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STEADY HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH SONIC BOUNDARY: (I) SUBSONIC DOPING PROFILE*

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Abstract. This series of papers concerns the structure of stationary solutions to the hydrodynamic model of semiconductors with sonic boundary represented by Euler–Poisson equations. The physical solutions are characterized according to different types of doping profiles. In the first part of the series, we consider the case of the subsonic doping profile and prove that the steady-state equations with sonic boundary possess a unique interior subsonic solution, at least one interior supersonic solution, infinitely many shock transonic solutions when the relaxation time is large, and infinitely many C^1-smooth transonic solutions when the relaxation time is small. In particular, the interior subsonic/supersonic solutions are proved to be globally $C^{\frac{1}{2}}$ Hölder continuous, and the Hölder exponent $\frac{1}{2}$ is optimal. The regularity of transonic solutions is dependent on the size of the relaxation time, equivalently, the effect of semiconductors. The proof of the existence of subsonic/supersonic solutions is the technical compactness analysis combining the energy method and the phase-plane analysis, while the approach for the existence of multiple shock/smooth transonic solutions is the artful construction. The results obtained significantly improve and develop existing studies.

Key words. Euler–Poisson equations, hydrodynamic model of semiconductors, sonic boundary, subsonic doping profile, subsonic solutions, supersonic solutions, transonic solutions with shock, C^1-smooth transonic solution

AMS subject classifications. 35R35, 35Q35, 76N10, 35J70

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1. Introduction.

The hydrodynamic model of semiconductors, first introduced by Bløtekjær in [5], is usually described for the charged fluid particles such as electrons and holes in semiconductor devices [5, 19, 24], and positively and negatively charged ions in plasma [28]. The governing equations are Euler–Poisson equations as follows [15, 16, 17, 20]:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = \rho E - \dfrac{\rho u}{\tau}, \\ E_x = \rho - b(x). \end{cases} \tag{1}$$

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Here ρ, u, and E represent the electron density, the velocity, and the electric field, respectively. $P(\rho)$ is the pressure function of the electron density. When the fluid is isothermal, the pressure function is physically represented by

$$P(\rho) = T\rho, \quad \text{with the constant temperature } T > 0. \tag{2}$$

The function $b(x) > 0$ is the doping profile standing for the density of impurities in semiconductor device. The constant $\tau > 0$ denotes the momentum relaxation time.

In this series of papers, we are mainly interested in investigating the existence and large-time behaviors of the solutions to (1) with sonic boundary condition. At the first, but important, stage, we focus on the existence and classification of all stationary solutions. Throughout this paper, we consider the following steady-state equations to (1) in the bounded domain $[0, 1]$ with subsonic doping profile. Denote $J = \rho u$, the current density; then we have the stationary equations of (1) as follows:

$$\begin{cases} J = \text{constant}, \\ \left(\dfrac{J^2}{\rho} + P(\rho)\right)_x = \rho E - \dfrac{J}{\tau}, \qquad x \in (0,1). \\ E_x = \rho - b(x). \end{cases} \tag{3}$$

Using the terminology from gas dynamics, we call $c := \sqrt{P'(\rho)} = \sqrt{T} > 0$ the sound speed for $P(\rho) = T\rho$ (see (2)). Thus, the stationary flow of (3) is called to be subsonic/sonic/supersonic if the fluid velocity satisfies

$$\text{fluid velocity: } u = \frac{J}{\rho} \lesseqqgtr c = \sqrt{P'(\rho)} = \sqrt{T}\text{: sound speed.} \tag{4}$$

We consider the current driven flow; thus the current density J is a prescribed constant. Note that if $(\rho(x), E(x))$ is a solution to (3) with a given constant current density J, then $(\rho(1-x), -E(1-x))$ is a solution to (3) with respect to $-J$ and $b(1-x)$. So, we may consider only the case of $J > 0$. Without loss of generality, let us assume throughout the paper that

$$T = J = 1.$$

Thus, (3) is transformed into

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x). \end{cases} \tag{5}$$

From (4), it can be identified that the flow is subsonic if $\rho > 1$, sonic if $\rho = 1$, and supersonic if $0 < \rho < 1$. Therefore, our sonic boundary conditions to (3) are proposed as follows:

$$\text{sonic boundary:} \quad \rho(0) = \rho(1) = 1. \tag{6}$$

Dividing the first equation of (5) by ρ and differentiating the resultant equation with respect to x, and substituting the second equation of (5) into this modified equation, we have

$$\begin{cases} \left[\left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x\right]_x + \dfrac{1}{\tau}\left(\dfrac{1}{\rho}\right)_x - [\rho - b(x)] = 0, \qquad x \in (0,1), \\ \rho(0) = \rho(1) = 1. \end{cases} \tag{7}$$

When $\rho(x) > 1$ or $0 < \rho(x) < 1$ for $x \in (0,1)$, equation (7) is elliptic but degenerate at the sonic boundary. When $\rho(x) > 0$ varies around the sonic line $\rho = 1$ for $x \in (0,1)$, the system then changes its property and the phenomena of phase transitions occur.

The existence of subsonic/supersonic/transonic solutions to the steady-state Euler–Poisson equations for the hydrodynamic model of semiconductors has been intensively studied. In 1990, Degond and Markowich [9] first showed the existence of the subsonic solution when the flow and its boundary are completely subsonic. The uniqueness was obtained with a very strongly subsonic background, namely, $|J| \ll 1$. Then, the steady subsonic flows were studied in great depth with different boundaries as well as the higher dimensions case in [2, 3, 10, 11, 15, 18, 25]; see the references therein. For the case of steady supersonic flows, Peng and Violet [26] obtained the existence and uniqueness of the supersonic solution when the flow and the boundary are strongly supersonic (i.e., $J \gg 1$). On the other hand, much attention has been given to the case of steady transonic flows. By a phase-plane analysis, Ascher et al. [1] first tested the existence of the transonic solution when the doping profile b is a supersonic constant, which was then extended by Rosini [27] to the nonisentropic flow. When the doping profile $b(x)$ is nonconstant, by using the method of vanishing viscosity, Gamba constructed 1-D transonic solutions with shocks in [12], and 2-D transonic solutions with shocks in [13], but the solutions as the limits of vanishing viscosity yield boundary layers. Recently, Luo and Xin [23] and Luo et al. [22] studied the Euler–Poisson equations without the effect of the semiconductor, namely, the momentum equation $(1)_2$ is missing the term of $-\frac{J}{\tau}$. This means either the current density $J = 0$ (the absence of the semiconductor effect for the device) or the relaxation time $\tau = \infty$ (the huge relaxation time). Some interesting results on the structure of steady solutions with nonsonic boundary condition are obtained. Precisely, based on phase-plane analysis, Luo and Xin [23] thoroughly studied the existence/nonexistence and the uniqueness/nonuniqueness of the transonic solutions with one side supersonic boundary and the other side subsonic boundary when the doping profile $b(x)$ is a constant either in the supersonic regime or the subsonic regime. Some restrictions on the boundary and the domain are also needed. Then, Luo et al. [22] showed the existence of transonic solutions with shocks in the case of variable supersonic doping profile $b(x)$, which is regarded as a small perturbation of a constant, and further proved the time-asymptotic stability of the transonic shock profiles.

In this paper, the subjected boundary is sonic, which is a critical case for boundary and causes the problem to be more complicated and challenging. We are interested in how the doping profile, the semiconductor effect, and the sonic boundary affect the structures of solutions to system (5). All these features not only cause us some essential difficulties in the study of well-posedness and regularity of the solutions, but also bring us fairly rich and interesting phenomena on the structure of solutions. In fact, we have the following: (1) the elliptic equation (7) is degenerate at the boundary, and hence the standard approaches on uniformly elliptic equations adopted in [9, 26, 22] do not work; (2) because of the degeneracy of the equation, all subsonic/supersonic solutions are expected only to be globally $C^{\frac{1}{2}}$ Hölder continuous, and the $C^{\frac{1}{2}}$ regularity is proved to be optimal; (3) unlike in the phase-plane analysis in [23], the presence of relaxation causes a difficulty in that the electric field E cannot be *explicitly* formulated as a function of ρ; and (4) most importantly, under the large effect of semiconductor (i.e., $\tau \ll 1$) in combination with the degeneracy at the boundary, the steady-state equations possess C^1 transonic solutions rather than transonic shock solutions, while, when $\tau \gg 1$, the equations admit transonic shock solutions, which coincides with the works of Luo and Xin [23] on Euler–Poisson equations without the effect of a semiconductor (i.e., $\tau = +\infty$). To get the transonic solutions, inspired by [23, 22], the approach adopted in this paper is also the artful construction method but with a new development due to the difficulty caused by the sonic boundary. Precisely, since there are many possibilities for the stationary solutions starting on the sonic boundary, it is impossible for us to construct a supersonic solution directly from the sonic boundary. Instead of the sonic boundary at $x = 0$, we first consider a supersonic boundary $\rho^\delta|_{x=0} = 1 - \delta$ for $0 < \delta \ll 1$. We can technically construct an approximate supersonic solution $\rho^\delta < 1$ in $[0, x_0)$, where the location x_0 will be specified such that the Rankine–Hugoniot condition and the entropy condition are satisfied. Connecting by the Rankine–Hugoniot condition and the entropy condition, we further construct a subsonic solution $\rho^\delta(x) > 1$ in $(x_0, 1]$ with the subsonic condition $\rho^\delta|_{x=1} = 1 + \delta$. By carrying out the compactness analysis, we can technically prove the weak convergence of the approximate transonic solution to the desired transonic solution $(\rho_{trans}, E_{trans})(x)$ with the sonic boundary condition. The procedure presented in this paper is technical, explicit, and instructive. Remarkably, different from the previous studies [23, 22] without the semiconductor effect (i.e., $\tau = \infty$), there also exist infinitely many C^1 transonic solutions when $\tau \ll 1$. As we know, this is a physical case for the semiconductor industry and represents the first attempt to show C^1 transonic solutions for semiconductor models.

We now state the main results of this paper. To do so, we need to make our notation and assumptions more precise. Throughout this paper we assume that the doping profile $b(x) \in L^\infty(0, 1)$ is subsonic:

$$\text{subsonic doping profile:} \quad b(x) > 1 \quad \text{for all } x \in [0, 1].$$

We denote

$$\underline{b} := \operatorname*{essinf}_{x\in(0,1)} b(x) \qquad \text{and} \qquad \bar{b} := \operatorname*{esssup}_{x\in(0,1)} b(x).$$

Because equation (7) is degenerate, we have to introduce the concepts of interior subsonic/supersonic/transonic solutions in the weak sense.

Definition 1.1. *$\rho(x)$ is called an interior subsonic (correspondingly, interior supersonic) solution of equation* (7) *if $\rho(0) = \rho(1) = 1$ but $\rho(x) > 1$ (correspondingly, $0 < \rho(x) < 1$) for $x \in (0, 1)$, and $(\rho(x) - 1)^2 \in H_0^1(0, 1)$, and it holds that for any $\varphi \in H_0^1(0, 1)$*

$$\int_0^1 \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x \varphi_x dx + \frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0,$$

which is equivalent to

$$\frac{1}{2}\int_0^1 \frac{\rho+1}{\rho^3}\left((\rho-1)^2\right)_x \varphi_x dx + \frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0. \tag{8}$$

Once $\rho = \rho(x)$ is determined by (7), in view of the first equation of (5), the electric field $E(x)$ can be solved by

$$E(x) = \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x + \frac{1}{\tau\rho} = \frac{(\rho+1)[(\rho-1)^2]_x}{2\rho^3} + \frac{1}{\tau\rho}.$$

In this way, we could obtain the interior subsonic/supersonic solutions to system (5)–(6).

DEFINITION 1.2. *$\rho(x) > 0$ is called a C^1 transonic solution of system (5)–(6) if $\rho(x) \in C^1(0,1)$ with $\rho(0) = \rho(1) = 1$ and there exists a number $x_0 \in (0,1)$ such that*

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in [0, x_0], \\ \rho_{sub}(x), & x \in [x_0, 1], \end{cases}$$

where $0 < \rho_{sup}(x) \leq 1$ on $(0, x_0)$, $\rho_{sub}(x) \geq 1$ on $(x_0, 1)$, and

$$\rho_{sup}(x_0) = \rho_{sub}(x_0) = 1 \quad \text{and} \quad \rho'_{sup}(x_0) = \rho'_{sub}(x_0). \tag{9}$$

$\rho(x) > 0$ is called a transonic shock solution of system (5)–(6) if $\rho(0) = \rho(1) = 1$ and it is separated by a point $x_0 \in (0,1)$ in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where $0 < \rho_{sup}(x) < 1$ and $\rho_{sub}(x) > 1$ satisfy the entropy condition at x_0,

$$0 < \rho_{sup}(x_0^-) < 1 < \rho_{sub}(x_0^+), \tag{10}$$

and the Rankine–Hugoniot condition,

$$\begin{aligned} \rho_{sup}(x_0^-) + \frac{1}{\rho_{sup}(x_0^-)} &= \rho_{sub}(x_0^+) + \frac{1}{\rho_{sub}(x_0^+)}, \\ E_{sup}(x_0^-) &= E_{sub}(x_0^+). \end{aligned} \tag{11}$$

Set $\rho_l = \rho_{sup}(x_0^-)$ and $\rho_r = \rho_{sub}(x_0^+)$; a simple computation from (11) shows that

$$\rho_l \rho_r = 1. \tag{12}$$

Our main results on the structures of solutions to (5)–(6) are as follows.

