phi(\pm\infty) = u_\pm$, if the initial data satisfy $0 \le u_0(x) \le 1$ and the initial perturbation $u_0(x) - \phi(x)$ is in $L^1_w(\mathbb{R}) \cap H^1(\mathbb{R})$, then the solution of* (4.5) *converges to the traveling wave $\phi(x + ct)$ in the sense that*

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x + ct)| \le Ce^{-\mu t}, \ \forall t \ge 0, \quad \textit{for } c > c_* \quad \textit{and} \quad \mu = \mu(c) > 0,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x + c_*t)| \le C(1 + t)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \textit{for } c = c_*.$$

Remark 6. It is interesting to compare Theorem 4.5 with the classical results for the Fisher–KPP equation. In [35], Sattinger proved the exponential stability for all noncritical waves by the spectral analysis method, but the stability of the critical wave was left open. The authors of [31, 15, 9] improved Sattinger's result and proved the stability for all waves including the critical wave. For the critical wave $\phi(x+c_*t)$, when the initial perturbation around the wave $\phi(x + c_*t)$ decays as

$$|u_0(x) - \phi(x)| = O(1)e^{-c_*|x|/2} \ \text{ as } x \to -\infty,$$

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Moet [31] applied the maximum principle to prove

$$\|u(t,x) - \phi(x+c_*t)\|_{L^\infty} = O(1)t^{-1/2},$$

and Kirchgassner [15] used the spectral analysis method to obtain

$$\|u(t,x) - \phi(x+c_*t)\|_{L^\infty} = O(1)t^{-1/4},$$

which was then improved by Gallay [9], using the renormalization group method, as

$$\|u(t,x) - \phi(x+c_*t)\|_{L^\infty} = O(1)t^{-3/2}.$$

But the initial perturbation around the critical wave needs to be is much faster than what we assumed, because his weight function is chosen as

$$w(x) = \begin{cases} e^{-\beta|x|} & \text{for } x \le 0, \\ (1+x)^3 & \text{for } x > 0. \end{cases}$$

It is also easy to see that the exponential stability of noncritical waves in [35] and the algebraic stability of the critical wave in [31] are the consequences of our main result. For the critical wave case, our obtained decay rate is faster than that in [15], but slower than that in [9].

5. A generalization. We consider a more general time-delayed reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + d(u) = F\left(\displaystyle\int_{\mathbb{R}} g(y)b(u(t-\tau, x-y))dy\right), & t > 0, x \in \mathbb{R}, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Here the death rate function $d(u)$, the birth rate function $b(u)$, the nonlinear function $F(u)$, and the kernel $g(x)$ satisfy the following conditions:

($\mathcal{H}1$) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = F(0) = 0$, $d(u_+) = F(b(u_+))$, $d \in C^2[0, u_+]$, $b \in C^2[0, u_+]$, $F \in C^2[0, b(u_+)]$;

($\mathcal{H}2$) $F'(0)b'(0) > 0$, $F'(0)b'(0) > d'(0)$, $F'(b(u_+))b'(u_+) < d'(u_+)$, and $d'(u_+)^2 > F'(0)^2 b'(0)b'(u_+)$;

($\mathcal{H}3$) For $0 \le u \le u_+$, $d'(u) \ge 0$, $b'(u) \ge 0$, $d''(u) \ge 0$, $b''(u) \le 0$ but at least one of $d''(u)$ and $|b''(u)|$ is strictly greater than 0;

($\mathcal{H}4$) $F \in C^2[0, b(u_+)]$, $F'(u) \ge 0$, and $F''(u) \le 0$ for $u \in [0, b(u_+)]$;

($\mathcal{H}5$) The kernel $g(x)$ is any integrable nonnegative function satisfying $g(-x) = g(x)$ and $\displaystyle\int_{\mathbb{R}} g(y)dy = 1$.

By the theory of spreading speeds and traveling waves developed in [21, 22] for monotone semiflows, we have the following result.

LEMMA 5.1 (existence of traveling waves). *Under the conditions ($\mathcal{H}1$)–($\mathcal{H}5$), there exist a minimum speed (also called the critical wave speed) $c_* > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying*

$$\mathcal{F}_{c_*}(\lambda_*) = \mathcal{G}_{c_*}(\lambda_*), \qquad \mathcal{F}'_{c_*}(\lambda_*) = \mathcal{G}'_{c_*}(\lambda_*),$$

where

$$\mathcal{F}_c(\lambda) = F'(0)b'(0)\int_{\mathbb{R}} e^{-\lambda(y+c\tau)} g(y)dy, \qquad \mathcal{G}_c(\lambda) = c\lambda - D\lambda^2 + d'(0),$$

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and (c_*, λ_*) *is the tangent point of* $\mathcal{F}_c(\lambda)$ *and* $\mathcal{G}_c(\lambda)$*, namely,*

$$c_*\lambda_* - D\lambda_*^2 + d'(0) = F'(0)b'(0)\int_{\mathbb{R}} e^{-\lambda_*(y+c_*\tau)}g(y)dy,$$
$$c_* - 2D\lambda_* = -F'(0)b'(0)\int_{\mathbb{R}}(y+c_*\tau)e^{-\lambda_*(y+c_*\tau)}g(y)dy,$$

such that for any $c \geq c_*$*, the traveling wavefront* $\phi(x+ct)$ *of* (5.1) *connecting* $u_\pm$ *exists, and for any* $c < c_*$*, no traveling wave* $\phi(x+ct)$ *exists. When* $c > c_*$*, there exist two numbers* $\lambda_1 > 0$ *and* $\lambda_2 > 0$*, as the solutions to the equation* $F_c(\lambda_i) = G_c(\lambda_i)$*, such that* $\mathcal{F}_c(\lambda) < \mathcal{G}_c(\lambda)$ *for* $\lambda_1 < \lambda < \lambda_2$*, and* $\lambda_1 < \lambda_* < \lambda_2$*. When* $c = c_*$*, we have* $\lambda_1 = \lambda_* = \lambda_2$*.*

