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Asymptotic profile of a parabolic–hyperbolic system with boundary effect arising from tumor angiogenesis

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Abstract

This paper concerns a parabolic–hyperbolic system on the half space $\mathbb{R}_+$ with boundary effect. The system is derived from a singular chemotaxis model describing the initiation of tumor angiogenesis. We show that the solution of the system subject to appropriate boundary conditions converges to a traveling wave profile as time tends to infinity if the initial data is a small perturbation around the wave which is shifted far away from the boundary but its amplitude can be arbitrarily large.

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1. Introduction

To model the dynamics and interaction between signaling molecules vascular endothelial growth factor (VEGF) and vascular endothelial cells during the initiation of tumor angiogenesis, the following PDE–ODE hybrid model was proposed in [12]

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$$\begin{cases} u_t = (Du_x - \xi u(\ln c)_x)_x, \\ c_t = -\mu uc, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $c(x,t)$ denote the density of vascular endothelial cells and concentration of VEGF, respectively. The parameter $D > 0$ is the diffusivity of endothelial cells, $\xi > 0$ is referred to as the chemotactic coefficient measuring the intensity of chemotaxis and μ denotes the degradation rate of the chemical c. Here the chemical diffusion is neglected since it is far less important than its interaction with endothelial cells as treated in [12].

The striking feature of model (1.1) is that the first equation contains a logarithmic sensitivity function $\ln c$ which is singular at $c = 0$. This singular logarithmic sensitivity was first used by Keller and Segel in their original seminal paper [10] to describe the propagation of traveling wave band formed by bacterial chemotaxis observed in the experiment of Adler [1]. Its mathematical derivation was later given in [28] and biological relevance was provided in [9] by both experimental measurements and model simulations. Therefore the logarithm is a meaningful chemotactic sensitivity representation though it causes great challenges in its mathematical analysis and numerical computations. Hence among other things, the foremost mathematical question is how to resolve the logarithmic singularity in order to being able to carry the analysis forward. Toward this end, a Cole–Hopf type transformation as follows was used in [11,32]

$$v = -\frac{1}{\mu}(\ln c)_x = -\frac{1}{\mu}\frac{c_x}{c} \tag{1.2}$$

which transforms the system (1.1) into a parabolic–hyperbolic system:

$$\begin{cases} u_t - \chi(uv)_x = Du_{xx}, \\ v_t - u_x = 0, \end{cases} \tag{1.3}$$

where $\chi = \mu\xi > 0$. Apparently the transformed system (1.3) is much more manipulable mathematically than the original singular system (1.1) since the singularity vanishes. Therefore the Cole–Hopf transformation (1.2) is the key to open a door to study the singular system (1.1). On the other hand, as a newly derived system of conservation laws from biology, the system (1.3) itself is of great interest to study. There has been an amount of interesting works carried out for the transformed system (1.3). In the one dimensional whole space $\mathbb{R}$, the existence of traveling wavefront solutions of (1.3) was obtained first in [32] and nonlinear stability of traveling wave solutions with large wave amplitude was subsequently established by the third author with his collaborators in a series of works [8,18,19]. The stability of composite waves of (1.3) in $\mathbb{R}$ was proved in [16]. For the bounded domain, there are a few results obtained in [17,31,33] which showed that the asymptotic profile of solutions of (1.3) is a constant in one- and multidimensions if zero-flux boundary conditions are imposed. However it is still unknown how to prescribe the suitable boundary conditions to obtain a non-constant asymptotic profile (such as wave-like solution) for the model (1.3). In this paper, we shall make a step forward to this question by considering the asymptotic behavior of solutions of initial–boundary problem (1.3) in the half-space $\mathbb{R}_+ = [0,\infty)$ with the following initial data

$$(u,v)(x,0) = (u_0, v_0)(x), \quad x \in \mathbb{R}_+ \tag{1.4}$$

and boundary conditions:

$$u(0,t) = u_-, \quad (u,v)(\infty,t) = (u_+, v_+), \quad t \in \mathbb{R}_+, \tag{1.5}$$

where $u_\pm > 0$ due to the biological relevance. The main goal of this paper is to show that the solution of (1.3)–(1.5) with $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ converges to a (shifted) traveling wave profile as time tends to infinity. Our results may shed light on the way how to prescribe appropriate boundary conditions to obtain a non-constant wave profile in the bounded domain, which remains an interesting open question up to date.

The problem of the stability of traveling waves in the half-space $\mathbb{R}_+$ with the boundary effect has been an important topic of PDEs arising from fluid mechanics and gas dynamics. Liu and Yu in [23] first studied the scalar Burgers equation, followed a generalization by Liu and Nishihara in [22]. For the case of system, Matsumura and Mei [24] solved the viscous p-system for the first time. For the other relevant studies on the asymptotic stability of solutions with boundary effects, we refer to [5–7,25–27] and references therein. In this paper, we shall first employ the idea of [24] to identify the appropriate asymptotic wave profile of solutions of (1.3)–(1.5) and then use the method of energy estimates to show that the solution of (1.3)–(1.5) converges to the identified wave profile with a shift under suitable initial perturbations. Compared to the results and analysis of [24], there are two essential differences. First the wave strength in [24] was subject to certain conditions, but our results do not impose any condition on the wave strength and particularly hold for arbitrarily large wave strength. Second the nonlinear advection term of p-system considered in [24] has no interactive nonlinearity as in the model (1.3). Due to these distinctions, the analysis and estimates in our paper are much more complicated than those in [24]. Furthermore the idea of "constructing total differential" in the higher-order energy estimates used in [8] for the whole space $\mathbb{R}$ no longer applies due to the presence of boundaries. In this paper, we develop new ideas of "cancelation" (see the proof of Lemma 3.7) to establish the higher-order estimates and achieve our goal.

Before concluding the Introduction, we briefly mention some other results related to the system (1.3) below. First in the one dimensional bounded interval $\Omega \subset \mathbb{R}$, the global existence of solutions of (1.3) was first established in [33] for small data, and later in [31] the asymptotic behavior of solutions was established for large data. In the multidimensional bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, the global existence and exponential decay rates of solutions under Neumann boundary conditions were obtained in [17] for small data. In the one dimensional whole space $\mathbb{R}$, except afore-mentioned traveling wave solutions studied in [8,18,19,32], the global well-posedness of (1.3) was established in [3] for large data under the condition that v_0 has a positive lower bound. For the multidimensional whole space $\mathbb{R}^d$ $(d \geq 2)$, when the initial data is close to the constant ground state $(\bar{u}, 0)$, there are a few studies on the system (1.3). First in [13], the global well-posedness and regularity criterion of classical solutions of (1.3) were obtained if $(u_0, v_0) \in H^s(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$ and $\|(u_0 - \bar{u}, v_0)\|_{H^s}$ is small. Later the global existence of mild solutions in critical Besov space $\dot{B}_{2,1}^{-\frac{1}{2}} \times (\dot{B}_{2,1}^{-\frac{1}{2}})^d$ with minimal regularity was established in [4] in the Chemin–Lerner space framework. The global well-posedness of strong solutions of (1.3) in $\mathbb{R}^3$ was recently established in [2] by the Fourier analysis if $\|(u_0 - \bar{u}, v_0)\|_{L^2 \times H^1}$ is small, where algebraic decay rate of solutions was given under the additional condition that $\|(u_0 - \bar{u}, v_0)\|_{H^2 \times H^1}$ is small. Finally, we refer the readers to the works [14,15,20,29,31] where the chemical diffusion is incorporated.

The rest of paper is organized as follows. In Section 2, the existence and properties of traveling wave solutions of (1.3) in the whole space $\mathbb{R}$ will be studied first. Then we identify the asymptotic wave profile of solutions to the initial–boundary value problem (1.3)–(1.5) in the half space $\mathbb{R}_+$ and state our main results. In Section 3, we show the nonlinear stability of wave profiles of (1.3)–(1.5) and prove our main results.

2. Preliminaries and main results

We first explain some conventions used throughout the paper. C denotes a generic positive constant which can change from one line to another depending on the context. $H^k(\mathbb{R}_+)$ denotes the usual k-th order Sobolev space on $\mathbb{R}_+$ with norm $\|f\|_{H^k(\mathbb{R}_+)} := \left(\sum_{j=0}^{k}\int_{\mathbb{R}_+}|\partial_x^j f|^2 dx\right)^{1/2}$. For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}_+)}$ and $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R}_+)}$.