THEOREM 1.3. *Let the doping profile be subsonic such that $b(x) \in L^\infty(0,1)$ and $\underline{b} > 1$. Then the steady-state Euler–Poisson equations (5)–(6) admit the following:*

1. *A unique pair of interior subsonic solutions $(\rho_{sub}, E_{sub})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ satisfying*

$$1 + m\sin(\pi x) \leq \rho_{sub}(x) \leq \bar{b}, \quad x \in [0,1], \tag{13}$$

and particularly,

$$\begin{cases} C_1(1-x)^{\frac{1}{2}} \leq \rho_{sub}(x) - 1 \leq C_2(1-x)^{\frac{1}{2}} \\ -C_3(1-x)^{-\frac{1}{2}} \leq \rho'_{sub}(x) \leq -C_4(1-x)^{-\frac{1}{2}} \end{cases} \quad \text{for } x \text{ near } 1, \tag{14}$$

where $m = m(\tau, \underline{b})$ is a positive constant and $C_2 > C_1 > 0$ and $C_3 > C_4 > 0$ are some positive constants.

2. *At least one pair of interior supersonic solutions $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ satisfying*

$$\begin{cases} C_5 x^{\frac{1}{2}} \leq 1 - \rho_{sup}(x) \leq C_6 x^{\frac{1}{2}} \\ -C_7 x^{-\frac{1}{2}} \leq \rho'_{sup}(x) \leq -C_8 x^{-\frac{1}{2}} \end{cases} \quad \text{for } x \text{ near } 0, \tag{15}$$

where $C_6 > C_5 > 0$ and $C_7 > C_8 > 0$ are some positive constants. ρ_{sup} has only one critical point z_0 over $(0,1)$, such that $(\rho_{sup})_x < 0$ on $(0, z_0)$ and $(\rho_{sup})_x > 0$ on $(z_0, 1)$.

3. *Assume further that τ is large and that $\bar{b}-\underline{b} \ll 1$; then* (5)–(6) *have infinitely many transonic solutions $(\rho_{trans}, E_{trans})(x)$ combining stationary shocks which satisfy the entropy condition* (10) *and the Rankine–Hugoniot jump condition* (11) *at different jump locations x_0, where x_0 can be uniquely determined when ρ_r satisfying $\rho_r - \rho_l \ll 1$ is fixed, but the choice of ρ_l can be infinitely many.*
4. *Assume further that $b(x) = b > 1$ is a constant; then when τ is small enough,* (5)–(6) *have infinitely many C^1 transonic solution; moreover, in this case there is no transonic shock solution.*

Remark 1.4.

1. In parts 1 and 2 of Theorem 1.3, the estimates (14) and (15) imply that $C^{\frac{1}{2}}[0,1]$ is the optimal Hölder space for the global regularity of the subsonic solution $\rho_{sub}(x)$ and the supersonic solution $\rho_{sup}(x)$. We notice that the same regularity $C^{\frac{1}{2}}$ was also obtained for the subsonic/sonic flow in nozzles in [29, 30]. Regarding the other interesting studies on the subsonic/sonic flow in nozzles, we refer interested readers to [4, 7, 8, 33].
2. The existence of infinitely many transonic shock solutions obtained in part 3 of Theorem 1.3 also holds for the Euler–Poisson equations without relaxation term (i.e., $\tau = \infty$) that were studied in [23].
3. Part 4 of Theorem 1.3 implies that if $b(x)$ is a constant and $\tau \ll 1$, the regularity of the interior subsonic solution on the left boundary, as well as the regularity for the interior supersonic solution on the right boundary, can be lifted up to C^1. To the best of our knowledge, such a C^1 regularity of transonic solutions is the first result obtained for semiconductor models so far. Essentially, the strong damping effect (the semiconductor effect) of $-\frac{J}{\tau}$ causes the transonic solutions to be C^1-smooth. Notice that the C^2 transonic flow also arises in the finite de Laval nozzles, where the geometry structure causes the transonic flow to be smooth. For details, we refer to the interesting works of Wang and Xin [31, 32].
4. When the doping profile b is supersonic, a further study on the interesting structures of solutions to (5)–(6) will be discussed in the second part of this series of papers [21].

The paper is organized as follows. Sections 2 and 3 are devoted to the existence of subsonic/supersonic solutions to (5)–(6), respectively. The proof is long and technical. To regularize the degeneracy of (7), for the fixed sonic boundary, by taking the current densities $j > 1$ for the subsonic case and $j < 1$ for the supersonic case, respectively, we then have the approximate equations for the subsonic/supersonic cases both to be uniformly elliptic, and hence the approximate solutions are always subsonic/supersonic, respectively. We observe that $(\rho_j - 1)^2$ can be estimated in H_0^1, where $\rho_j(x)$ are the approximate subsonic/supersonic solutions for $j > 1$ and $j < 1$, respectively. By using a compactness analysis, after taking the limits as $j \to 1^-$ for the subsonic case and $j \to 1^+$ for the supersonic case, we finally obtain the existence of the interior subsonic/supersonic solution. Furthermore, we analyze the regularities of these two types of solutions and show their optimal regularity as $C^{1/2}$. In sections 4 and 5, we artfully construct infinitely many transonic shock solutions when $\tau \gg 1$, and infinitely many C^1-smooth transonic solutions when $\tau \ll 1$, respectively. For $\tau \gg 1$, to regularize the degeneracy, different from the skill mentioned before, here we keep the equations fixed but regularize the boundary condition as $\rho^{\delta}_{tran}(0) = 1-\delta$ and $\rho^{\delta}_{tran}(1) = 1+\delta$ for any small enough $\delta > 0$. Then by the shooting method, we obtain approximate supersonic solutions first, and then the approximate transonic solutions can be easily constructed by matching a subsonic solution to the obtained supersonic solution satisfying the entropy condition and Rankine–Hugoniot condition. Letting $\delta \to 0^+$, the diagonal argument finally picks up a transonic solution to (5)–(6). Since the left side ρ_- for the Rankine–Hugoniot condition can be arbitrarily chosen, we get the infinitely many shock transonic solutions. For $\tau \ll 1$, we recognize that the semiconductor effect $\frac{J}{\tau} \gg 1$ makes the transonic solutions C^1-smooth, and no shock transonic solutions exists. Finally, at the end of the paper, in order to understand better the structure of all solutions as shown in Theorem 1.3, we present some examples in section 6.

2. Existence and uniqueness of interior subsonic solution. First, we prove that there exists a unique interior subsonic solution to (7). The adopted approach is the technical compactness method, which is inspired by the vanishing viscosity method.

THEOREM 2.1. *Assume that $b \in L^\infty(0,1)$ and $\underline{b} > 1$; then (7) has a unique interior subsonic solution ρ_{sub} satisfying*

$$1 + m\sin(\pi x) \le \rho_{sub} \le \bar{b}, \quad x \in [0,1], \tag{16}$$

where $m = m(\tau, \underline{b})$ is a positive constant.

Since (7) is partially elliptic but degenerates at the boundary, the corresponding solution to (7) will lack the necessary regularity, and we cannot directly work on (7). In order to prove Theorem 2.1, we consider the following approximate equation:

$$\begin{cases} \left[\left(\dfrac{1}{\rho_j} - \dfrac{j^2}{(\rho_j)^3}\right)(\rho_j)_x\right]_x + \left(\dfrac{j}{\tau\rho_j}\right)_x - [\rho_j - b(x)] = 0, \quad x \in (0,1), \\ \rho_j(0) = \rho_j(1) = 1, \end{cases} \tag{17}$$

where the parameter j is a constant such that $0 < j < 1$. Thus, (17) is expected to be uniformly elliptic in $[0,1]$, because $\frac{1}{\rho_j} - \frac{j^2}{\rho_j^3} = \frac{1}{\rho_j^3}(\rho_j + j)(\rho_j - j) > 0$ for the expected solution $\rho_j \ge 1$. To show the well-posedness of the approximate equation (17) and to establish the lower bound estimate in (16), we need the following comparison principle.

LEMMA 2.2 (comparison principle). *Let $U \in C^1[0,1]$ be a weak solution of (17) satisfying $U \ge 1$ on $[0,1]$, and that*

$$\int_0^1 \left[\left(\frac{1}{U} - \frac{j^2}{U^3}\right)U_x + \frac{j}{\tau U}\right]\varphi_x dx + \int_0^1 (U - b)\varphi dx = 0 \quad \text{for any } \varphi \in H_0^1(0,1), \tag{18}$$

where $0 < j < 1$ is a constant, and let $V \in C^1[0,1]$ be such that $V(x) > 0$ for $x \in [0,1]$, $V(0) \le 1$, $V(1) \le 1$, and

$$\int_0^1 \left[\left(\frac{1}{V} - \frac{j^2}{V^3}\right)V_x + \frac{j}{\tau V}\right]\varphi_x dx + \int_0^1 (V - b)\varphi dx \le 0 \quad \text{for any } \varphi \ge 0,\ \varphi \in H_0^1(0,1).$$

Then $U(x) \ge V(x)$ over $[0,1]$.

Proof. Inspired by the textbook [14] (see Theorem 2.7 in section 10.4), we can prove this comparison principle. Let us denote

$$A(z,p) := \left(\frac{1}{z} - \frac{j^2}{z^3}\right)p + \frac{j}{\tau z}$$

for simplicity. Then, for any $\varphi \in H_0^1(0,1)$, $\varphi \geq 0$, we have

$$\int_0^1 [A(V,V_x) - A(U,U_x)]\varphi_x dx + \int_0^1 (V-U)\varphi dx \leq 0. \tag{19}$$

Set $e(x) := V(x) - U(x)$. A simple calculation gives

$$\begin{aligned}A(V,V_x) - A(U,U_x) &= A(V,V_x) - A(U,V_x) + A(U,V_x) - A(U,U_x)\\ &= \int_0^1 \frac{\partial A}{\partial z}(V_t,V_x)dt \cdot e(x) + \int_0^1 \frac{\partial A}{\partial p}(U,(V_t)_x)dt \cdot e_x(x),\end{aligned}$$

where $V_t(x) := tV(x) + (1-t)U(x)$. Taking $\varphi(x) = \frac{e^+(x)}{e^+(x)+h}$ with $e^+(x) := \max\{0, e(x)\}$ and $h > 0$ a constant, a straightforward computation yields

$$\left[\ln\left(1 + \frac{e^+(x)}{h}\right)\right]_x = \frac{e_x^+(x)}{e^+(x)+h} \quad \text{and} \quad \varphi_x = \frac{h}{e^+(x)+h}\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x.$$

Since $0 < j < 1$, $V \in C^1[0,1]$, and $\min_{x\in[0,1]} V(x) > 0$, it is easy to see that

$$\begin{aligned}&\int_0^1 \frac{\partial A}{\partial p}(U,(V_t)_x)dt = \frac{1}{U} - \frac{1}{U^3} + \frac{1-j^2}{U^3} \geq \frac{1-j^2}{\|U\|_{L^\infty}^3},\\ &\int_0^1 \frac{\partial A}{\partial z}(V_t,V_x)dt \leq C\|V_x\|_{C[0,1]} + \frac{Cj}{\tau} \leq C.\end{aligned}$$

It then follows from (19) that

$$\begin{aligned}&\frac{h(1-j^2)}{\|U\|_{L^\infty}^3}\int_0^1 \left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x\right|^2 dx + \int_0^1 \frac{(e^+(x))^2}{e^+(x)+h}dx\\ &\leq Ch\int_0^1 \frac{e^+(x)}{e^+(x)+h}\left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x\right| dx\\ &\leq \frac{h(1-j^2)}{2\|U\|_{L^\infty}^3}\int_0^1 \left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x\right|^2 dx + \frac{C^2 h\|U\|_{L^\infty}^3}{2(1-j^2)},\end{aligned}$$

where we have used Young's inequality in the second inequality. Thus,

$$\int_0^1 \left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x\right|^2 dx \leq \frac{C^2\|U\|_{L^\infty}^6}{(1-j^2)^2} \quad \text{for any } h > 0.$$

This inequality together with Poincaré's inequality leads to

$$\begin{aligned}\int_0^1 \left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]^2 dx &\leq \int_0^1 \left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]_x\right|^2 dx\\ &\leq \frac{C^2\|U\|_{L^\infty}^6}{(1-j^2)^2} \quad \text{for any } h > 0.\end{aligned} \tag{20}$$

Now letting $h \to 0^+$, one can see that if $e^+(x) \neq 0$ for some $x \in (0,1)$, then

$$\lim_{h\to 0^+}\int_0^1 \left|\left[\ln\left(1+\frac{e^+(x)}{h}\right)\right]\right|^2 dx = \infty,$$

which is a contradiction to (20). Therefore, $U(x) \geq V(x)$ over $[0,1]$. $\square$

Let us now prove the well-posedness of (17).

LEMMA 2.3. *Assume that* $b(x) \in L^\infty(0,1)$ *and* $\underline{b} > 1$*; then* (17) *admits a unique weak solution* ρ_j *satisfying* $\rho_j - 1 \in H_0^1(0,1)$ *and*

$$1 + m\sin(\pi x) \le \rho_j(x) \le \bar{b}, \quad x \in [0,1], \tag{21}$$

where $m = m(\tau, \underline{b})$ *is a positive constant independent of* j.

Remark 2.4. In [9], Degond and Markowich also obtained the uniqueness of the subsonic solution, but they needed to restrict the current density to be sufficiently small $j \ll 1$ (the completely subsonic case). Here, we still have the uniqueness of the subsonic solution for any j with $0 < j < 1$.