Let $\eta_1 > 0$ be a number such that

$$0 < \frac{d'(u_+) - \sqrt{d'(u_+)^2 - F'(0)^2 b'(0)b'(u_+)}}{F'(0)b'(0)} < \eta_1$$
$$< \frac{d'(u_+) + \sqrt{d'(u_+)^2 - F'(0)^2 b'(0)b'(u_+)}}{F'(0)b'(0)},$$

and let x_1 be sufficiently large so that

$$2d'(\phi(x_1)) - \frac{1}{\eta_1}F'(0)b'(\phi(x_1)) - \eta_1 F'(0)b'(0) > 0.$$

For any given $c \geq c_*$, we define the weight function as follows:

$$w_2(x) = \begin{cases} e^{-\lambda(x-x_1)} & \text{for } x \leq x_1, \\ 1 & \text{for } x > x_1, \end{cases} \tag{5.2}$$

where λ is any fixed number in $(\lambda_1, \lambda_*]$ when $c > c_*$, but $\lambda = \lambda_*$ when $c = c_*$.

By similar arguments as in section 3, we can prove the following result on the global stability of all traveling wavefronts, including the critical traveling wavefront.

THEOREM 5.2 (global stability). *Let the hypotheses* $(\mathcal{H}1)$–$(\mathcal{H}5)$ *hold. For a traveling wave* $\phi(x+ct)$ *of* (5.1) *with* $c \geq c_*$*, if the initial data satisfy*

$$0 = u_- \leq u_0(s,x) \leq u_+, \quad \textit{for } (s,x) \in [-\tau, 0] \times \mathbb{R},$$

and the initial perturbation $u_0(s,x) - \phi(x+cs)$ *is in* $C([-\tau,0]; L^1_{w_2}(\mathbb{R}) \cap H^1(\mathbb{R}))$*, then the solution of* (5.1) *uniquely exists and satisfies*

$$0 = u_- \leq u(t,x) \leq u_+, \quad \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

and

$$u(t,x) - \phi(x+ct) \in C([0,\infty)L^1_{w_2}(\mathbb{R}) \cap H^1(\mathbb{R})),$$

and, in particular, the solution $u(t,x)$ *converges to the traveling wave* $\phi(x+ct)$ *in the sense that*

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \leq Ce^{-\hat{\mu}t}, \ \forall t \geq 0, \quad \textit{for } c > c_* \textit{ and } \hat{\mu} = \hat{\mu}(c) > 0,$$

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and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le C(t+1)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \textit{for } c = c_*.$$

If we replace the condition $(\mathcal{H}4)$ with the following weaker one

$(\mathcal{H}4')$ $F \in C^2[0, b(u_+)]$ with $F'(u) \ge 0$, $\forall u \in [0, b(u_+)]$,

then we can prove the following local stability of traveling waves for a class of nonlocal time-delayed reaction-diffusion equations.

THEOREM 5.3 (local stability). *Let the hypotheses* $(\mathcal{H}1)$*–*$(\mathcal{H}3)$*,* $(\mathcal{H}4')$*, and* $(\mathcal{H}5)$ *hold. For a traveling wave* $\phi(x+ct)$ *of* (5.1) *with* $c \ge c_*$*, if the the initial perturbation* $u_0(s,x) - \phi(x+cs)$ *is in* $C([-\tau,0]; L^1_{w_2}(\mathbb{R}) \cap H^1(\mathbb{R}))$ *and*

$$\|u_0(s,\cdot) - \phi(\cdot + cs)\|_{L^1_{w_2}} + \|u_0(s,\cdot) - \phi(\cdot + cs)\|_{H^1} \ll 1, \quad s \in [-\tau, 0],$$

then the solution of (5.1) *uniquely exists and satisfies*

$$u(t,x) - \phi(x+ct) \in C([0,\infty) L^1_{w_2}(\mathbb{R}) \cap H^1(\mathbb{R})),$$

and, in particular, the solution $u(t,x)$ *converges to the traveling wave* $\phi(x+ct)$ *in the sense that*

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le Ce^{-\hat{\mu}t}, \ \forall t \ge 0, \quad \textit{for } c > c_* \textit{ and } \hat{\mu} = \hat{\mu}(c) > 0,$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+ct)| \le C(t+1)^{-\frac{1}{2}}, \ \forall t \ge 0, \quad \textit{for } c = c_*.$$

As a final remark, we consider a nonlocal vector disease model

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} + d(u) = F\left(h(u(t,x)) \displaystyle\int_{\mathbb{R}} g(y) u(t-\tau, x-y) dy\right), \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}, \end{cases} \tag{5.3}$$

which is a generalized version of the model presented in [34]. Under appropriate assumptions, the spreading speed and its coincidence with the minimal wave speed can be established for system (5.3) in the same way as in [48, 49]. In the case where $F(0) = 0$, $F'(0) > 0$, and $h(0) > 0$, e.g., $F(u) = u$ and $h(u) = 1 - u$ as in [48, 49], the linearization of (5.3) at $u = 0$ is

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + d'(0)u = F'(0)h(0) \int_{\mathbb{R}} g(y) u(t-\tau, x-y) dy,$$

which is essentially the same as the linearized equation of (1.1) at $u = 0$. It then follows that we can use the similar arguments as in this paper to obtain sufficient conditions for the exponential stability of noncritical traveling waves and the algebraic stability of the critical traveling wave for model system (5.3).

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