In this section, we shall present our main results concerning the asymptotic behavior of solutions of the initial boundary value problem (1.3)–(1.5). To this end, we first identify the appropriate asymptotic profile of solutions. We depart form the existence of traveling wave solutions of (1.3) in the whole space $\mathbb{R}$.

2.1. Traveling wave profiles

The traveling wave solution of (1.3) on $\mathbb{R}$ is a non-constant special solution $(U, V) \in C^\infty(\mathbb{R})$ in the form of

$$(u, v)(x, t) = (U, V)(z), \; z = x - st,$$

which satisfies

$$\begin{cases} -sU' - \chi(UV)' = DU'', \\ -sV' - U' = 0, \end{cases} \tag{2.1}$$

with boundary condition

$$U(\pm\infty) = u_\pm, \; V(\pm\infty) = v_\pm, \tag{2.2}$$

where $' = \frac{d}{dz}$ and s is the wave speed. Here we require $u_\pm \geq 0$ due to the biological interest. Integrating (2.1) in z over $\mathbb{R}$ yields the Rankine–Hugoniot condition as follows

$$\begin{cases} -s(u_+ - u_-) - \chi(u_+v_+ - u_-v_-) = 0, \\ -s(v_+ - v_-) - (u_+ - u_-) = 0, \end{cases} \tag{2.3}$$

which gives

$$s^2 + \chi v_+ s - \chi u_- = 0. \tag{2.4}$$

In this paper, we only consider the case $s > 0$ and the results for $s < 0$ follow similarly. Solving (2.4) for s yields that

$$s = \frac{-\chi v_+ + \sqrt{(\chi v_+)^2 + 4\chi u_-}}{2}. \tag{2.5}$$

Then the existence of traveling wave solutions of (1.3) in $\mathbb{R}$ is given as follows.

Proposition 2.1. *Assume that $u_\pm$ and $v_\pm$ satisfy (2.3). Then the system (2.1) admits a unique (up to a translation) monotone traveling wave solution $(U, V)(x - st)$ with the wave speed s given by (2.5), which satisfies:*

$$U' < 0, \quad V' > 0, \tag{2.6}$$

and the following asymptotic decay rates at far field:

$$\begin{aligned} |U(z) - u_\pm| &\sim |u_- - u_+|e^{-\lambda|z|}, \quad z \to \pm\infty, \\ |V(z) - v_\pm| &\sim |u_- - u_+|e^{-\lambda|z|}, \quad z \to \pm\infty, \end{aligned} \tag{2.7}$$

where

$$\lambda = \frac{\chi(u_- - u_+)}{Ds} > 0. \tag{2.8}$$

Proof. Integrating the second equation of (2.1), one has that

$$sV + U = \varrho_1 = sv_+ + u_+ = sv_- + u_-. \tag{2.9}$$

Substituting (2.9) into the first equation of (2.1), we obtain a unique solution $(U(z), V(z))$ up to a translation which is explicitly given as (see the details in [8])

$$U(z) = u_+ - \frac{u_+ - u_-}{e^{\lambda z} + 1}, \quad V(z) = \frac{\varrho_1 - U}{s} = v_- + \frac{(u_- - u_+)e^{\lambda z}}{s(e^{\lambda z} + 1)}, \tag{2.10}$$

where λ is defined in (2.8). Further calculations give rise to

$$U' = \frac{\lambda(u_+ - u_-)e^{\lambda z}}{(e^{\lambda z} + 1)^2}, \qquad V' = -\frac{U'}{s}.$$

Noticing that $s > 0$, we can easily find that $U' < 0$, $V' > 0$, which leads to

$$u_+ < U(z) < u_-, \qquad v_- < V(z) < v_+. \tag{2.11}$$

Moreover, simple calculations yield

$$\begin{aligned} |U(z) - u_+| &= \left|\frac{u_+ - u_-}{e^{\lambda z} + 1}\right| \sim |u_- - u_+|e^{-\lambda z}, \quad \text{as } z \to \infty, \\ |U(z) - u_-| &= \left|\frac{(u_- - u_+)e^{\lambda z}}{e^{\lambda z} + 1}\right| \sim |u_- - u_+|e^{\lambda z}, \quad \text{as } z \to -\infty. \end{aligned}$$

The above two results can be combined as

$$|U(z) - u_\pm| \sim |u_- - u_+|e^{-\lambda|z|}, \quad \text{as } z \to \pm\infty. \tag{2.12}$$

In a similar way, we get that

$$|V(z) - v_\pm| \sim |u_- - u_+|e^{-\lambda|z|}, \quad \text{as } z \to \pm\infty.$$

The proof of lemma is complete. □

Remark 2.1. The traveling wave solutions of the parabolic–hyperbolic system (1.3) in $\mathbb{R}$ obtained in Proposition 2.1 are mathematically valid for any $u_- > 0$ and $u_+, v_\pm \in \mathbb{R}$. In this paper, we shall consider the case $u_- > 0, u_+ > 0, v_- = 0$ and explore the asymptotic behavior of solutions to the transformed system (1.3) in the half space with boundary conditions given by (1.5). The initial–boundary value problem of (1.3)–(1.5) in the half space $\mathbb{R}_+$ for other values of u_+ and $v_\pm$ remains unsolved in the present paper due to the technical difficulty. However if the results were transferred to original system (1.1) via the Cole–Hopf transformation (1.2), one finds that the biologically meaningful traveling wave solutions of (1.1) exist if and only if $u_- > 0, v_- < 0$ and $u_+ = v_+ = 0$ (see the details in [21]).

2.2. *Asymptotic profile*

In [8], it was shown that if the initial date is a small perturbation of the traveling wave profile $(V, U)(x - st)$, then the solution of (1.3) in $\mathbb{R}$ approaches the shifted wave profile $(V, U)(x - x_0 + st)$ as time tends to infinity where the shift x_0 is determined by the initial date. For the problem in the half space $\mathbb{R}_+$ considered in the present paper, the value of traveling profile $(V, U)(x - st)$ at the boundary $x = 0$ is always less than u_-. This generates an initial boundary layer $(u - U)|_{(x,t)=(0,0)} = u_- - U(0)$, which could make the solutions fail to converge to a shifted wave profile $(V, U)(x - st - x_0)$ in general. In order to get convergence, it is natural to take a shift $\beta \gg 1$ such that the initial boundary layer around the shifted wave $|(u(x, t) - U(x - st - \beta)|_{(x,t)=(0,0)}| = |u_- - U(-\beta)| \ll 1$. With this treatment, one may expect that the solution of (1.3) will asymptotically approach the wave profile $U(x - st - \beta - \alpha)$ with a shift α (comparable with x_0 above) if the initial data is a small perturbation of the shifted wave profile $U(x - st - \beta)$. One key question in this argument is how to determine the shift α for a given sufficiently large shift β. In the following, inspired by the idea in [24], we shall clarify the relation between α and β.

First from the second equation of (1.3), we have

$$(v - V)_t = (u - U)_x, \quad (U, V) = (U, V)(x - st + \alpha - \beta). \tag{2.13}$$

Integrating (2.13) over $\mathbb{R}_+$ with respect to x and using the boundary condition (1.5), we have

$$\frac{d}{dt}\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx = (u - U)|_{x=0}^\infty = U(-st + \alpha - \beta) - u_-. \tag{2.14}$$

Integrating (2.14) with respect to t, we get

$$\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx$$

$$= \int_0^\infty [v_0(x) - V(x + \alpha - \beta)]dx + \int_0^t [U(-s\tau + \alpha - \beta) - u_-]d\tau. \tag{2.15}$$

By the idea of conservation of mass principle (e.g. see [30]), we are looking for α such that

$$\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx \to 0 \quad \text{as} \quad t \to \infty. \tag{2.16}$$

Then, we set

$$I(\alpha) := \int_0^\infty [v_0(x) - V(x + \alpha - \beta)]dx + \int_0^\infty [U(-st + \alpha - \beta) - u_-]dt. \tag{2.17}$$