Proof. Because $0 < j < 1$, the fluid velocity of (17) is j/ρ_j, which is subsonic if $\rho_j \ge 1$. In other words, (17) is uniformly elliptic for $\rho_j \ge 1$. Recalling Theorem 1 of [9], (17) has a subsonic weak solution $\rho_j \in H^2(0,1)$ satisfying $1 \le \rho_j(x) \le \bar{b}$. Thus, we only need to show that such ρ_j is unique for any $0 < j < 1$, and to establish the lower bound estimate in (21).

Suppose that there are two solutions u and v satisfying $u, v \ge 1$, $u, v \in H^2(0,1)$. By the Sobolev imbedding theorem, u, $v \in C^1[0,1]$. Hence, the comparison principle (Lemma 2.2) gives $u(x) = v(x)$ over $[0,1]$.

We now derive the lower bound estimate for $\rho_j(x)$. Denote

$$q(x) := 1 + m\sin(\pi x),$$

where $m > 0$ is a constant to be determined later. Since $0 < j < 1$, it is easy to calculate that

$$-\left[\left(\frac{1}{q} - \frac{j^2}{q^3}\right) q_x\right]_x - \left(\frac{j}{\tau q}\right)_x + (q-b) \le C(m^2+m) + (1-b) \le C(m^2+m) + (1-\underline{b}) < 0$$

if m is small enough such that $C(m^2 + m) < (\underline{b} - 1)$. Here $C = C(\tau)$ is a positive constant independent of j. Thus, by Lemma 2.2 again, we have $\rho_j(x) \ge q(x) = 1 + m\sin(\pi x)$ on $[0,1]$. ∎

Proof of Theorem 2.1. Multiplying (17) by $(\rho_j - 1)$, we have

$$\begin{aligned}(1-j^2)\int_0^1 \frac{|(\rho_j)_x|^2}{(\rho_j)^3}dx + \frac{4}{9}\int_0^1 \frac{(\rho_j+1)}{(\rho_j)^3}\cdot |((\rho_j-1)^{\frac{3}{2}})_x|^2 dx \\ + \frac{j}{\tau}\int_0^1 \frac{(\rho_j)_x}{\rho_j}dx + \int_0^1 (\rho_j - b)(\rho_j - 1)dx = 0.\end{aligned} \tag{22}$$

Noting that

$$\frac{j}{\tau}\int_0^1 \frac{(\rho_j)_x}{\rho_j}dx = \frac{j}{\tau}\int_0^1 (\ln\rho_j)_x dx = 0,$$

$$\begin{aligned}\int_0^1 (\rho_j - b)(\rho_j - 1)dx &= \int_0^1 (\rho_j - 1)^2 dx + \int_0^1 (1-b)(\rho_j - 1)dx \\ &\ge \frac{1}{2}\int_0^1 (\rho_j - 1)^2 dx - \frac{1}{2}\int_0^1 (b-1)^2 dx,\end{aligned}$$

$0 < j < 1$, and $1 \le \rho_j \le \bar{b}$, it follows from (22) that

$$\frac{(1-j^2)}{\bar{b}^3}\int_0^1 |(\rho_j)_x|^2 dx + \frac{8}{9\bar{b}^3}\int_0^1 \left|((\rho_j-1)^{\frac{3}{2}})_x\right|^2 dx + \frac{1}{2}\int_0^1 (\rho_j-1)^2 dx \le \frac{1}{2}\int_0^1 [b(x)-1]^2 dx,$$

which gives

$$\left\|(\rho_j-1)^{\frac{3}{2}}\right\|_{H^1} \le C \quad \text{and} \quad \left\|(1-j^2)(\rho_j)_x\right\|_{L^2} \le C(1-j^2)^{\frac{1}{2}}. \tag{23}$$

Thus, by the compact imbedding $H^1(0,1) \hookrightarrow C^{1/2}[0,1]$, there exists a function ρ such that, as $j \to 1^-$, up to a subsequence,

$$(\rho_j-1)^{\frac{3}{2}} \rightharpoonup (\rho-1)^{\frac{3}{2}} \text{ weakly in } H^1(0,1), \tag{24}$$

$$(\rho_j-1)^{\frac{3}{2}} \to (\rho-1)^{\frac{3}{2}} \text{ strongly in } C^{\frac{1}{2}}[0,1], \tag{25}$$

$$(1-j^2)(\rho_j)_x \to 0 \text{ strongly in } L^2(0,1). \tag{26}$$

Observing that $((\rho_j-1)^2)_x = \frac{4}{3}(\rho_j-1)^{\frac{1}{2}}((\rho_j-1)^{\frac{3}{2}})_x$, we get from (23) that

$$\left\|(\rho_j-1)^2\right\|_{H^1} = \left\|(\rho_j-1)^2\right\|_{L^2} + \left\|((\rho_j-1)^2)_x\right\|_{L^2} \le C\left\|(\rho_j-1)^{\frac{3}{2}}\right\|_{H^1} \le C,$$

which leads to

$$(\rho_j-1)^2 \rightharpoonup (\rho-1)^2 \text{ weakly in } H^1(0,1) \text{ as } j \to 1^-. \tag{27}$$

Now we multiply (17) by $\varphi \in H_0^1(0,1)$ to derive

$$\frac{1}{2}\int_0^1 \frac{\rho_j+1}{\rho_j^3}[(\rho_j-1)^2]_x\varphi_x dx + \int_0^1 \frac{1}{\rho_j^3}(1-j^2)(\rho_j)_x\varphi_x dx + \frac{j}{\tau}\int_0^1 \frac{\varphi_x}{\rho_j}dx + \int_0^1 [\rho_j(x)-b(x)]\varphi dx = 0.$$

Letting $j \to 1^-$, and applying (25)–(27), we prove the existence of weak solution $\rho(x) = \rho_{sub}(x)$ satisfying (8). Since m presented in (21) is independent of j, then the lower bound estimate in (16) immediately follows from (21) and (25).

To prove the uniqueness of the interior subsonic solution, we first need to investigate the regularity of $w(x)$ defined by $w(x) := (\rho(x)-1)^2$. Clearly, $w \in H_0^1(0,1)$. From (7), it can be verified that w satisfies

$$\left(\frac{(2+\sqrt{w(x)})w_x}{2(1+\sqrt{w(x)})^3} + \frac{1}{\tau(1+\sqrt{w(x)})}\right)_x - (\sqrt{w(x)}+1-b) = 0, \quad x \in (0,1). \tag{28}$$

For simplicity, we set

$$f_1(x) := \frac{2+\sqrt{w(x)}}{(1+\sqrt{w(x)})^3}, \quad f_2(x) := \frac{1}{1+\sqrt{w(x)}}, \quad f_3(x) := \frac{f_1(x)w_x(x)}{2} + \frac{f_2(x)}{\tau}.$$

Because (28) holds in the sense of distribution, we have $f_3 \in H^1(0,1)$. By the Sobolev imbedding theorem, we have $w, f_3 \in C^{1/2}[0,1]$. Since $w \ge 0$ on $[0,1]$, then

$$|\sqrt{w(y)} - \sqrt{w(x)}| = \frac{|w(y)-w(x)|}{\sqrt{w(y)}+\sqrt{w(x)}} \le \frac{|w(y)-w(x)|}{\sqrt{|w(y)-w(x)|}} \le C|y-x|^{1/4}.$$

On the other hand, for any $x, y \in [0,1]$, it holds that

$$f_2(x) - f_2(y) = \frac{1}{1+\sqrt{w(x)}} - \frac{1}{1+\sqrt{w(y)}} = \frac{\sqrt{w(y)} - \sqrt{w(x)}}{(1+\sqrt{w(x)})(1+\sqrt{w(y)})}.$$

Thus,

$$|f_2(x) - f_2(y)| \le |\sqrt{w(y)} - \sqrt{w(x)}| \le C|y-x|^{1/4}.$$

This means $f_2 \in C^{1/4}[0,1]$. Similarly, we have $f_1 \in C^{1/4}[0,1]$. Notice that $w_x = \frac{2f_3 - 2f_2/\tau}{f_1} \in C^{1/4}[0,1]$; then

$$w \in C^{1+1/4}[0,1]. \tag{29}$$

Now, integrating (28) over $[0,x]$ and setting

$$G_w(x) := \frac{(2+\sqrt{w(x)})w_x(x)}{2(1+\sqrt{w(x)})^3} + \frac{1}{\tau(1+\sqrt{w(x)})},$$

we have

$$\begin{cases} \dfrac{(2+\sqrt{w})w_x}{2(1+\sqrt{w})^3} = G_w - \dfrac{1}{\tau(1+\sqrt{w})}, \\ G_w(x) = G_w(0) + \displaystyle\int_0^x [\sqrt{w(s)} + 1 - b(s)]ds. \end{cases} \tag{30}$$

We are now ready to prove the uniqueness of the interior subsonic solution. Suppose $\rho_1(x)$ and $\rho_2(x)$ are two different interior subsonic solutions to (7). Thus, there exists at least a number $z \in (0,1)$ such that $\rho_1(z) \neq \rho_2(z)$. Without loss of generality, we may assume that $\rho_1(z) > \rho_2(z)$; then $w_1(z) > w_2(z)$. Since $w_1, w_2 \in C^{1+1/4}[0,1]$, there exists a maximal interval $[a,c] \subset [0,1]$ such that $z \in (a,c)$,

$$w_1(a) = w_2(a), \ w_1(c) = w_2(c), \quad \text{and} \quad w_1(x) > w_2(x), \ x \in (a,c).$$

Obviously, it holds that

$$(w_1)_x(a) = \lim_{x\to a^+} \frac{w_1(x) - w_1(a)}{x-a} \ge \lim_{x\to a^+} \frac{w_2(x) - w_2(a)}{x-a} = (w_2)_x(a), \tag{31}$$

$$(w_1)_x(c) = \lim_{x\to c^-} \frac{w_1(x) - w_1(c)}{x-c} \le \lim_{x\to c^-} \frac{w_2(x) - w_2(c)}{x-c} = (w_2)_x(c). \tag{32}$$

Owing to (32) and the first equation of (30),

$$G_{w_1}(c) \le G_{w_2}(c).$$

Substituting this inequality into the second equation of (30), we have

$$G_{w_1}(a) + \int_a^c \left[\sqrt{w_1(x)} + 1 - b(x)\right]dx \le G_{w_2}(a) + \int_a^c \left[\sqrt{w_2(x)} + 1 - b(x)\right]dx.$$

Since $w_1(x) > w_2(x)$ over (a,c), then

$$G_{w_1}(a) < G_{w_2}(a).$$

Using the first equation of (30) again, we obtain

$$(w_1)_x(a) < (w_2)_x(a),$$

which contradicts (31). Therefore, $\rho_1(x) = \rho_2(x)$ over $[0,1]$, namely, the interior subsonic solution $\rho_{sub}(x)$ is unique. ∎

We proceed to study the regularity of this interior subsonic solution.

PROPOSITION 2.5. $\rho_{sub} \in C^{1/2}[0,1]$, *and there exist* $0 < s_1 < 1$, C_i $(i = 1,2,3,4)$ *such that*

$$\begin{aligned} C_1(1-x)^{1/2} < \rho_{sub}(x) - 1 < C_2(1-x)^{1/2} \\ -C_3(1-x)^{-1/2} < (\rho_{sub})_x(x) < -C_4(1-x)^{-1/2} \end{aligned} \quad \text{for } x \in [1-s_1, 1]. \tag{33}$$

Remark 2.6. This proposition indicates that $\frac{1}{2}$ is the optimal exponent in Hölder space for the global regularity of the unique interior subsonic solution $\rho_{sub}(x)$. And the derivative of the approximate subsonic solution sequence $\{\rho_j\}_{0<j<1}$ constructed in Lemma 2.3 blows up as $j \to 1^-$ at $x = 1$, namely, $\lim_{j\to 1^-} \rho_j'(1) = -\infty$.

Proof. For convenience, we denote by ρ the interior subsonic solution of (5). By (29), we have $(\rho-1)^2 = w \in C^1[0,1]$. Since $\rho \geq 1$ on $[0,1]$, then

$$|\rho(x)-1+\rho(y)-1| = |\rho(x)-1|+|\rho(y)-1| \geq |(\rho(x)-1)-(\rho(y)-1)| = |\rho(x)-\rho(y)|.$$

Thus, we have

$$\frac{|\rho(x)-\rho(y)|^2}{|x-y|} = \frac{|\rho(x)-\rho(y)||(\rho(x)-1)^2-(\rho(y)-1)^2|}{|x-y||\rho(x)-1+\rho(y)-1|} \leq \frac{|w(x)-w(y)|}{|x-y|} \leq C$$

for any $x, y \in [0,1]$, which indicates that $\rho \in C^{1/2}[0,1]$.

Now we are going to prove the estimates in (33). We first claim $E(1) < \frac{1}{\tau}$. Otherwise, if $E(1) \geq \frac{1}{\tau}$, then it will imply a contradiction. In fact, since $\rho \in C[0,1]$ and $\rho(1) = 1 < \underline{b} \leq b(x)$ for $x \in [0,1]$, there exists $\hat{\epsilon} > 0$ such that $\rho(x) - b(x) < 0$ for a.e. $x \in [1-\hat{\epsilon}, 1]$. By integrating the second equation of (5) over $[x,1]$ for $x \in [1-\hat{\epsilon},1]$, we have

$$E(x) = E(1) - \int_x^1 [\rho(s) - b(s)]ds > E(1) \geq \frac{1}{\tau} \quad \text{for } x \in [1-\hat{\epsilon}, 1].$$

Noting $\rho(x) > 1$ over $(0,1)$, we have $E(x) - \frac{1}{\tau\rho(x)} > \frac{1}{\tau}\big(1 - \frac{1}{\rho(x)}\big) \geq 0$ for $x \in [1-\hat{\epsilon}, 1]$. It then follows from the first equation of (5) that $\rho_x(x) > 0$ on $[1-\hat{\epsilon}, 1]$, which contradicts the fact that $\rho(1) = 1$ and $\rho(x) > 1$ over $(0,1)$.