From (2.15) and (2.16), we see that the shift α satisfies $I(\alpha) = 0$. Differentiating (2.17) with respect to α, we have

$$\begin{aligned} I'(\alpha) &= -\int_0^\infty V'(x + \alpha - \beta)dx + \int_0^\infty U'(-st + \alpha - \beta)dt \\ &= -[v_+ - V(\alpha - \beta)] - \frac{1}{s}[u_- - U(\alpha - \beta)] \\ &= -v_+ - \frac{u_-}{s} + v_- + \frac{u_-}{s} = -v_+, \end{aligned} \tag{2.18}$$

where we have used (2.9) and $v_- = 0$. Then, integrating (2.18) in α over $(0, \alpha)$ gives

$$I(\alpha) = I(0) - v_+\alpha = \int_0^\infty [v_0(x) - V(x - \beta)]dx + \int_0^\infty [U(-st - \beta) - u_-]dt - v_+\alpha. \tag{2.19}$$

Note that $I(\alpha) = 0$. Then the shift $\alpha = \alpha(\beta)$ is determined explicitly by

$$\alpha := \frac{1}{v_+}\left\{\int_0^\infty [v_0(x) - V(x - \beta)]dx + \int_0^\infty [U(-st - \beta) - u_-]dt\right\}. \tag{2.20}$$

This, combined with (2.15) and (2.16), gives

$$\begin{aligned} &\int_0^\infty [v(x,t) - V(x - st + \alpha - \beta)]dx \\ &= I(\alpha) - \int_t^\infty [U(-s\tau + \alpha - \beta) - u_-]d\tau \end{aligned}$$

$$= -\int_t^{\infty} [U(-s\tau + \alpha - \beta) - u_-] d\tau \to 0 \quad \text{as} \quad t \to \infty. \tag{2.21}$$

This implies from (2.15) that

$$\int_0^{\infty} [v_0(x) - V(x + \alpha - \beta)] dx = -\int_0^{\infty} [U(-s\tau + \alpha - \beta) - u_-] d\tau.$$

Thus, by such a heuristical analysis, the expected asymptotic profile for the IBVP (1.3)–(1.5) is the selected pair of traveling waves $(V, U)(x - st + \alpha - \beta)$ with $\beta \gg 1$ and $\alpha = \alpha(\beta) \ll 1$. In fact, this is true as given in the following theorem which is the main result of this paper.

Theorem 2.2. *Let $u_+ > 0$, $v_- = 0$ and β be a positive constant. Then there exists a constant $\delta_0 > 0$ such that if*

$$\|\Phi_0\|_2 + \|\Psi_0\|_2 + \beta^{-1} \le \delta_0, \tag{2.22}$$

where

$$(\Phi_0, \Psi_0)(x) = -\int_x^{\infty} (u_0(y) - U(y - \beta), v_0(y) - V(y - \beta)) dy, \tag{2.23}$$

the initial–boundary value problem (1.3)–(1.5) has a unique global solution $(u, v)(x, t)$ satisfying

$$u(x,t) - U(x - st + \alpha - \beta) \in C([0, \infty); H^1) \cap L^2((0, \infty); H^2),$$
$$v(x,t) - V(x - st + \alpha - \beta) \in C([0, \infty); H^1) \cap L^2((0, \infty); H^1)$$

where α is a shift constant determined by (2.20). Furthermore, the solution has the following asymptotic stability:

$$\sup_{x \in \mathbb{R}_+} |(u, v)(x,t) - (U, V)(x - st + \alpha - \beta)| \to 0, \quad as \quad t \to \infty.$$

3. Proof of Theorem 2.2

3.1. Reformulation of the problem

This section is devoted to proving Theorem 2.2. Since (1.3) is a system of conservation, we employ the technique of taking antiderivative to define the perturbation functions as follows:

$$(\phi(x,t), \psi(x,t)) = -\int_x^{\infty} (u(y,t) - U(y - st + \alpha - \beta), v(y,t) - V(y - st + \alpha - \beta)) dy$$

for $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$. That is

$$(u, v)(x, t) = (U, V)(x - st + \alpha - \beta) + (\phi_x, \psi_x)(x, t). \tag{3.1}$$

Substituting (3.1) into (1.3), using (2.1) and integrating the system with respect to x, we obtain that $(\phi, \psi)(x, t)$ satisfies

$$\begin{cases} \phi_t = D\phi_{xx} + \chi V \phi_x + \chi U \psi_x + \chi \phi_x \psi_x, \quad t > 0, \quad x \in \mathbb{R}_+, \\ \psi_t = \phi_x, \end{cases} \tag{3.2}$$

with initial perturbation

$$(\phi_0, \psi_0)(x) = -\int_x^\infty (u_0(y) - U(y + \alpha - \beta), v_0(y) - V(y + \alpha - \beta))dy, \tag{3.3}$$

and boundary condition

$$\psi|_{x=0} = \int_t^\infty [U(-st + \alpha - \beta) - u_-]d\tau = A(t), \tag{3.4}$$

where (2.21) has been used.

We look for solutions of the system (3.2) in the following solution space:

$$\begin{aligned} X(0, T) := \{\, (\phi(x, t), \psi(x, t)) \big| &\phi \in C([0, T]; H^2), \phi_x \in L^2((0, T); H^2) \\ &\psi \in C([0, T]; H^2), \psi_x \in C([0, T]; H^1) \cap L^2((0, T); H^1)\}. \end{aligned}$$

Set

$$N(t) := \sup_{\tau \in [0,t]} (\|\psi(\cdot, \tau)\|_2 + \|\phi(\cdot, \tau)\|_2).$$

By the Sobolev embedding theorem, we have

$$\sup_{\tau \in [0,t]} \{\|\phi(\cdot, \tau)\|_{L^\infty}, \|\phi_x(\cdot, \tau)\|_{L^\infty}, \|\psi(\cdot, \tau)\|_{L^\infty}, \|\psi_x(\cdot, \tau)\|_{L^\infty}\} \le N(t). \tag{3.5}$$

For the problem (3.2)–(3.4), we have the following results.

Theorem 3.1. *Let $u_+ > 0$, $v_- = 0$. Then there exists a positive constant ε_0, such that if $N(0) + \beta^{-1} \le \varepsilon_0$, then the problem (3.2)–(3.4) has a unique global solution $(\phi, \psi) \in X([0, \infty))$ such that*

$$\begin{aligned} \|\phi\|_2^2 + \|\psi\|_2^2 + \int_0^t (\|\phi_x(\tau)\|_2^2 + \|\psi_x(\tau)\|_1^2)d\tau &\le C\Big(\|\phi_0\|_2^2 + \|\psi_0\|_2^2 + e^{-\lambda\beta}\Big) \\ &\le C(N^2(0) + e^{-\lambda\beta}), \end{aligned} \tag{3.6}$$

for any $t \in [0, \infty)$. *Moreover, it follows that*

$$\sup_{x \in \mathbb{R}_+} |(\phi_x, \psi_x)(x,t)| \to 0 \ \text{ as } \ t \to \infty. \tag{3.7}$$

Note that the initial conditions in Theorem 2.2 and Theorem 3.1 are slightly different. The following lemmas reveal the relation between them.

Lemma 3.2. *Let* (2.22) *hold. Then* $\alpha \to 0$ *provided that* $\|\Psi_0\|_2 \to 0$ *and* $\beta \to \infty$.

Proof. From (2.10) and (2.12), it follows that

$$0 < u_- - U(-st - \beta) \le Ce^{-\lambda(st+\beta)}.$$

This gives $|\int_0^\infty [u_- - U(-st-\beta)]dt| \le Ce^{-\lambda\beta}$. It follows from (2.20) that

$$\begin{aligned} |\alpha| &\le \frac{1}{v_+}\left\{\left|\int_0^\infty [v_0(x) - V(x-\beta)]\right| dx + \left|\int_0^t [U(-st-\beta) - u_-]dt\right|\right\} \\ &\le C(|\Psi_0(0)| + e^{-\lambda\beta}) \le C(\|\Psi_0\|_2 + e^{-\lambda\beta}) \to 0, \end{aligned}$$

as $\|\Psi_0\|_2 \to 0$ and $\beta \to \infty$. $\square$

Lemma 3.3. *Let* (2.22) *hold. Then* $\|\phi_0\|_2 + \|\psi_0\|_2 \to 0$ *if* $\|\Phi_0\|_2 + \|\Psi_0\|_2 \to 0$ *and* $\beta \to \infty$.

Proof. By (2.23) and (3.3), we have

$$\begin{aligned} \phi_0(x) &= -\int_x^\infty [u_0(y) - U(y+\alpha-\beta)]dy \\ &= \Phi_0(x) + \int_x^\infty [U(y+\alpha-\beta) - U(y-\beta)]dy \\ &= \Phi_0(x) + \int_x^\infty \int_0^\alpha U'(y+\theta-\beta)d\theta dy \\ &= \Phi_0(x) + \int_0^\alpha [u_+ - U(x+\theta-\beta)]d\theta. \end{aligned} \tag{3.8}$$

Notice that (2.7) yields

$$|u_+ - U(x+\theta-\beta)| \le Ce^{-\lambda|x+\theta-\beta|} \le Ce^{-\lambda|x-\beta|}.$$

Set $B(x) = \int_0^\alpha [u_+ - U(x+\theta-\beta)]d\theta$. Then we have

$$
\begin{aligned}
\|B\|^2 &\le C\alpha^2 \int_0^\infty e^{-2\lambda|x-\beta|}dx \\
&\le C\alpha^2 \int_0^\beta e^{-2\lambda(\beta-x)}dx + C\alpha^2 \int_\beta^\infty e^{-2\lambda(x-\beta)}dx \\
&\le \frac{C\alpha^2}{2\lambda}(2-e^{-2\lambda\beta}) \\
&\le C\alpha^2,
\end{aligned} \tag{3.9}
$$

where Lemma 3.2 has been used and C is independent of α and β. Similarly, we can obtain that $\|B\|_2 \le C|\alpha|$. This, together with (3.8) and Lemma 3.2 gives

$$\|\phi_0\|_2 \le C(\|\Phi_0\|_2 + \|B\|_2) \le C(\|\Phi_0\|_2 + |\alpha|),$$

which goes to zero as $\beta \to \infty$ and $\|\Phi_0(x)\|_2 \to 0$. In the same way, we can get that

$$\|\psi_0\|_2 \to 0$$

provided $\|\Psi_0\|_2 \to 0$ and $\beta \to \infty$. Thus, the proof of Lemma 3.3 is complete. $\square$

Theorem 2.2 is a consequence of Theorem 3.1 and Lemma 3.3. Hence it remains to prove Theorem 3.1 which follows from the local existence theorem and the a *priori* estimates given below.

Proposition 3.4 *(Local existence). Suppose that the assumptions in Lemma 3.1 hold. For any $\varepsilon_1 > 0$, there exists a positive constant T_0 depending on ε_1 such that if $(\phi_0, \psi_0) \in H^2$ with $N(0) + \beta^{-1} \le \varepsilon_1$, then the problem (3.2)–(3.4) has a unique solution $(\phi, \psi) \in X(0, T_0)$ satisfying $N(t) \le 2\varepsilon_1$ for any $0 \le t \le T_0$.*

Proposition 3.5 *(A priori estimate). Assume that $(\phi, \psi) \in X(0, T)$ is a solution obtained in Proposition 3.4 for a positive constant T. Then there is a positive constant $\varepsilon_2 > 0$, independent of T, such that if*

$$N(t) \le \varepsilon_2$$

for any $0 \le t \le T$, then the solution (ϕ, ψ) of (3.2)–(3.4) satisfies (3.6) for any $0 \le t \le T$.

The local existence in Proposition 3.4 can be proved using the standard fixed point theorem and we omit the details for brevity. Proposition 3.5 is the key to establish Theorem 3.1. Next we focus on proving Proposition 3.5 by the energy estimates.

Due to Lemma 3.2 and the conditions in Theorem 2.2, in the sequel we may assume, without loss of generality, that $\beta > 1$ and $|\alpha| < 1$. Since $N(t)$ is small (see Proposition 3.5), we assume that $N(t) < 1$ in the following.

3.2. Boundary estimates

To derive the *a priori* estimate, we first give boundary estimates.

Lemma 3.6. *Assume that $u_+ > 0, v_- = 0$. Let (ϕ, ψ) be a solution of (3.2)–(3.4). Then the following boundary estimates hold*

$$\left| \int_0^t \left(\chi\phi\psi + \frac{D\phi\phi_x}{U} + \frac{DU_x\phi^2}{2U^2} + \frac{\chi V\phi^2}{2U} \right) \Big|_{x=0} d\tau \right| \le Ce^{-\lambda\beta}, \tag{3.10}$$

$$\left| \int_0^t \left(\frac{\phi_t\phi_x}{U} + \chi\psi_t\psi_x + \frac{DU_x\phi_x^2}{2U^2} - \chi\phi_x\psi_x - \frac{\chi V\phi_x^2}{2U} \right) \Big|_{x=0} d\tau \right| \le Ce^{-\lambda\beta}, \tag{3.11}$$

$$\left| \int_0^t \left(\phi_x\phi_{xx} + \frac{\phi_{xt}\phi_{xx}}{U} \right) |_{x=0} d\tau \right| \le Ce^{-\lambda\beta}, \tag{3.12}$$

where λ is defined in (2.8).

Proof. From the second equation of (3.2) and the boundary condition (3.4), we have

$$\psi_t|_{x=0} = \phi_x|_{x=0} = u_- - U(-st + \alpha - \beta). \tag{3.13}$$

Thus, by the facts $|-st + \alpha - \beta| = st + \beta - \alpha$ because $s > 0$ and $\beta > \alpha$, and Proposition 2.1, we have

$$|U(-s\tau + \alpha - \beta) - u_-| \le Ce^{-\lambda|-st+\alpha-\beta|} \le Ce^{-\lambda(\beta-\alpha)}e^{-\lambda st} \le Ce^{-\lambda\beta}e^{-\lambda st},$$

and

$$|\psi(0,t)| = |\psi|_{x=0}| = |A(t)| \le Ce^{-\lambda\beta}e^{-\lambda st}. \tag{3.14}$$

Since $\phi_x|_{x=0} = \psi_t|_{x=0} = A'(t)$, we have $\phi_{xt}|_{x=0} = A''(t)$ and conclude that $A(t) \in W^{3,1}(0,\infty)$ and

$$\left| \frac{d^k}{dt^k} A(t) \right| \le Ce^{-\lambda\beta}e^{-\lambda st}, \quad k = 0, 1, 2, 3,$$
$$\|A(t)\|_{W^{3,1}(0,\infty)} \le Ce^{-\lambda\beta}. \tag{3.15}$$

It follows from (3.5) that

$$|\phi(0,t)| \le \sup_{x\in\mathbb{R}^+} |\phi(x,t)| \le CN(t),$$
$$|\phi_x(0,t)| + |\psi_x(0,t)| \le \sup_{x\in\mathbb{R}^+} |\phi_x(x,t)| + \sup_{x\in\mathbb{R}^+} |\psi_x(x,t)| \le CN(t). \tag{3.16}$$

On the other hand, since $-st+\alpha-\beta<0$ by $\beta>|\alpha|$ and $u_\pm>0$ then

$$U(-st+\alpha-\beta)>U(0)=\frac{u_-+u_+}{2}>0,$$

where we have used the monotonicity of $U(z)$ and (2.10). This means

$$\frac{1}{U(-st+\alpha-\beta)}\le\frac{1}{U(0)}\le C. \tag{3.17}$$

Furthermore, using (2.10) with $v_-=0$, we have

$$U'(-st+\alpha-\beta)=\frac{\lambda(u_+-u_-)e^{\lambda(-st+\alpha-\beta)}}{(e^{\lambda(-st+\alpha-\beta)}+1)^2}\le Ce^{-\lambda\beta}e^{-\lambda st}$$

and

$$V(-st+\alpha-\beta)=\frac{(u_--u_+)e^{\lambda(-st+\alpha-\beta)}}{s(e^{\lambda(-st+\alpha-\beta)}+1)}\le Ce^{-\lambda\beta}e^{-\lambda st}.$$

The above two inequalities lead to

$$\int_0^t|U'(-s\tau+\alpha-\beta)|d\tau\le Ce^{-\lambda\beta} \tag{3.18}$$

and

$$\int_0^t|V(-s\tau+\alpha-\beta)|d\tau\le Ce^{-\lambda\beta}. \tag{3.19}$$

Next, let us give the proofs of (3.10)–(3.12). Using (3.14)–(3.16), we have

$$\left|\int_0^t\phi\psi|_{x=0}d\tau\right|\le C\int_0^t|A(\tau)|\,|\phi(0,\tau)|\,d\tau\le Ce^{-\lambda\beta}. \tag{3.20}$$