Now let $q := E(1) - \frac{1}{\tau}$; then $q < 0$. Based on the continuity of the function $\big(E(x) - \frac{1}{\tau\rho(x)}\big)$, there exists a number $0 < s_1 < \hat{\epsilon}$ such that

$$\frac{3q}{2} \leq E(x) - \frac{1}{\tau\rho(x)} \leq \frac{q}{2} < 0 \quad \text{for } x \in [1-s_1, 1]. \tag{34}$$

From the first equation of (5), we have

$$E(x) - \frac{1}{\tau\rho(x)} = \Big(1 - \frac{1}{\rho^2}\Big)\frac{\rho_x}{\rho} = \frac{\rho+1}{\rho^3}(\rho-1)\rho_x = \frac{\rho+1}{2\rho^3}\Big((\rho-1)^2\Big)_x.$$

Applying (34) to the above equation, we then have

$$\frac{3q\rho^3(x)}{\rho(x)+1} \leq \Big((\rho-1)^2\Big)_x = \Big[E(x) - \frac{1}{\tau\rho(x)}\Big]\frac{2\rho^3(x)}{\rho(x)+1} \leq \frac{q\rho^3(x)}{\rho(x)+1} < 0 \text{ for } x \in [1-s_1, 1].$$

Applying (16) to the above inequalities, we can estimate

$$\frac{3q\bar{b}^3}{2} < \Big((\rho(x)-1)^2\Big)_x < \frac{q}{\bar{b}+1} < 0 \quad \text{for } x \in [1-s_1, 1]. \tag{35}$$

Integrating (35) over $[x, 1]$ for $x \in [1 - s_1, 1]$, we get

$$C_1(1-x)^{\frac{1}{2}} < \rho(x) - 1 < C_2(1-x)^{\frac{1}{2}} \quad \text{for } x \in [1-s_1, 1], \tag{36}$$

with

$$C_1 := \sqrt{\frac{|q|}{\bar{b}+1}} \quad \text{and} \quad C_2 := \sqrt{\frac{3|q|\bar{b}^3}{2}}.$$

Furthermore, from (35), we have

$$\frac{3q\bar{b}^3}{4(\rho(x)-1)} < \rho_x(x) < \frac{q}{2(\bar{b}+1)(\rho(x)-1)} < 0 \quad \text{for } x \in [1-s_1, 1].$$

This with (36) together implies

$$-C_3(1-x)^{-\frac{1}{2}} < \rho_x(x) < -C_4(1-x)^{-\frac{1}{2}}, \quad x \in [1-s_1, 1],$$

for some positive constants C_3 and C_4. The proof is complete. ∎

3. Existence of interior supersonic solutions.

We next prove the existence of interior supersonic solutions of (7).

THEOREM 3.1. *Assume that $b \in L^\infty(0,1)$ and $\underline{b} > 1$; then (7) admits an interior supersonic solution $\rho_{sup}(x)$ satisfying $\ell \le \rho_{sup}(x) \le 1$ over $[0,1]$ for some positive constant ℓ. Moreover, ρ_{sup} satisfies the following properties:*

1. *For any $\frac{1}{2} > \epsilon > 0$, there exists a number $\delta > 0$ such that $\rho_{sup}(x) \le 1 - \delta$ for any $x \in [\epsilon, 1-\epsilon]$.*
2. *ρ_{sup} has only one critical point z_0 over $(0,1)$ such that $(\rho_{sup})_x < 0$ on $(0, z_0)$ and $(\rho_{sup})_x > 0$ on $(z_0, 1)$; i.e., z_0 is the minimal point.*

As shown in the proof of Theorem 2.1, we consider the approximate equation

$$\begin{cases} \left[\left(\frac{1}{\rho_k} - \frac{k^2}{(\rho_k)^3}\right)(\rho_k)_x\right]_x + \left(\frac{k}{\tau \rho_k}\right)_x - [\rho_k(x) - b(x)] = 0, \quad x \in (0,1), \\ \rho_k(0) = \rho_k(1) = 1, \end{cases} \tag{37}$$

but with the parameter $1 < k < \infty$.

LEMMA 3.2. *Let the doping profile be subsonic with $b(x) \in L^\infty(0,1)$ and $\underline{b} > 1$. Then (37) admits a weak solution $\rho_k(x)$ satisfying*

$$\rho_k \in H^1(0,1) \text{ and } 0 < \rho_k(x) \le 1 \text{ over } [0,1]. \tag{38}$$

Remark 3.3. Peng and Violet [26] showed that if k is large enough, then (37) has a supersonic solution. Our Lemma 3.2 further shows that, in the case of the subsonic doping profile, for all $1 < k < \infty$, (37) has a supersonic solution. Thus, our result essentially improves the previous study in [26].

Proof. The velocity $u_k(x) = \frac{k}{\rho_k(x)}$ satisfies

$$\begin{cases} \left[\left(u_k - \frac{1}{u_k}\right)(u_k)_x\right]_x + \frac{(u_k)_x}{\tau} - \left(\frac{k}{u_k} - b\right) = 0, \quad x \in (0,1). \\ u_k(0) = u_k(1) = k. \end{cases} \tag{39}$$

So we only need to show that (39) has a weak solution $u_k \in H^1(0,1)$ satisfying $k \le u_k < \infty$. To this end, we define an operator $\mathcal{T} : \psi \to u$ by solving the following linear elliptic equation:

$$\begin{cases} \left[\left(\psi - \frac{1}{\psi}\right) u_x\right]_x + \frac{u_x}{\tau} - \left(\frac{k}{\psi} - b\right) = 0, \quad x \in (0,1), \\ u(0) = u(1) = k. \end{cases} \tag{40}$$

Set

$$\mathcal{X} := \big\{\psi(x) : \psi \in C^1[0,1],\ k \le \psi(x) \le M,\ \psi(0) = \psi(1) = k, \\ \|\psi\|_{C^\alpha[0,1]} \le \Lambda,\ \|\psi\|_{C^1[0,1]} \le \Upsilon(\Lambda)\big\},$$

where $0 < \alpha < 1/2$, M, Λ, and $\Upsilon(\Lambda)$ are some positive constants to be determined later. Suppose that $\psi \in \mathcal{X}$. By the L^2 theory of elliptic equations and the Sobolev imbedding theorem, we see that (40) has a unique solution $u \in C^{1+\alpha}[0,1]$ for $0 < \alpha < 1$. Multiplying (40) by $(u-k)^-(x) := \min\{0, (u-k)(x)\}$, we have

$$\int_0^1 \left(\psi - \frac{1}{\psi}\right) |[(u-k)^-]_x|^2 dx - \frac{1}{\tau}\int_0^1 u_x (u-k)^- dx \\ + \int_0^1 \left(\frac{k}{\psi} - b\right)(u-k)^- dx = 0. \tag{41}$$

Because $k > 1$ and $\psi \ge k$, we have $\psi - \frac{1}{\psi} \ge k - 1 > 0$, and noting that

$$\frac{1}{\tau}\int_0^1 u_x (u-k)^- dx = \frac{1}{2\tau}\int_0^1 ([(u-k)^-]^2)_x dx = 0,$$

it follows from (41) that

$$(k-1)\int_0^1 |[(u-k)^-]_x|^2 dx + \int_0^1 \left(\frac{k}{\psi} - b\right)(u-k)^- dx \le 0. \tag{42}$$

This inequality in combination with the fact that $\frac{k}{\psi(x)} - b(x) < 0$ gives $(u-k)^-(x) = 0$ for all $x \in [0,1]$. Thus, $u(x) \ge k$ over $[0,1]$. Now multiplying (40) by $(u-k)$, just as shown in (42), and using Young's inequality and Poincaré's inequality, we get

$$\begin{aligned} (k-1)\int_0^1 |(u-k)_x|^2 dx &\le \int_0^1 \left(b - \frac{1}{\psi}\right)(u-k) dx \\ &\le \frac{k-1}{2}\int_0^1 (u-k)^2 dx + \frac{1}{2(k-1)}\int_0^1 \left(b(x) - \frac{k}{\psi}\right)^2 dx \\ &\le \frac{k-1}{2}\int_0^1 |(u-k)_x|^2 dx + \frac{1}{2(k-1)}\int_0^1 b^2(x) dx. \end{aligned}$$

It then follows that

$$\|u_x\|_{L^2(0,1)} \le \frac{\|b\|_{L^2}}{k-1}.$$

Furthermore, a straightforward computation yields

$$0 < u(x) \le k + \frac{\|b\|_{L^2}}{k-1}.$$

Thus, the compact imbedding of $H^1(0,1)$ into $C^{\alpha_0}[0,1]$ with $0<\alpha_0<1/2$ gives

$$\|u\|_{C^{\alpha_0}[0,1]} \le C_0(k, \|b\|_{L^2}) \text{ for a constant } C_0 > 0.$$

Hence we determine $M = 1 + \frac{\|b\|_{L^2}}{k-1}$, $\alpha = \alpha_0$, and $\Lambda = C_0(k, \|b\|_{L^2})$. By the Hölder estimate for the first order derivative of divergence-form elliptic equation [14], we derive

$$\|u\|_{C^{1+\alpha}[0,1]} \le C_1(k, \|b\|_{L^2}, \Lambda).$$

Now we take $\Upsilon(\Lambda) = C_1(k, \|b\|_{L^2}, \Lambda)$ with $\Lambda = C_0(k, \|b\|_{L^2})$. Then it is easy to see that $u \in \mathcal{X}$ and $\mathcal{X}$ is a nonempty bounded and closed convex set in $C^1[0,1]$. On the other hand, by the Arzelà–Ascoli theorem, the imbedding $C^{1+\alpha}[0,1] \hookrightarrow C^1[0,1]$ is compact. Thus, the operator $\mathcal{T}$ is a compact map of $\mathcal{X}$ into itself. By the Schauder fixed point theorem (see Corollary 2.3.10 in [6]), there exists a fixed point $u \in \mathcal{X}$ such that

$$\mathcal{T}(u) = u.$$

Therefore, (39) has a weak solution $u_k \in C^1[0,1]$, and $\rho_k(x) = k/u_k(x)$ is a desired weak supersonic solution of (37). ∎

Proof of Theorem 3.1. Multiplying (39) by $(u_k - k)$ and using Young's inequality, we have

$$\begin{aligned}
&(k-1)\int_0^1 \frac{u_k+1}{u_k}|(u_k)_x|^2dx + \frac{4}{9}\int_0^1 \frac{u_k+1}{u_k}|[(u_k-k)^{3/2}]_x|^2dx \\
&= \int_0^1 \left(b - \frac{k}{u_k}\right)(u_k-k)dx \\
&\le \frac{1}{3}\int_0^1 (u_k-k)^3dx + \frac{2}{3}\int_0^1 \left(b-\frac{k}{u_k}\right)^{3/2}dx \\
&\le \frac{1}{3}\int_0^1 |[(u_k-k)^{3/2}]_x|^2dx + \frac{2}{3}\int_0^1 b^{3/2}(x)dx.
\end{aligned}$$

Thus, we have

$$\|(k-1)^{\frac{1}{2}}(u_k)_x\|_{L^2} + \|(u_k-k)^{\frac{3}{2}}\|_{H^1} \le C \tag{43}$$

for a constant C independent of k, where we have used $k>1$ and $u_k \ge k$. This inequality together with the Sobolev imbedding theorem yields

$$\|u_k\|_{L^\infty} \le k + C^{\frac{2}{3}}. \tag{44}$$

Hence

$$\rho_k(x) = \frac{k}{u_k(x)} \ge \frac{k}{\|u_k\|_{L^\infty}} \ge \frac{k}{k+C^{\frac{2}{3}}} \ge \frac{1}{1+C^{\frac{2}{3}}} \triangleq \ell \quad \text{for all } x \in [0,1]. \tag{45}$$

A direct calculation yields

$$(\rho_k)_x = -\frac{k(u_k)_x}{u_k^2} \quad \text{and} \quad ((1-\rho_k)^2)_x = \frac{4k(u_k-1)^{\frac{1}{2}}((u_k-1)^{\frac{3}{2}})_x}{3u_k^3}.$$

It then follows from (43) and (44) that

$$\|(1-\rho_k)^2\|_{H^1} + \|(1-\rho_k)^{3/2}\|_{H^1} \le C_1,$$
$$\|(k-1)(\rho_k)_x\|_{L^2} \le C_1(k-1)^{\frac{1}{2}}.$$

Thus, there exists a function $\rho_{sup}(x)$ such that, as $k \to 1^+$, up to a subsequence,

$$\begin{aligned}
&(1-\rho_k)^2 \rightharpoonup (1-\rho_{sup})^2 \ \text{ weakly in } H^1(0,1),\\
&(1-\rho_k)^{3/2} \rightharpoonup (1-\rho_{sup})^{3/2} \ \text{ weakly in } H^1(0,1),\\
&(1-\rho_k)^{3/2} \to (1-\rho_{sup})^{3/2} \ \text{ strongly in } C^{\frac{1}{2}}[0,1],\\
&(k-1)(\rho_k)_x \to 0 \ \text{ strongly in } L^2(0,1).
\end{aligned} \tag{46}$$

Applying the same procedure as in the proof of Theorem 2.1, one can show that ρ_{sup} satisfies (8). The lower bound of ρ_{sup} follows from (45) and the third convergence of (46).