In a similar way, we get

$$\left|\int_0^t(\phi\phi_x)|_{x=0}d\tau\right|\le C\int_0^t\left|A'(\tau)\right|\,|\phi(0,\tau)|\,d\tau\le Ce^{-\lambda\beta} \tag{3.21}$$

and

$$\left|\int_0^t(\psi_t\psi_x)|_{x=0}d\tau\right|=\left|\int_0^t(\phi_x\psi_x)|_{x=0}d\tau\right|\le C\int_0^t\left|A'(\tau)\right|\,|\psi_x(0,\tau)|\,d\tau\le Ce^{-\lambda\beta}. \tag{3.22}$$

Using (3.15)–(3.19), we get

$$\left|\int_0^t \frac{\phi\phi_x}{U}|_{x=0}d\tau\right| \le C\int_0^t \left|A'(\tau)\right| |\phi(0,\tau)|\,d\tau \le Ce^{-\lambda\beta}, \tag{3.23}$$

$$\left|\int_0^t \frac{V\phi^2}{U}|_{x=0}d\tau\right| \le C\int_0^t |V(-s\tau+\alpha-\beta)| \left|\phi^2(0,\tau)\right| d\tau \le Ce^{-\lambda\beta}, \tag{3.24}$$

$$\left|\int_0^t \frac{U_x\phi^2}{U^2}|_{x=0}d\tau\right| \le C\int_0^t \left|U'(-s\tau+\alpha-\beta)\right| \left|\phi^2(0,\tau)\right| d\tau \le Ce^{-\lambda\beta}, \tag{3.25}$$

$$\left|\int_0^t \frac{V\phi_x^2}{U}|_{x=0}d\tau\right| \le C\int_0^t |V(-s\tau+\alpha-\beta)| \left|\phi_x^2(0,\tau)\right| d\tau \le Ce^{-\lambda\beta}, \tag{3.26}$$

$$\left|\int_0^t \frac{U_x\phi_x^2}{U^2}|_{x=0}d\tau\right| \le C\int_0^t \left|U'(-s\tau+\alpha-\beta)\right| \left|\phi_x^2(0,\tau)\right| d\tau \le Ce^{-\lambda\beta}. \tag{3.27}$$

Then (3.10) follows from (3.20) and (3.22)–(3.25). To prove other boundary estimates, we make use of $\psi_{xt} = \phi_{xx}$ (see the second equation of (3.2)), integration by parts, and (3.15)–(3.17) to get

$$\begin{aligned}
\left|\int_0^t \frac{\phi_t\phi_x}{U}|_{x=0}d\tau\right| &\le C\left|\int_0^t \frac{A'(\tau)\phi_t(0,\tau)}{U(-s\tau+\alpha-\beta)}d\tau\right| \\
&\le C\left|\int_0^t \left\{\frac{A'(\tau)\phi(0,\tau)}{U(-s\tau+\alpha-\beta)}\right\}_t d\tau\right| + C\left|\int_0^t \frac{A''(\tau)\phi(0,\tau)}{U(-s\tau+\alpha-\beta)}d\tau\right| \\
&\quad + C\left|\int_0^t \frac{s\phi(0,\tau)A'(\tau)U'(-s\tau+\alpha-\beta)}{U^2(-s\tau+\alpha-\beta)}d\tau\right| \\
&\le C\left|\frac{A'(t)\phi(0,t)}{U(-st+\alpha-\beta)}\right| + C\left|\frac{A'(0)\phi_0(0)}{U(\alpha-\beta)}\right| \\
&\quad + C\int_0^t \left(\left|A''(\tau)\right| + \left|A'(\tau)\right|\right) |\phi(0,\tau)|\,d\tau \\
&\le CN(t)\left[|A'(t)| + |A'(0)| + \int_0^t \left(\left|A''(\tau)\right| + \left|A'(\tau)\right|\right) d\tau\right] \\
&\le Ce^{-\lambda\beta}.
\end{aligned} \tag{3.28}$$

Thus (3.11) results from (3.22) and (3.26)–(3.28). Following a process similar to (3.28), we can derive

$$\left|\int_0^t (\phi_x\phi_{xx})|_{x=0}d\tau\right| \le Ce^{-\lambda\beta} \quad \text{and} \quad \left|\int_0^t \frac{\phi_{xt}\phi_{xx}}{U}|_{x=0}d\tau\right| \le Ce^{-\lambda\beta},$$

which lead to (3.12). This completes the proof of Lemma 3.6. □

Then the L^2-estimate is given as follows.

Lemma 3.7. *Let the assumptions in Proposition 3.5 hold. Then there exists a constant $C > 0$ such that*

$$\|\phi\|^2 + \|\psi\|^2 + \int_0^t \|\phi_x\|^2\, d\tau \le C\left(\|\phi_0\|^2 + \|\psi_0\|^2 + e^{-\lambda\beta} + N(t)\int_0^t\int_0^\infty \psi_x^2 dxd\tau\right). \quad (3.29)$$

Proof. Multiplying the first equation of (3.2) by ϕ/U and the second by $\chi\psi$ and adding these equalities, we obtain

$$\frac{1}{2}\left(\frac{\phi^2}{U}\right)_t - \frac{\phi^2}{2}\left(\frac{1}{U}\right)_t + \left(\frac{\chi\psi^2}{2}\right)_t = \frac{D\phi\phi_{xx}}{U} + \chi\,(\phi\psi)_x + \frac{\chi V\phi\phi_x}{U} + \frac{\chi\phi\phi_x\psi_x}{U}$$

Noting that

$$\frac{\phi^2}{2}\left(\frac{1}{U}\right)_t = -\frac{s\phi^2}{2}\left(\frac{1}{U}\right)_x,$$

$$\frac{\phi\phi_{xx}}{U} = \left(\frac{\phi\phi_x}{U}\right)_x - \frac{\phi_x^2}{U} - \phi\phi_x\left(\frac{1}{U}\right)_x = \left(\frac{\phi\phi_x}{U}\right)_x - \frac{\phi_x^2}{U} - \left(\frac{\phi^2}{2}\left(\frac{1}{U}\right)_x\right)_x + \frac{\phi^2}{2}\left(\frac{1}{U}\right)_{xx},$$

$$\frac{V\phi\phi_x}{U} = \frac{1}{2}\left(\frac{V\phi^2}{U}\right)_x - \frac{\phi^2}{2}\left(\frac{V}{U}\right)_x,$$

we get

$$\begin{aligned}\frac{1}{2}\left(\frac{\phi^2}{U} + \chi\psi^2\right)_t + \frac{D\phi_x^2}{U} &= \left(\chi\phi\psi + \frac{D\phi\phi_x}{U} + \frac{DU_x\phi^2}{2U^2} + \frac{\chi V\phi^2}{2U}\right)_x \\ &\quad + \frac{\phi^2}{2}\left[\left(\frac{D}{U}\right)_{xx} - \left(\frac{s+\chi V}{U}\right)_x\right] + \frac{\chi\phi\phi_x\psi_x}{U}. \end{aligned} \quad (3.30)$$

By using (2.1) and the fact that $U_x < 0$ and $0 < u_+ \le U \le u_-$, it can be checked that

$$\left(\frac{D}{U}\right)_{xx} - \left(\frac{s+\chi V}{U}\right)_x = \frac{2u_+}{U^3}(s + \chi v_+)\cdot U_x < 0. \quad (3.31)$$

Substituting (3.31) into (3.30) and integrating the equation over $[0, \infty) \times [0, t]$, we derive

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty \left(\frac{\phi^2}{U}+\chi\psi^2\right)dx + D\int_0^t\int_0^\infty \frac{\phi_x^2}{U}dxd\tau \\
&= \frac{1}{2}\int_0^\infty \left(\frac{\phi_0^2}{U}+\chi\psi_0^2\right)dx - \int_0^t \left(\chi\phi\psi + \frac{D\phi\phi_x}{U} + \frac{DU_x\phi^2}{2U^2} + \frac{\chi V\phi^2}{2U}\right)|_{x=0}d\tau \\
&\quad + \chi\int_0^t\int_0^\infty \frac{\phi_x\psi_x\phi}{U}dxd\tau \\
&\le \frac{\chi}{2}\|\psi_0\|^2 + C\|\phi_0\|^2 + \left|\int_0^t \left(\chi\phi\psi + \frac{D\phi\phi_x}{U} + \frac{DU_x\phi^2}{2U^2} + \frac{\chi V\phi^2}{2U}\right)|_{x=0}d\tau\right| \\
&\quad + \frac{DN(t)}{2}\int_0^t\int_0^\infty \frac{\phi_x^2}{U}dxd\tau + \frac{N(t)\chi^2}{2D}\int_0^t\int_0^\infty \frac{\psi_x^2}{U}dxd\tau,
\end{aligned}
$$

where we have used the fact that $\|\phi(\cdot,t)\|_{L^\infty} \le N(t)$ by (3.5). Then, using (3.10) and $0 < u_+ \le U \le u_-$, we obtain (3.29) and the proof of Lemma 3.7 is complete. □

The next lemma gives the estimate of the first order derivatives of (ϕ, ψ).

Lemma 3.8. *Let the assumptions in Proposition 3.5 hold. Then there exists a constant $C > 0$ such that*

$$
\|\phi\|_1^2 + \|\psi\|_1^2 + \int_0^t \left(\|\phi_x\|_1^2 + \|\psi_x\|^2\right)d\tau \le C\left(\|\phi_0\|_1^2 + \|\psi_0\|_1^2 + e^{-\lambda\beta}\right). \tag{3.32}
$$

Proof. Multiplying the first equation of (3.2) by $-\phi_{xx}/U$ and the second by $-\chi\psi_{xx}$ and adding these equalities, we obtain

$$
-\frac{\phi_t\phi_{xx}}{U} - \chi\psi_t\psi_{xx} = -\frac{D\phi_{xx}^2}{U} - \chi(\phi_x\psi_x)_x - \frac{\chi V\phi_x\phi_{xx}}{U} - \frac{\chi\phi_x\psi_x\phi_{xx}}{U}.
$$

Simple calculations give us that

$$
\begin{aligned}
-\frac{\phi_t\phi_{xx}}{U} &= -\left(\frac{\phi_t\phi_x}{U}\right)_x + \left(\frac{\phi_t}{U}\right)_x\phi_x \\
&= -\left(\frac{\phi_t\phi_x}{U}\right)_x + \frac{\phi_{xt}\phi_x}{U} + \left(\frac{1}{U}\right)_x\phi_t\phi_x \\
&= -\left(\frac{\phi_t\phi_x}{U}\right)_x + \left(\frac{\phi_x^2}{2U}\right)_t + \left(\frac{1}{U}\right)_x\frac{s\phi_x^2}{2} + \underbrace{\left(\frac{1}{U}\right)_x\phi_t\phi_x}_{I},
\end{aligned}
$$

$$
\begin{aligned}
I &= \left(\frac{1}{U}\right)_x \phi_x \left(D\phi_{xx} + \chi V\phi_x + \chi U\psi_x + \chi\phi_x\psi_x\right) \\
&= \left(\frac{D\phi_x^2}{2}\left(\frac{1}{U}\right)_x\right)_x - \frac{D\phi_x^2}{2}\left(\frac{1}{U}\right)_{xx} + \chi V\left(\frac{1}{U}\right)_x \phi_x^2 \\
&\quad + \chi U\left(\frac{1}{U}\right)_x \psi_x\phi_x + \chi\left(\frac{1}{U}\right)_x \phi_x^2\psi_x,
\end{aligned}
$$

$$
-\psi_t\psi_{xx} = -(\psi_t\psi_x)_x + \left(\frac{\psi_x^2}{2}\right)_t,
$$

$$
-\frac{V\phi_x\phi_{xx}}{U} = -\frac{1}{2}\left(\frac{V\phi_x^2}{U}\right)_x + \frac{\phi_x^2}{2}\left(\frac{V}{U}\right)_x.
$$

Thus we get from the above inequalities that

$$
\begin{aligned}
\frac{1}{2}\left(\frac{\phi_x^2}{U} + \chi\psi_x^2\right)_t + \frac{D\phi_{xx}^2}{U} &= \left(\frac{\phi_t\phi_x}{U} + \chi\psi_t\psi_x + \frac{DU_x\phi_x^2}{2U^2} - \chi\phi_x\psi_x - \frac{\chi V\phi_x^2}{2U}\right)_x \\
&\quad + \frac{\phi_x^2}{2}\left[\left(\frac{D}{U}\right)_{xx} - \left(\frac{s+\chi V}{U}\right)_x\right] + \frac{\chi V_x\phi_x^2}{U} \\
&\quad - \chi U\left(\frac{1}{U}\right)_x \psi_x\phi_x - \chi\left(\frac{1}{U}\right)_x \phi_x^2\psi_x - \frac{\chi\phi_x\psi_x\phi_{xx}}{U}.
\end{aligned} \tag{3.33}
$$

Integrating (3.33) over $[0,\infty)\times[0,t]$ and using (3.31), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty \left(\frac{\phi_x^2}{U} + \chi\psi_x^2\right)dx + D\int_0^t\int_0^\infty \frac{\phi_{xx}^2}{U}dxd\tau \\
&= \frac{1}{2}\int \left(\frac{\phi_{x0}^2}{U} + \chi\psi_{x0}^2\right)dx - \int_0^t \left(\frac{\phi_t\phi_x}{U} + \chi\psi_t\psi_x + \frac{DU_x\phi_x^2}{2U^2} - \chi\phi_x\psi_x - \frac{\chi V\phi_x^2}{2U}\right)|_{x=0}d\tau \\
&\quad + \chi\int_0^t\int_0^\infty \frac{V_x\phi_x^2}{U}dxd\tau + \chi\int_0^t\int_0^\infty \frac{U_x\psi_x\phi_x}{U}dxd\tau \\
&\quad + \chi\int_0^t\int_0^\infty \frac{U_x\phi_x^2\psi_x}{U^2}dxd\tau - \chi\int_0^t\int_0^\infty \frac{\phi_{xx}\phi_x\psi_x}{U}dxd\tau.
\end{aligned}
$$

Duo to (2.10), it is easy to get that $U_x = U' = \frac{\lambda(u_+ - u_-)e^{\lambda z}}{(e^{\lambda z}+1)^2}$ and $V_x = -\frac{U_x}{s}$ which imply

$$
|U_x| \le \lambda(u_- - u_+), \quad |V_x| \le \lambda v_+. \tag{3.34}
$$

Using the Cauchy–Schwarz inequality, the boundary estimate (3.11), (3.34) and $\|\psi_x(\cdot,t)\|_{L^\infty} \le N(t) < 1$ for any $t\in[0,T]$ by (3.5), we have

$$\int_0^\infty \left(\frac{\phi_x^2}{U} + \chi\psi_x^2\right) dx + D\int_0^t\int_0^\infty \frac{\phi_{xx}^2}{U} dx d\tau$$
$$\leq \int_0^\infty \left(\frac{\phi_{x0}^2}{U} + \chi\psi_{x0}^2\right) dx + Ce^{-\lambda\beta} + C\int_0^t\int_0^\infty \frac{\phi_x^2}{U} dx d\tau + C\int_0^t\int_0^\infty U\psi_x^2 dx d\tau$$
$$+ \frac{DN(t)}{2}\int_0^t\int_0^\infty \frac{\phi_{xx}^2}{U} dx d\tau + CN(t)\int_0^t\int_0^\infty \frac{\phi_x^2}{U} dx d\tau,$$

which together with (3.29) yields

$$\int_0^\infty \left(\frac{\phi_x^2}{U} + \chi\psi_x^2\right) dx + D\left(1 - \frac{N(t)}{2}\right)\int_0^t\int_0^\infty \frac{\phi_{xx}^2}{U} dx d\tau$$
$$\leq C\left(\|\phi_0\|_1^2 + \|\psi_0\|_1^2 + e^{-\lambda\beta} + N(t)\int_0^t\int_0^\infty \psi_x^2 dx d\tau + \int_0^t\int_0^\infty U\psi_x^2 dx d\tau\right). \tag{3.35}$$

Now we claim

$$\int_0^t\int_0^\infty \psi_x^2 dx d\tau \leq C\left(\|\psi_0\|_1^2 + \|\phi_0\|^2 + e^{-\lambda\beta}\right). \tag{3.36}$$