Let us now prove that $\rho_{sup}(x) < 1$ for any interior point $x \in (0,1)$. We observe that if a function ρ satisfies $\rho(x) \equiv 1$ on an interval $[\hat{a},\hat{c}] \subset [0,1]$, then ρ is not a solution of (7) because $\underline{b} > 1$. Thus, for any $1 \gg \epsilon > 0$, there exist a $\delta > 0$ and two points $\hat{a}_\epsilon \in (0,\epsilon]$ and $\hat{c}_\epsilon \in [1-\epsilon, 1)$ such that $\rho_{sup}(\hat{a}_\epsilon), \rho_{sup}(\hat{c}_\epsilon) \le 1-\delta < 1$. We only need to show that $\rho_{sup}(x) \le 1-\delta$ over $[\hat{a}_\epsilon,\hat{c}_\epsilon]$. Actually, set $w := (1-\rho_{sup})^2$; then $w \in H_0^1(0,1)$, $w(\hat{a}_\epsilon), w(\hat{c}_\epsilon) \ge \delta^2$, and it follows from (8) that for any $\varphi \in H_0^1(\hat{a}_\epsilon,\hat{c}_\epsilon)$

$$\frac{1}{2}\int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{2-\sqrt{w}}{(1-\sqrt{w})^3} w_x\varphi_x dx + \frac{1}{\tau}\int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{\varphi_x}{1-\sqrt{w}} dx + \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} (1-\sqrt{w}-b)\varphi dx = 0.$$

Taking $\varphi(x) = (w-\delta^2)^-(x)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{2-\sqrt{w}}{(1-\sqrt{w})^3} |[(w-\delta^2)^-]_x|^2 dx + \frac{1}{\tau}\int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} \frac{[(w-\delta^2)^-]_x}{1-\sqrt{w}} dx\\
&+ \int_{\hat{a}_\epsilon}^{\hat{c}_\epsilon} (1-\sqrt{w}-b)(w-\delta^2)^- dx = 0.
\end{aligned}$$

Observing that $\rho_{sup} \ge \ell$, hence $2-\sqrt{w} > 1-\sqrt{w} \ge \ell > 0$. This implies that the first term of the equality is nonnegative. Because $b > \underline{b} > 1$, the third term is also nonnegative. On the other hand, a simple computation gives $-2(\sqrt{w}+\ln(1-\sqrt{w}))_x = \frac{w_x}{1-\sqrt{w}}$, which implies that the second term is zero. Thus, $(w-\delta^2)^-(x) = 0$ over $[\hat{a}_\epsilon,\hat{c}_\epsilon]$. And as a result, $\rho_{sup}(x) \le 1-\delta$ over $[\hat{a}_\epsilon,\hat{c}_\epsilon]$.

It remains to show part 2 of Theorem 3.1. We only need to show that if $z_0 \in (0,1)$ is a critical point of ρ_{sup}, then it must be a local minimal point. Because $\rho_{sup} \in C[0,1]$ and $\rho_{sup} < 1$ over $(0,1)$, by the interior regularity theory of elliptic equations and the Sobolev imbedding, for any $z_0 \in (0,1)$, there exists an interval $I \subset (0,1)$ such that $z_0 \in I$, $\rho_{sup} \in W^{2,p}(I)$ for any $1 < p < \infty$, and $\rho_{sup} \in C^1(\bar{I})$. Now if z_0 is a critical point, then $(\rho_{sup})_x(z_0) = 0$. Since $\rho_{sup} \in C^1(\bar{I})$, there exists a $\delta > 0$ such that

$$|(\rho_{sup})_x(x)| < \frac{\tau(\underline{b}-1)}{2} \ \text{ for any } x \in (z_0-\delta, z_0+\delta).$$

If $x \in (z_0, z_0+\delta)$, we integrate (7) over (z_0, x) to derive

$$\begin{aligned}\left(\frac{1}{\rho_{sup}} - \frac{1}{\rho_{sup}^3}\right)(\rho_{sup})_x &= \int_{z_0}^{x}\left[\rho_{sup} - b + \frac{(\rho_{sup})_x}{\tau\rho_{sup}^2}\right] ds\\ &< \int_{z_0}^{x}\left(1 - \underline{b} + \frac{|(\rho_{sup})_x|}{\tau\rho_{sup}^2}\right) ds\\ &< \int_{z_0}^{x}\left(1 - \underline{b} + \frac{\underline{b}-1}{2}\right) ds\\ &= \frac{(1-\underline{b})(x-z_0)}{2} < 0,\end{aligned}$$

where we have used $(\rho_{sup})_x(z_0) = 0$ and $\rho_{sup} < 1$. Thus,

$$(\rho_{sup})_x(x) > 0 \text{ on } (z_0, z_0+\delta).$$

Similarly, integrating (7) over (x, z_0) with $x \in (z_0-\delta, z_0)$, one can get that

$$(\rho_{sup})_x(x) < 0 \text{ on } (z_0-\delta, z_0).$$

Therefore, z_0 is a local minimal point of ρ_{sup}. The proof is complete. ∎

As in Proposition 2.5, we also study the optimal global regularity of the interior supersonic solution.

PROPOSITION 3.4. $\rho_{sup} \in C^{1/2}[0,1]$, *and there exist* $s_2 \ll 1$, C_i $(i = 5,6,7,8)$ *such that*

$$\begin{aligned}-C_5x^{1/2} < \rho_{sup} - 1 < -C_6x^{1/2}\\ -C_7x^{-1/2} < (\rho_{sup})_x < -C_8x^{-1/2}\end{aligned} \quad \text{for } x \in [0, s_2]. \tag{47}$$

Proof. The proof is similar to that of Proposition 2.5. Here for supersonic solutions, we need the local analysis for the solution near $x = 0$. We omit the details. ∎

4. Infinitely many transonic shock solutions. We turn to study the existence of transonic solutions of (5)–(6). We first consider Euler–Poisson equations (5) with constant doping profile $\underline{b}$ but without the semiconductor effect (namely, $\frac{1}{\tau} = 0$, or, say, $\tau = \infty$), and the imposed boundary condition is completely supersonic. That is,

$$\begin{cases}\left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E,\\ E_x = \rho - \underline{b},\\ \rho(0) = \rho(L) = 1-\delta \quad \text{(supersonic boundary)},\end{cases} \tag{48}$$

where $L \geq \frac{1}{4}$ is the parameter of length and $\delta > 0$ is a small constant. As shown in the proof of Theorem 3.1, for any $\delta > 0$, (48) has a supersonic solution. We have the following uniform estimates with respect to δ for the supersonic solutions of (48).

LEMMA 4.1. *Assume that* $\underline{b} > 1$, *and that* $(\rho_L, E_L)(x)$ *are supersonic solutions of* (48). *Then*

$$\beta(L, \underline{b}) \leq \min_{x\in[0,L]} \rho_L(x) \leq \gamma(L, \underline{b}) \quad \textit{and} \quad E_L(0) \geq C(L, \underline{b}),$$

where $\beta(L, \underline{b})$, $\gamma(L, \underline{b})$, *and* $C(L, \underline{b})$ *are positive constants independent of* δ.

Proof. For convenience, we denote (ρ_L, E_L) by (ρ, E). In the phase-plane (ρ, E), we have

$$\frac{dE}{d\rho} = \frac{(\rho+1)(\rho-\underline{b})(\rho-1)}{E\rho^3}.$$

Integrating the above equation with respect to ρ, we obtain the part of trajectory through $(1-\delta, E(0))$ as follows:

$$\frac{E^2(x)}{2} = \frac{E^2(0)}{2} - \frac{2\rho(0)-\underline{b}}{2\rho^2(0)} - \rho(0) + \underline{b}\ln\rho(0) + \frac{2\rho(x)-\underline{b}}{2\rho^2(x)} + \rho(x) - \underline{b}\ln\rho(x) \tag{49}$$

and

$$E(x) = \pm\sqrt{2}\sqrt{\frac{E^2(0)}{2} - \frac{2\rho(0)-\underline{b}}{2\rho^2(0)} - \rho(0) + \underline{b}\ln\rho(0) + \frac{2\rho(x)-\underline{b}}{2\rho^2(x)} + \rho(x) - \underline{b}\ln\rho(x)}.$$

Thus, the supersonic solution obtained satisfies $0 < \rho(x) < 1-\delta$ and is symmetric in $x \in (0, L)$. Set $\underline{\rho} := \min_{x\in[0,L]}\rho(x)$. By the symmetry of $\rho(x)$ in $(0, L)$, we know that $\rho(x)$ reaches its minimum at $x = \frac{L}{2}$. Thus,

$$\underline{\rho} = \rho(L/2) \quad \text{and} \quad \rho'(L/2) = 0. \tag{50}$$

We next estimate $\underline{\rho}$. The velocity $u(x) = 1/\rho(x)$ satisfies $u(x) \geq \frac{1}{1-\delta}$ and

$$\left(\left(u - \frac{1}{u}\right)u_x\right)_x = \frac{1-\underline{b}u}{u}, \quad u(0) = u(L) = \frac{1}{1-\delta}. \tag{51}$$

Multiplying (51) by $(u - \frac{1}{1-\delta})^2$, we get

$$2\int_0^L \left(u - \frac{1}{u}\right)\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx = \int_0^L \frac{\underline{b}u - 1}{u}\left(u - \frac{1}{1-\delta}\right)^2 dx. \tag{52}$$

Artfully, we can reduce the left-hand side of (52) to

$$\begin{aligned}
&2\int_0^L \left(u - \frac{1}{u}\right)\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx \\
&= 2\int_0^L \frac{u+1}{u}(u-1)\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx \\
&= 2\int_0^L \frac{u+1}{u}\left(\frac{\delta}{1-\delta} + u - \frac{1}{1-\delta}\right)\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx \\
&= \frac{2\delta}{1-\delta}\int_0^L \frac{u+1}{u}\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx \\
&\quad + 2\int_0^L \frac{u+1}{u}\left(u - \frac{1}{1-\delta}\right)^2 (u_x)^2 dx \\
&= \frac{2\delta}{1-\delta}\int_0^L \frac{u+1}{u}\left(u - \frac{1}{1-\delta}\right)(u_x)^2 dx \\
&\quad + \frac{1}{2}\int_0^L \frac{u+1}{u}\left|\left(\left(u - \frac{1}{1-\delta}\right)^2\right)_x\right|^2 dx.
\end{aligned} \tag{53}$$

Further, by using the Cauchy–Schwarz inequality, $|ab| \le \mu a^2 + \frac{1}{4\mu} b^2$ with any $\mu > 0$, and by using Poincaré's inequality, $\|g\|_{L^2(0,L)} \le \frac{L}{\pi}\|g_x\|_{L^2(0,L)}$, where $\frac{L}{\pi}$ is the best number for Poincaré's inequality, because $\lambda := \frac{\pi^2}{L^2}$ is the minimum eigenvalue of $-g_{xx} = \lambda g$ with $g(0) = g(L) = 0$, we can estimate the right-hand side of (52) as follows:

$$
\begin{aligned}
\int_0^L \frac{\underline{b}u-1}{u}\Big(u - \frac{1}{1-\delta}\Big)^2 dx &\le \frac{1}{2L^2}\int_0^L \Big(u - \frac{1}{1-\delta}\Big)^4 dx + \frac{\underline{b}^2 L^3}{2} \\
&\le \frac{1}{2\pi^2}\int_0^L \Big|\Big(\Big(u - \frac{1}{1-\delta}\Big)^2\Big)_x\Big|^2 dx + \frac{\underline{b}^2 L^3}{2}.
\end{aligned} \tag{54}
$$

Substituting (53) and (54) to (52), we then have

$$
\frac{2\delta}{1-\delta}\int_0^L \frac{u+1}{u}\Big(u - \frac{1}{1-\delta}\Big)u_x^2 dx + \int_0^L \frac{(\pi^2-1)u+\pi^2}{2\pi^2 u}\Big|\Big(\Big(u - \frac{1}{1-\delta}\Big)^2\Big)_x\Big|^2 \le \frac{\underline{b}^2 L^3}{2},
$$

which gives

$$
\Big\|\Big(\Big(u - \frac{1}{1-\delta}\Big)^2\Big)_x\Big\|_{L^2(0,L)} \le \pi \underline{b} L\sqrt{L}. \tag{55}
$$

Notice that, for $\phi \in H_0^1(0,L)$, it holds that

$$
\|\phi\|_{L^\infty} \le \sqrt{L}\|\phi_x\|_{L^2}.
$$

Thus, from (55) we have

$$
\Big(u(x) - \frac{1}{1-\delta}\Big)^2 \le \sqrt{L}\Big\|\Big(\Big(u - \frac{1}{1-\delta}\Big)^2\Big)_x\Big\|_{L^2(0,L)} \le \pi\underline{b}L^2,
$$

which gives

$$
u(x) \le \frac{1}{1-\delta} + \sqrt{\pi\underline{b}}\cdot L.
$$

Thus, we can estimate the minimum of $\rho(x)$ by

$$
\underline{\rho} \ge \Big(\frac{1}{1-\delta} + \sqrt{\pi\underline{b}}\cdot L\Big)^{-1} \ge \Big(2 + \sqrt{\pi\underline{b}}\cdot L\Big)^{-1} \triangleq \beta(L), \text{ when } \delta \le \frac{1}{2}.
$$