Indeed multiplying the first equation of (3.2) by ψ_x, we get

$$\chi U\psi_x^2 = \phi_t\psi_x - D\phi_{xx}\psi_x - \chi V\phi_x\psi_x - \chi\phi_x\psi_x^2. \tag{3.37}$$

Integrating (3.37) over $[0,\infty) \times [0,t]$, using the fact $\psi_{xt} = \phi_{xx}$ and following results

$$\phi_t\psi_x = (\phi\psi_x)_t - \phi\psi_{xt} = (\phi\psi_x)_t - \phi\phi_{xx} = (\phi\psi_x)_t - (\phi\phi_x)_x + \phi_x^2,$$
$$\phi_{xx}\psi_x = \psi_{xt}\psi_x = \frac{1}{2}(\psi_x^2)_t,$$

we obtain

$$\frac{D}{2}\int_0^\infty \psi_x^2 dx + \chi\int_0^t\int_0^\infty U\psi_x^2 dx d\tau$$
$$= \frac{D}{2}\int_0^\infty \psi_{0x}^2 dx + \int_0^\infty \phi\psi_x dx - \int_0^\infty \phi_0\psi_{0x} dx + \int_0^t (\phi\phi_x)|_{x=0} d\tau$$

$$
\begin{aligned}
&+\int_0^t\int_0^\infty \phi_x^2 dxd\tau - \chi\int_0^t\int_0^\infty V\phi_x\psi_x dxd\tau - \chi\int_0^t\int_0^\infty \phi_x\psi_x^2 dxd\tau \\
&\le \frac{D+1}{2}\int_0^\infty \psi_{0x}^2 dx + \frac{1}{2}\int \phi_0^2 dx + Ce^{-\lambda\beta} + \frac{1}{D}\int_0^\infty \phi^2 dx + \frac{D}{4}\int_0^\infty \psi_x^2 dx + \int_0^t\int_0^\infty \phi_x^2 dxd\tau \\
&\quad + C(1+N(t))\int_0^t\int_0^\infty \frac{\phi_x^2}{U} dxd\tau + \frac{(1+N(t))\chi}{4}\int_0^t\int_0^\infty U\psi_x^2 dxd\tau,
\end{aligned}
$$

where we have used Young's inequality and the fact $\|\psi_x(\cdot,t)\|_{L^\infty} \le N(t)$, $|V| \le C$. From this inequality and using $0 < u_+ \le U \le u_-$ and (3.29), it follows that

$$
\begin{aligned}
&\int_0^\infty \psi_x^2 dx + \int_0^t\int_0^\infty U\psi_x^2 dxd\tau \\
&\le C\left(\int_0^\infty \psi_{0x}^2 dx + \int_0^\infty \phi_0^2 dx + Ce^{-\lambda\beta} + \int_0^\infty \frac{\phi^2}{U} dx + \int_0^t\int_0^\infty \frac{\phi_x^2}{U} dxd\tau\right) \\
&\le C\left(\|\psi_0\|_1^2 + \|\phi_0\|^2 + e^{-\lambda\beta} + N(t)\int_0^t\int_0^\infty \psi_x^2 dxd\tau\right).
\end{aligned}
\tag{3.38}
$$

Choosing $N(t)$ sufficiently small and using $0 < u_+ \le U \le u_-$, we get (3.36) from (3.38). Then substituting (3.36) into (3.35) yields

$$
\int_0^\infty \psi_x^2 dx + \int_0^\infty \frac{\phi_x^2}{U} dx + D\int_0^t\int_0^\infty \frac{\phi_{xx}^2}{U} dxd\tau \le C\left(\|\psi_0\|_1^2 + \|\phi_0\|_1^2 + Ce^{-\lambda\beta}\right). \tag{3.39}
$$

Thus, by $0 < u_+ \le U \le u_-$, (3.29) and (3.39), we derive (3.32). □

Next, we give the estimates of the second order derivative of (ϕ, ψ).

Lemma 3.9. *Let the assumptions in Proposition 3.5 hold. Then there exists a constant $C > 0$ such that*

$$
\|\phi_{xx}\|^2 + \|\psi_{xx}\|^2 + \int_0^t \left(\|\phi_{xxx}\|^2 + \|\psi_{xx}\|^2\right) d\tau \le C\left(\|\phi_0\|_2^2 + \|\psi_0\|_2^2 + e^{-\lambda\beta}\right). \tag{3.40}
$$

Proof. We differentiate (3.2) with respect to x to get

$$
\begin{cases}
\phi_{xt} = D\phi_{xxx} + \chi U_x\psi_x + \chi U\psi_{xx} + \chi V_x\phi_x + \chi V\phi_{xx} + \chi\phi_{xx}\psi_x + \chi\phi_x\psi_{xx}, \\
\psi_{xt} = \phi_{xx}.
\end{cases}
\tag{3.41}
$$

Multiplying the first equation of (3.41) by $-\phi_{xxx}/U$, one gets

$$-\frac{\phi_{xt}\phi_{xxx}}{U}=-\frac{D\phi_{xxx}^2}{U}-\chi\psi_{xx}\phi_{xxx}-\frac{\chi\phi_{xxx}}{U}(U_x\psi_x+V_x\phi_x+V\phi_{xx}+\phi_{xx}\psi_x+\phi_x\psi_{xx}). \tag{3.42}$$

If we follow the standard procedure to integrate (3.42) with respect to x over $\mathbb{R}_+$ to derive the estimate of $\|\phi_{xx}\|^2$, the boundary term $\int_0^t \phi_{xx}\psi_{xx}|_{x=0}d\tau$ will be present, which is out of control in our problem. Hence to avoid this boundary estimate, below we shall develop a new idea by constructing the term $\chi\psi_{xx}\phi_{xxx}$ from the second equation of (3.41) and canceling the term $-\chi\psi_{xx}\phi_{xxx}$ in (3.42) which causes the boundary estimates. By doing this, new boundary estimate arising is $\int_0^t \phi_{xt}\phi_{xx}|_{x=0}d\tau$ which is however under control (see (3.12)). To this end, we differentiate the second equation of (3.41) and multiply the resultant equation by $\chi\psi_{xx}$ to get

$$\left(\frac{\chi\psi_{xx}^2}{2}\right)_t=\chi\psi_{xx}\phi_{xxx}. \tag{3.43}$$

Adding (3.42) and (3.43) up and noticing that

$$\begin{aligned}
-\frac{\phi_{xt}\phi_{xxx}}{U}&=-\left(\frac{\phi_{xt}\phi_{xx}}{U}\right)_x+\frac{\phi_{xxt}\phi_{xx}}{U}-\frac{U_x\phi_{xt}\phi_{xx}}{U^2}\\
&=-\left(\frac{\phi_{xt}\phi_{xx}}{U}\right)_x+\left(\frac{\phi_{xx}^2}{2U}\right)_t-\frac{sU_x\phi_{xx}^2}{2U^2}\\
&\quad-\frac{U_x\phi_{xx}}{U^2}(D\phi_{xxx}+\chi U_x\psi_x+\chi U\psi_{xx}+\chi V_x\phi_x+\chi V\phi_{xx}+\chi\phi_{xx}\psi_x+\chi\phi_x\psi_{xx}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\phi_{xx}^2}{U}+\chi\psi_{xx}^2\right)_t+\frac{D\phi_{xxx}^2}{U}\\
&=\left(\frac{\phi_{xt}\phi_{xx}}{U}\right)_x+\frac{sU_x\phi_{xx}^2}{2U^2}-\frac{\chi\phi_{xxx}}{U}(U_x\psi_x+V_x\phi_x+V\phi_{xx}+\phi_{xx}\psi_x+\phi_x\psi_{xx})\\
&\quad+\frac{U_x\phi_{xx}}{U^2}(D\phi_{xxx}+\chi U_x\psi_x+\chi U\psi_{xx}+\chi V_x\phi_x+\chi V\phi_{xx}+\chi\phi_{xx}\psi_x+\chi\phi_x\psi_{xx}).
\end{aligned} \tag{3.44}$$