On the other hand, by (48), since $\underline{b} \ge 1 > \rho$, we have

$$
\rho_{xx} = \frac{\rho^3}{\rho+1}\left[\frac{1}{\rho^2(1-\rho)}\left(\frac{3}{\rho^2} - 1\right)\rho_x^2 + \frac{\underline{b}-\rho}{(1-\rho)}\right] \ge \frac{\underline{\rho}^3}{2} \ge \frac{\beta^3(L)}{2} \text{ on } [0,L]. \tag{56}
$$

By Taylor expansion

$$
\rho(0) = \rho(L/2) - \rho'(L/2)L/2 + \rho''(\xi)(L/2)^2/2 \quad \text{with } \xi \in [0, L/2],
$$

it then follows from (50) and (56) that

$$
\underline{\rho} \le 1 - \delta - \frac{L^2\beta^3(L)}{2^4} \le 1 - \frac{L^2}{2^4}\cdot\frac{1}{\big(2 + \sqrt{\pi\underline{b}}\cdot L\big)^3} \triangleq \gamma(L). \tag{57}
$$

Since $\underline{\rho} = \rho(L/2)$ is the minimum value, from (48) and the fact $\rho_x(L/2) = 0$, we have $E(L/2) = 0$. Thus, in view of (49), we further obtain

$$\begin{aligned}\frac{E^2(0)}{2} &= \frac{2-\underline{b}-2\delta}{2(1-\delta)^2} + 1 - \delta - \underline{b}\ln(1-\delta) - \left[\frac{2\underline{\rho}-\underline{b}}{2\underline{\rho}^2} + \underline{\rho} - \underline{b}\ln\underline{\rho}\right]\\ &= \frac{\delta[2-2\underline{b}-(2-\underline{b})\delta]}{2(1-\delta)^2} - \delta - \underline{b}\ln(1-\delta) + 1\\ &\quad + \frac{(2-\underline{b})(\underline{\rho}-1)^2 + (2-2\underline{b})(\underline{\rho}-1)}{2\underline{\rho}^2} - \underline{\rho} + \underline{b}\ln\underline{\rho}\\ &\geq \frac{\delta[2-2\underline{b}-(2-\underline{b})\delta]}{2(1-\delta)^2} - \delta + f(\underline{\rho}),\end{aligned}$$

where

$$f(s) := 1 + \frac{(2-\underline{b})(s-1)^2 + (2-2\underline{b})(s-1)}{2s^2} - s + \underline{b}\ln s, \quad s \in (0,1).$$

Notice that $f(1) = 0$ and $f'(s) := -\frac{(\underline{b}-s)(1-s^2)}{s^3} < 0$ for $s \in (0,1)$, namely, $f(s)$ is decreasing and positive for $s \in (0,1)$. Using the boundness estimates carried out in (57), i.e., $\underline{\rho} \leq \gamma(L)$, when δ is small such that $\frac{\delta((2-\delta)(\underline{b}-1)+\delta)}{2(1-\delta)^2} + \delta \leq \frac{f(\gamma(L))}{2}$, we have

$$\begin{aligned}E^2(0) &\geq 2\left[\frac{-\delta((2-\delta)(\underline{b}-1)+\delta)}{2(1-\delta)^2} - \delta + f(\underline{\rho})\right]\\ &\geq 2\left[\frac{-\delta((2-\delta)(\underline{b}-1)+\delta)}{2(1-\delta)^2} - \delta + f(\gamma(L))\right]\\ &\geq f(\gamma(L)).\end{aligned} \tag{58}$$

Integrating the second equation of (48) over $[0, L/2]$, we get

$$E(0) = E(L/2) + \int_0^{L/2} (\underline{b} - \rho)dt = \int_0^{L/2} (\underline{b} - \rho)dt > 0.$$

Hence, it follows from (58) that $E(0)$ has a positive lower bound,

$$E(0) \geq \sqrt{f(\gamma(L))}, \tag{59}$$

which is independent of δ. ∎

Theorem 4.2. *If $\underline{b} > 1$, $\tau \gg 1$, and $0 \leq \bar{b} - \underline{b} \ll 1$, then system* (5)–(6) *has infinitely many transonic shock solutions over* $[0, 1]$.

Proof. The proof is technical and longer. We divide it into seven steps.

Step 1. Let η be a small number to be determined later such that $\delta < \eta \ll 1$. Denote by $(\rho_1, E_1)(x)$ the solution of (48) with $L = \frac{1}{2}$. Then by (59),

$$E_1(0) \geq \sqrt{f(\gamma(1/2))} \triangleq \Lambda_1. \tag{60}$$

Let us consider system (5) with the supersonic initial value:

$$\begin{cases}\left(1 - \frac{1}{\rho^2}\right)\rho_x = \rho E - \frac{1}{\tau},\\ E_x = \rho - b(x),\\ (\rho(0), E(0)) = (1-\delta, E_1(0)).\end{cases} \tag{61}$$

In this step, we will show that when $\tau \gg 1$, there exists a number $x_1 \leq C\eta$ such that $\rho(x_1) = 1-\eta$, and $E(x_1) \geq E_1(0) - C\eta^2$, where $C > 0$ is a constant independent of τ, δ and η.

It is easy to see that if $\tau \geq \frac{4}{\Lambda_1} \geq \frac{4}{E_1(0)}$ and $\delta \leq \frac{1}{4}$, then the initial data of (61) satisfies

$$\rho(0)E(0) - \frac{1}{\tau} = (1-\delta)E_1(0) - \frac{1}{\tau} \geq \frac{E_1(0)}{2} > 0.$$

Observing that (61) is a standard initial value problem for the ODE system without degeneracy, it follows that (61) has a unique supersonic solution on some interval. Because $b \geq \underline{b} > 1 > \rho$, the solution component E keeps decreasing. Using result 2 of Theorem 3.1, ρ is decreasing until it attains the unique critical point, after which ρ keeps increasing. Denote by x_1 the first number for which $\rho(x)$ attains $1-\eta$, namely, $\rho(x_1) = 1-\eta$. By the second equation of (61),

$$E(x) = E_1(0) + \int_0^x (\rho - b)ds \geq E_1(0) - \bar{b}x \quad \text{for } x \in (0, x_1).$$

Since $\rho \in (1-\eta, 1-\delta)$ on $(0, x_1)$, if $\eta \leq \frac{1}{2}$ and $\tau \geq \frac{4}{\Lambda_1} \geq \frac{4}{E_1(0)}$, then

$$\rho E - \frac{1}{\tau} \geq (1-\eta)(E_1(0) - \bar{b}x) - \frac{E_1(0)}{4} \geq (1-\eta)\left(\frac{E_1(0)}{4} - \bar{b}x\right) \quad \text{for } x \in (0, x_1),$$

which in combination with the first equation of (61) leads to

$$x_1 = \frac{\rho(x_1) - \rho(0)}{\rho_x(\xi)} = \frac{(\eta - \delta)(1-\rho^2(\xi))}{(\rho(\xi)E(\xi) - \frac{1}{\tau})\rho^2(\xi)} \leq \frac{2\eta^2}{(1-\eta)^3(\frac{E_1(0)}{4} - \bar{b}x_1)} \quad \text{with } \xi \in (0, x_1). \tag{62}$$

To solve this inequality, we notice that when η is small such that $\eta \leq \min\left\{\frac{E_1(0)}{2^4\sqrt{\bar{b}}}, \frac{1}{2}\right\}$, then

$$\frac{E_1^2(0)}{4} - \frac{8\bar{b}\eta^2}{(1-\eta)^3} \geq \frac{E_1^2(0)}{4} - 2^6\bar{b}\eta^2 \geq \frac{1}{4}(E_1^2(0) - 2^8\bar{b}\eta^2) \geq 0.$$

Thus, we get from inequality (62) that

$$\begin{aligned} x_1 &\leq \frac{1}{2\bar{b}}\left(\frac{E_1(0)}{4} - \left(\frac{E_1^2(0)}{4^2} - \frac{8\bar{b}\eta^2}{(1-\eta)^3}\right)^{1/2}\right) \\ &= \frac{4\eta^2}{(1-\eta)^3\left(\frac{E_1(0)}{4} + (\frac{E_1^2(0)}{4^2} - \frac{8\bar{b}\eta^2}{(1-\eta)^3})^{1/2}\right)} \\ &\leq \frac{16\eta^2}{(1-\eta)^3 E_1(0)} \leq \frac{2^7\eta^2}{\Lambda_1}, \end{aligned}$$

where we have used (60) in the last inequality. In view of the second equation of (61), we further get

$$E(x_1) = E_1(0) + \int_0^{x_1} (\rho - b)ds \geq E_1(0) - \bar{b}x_1 \geq E_1(0) - \frac{2^7\bar{b}\eta^2}{\Lambda_1}. \tag{63}$$

Step 2. Now let us consider the initial value problem for the ODE system without semiconductor effect

$$\begin{cases} \left(1 - \frac{1}{\hat{\rho}^2}\right)\hat{\rho}_x = \hat{\rho}\hat{E}, \\ \hat{E}_x = \hat{\rho} - \underline{b}, \\ (\hat{\rho}(0), \hat{E}(0)) = (1-\delta, \hat{E}_0). \end{cases} \tag{64}$$

In this step, we prove that there exist numbers $x_2 > 0$ and $\hat{E}_0 > 0$ such that $x_2 \le C\eta^2$ and the solution of (64) satisfies $\hat{\rho}(x_2) = 1-\eta$ and $\hat{E}(x_2) = E(x_1)$. Here E and x_1 are given by Step 1, and $C > 0$ is a constant independent of τ, δ, and η.

We argue by the shooting method. Using phase-plane analysis, it is easy to see that, for any $\hat{E}_0 > 0$, there exists $\hat{L}(\hat{E}_0) > 0$, such that (64) has a symmetric supersonic solution on $[0, \hat{L}(\hat{E}_0)]$ satisfying

$$\hat{\rho}(0) = \hat{\rho}(\hat{L}(\hat{E}_0)) = 1-\delta, \quad \hat{E}(0) = -\hat{E}(\hat{L}(\hat{E}_0)) = \hat{E}_0.$$

Now taking $\hat{E}_0 = 2E_1(0)$, suppose $\bar{x}_2$ is the first number for which $\hat{\rho}$ attains $1-\eta$. Since $\hat{\rho} \in (1-\eta, 1-\delta)$ on $(0, \bar{x}_2)$, by the second equation of (64),

$$\hat{E}(x) = \hat{E}(0) + \int_0^x (\hat{\rho} - b)ds \ge 2E_1(0) - \bar{b}x \quad \text{for } x \in (0, \bar{x}_2). \tag{65}$$

Hence

$$\hat{\rho}\hat{E}(x) \ge (1-\eta)(2E_1(0) - \bar{b}x),$$

which in combination with the first equation of (64) leads to

$$\bar{x}_2 = \frac{\hat{\rho}(\bar{x}_2) - \hat{\rho}(0)}{\hat{\rho}_x(\hat{\xi})} = \frac{(\eta-\delta)(1-\hat{\rho}^2(\hat{\xi}))}{\hat{\rho}^3(\hat{\xi})\hat{E}(\hat{\xi})} \le \frac{2\eta^2}{(1-\eta)^3(2E_1(0) - \bar{b}\bar{x}_2)}. \tag{66}$$

Notice that when $\eta \le \min\left\{\frac{E_1(0)}{4\sqrt{\bar{b}}}, \frac{1}{2}\right\}$, it holds that

$$4E_1^2(0) - \frac{8\bar{b}\eta^2}{(1-\eta)^3} \ge 4(E_1^2(0) - 2^4\bar{b}\eta^2) \ge 0.$$

It then follows from (66) that

$$\begin{aligned} \bar{x}_2 &\le \frac{1}{2\bar{b}}\left(2E_1(0) - \left(4E_1^2(0) - \frac{8\bar{b}\eta^2}{(1-\eta)^3}\right)^{1/2}\right) \\ &= \frac{2\eta^2}{(1-\eta)^3\left(E_1(0) + (E_1^2(0) - \frac{2\bar{b}\eta^2}{(1-\eta)^3})^{1/2}\right)} \\ &\le \frac{2\eta^2}{(1-\eta)^3E_1(0)} \le \frac{2^4\eta^2}{\Lambda_1}, \end{aligned} \tag{67}$$

where we have used (60) in the last inequality. This inequality together with (65) gives

$$\hat{E}(\bar{x}_2) \ge 2E_1(0) - \bar{b}\bar{x}_2 \ge 2E_1(0) - \frac{2^4\bar{b}\eta^2}{E_1(0)} \ge 2E_1(0) - \frac{2^4\bar{b}\eta^2}{E_1(0)} \cdot \frac{E_1^2(0)}{16\bar{b}\eta^2} = E_1(0) > E(x_1).$$

Here we have used $E_1(0) = E(0) > E(x_1)$ because E is decreasing.

On the other hand, if $\hat{E}_0 = \frac{E_1(0)}{2}$, by (63), one can easily see that if $\eta < \frac{1}{2^4}\sqrt{\frac{\Lambda_1}{\bar{b}}}$, it holds that $E(x_1) > \frac{E_1(0)}{2}$. Thus, $\hat{E}(x) < \hat{E}_0 = \frac{E_1(0)}{2} < E(x_1)$ for any $x > 0$ because $\hat{E}$ is decreasing. Now by the continuity of the solution with respect to the initial data, there exist $\hat{E}_0 \in (\frac{E_1(0)}{2}, 2E_1(0))$ and length $\hat{L} > 0$ such that (64) has a supersonic solution $(\hat{\rho}, \hat{E})$ satisfying

$$\hat{\rho}(0) = \hat{\rho}(\hat{L}) = 1 - \delta, \quad \hat{E}(0) = -\hat{E}(\hat{L}) = \hat{E}_0.$$

Moreover, as in (67), there exists a number $x_2 \leq C\eta^2$ such that

$$\hat{\rho}(x_2) = 1 - \eta \quad \text{and} \quad \hat{E}(x_2) = E(x_1). \tag{68}$$

Thus,

$$0 < \hat{E}_0 - E(x_1) = \hat{E}_0 - \hat{E}(x_2) = -\int_0^{x_2} \hat{E}_x dx = \int_0^{x_2} (b - \hat{\rho}) dx < \bar{b} x_2 < C_1 \eta^2,$$

which in combination with (63) yields

$$\begin{aligned} &\hat{E}_0 - E_1(0) = \hat{E}_0 - E(x_1) + E(x_1) - E_1(0) > E(x_1) - E_1(0) > -C_2\eta^2, \\ &\hat{E}_0 - E_1(0) = \hat{E}_0 - E(x_1) + E(x_1) - E_1(0) < C_1\eta^2, \end{aligned}$$

where we have used the fact that E is decreasing. Thus,

$$|\hat{E}_0 - E_1(0)| \leq C_3\eta^2 \text{ with } C_3 = \min\{C_1, C_2\}.$$

Observing that the length $\hat{L}$ of solution is also continuous with respect to the initial data, since the length of the solution (ρ_1, E_1) to (48) with initial data $(1 - \delta, E_1(0))$ is $\frac{1}{2}$, there exists $l_0 > 0$ independent of τ, δ, and η, such that if $C_3\eta^2 < l_0$, then

$$\frac{1}{4} \leq \hat{L} \leq \frac{3}{4}.$$

Step 3. In this step, we show that when $\tau \gg 1$ and $\bar{b} - \underline{b} \ll 1$, system (61) has a unique solution (ρ, E) on $[0, x_4]$ with

$$\frac{1}{4} - C\eta^2 \leq x_4 \leq \frac{3}{4} + C\eta^2, \quad \rho(0) = \rho(x_4) = 1 - \delta$$

for some constant C independent of τ, δ, and η. Set $(\bar{\rho}, \bar{E})(x) := (\hat{\rho}, \hat{E})(x - x_1 + x_2)$. Then $(\bar{\rho}, \bar{E})$ satisfies (64) with initial-boundary data

$$(\bar{\rho}, \bar{E})(x_1) = (1 - \eta, \hat{E}(x_2)) = (\rho, E)(x_1) \quad \text{and} \quad \bar{\rho}(x_3) = 1 - \eta$$

with $x_3 := \hat{L} + x_1 - 2x_2$, where we have used the symmetry of $(\hat{\rho}, \hat{E})$, and hence $\hat{\rho}(\hat{L} - x_2) = \hat{\rho}(x_2) = 1 - \eta$. Set $\phi := \bar{\rho} - \rho$, $\psi := \bar{E} - E$. Then by (61) and (64), (ϕ, ψ) satisfies

$$\begin{cases} \phi_x = \frac{\bar{\rho}^3 \psi}{(\bar{\rho}+1)(\bar{\rho}-1)} + \frac{(\bar{\rho}^2\rho^2 - \bar{\rho}^2 - \bar{\rho}\rho - \rho^2)\phi E}{(\bar{\rho}+1)(\bar{\rho}-1)(\rho+1)(\rho-1)} + \frac{\rho^2}{\tau(\rho+1)(\rho-1)}, \\ \psi_x = \phi + b - \underline{b}, \\ (\phi(x_1), \psi(x_1)) = 0. \end{cases} \tag{69}$$

Define the solution space $X_T := \{(\phi, \psi) \in C[x_1, T] \,|\, \phi(x_1) = \psi(x_1) = 0, |\phi| \leq \eta/2, |\psi| \leq \eta/2\}$. We only need to show the a priori estimate

$$\phi^2(x) + \psi^2(x) \leq \eta^2/4 \;\text{ on } x \in [x_1, x_3]. \tag{70}$$

Multiplying the first equation of (69) by ϕ and the second one by ψ and adding them, and noting $|\rho-\bar\rho|\le\eta/2$, by Young's inequality, one can easily get

$$(\phi^2+\psi^2)_x \le \frac{C}{\eta^2}(\phi^2+\psi^2)+\frac{C}{\tau^2}+C(\bar b-\underline{b})^2,$$

where C is a constant independent of τ, δ, and η. It then follows from Gronwall's inequality that

$$\phi^2+\psi^2 \le C\left[\frac{C}{\tau^2}+(\bar b-\underline{b})^2\right]\eta^2 e^{Cx/\eta^2} \le C\left[\frac{C}{\tau^2}+(\bar b-\underline{b})^2\right]\eta^2 e^{C/\eta^2} \quad \text{for } x\in[x_1,x_3].$$

Now taking $\tau\gg1$ and $\bar b-\underline{b}\ll1$ such that $\left[\frac{C}{\tau^2}+(\bar b-\underline{b})^2\right]e^{C/\eta^2}\le\frac14$, we derive (70). Moreover, we also get

$$|\rho-\bar\rho|\le\eta/2 \quad\text{and}\quad |E-\bar E|\le\eta/2,$$

which gives $\rho(x_3)\le\bar\rho(x_3)+\frac{\eta}{2}=1-\frac{\eta}{2}$, and further by (68) and (63),

(71)
$$E(x_3)\le\bar E(x_3)+\frac{\eta}{2}=\hat E(\hat L-x_2)+\frac{\eta}{2}=-\hat E(x_2)+\frac{\eta}{2}=-E(x_1)+\frac{\eta}{2}\le -E_1(0)+C\eta.$$

Now taking x_3 as the initial data, we can extend (ρ,E), the solution of (61), to the state satisfying $\rho=1-\delta$. Denote by x_4 the number for which $\rho(x_4)=1-\delta$. As in the proof of Step 2, we have

$$x_4-x_3\le C\eta^2$$

for some constant C independent of τ, δ, and η. And then by (71),

$$E(x_4)\le E(x_3)\le -E_1(0)+C\eta.$$

Now we obtain a solution of (61) on $[0,x_4]$ satisfying

$$\rho(0)=\rho(x_4)=1-\delta,\quad E(0)=E_1(0),\quad E(x_4)\le -E_1(0)+C\eta. \tag{72}$$

Moreover,

$$\frac14-C\eta^2\le\hat L+x_1-2x_2=x_3\le x_4\le x_3+C\eta^2=\hat L+x_1-2x_2+C\eta^2\le\frac34+C\eta^2. \tag{73}$$

Step 4. In this step, we construct a transonic solution of (61) on an interval $[0,x_5]$ with

$$\frac14-C\eta\le x_5\le\frac34+C\eta,\quad \rho(0)=1-\delta,\quad \rho(x_5)=1+\delta.$$

Set $\rho_l=1-\eta$; then $\rho_r=1/\rho_l>1$. We take the jump location $\bar x_0\in(0,x_4)$ as the last number for which $\rho(\bar x_0)=\rho_l$, and restrict our supersonic solution $(\rho_{sup},E_{sup})(x)$ only on $[0,\bar x_0]$. We denote $E_{sup}(\bar x_0)\triangleq E_l$. As in the proof of Step 2,

$$x_4-\bar x_0\le C\eta. \tag{74}$$

Thus, owing to the inequality of (72), the supersonic solution satisfies

$$\rho_l=1-\eta,\quad E_l\le E(x_4)+C\eta\le -E_1(0)+C\eta. \tag{75}$$

It is then easy to see that

$$\rho_l E_l - \frac{1}{\tau} \leq (1-\eta)(-E_1(0) + C\eta) \leq -E_1(0) + (C + E_1(0))\eta.$$

Thus, when $\eta \ll 1$ such that $(C + E_1(0))\eta \leq \frac{E_1(0)}{2}$, it holds that

$$\rho_l E_l - \frac{1}{\tau} \leq -\frac{E_1(0)}{2} < 0 \quad \text{and} \quad E_l < 0. \tag{76}$$

Next we construct the corresponding subsonic solution. For $x \geq \bar{x}_0$, let us consider the system (61) with initial data

$$\rho(\bar{x}_0) = \rho_r, \quad E(\bar{x}_0) = E_r = E_l.$$

By the standard ODE theory, the initial value problem admits a unique solution $(\rho, E)(x)$ for $x > \bar{x}_0$. By (76), a simple calculation gives

$$\begin{aligned} \rho_r E_r - \frac{1}{\tau} &= \rho_l E_l - \frac{1}{\tau} + \left(\frac{1}{\rho_l} - \rho_l\right) E_l \\ &\leq -\frac{E_1(0)}{2} + \left[\frac{1}{1-\eta} - (1-\eta)\right](-E_1(0) + C\eta) \\ &\leq -\frac{E_1(0)}{2} + C\eta^2. \end{aligned}$$

It hence follows that when $C\eta^2 < \frac{E_1(0)}{4}$,

$$\rho_r E_r - \frac{1}{\tau} \leq -\frac{E_1(0)}{4} < 0.$$

From the first equation of (61), we know the component ρ of such a solution is decreasing in a neighborhood of $\bar{x}_0^+$. We denote this subsonic solution by $(\rho_{sub}, E_{sub})(x)$. If $\eta < 1 - \frac{1}{\underline{b}}$, then

$$\begin{aligned} E_{sub}(x) &= E_r + \int_{\bar{x}_0}^{x} (\rho_{sub} - b)dx \\ &\leq E_r + \int_{\bar{x}_0}^{x} \Big(\frac{1}{1-\eta} - b\Big)dx \\ &< E_r < 0, \end{aligned}$$

where we have used the second inequality of (76) and $E_r = E_l$. Noting that the function $g(s) = \frac{s^3}{s^2-1}$ is monotone decreasing on $(1, \sqrt{3})$, we thus get from (75) that

$$(\rho_{sub})_x = \frac{\rho_{sub}E_{sub} - \frac{1}{\tau}}{1 - \frac{1}{\rho_{sub}^2}} \leq \frac{\rho_r^3 E_r}{\rho_r^2 - 1} = \frac{E_r}{\eta(1-\eta)(2-\eta)} \leq \frac{-E_1(0) + C\eta}{\eta(1-\eta)(2-\eta)} < -\frac{E_1(0)}{2}$$

if $\eta < \min\left\{\frac{E_1(0)}{2C}, \frac{1}{2}\right\}$. This inequality implies ρ_{sub} will keep decreasing and attain $1+\delta$ at a finite number x_5 and

$$x_5 - \bar{x}_0 = \frac{\delta - \frac{\eta}{1-\eta}}{\int_0^1 (\rho_{sub})_x (s x_5 + (1-s)\bar{x}_0) ds} \leq C\eta \quad \text{if } \eta < \min\left\{\frac{E_1(0)}{2C}, \frac{1}{2}\right\}. \tag{77}$$

Now we have constructed the transonic solution to (61) in $[0, x_5]$ as follows:

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \bar{x}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\bar{x}_0, x_5], \end{cases}$$

which satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(x_5) = 1 + \delta,$$

the entropy condition at $\bar{x}_0$

$$0 < \rho_{sup}(\bar{x}_0^-) = 1 - \eta < 1 < \rho_{sub}(\bar{x}_0^+),$$

and the Rankine–Hugoniot condition (11) at $\bar{x}_0$. Furthermore, it follows from (73), (74), and (77) that

$$\frac{1}{4} - C\eta \le x_5 \le \frac{3}{4} + C\eta.$$

Step 5. In this step, we construct a transonic solution of (61) on an interval $[0, x_7]$ with $\frac{5}{4} - C\eta \le x_7 \le \frac{7}{4} + C\eta$, $\rho(0) = 1 - \delta$, and $\rho(x_7) = 1 + \delta$.

We take $L = \frac{3}{2}$ in (48) and denote by (ρ_2, E_2) its solution. As shown in Steps 1–3, we know that there exists an interval $[0, x_6]$ with

$$\frac{5}{4} - C\eta^2 \le x_6 \le \frac{7}{4} + C\eta^2,$$

such that system (61) has a supersonic solution on $[0, x_6]$ satisfying

$$\rho(0) = \rho(x_6) = 1 - \delta, \quad E(0) = E_2(0), \quad E(x_6) \le -E_2(0) + C\eta.$$

As in Step 4, we may construct another transonic solution for (61) in the form of

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \tilde{x}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\tilde{x}_0, x_7], \end{cases}$$

where $\tilde{x}_0 \in (0, x_6)$ and $\frac{5}{4} - C\eta^2 \le x_7 \le \frac{7}{4} + C\eta^2$ are some determined numbers. This transonic solution satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad \rho_{sub}(x_7) = 1 + \delta,$$

the entropy condition at $\tilde{x}_0$

$$0 < \rho_{sup}(\tilde{x}_0^-) = 1 - \eta < 1 < \rho_{sub}(\tilde{x}_0^+),$$

and the Rankine–Hugoniot condition (11) at $\tilde{x}_0$.