Integrating (3.44) with respect to x over $[0,\infty)$ and rearranging the resulting equation, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^\infty\left(\frac{\phi_{xx}^2}{U}+\chi\psi_{xx}^2\right)dx+D\int_0^\infty\frac{\phi_{xxx}^2}{U}dx\\
&=-\frac{\phi_{xt}\phi_{xx}}{U}|_{x=0}-\underbrace{\chi\int_0^\infty\frac{\phi_{xxx}}{U}\left(U_x\psi_x+V_x\phi_x+V\phi_{xx}+\phi_{xx}\psi_x+\phi_x\psi_{xx}-\frac{DU_x\phi_{xx}}{\chi U}\right)dx}_{I_1}
\end{aligned}$$

$$+\underbrace{\int_0^\infty \frac{U_x\phi_{xx}}{U^2}\left(\frac{s\phi_{xx}}{2}+\chi U_x\psi_x+\chi U\psi_{xx}+\chi V_x\phi_x+\chi V\phi_{xx}+\chi\phi_{xx}\psi_x+\chi\phi_x\psi_{xx}\right)dx}_{I_2}. \tag{3.45}$$

Because $\left|\frac{U_x}{U}\right|$, $|U_x|$ and $|V_x|$ are all bounded, $\|\psi_x(\cdot,t)\|_{L^\infty}\le N(t)<1$ and $\|\phi_x(\cdot,t)\|_{L^\infty}\le N(t)<1$ for any $t\in[0,T]$, we get by the Cauchy–Schwartz inequality that

$$\begin{aligned}I_1 &\le \frac{D+N(t)}{2}\int_0^\infty \frac{\phi_{xxx}^2}{U}dx + C\int_0^\infty U\psi_x^2dx + C\int_0^\infty \frac{\phi_x^2}{U}dx \\ &\quad + C(1+N(t))\int_0^\infty \frac{\phi_{xx}^2}{U}dx + CN(t)\int_0^\infty \frac{\psi_{xx}^2}{U}dx, \\ I_2 &\le C(1+N(t))\int_0^\infty \frac{\phi_{xx}^2}{U}dx + C\int_0^\infty U\psi_x^2dx + C\int_0^\infty \frac{\phi_x^2}{U}dx \\ &\quad + C\int_0^\infty U\psi_{xx}^2dx + CN(t)\int_0^\infty \frac{\psi_{xx}^2}{U}dx.\end{aligned}$$

Substituting the above two inequalities into (3.45), integrating the resultant inequality over $[0,t]$ and using (3.32), one has

$$\begin{aligned}&\int_0^\infty \frac{\phi_{xx}^2}{U}dx + \int_0^\infty \psi_{xx}^2dx + \int_0^t\int_0^\infty \frac{\phi_{xxx}^2}{U}dxd\tau \\ &\le C\left(\|\phi_0\|_2^2+\|\psi_0\|_2^2+e^{-\lambda\beta}+\int_0^t\int_0^\infty U\psi_{xx}^2dxd\tau + N(t)\int_0^t\int_0^\infty \psi_{xx}^2dxd\tau\right). \end{aligned}\tag{3.46}$$

Next we estimate the term $\int_0^t\int_0^\infty U\psi_{xx}^2dxd\tau$. Multiplying the first equation of (3.41) by ψ_{xx}, we obtain

$$\chi U\psi_{xx}^2 = \phi_{xt}\psi_{xx} - (D\phi_{xxx}+\chi U_x\psi_x+\chi V_x\phi_x+\chi V\phi_{xx}+\chi\phi_{xx}\psi_x+\chi\phi_x\psi_{xx})\psi_{xx}.$$

With the following identities

$$\begin{aligned}\phi_{xxt} &= \psi_{xxx}, \\ \phi_{xt}\psi_{xx} &= (\phi_x\psi_{xx})_t - \phi_x\psi_{xxt} = (\phi_x\psi_{xx})_t - \phi_x\phi_{xxx} = (\phi_x\psi_{xx})_t - (\phi_x\phi_{xx})_x + \phi_{xx}^2, \\ \phi_{xxx}\psi_{xx} &= \psi_{xxt}\psi_{xx} = \frac{1}{2}(\psi_{xx}^2)_t,\end{aligned}$$

we have

$$\frac{D}{2}(\psi_{xx}^2)_t + \chi U \psi_{xx}^2 = (\phi_x \psi_{xx})_t - (\phi_x \phi_{xx})_x + \phi_{xx}^2 \\ - (\chi U_x \psi_x + \chi V_x \phi_x + \chi V \phi_{xx} + \chi \phi_{xx} \psi_x + \chi \phi_x \psi_{xx}) \psi_{xx}.$$

Thus, integrating the above equation over $[0, \infty) \times [0, t]$ and using the Cauchy–Schwartz inequality, we have

$$\begin{aligned} &\frac{D}{2}\int_0^\infty \psi_{xx}^2 dx + \chi \int_0^t \int_0^\infty U \psi_{xx}^2 dx d\tau \\ &\le \frac{1}{D}\int_0^\infty \phi_x^2 dx + \frac{D}{4}\int_0^\infty \psi_{xx}^2 dx + \left(\frac{D+1}{2}\right)\int_0^\infty \psi_{0xx}^2 dx + \frac{1}{2}\int_0^\infty \phi_{0x}^2 dx \\ &\quad + \int_0^t (\phi_x \phi_{xx})|_{x=0} dx + \int_0^t \int_0^\infty \phi_{xx}^2 dx d\tau + \frac{(1+N(t))\chi}{4}\int_0^t \int_0^\infty U \psi_{xx}^2 dx d\tau \\ &\quad + C\left(\int_0^t \int_0^\infty \frac{\psi_x^2}{U} dx d\tau + \int_0^t \int_0^\infty \frac{\phi_x^2}{U} dx d\tau + \int_0^t \int_0^\infty \frac{\phi_{xx}^2}{U} dx d\tau\right) + N(t)\int_0^t \int_0^\infty \psi_{xx}^2 dx d\tau. \end{aligned}$$

Then it follows from (3.12) and (3.32) that

$$\begin{aligned} &\int_0^\infty \psi_{xx}^2 dx + \int_0^t \int_0^\infty U \psi_{xx}^2 dx d\tau \\ &\le C\left(\|\phi_0\|_1^2 + \|\psi_0\|_2^2 + e^{-\lambda\beta} + N(t)\int_0^t \int_0^\infty \psi_{xx}^2 dx d\tau\right). \end{aligned} \tag{3.47}$$

When $N(t)$ is small enough, the above inequality gives

$$\int_0^\infty \psi_{xx}^2 dx + \int_0^t \int_0^\infty \psi_{xx}^2 dx d\tau \le C\left(\|\phi_0\|_1^2 + \|\psi_0\|_2^2 + e^{-\lambda\beta}\right), \tag{3.48}$$

where $0 < u_+ \le U \le u_-$ has been used. This together with (3.46) leads to

$$\int_0^\infty \psi_{xx}^2 dx + \int_0^\infty \frac{\phi_{xx}^2}{U} dx + \int_0^t \int_0^\infty \frac{\phi_{xxx}^2}{U} dx d\tau \le C\left(\|\phi_0\|_2^2 + \|\psi_0\|_2^2 + e^{-\lambda\beta}\right), \tag{3.49}$$

which in combination with (3.48) gives (3.40). The proof of Lemma 3.9 is finished. □

Finally, the desired estimate (3.6) follows from (3.32) and (3.40), and the proof of Proposition 3.5 is complete.

3.3. *Proof of Theorem 3.1*

To complete the proof of Theorem 3.1, we only need to prove (3.7) since the rest has been implied by Proposition 3.5. From (3.6), we have

$$\|\phi_x(\cdot,t),\psi_x(\cdot,t)\|_1 \to 0 \ \text{ as } \ t\to\infty.$$

Hence, for all $x\in\mathbb{R}_+$,

$$\begin{aligned}\phi_x^2(x,t) &= 2\left|\int_x^\infty \phi_x\phi_{xx}(y,t)dy\right| \\ &\le 2\left(\int_0^\infty \phi_x^2 dy\right)^{1/2}\left(\int_0^\infty \phi_{xx}^2 dy\right)^{1/2} \\ &\le \|\phi_x(\cdot,t)\|_1 \to 0 \ \text{ as } \ t\to\infty.\end{aligned}$$

Similarly, we have

$$\psi_x(x,t)\to 0 \ \text{ as } \ t\to\infty \quad \text{for all } \ x\in\mathbb{R}_+.$$

Hence (3.7) is proved and the proof of Theorem 3.1 is complete.

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