Step 6. We next construct transonic solutions of (61) on $[0, 1]$. Without loss of generality, we assume that $E_1(0) < E_2(0)$. As in Step 4, one can see that when $0 < \delta < \eta \ll 1$, $\tau \gg 1$, for any $E_0 \in (E_1(0), E_2(0))$, there exist a number $x_8 > 0$ and a transonic solution of (61) on the interval $[0, x_8]$ satisfying the boundary condition

$$\rho_{sup}(0) = 1 - \delta, \quad E_{sup}(0) = E_0, \quad \rho_{sub}(x_8) = 1 + \delta,$$

the entropy condition at $\tilde{\tilde{x}}_0$

$$0 < \rho_{sup}(\tilde{\tilde{x}}_0^-) = 1 - \eta < 1 < \rho_{sub}(\tilde{\tilde{x}}_0^+),$$

and the Rankine–Hugoniot condition. Applying the continuation argument in the length of the interval L, we realize that system (61) has some transonic solutions $(\rho_{trans}, E_{trans})(x)$ for $x \in [0,1]$ and satisfies the boundary condition

$$\rho_{sup}(0) = 1 - \delta,\ \ \rho_{sub}(1) = 1 + \delta,$$

the entropy condition

$$0 < \rho_{sup}(x_0^\delta) = 1 - \eta < 1 < \rho_{sub}(x_0^\delta),$$

and the Rankine–Hugoniot condition at some jump location x_0^δ in $(0,1)$.

Step 7. Let us now prove the existence of transonic solutions of (5)–(6) on $[0,1]$.

For any $\delta > 0$, denote by (ρ^δ, E^δ) the transonic solution of (61) on $[0,1]$ obtained in Step 6. Multiplying the first equation of (61) by $\frac{1}{\rho^\delta}((1-\delta-\rho^\delta)^2)_x$, integrating the resultant equation on $(0, x_0^\delta)$, and using the second equation of (61), noting

$$\frac{((1-\delta-\rho^\delta)^2)_x}{\rho^\delta} = (-2(1-\delta)\ln\rho^\delta + 2\rho^\delta)_x,$$

$$\begin{aligned}\int_0^{x_0^\delta}(b-\rho^\delta)(1-\delta-\rho^\delta)^2dx &\le \int_0^{x_0^\delta} b(1-\delta-\rho^\delta)^2dx\\ &\le \frac{1}{4}\int_0^{x_0^\delta}(1-\delta-\rho^\delta)^4dx + \int_0^{x_0^\delta} b^2dx\\ &\le \frac{1}{4}\int_0^{x_0^\delta}|((1-\delta-\rho^\delta)^2)_x|^2dx + b^2,\end{aligned}$$

we have

$$\begin{aligned}&\int_0^{x_0^\delta}\frac{2\delta(\rho^\delta+1)(1-\delta-\rho^\delta)(\rho_x)^2}{(\rho^\delta)^3} + \frac{(\rho^\delta+1)|((1-\delta-\rho^\delta)^2)_x|^2}{2(\rho^\delta)^3}dx\\ &\le \frac{1}{4}\int_0^{x_0^\delta}|((1-\delta-\rho^\delta)^2)_x|^2dx + b^2 + E_l(1-\delta-\rho_l)^2\\ &\quad - \frac{2}{\tau}[-(1-\delta)\ln\rho_l + \rho_l + (1-\delta)\ln(1-\delta) - (1-\delta)].\end{aligned} \tag{78}$$

Similarly, multiplying the first equation of (61) by $\frac{1}{\rho^\delta}((\rho^\delta-1-\delta)^2)_x$, integrating the resultant equation on $(x_0^\delta, 1)$, we have

$$\begin{aligned}&\int_{x_0^\delta}^1\frac{2\delta(\rho^\delta+1)(\rho^\delta-1-\delta)((\rho^\delta)_x)^2}{\rho^3} + \frac{(\rho^\delta+1)|((\rho^\delta-1-\delta)^2)_x|^2}{2(\rho^\delta)^3}dx\\ &\le \frac{1}{4}\int_{x_0^\delta}^1|((\rho^\delta-1-\delta)^2)_x|^2dx + b^2 - E_r(\rho^\delta-1-\delta)^2\\ &\quad + \frac{2}{\tau}[\rho_r - (1+\delta)\ln\rho_r - 1 + (1+\delta)\ln(1+\delta)].\end{aligned}$$

Substituting this inequality into (78), we get

$$\|(1-\delta-\rho_{sup}^\delta)^2\|_{H^1(0,x_0^\delta)} \le C, \quad \|(\rho_{sub}^\delta-1-\delta)^2\|_{H^1(x_0^\delta,1)} \le C.$$

Since $\eta > 0$, as $\delta \to 0^+$, up to a subsequence, $x_0^\delta \to x_0 \in (0,1)$. Thus, for integer k large enough, there exists a subsequence, still denoted by $\{\rho^\delta\}$, such that

$$\begin{aligned} &(1-\delta-\rho_{sup}^\delta)^2 \rightharpoonup (1-\rho_{sup}^0)^2 \quad \text{weakly in } H^1(0, x_0 - 1/k),\\ &(\rho_{sub}^\delta - 1 - \delta)^2 \rightharpoonup (\rho_{sub}^0 - 1)^2 \quad \text{weakly in } H^1(x_0 + 1/k, 1). \end{aligned}$$

Applying the diagonal argument for $(\rho_{trans}^\delta, E_{trans}^\delta)$, we know that (5)–(6) has a transonic solution $(\rho_{trans}, E_{trans})(x)$ for $x \in [0,1]$ that satisfies the sonic boundary condition, the entropy condition, and the Rankine–Hugoniot condition at the jump location x_0 in $(0,1)$.

Because τ depends only on $(E_1(0), E_2(0), \eta)$, and η depends only on $(E_1(0), E_2(0))$, there exists a $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, there exists a transonic solution jump location at $\rho_l = 1 - \eta$. Thus, such transonic solutions are infinitely many due to an arbitrary choice of $0 < \eta < \eta_0$. The proof is complete. ∎

5. Infinitely many C^1 transonic solutions.

In this subsection, we assume that the doping profile $b(x) = b > 1$ is a given constant. We will construct C^1-smooth transonic solutions on the basis of refined local analysis of the interior subsonic solutions and interior supersonic solutions on the boundary. The approach relies on the phase-plane analysis.

We first study the structure of the interior subsonic solution. For convenience, we set

$$F = E - \frac{1}{\tau\rho} \quad \text{and} \quad n = \rho - 1. \tag{79}$$

Then system (5) is transformed into

$$\begin{cases} n_x = \dfrac{(1+n)^3 F}{(2+n)n}, \\ F_x = n + 1 - b + \dfrac{(1+n)F}{\tau(2+n)n}. \end{cases} \tag{80}$$

Clearly, $(b-1, 0)$ is a saddle point of (80). In the (n, F) plane, all trajectories satisfy

$$\frac{dF}{dn} = \frac{(n+1-b)(2+n)}{(1+n)^3} \cdot \frac{n}{F} + \frac{1}{\tau(1+n)^2} \triangleq H_1(n, F). \tag{81}$$

Here and in what follows, to avoid confusion, $F = F(n)$ denotes the function of the trajectory. The equation $H_1(n, F) = 0$ determines a curve

$$\Xi = \Xi(n) = -\frac{\tau(n+1-b)(2+n)n}{1+n}. \tag{82}$$

Obviously, if a trajectory interacts with the curve $\Xi = \Xi(n)$, then the interacting point is a critical point of the trajectory, and all critical points of a trajectory lie on the curve $\Xi(n)$. We draw the phase-plane of (n, F) in Figure 1 with $\tau = 0.5$ and $b = 1.5$. To state our results more precisely, we need the following definition.

DEFINITION 5.1. *If (ρ, E) is an interior subsonic (resp., interior supersonic) solution to system* (5) *on an interval $[0, L]$ satisfying $\rho(0) = \rho(L) = 1$, then the corresponding trajectory $E = E(\rho)$ in the phase-plane (ρ, E) is called an interior subsonic (resp., interior supersonic) trajectory to system* (5)*. And the transformed trajectory $F = F(n)$ in the (n, F) plane is called an interior positive (resp., interior negative) trajectory to system* (80).

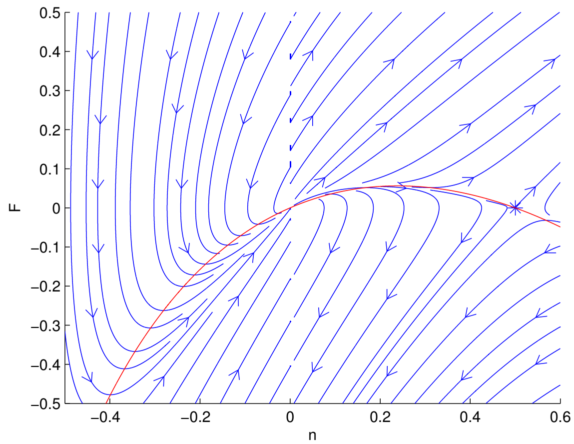

FIG. 1. *Phase-plane of* (n, F) *with* $\tau = 0.5$ *and* $b = 1.5$. $*$ *is the saddle point* $(0.5, 0)$; *the red line is the function* $\Xi(n) = -\frac{\tau(n+1-b)(2+n)n}{1+n}$.

Clearly, an interior subsonic (resp., interior supersonic) trajectory corresponds to an interior subsonic (resp., interior supersonic) solution to system (5) on some interval. Instead of studying system (5) directly, we turn to analyzing the structure of solutions to the transformed system (80). Based on the analysis of the relation between $F(n)$ and $\Xi(n)$, we first obtain the following important lemma.

LEMMA 5.2. *When* $0 < \tau < \frac{1}{2\sqrt{b^3+b}}$, *all interior positive trajectories to system* (80) *start from the point* $(0, 0)$.

Proof. It is easy to see that there are two zero points of $\Xi(n)$ on $[0, +\infty)$: $n_1 = 0$, $n_2 = b - 1$, and

$$\Xi'(n) = -\tau\left(2 - b + 2n - \frac{b}{(1+n)^2}\right) \text{ for } n \geq 0, \tag{83}$$

$$\Xi''(n) = -2\tau\left(1 + \frac{b}{(1+n)^3}\right) < 0 \text{ for } n \geq 0, \tag{84}$$

$$\Xi'(0) = 2(b-1)\tau > 0 \quad \text{and} \quad \Xi'(b-1) = -\tau\left(b - \frac{1}{b}\right) < 0.$$

Thus, $\Xi(n)$ is concave on $[0, \infty)$ and has only one maximal point denoted by n^* that depends only on b. We just focus on the region $F \geq 0$. By (81) and (82),

$$\frac{dF}{dn} = -\frac{\Xi}{\tau(1+n)^2 F} + \frac{1}{\tau(1+n)^2}, \tag{85}$$

which is equivalent to

$$\frac{dF}{dn} = \frac{1}{\tau(1+n)^2} \cdot \frac{F - \beta\Xi}{F} + \frac{(\beta-1)\Xi}{\tau(1+n)^2 F}, \tag{86}$$

where $\beta > 0$ is a constant to be determined later. This equation in combination with

(83) leads to

$$\begin{aligned}\left(F^2-\beta^2\Xi^2\right)' &= \frac{2(F-\beta\Xi)}{\tau(1+n)^2}+2\Xi\left[\frac{\beta-1}{\tau(1+n)^2}+\tau\beta^2\left(2-b+2n-\frac{b}{(1+n)^2}\right)\right]\\ &= \left(F^2-\beta^2\Xi^2\right)\cdot\frac{2}{\tau(1+n)^2(F+\beta\Xi)}+2\Xi\cdot I,\end{aligned}\tag{87}$$

where $I := \frac{\beta-1}{\tau(1+n)^2}+\tau\beta^2\big(2-b+2n-\frac{b}{(1+n)^2}\big)$. Since $\Xi(0)=0$, if $F(0)=h>0$, then we have $F^2(0)-\beta^2\Xi^2(0)=h^2>0$ for any $\beta>0$. We next determine β such that $I>0$ for $n\in[0,b-1]$. To do this, we set $\beta=\frac{c_0}{\tau^2}$ with $c_0=\frac{1}{2(b^3+b)}$. When $\tau^2<\frac{c_0}{2}$, we have for $n\in[0,b-1]$

$$\begin{aligned}I &= \frac{1}{\tau(1+n)^2}\cdot\left[\frac{c_0}{\tau^2}-1+\frac{c_0^2}{\tau^2}\cdot(2(1+n)^3-b(1+n)^2-b)\right]\\ &\geq \frac{1}{\tau(1+n)^2}\cdot\left[\frac{c_0}{\tau^2}-1-\frac{c_0^2}{\tau^2}\cdot(b^3+b)\right]\\ &= \frac{1}{\tau(1+n)^2}\cdot\left[\frac{c_0}{\tau^2}\cdot(1-c_0(b^3+b))-1\right]\\ &= \frac{1}{\tau(1+n)^2}\cdot\left(\frac{c_0}{2\tau^2}-1\right)\\ &> 0.\end{aligned}$$

Noting $\Xi(n)>0$ on $(0,b-1)$, it then follows from (87) that

$$F^2(n)>\beta^2\Xi^2(n)\quad\text{for } n\in[0,b-1].\tag{88}$$

Since $(b-1,0)$ is a saddle point lying on the curve $\Xi=\Xi(n)$, the trajectories starting from $(0,h)$ with $h>0$ cannot go back to the line $n=0$, but go to infinity. Obviously, a trajectory cannot start from $(0,-h)$. Therefore, when $\tau<\frac{1}{2\sqrt{b^3+b}}$, all interior positive trajectories to system (80) must start from $(0,0)$. ∎

LEMMA 5.3. *When* $0<\tau<\frac{1}{3\sqrt{b^3+b}}$, *all interior positive trajectories to system* (80) *satisfy*

$$F(n)\leq\frac{3}{2}\cdot\Xi(n)\quad\textit{for } n\geq 0.\tag{89}$$

Proof. Taking $\beta=\frac{3}{2}$ in (87), when $\tau^2<\frac{1}{9(b^3+b)}$, we have for $n\in[0,b-1]$

$$\begin{aligned}I &= \frac{1}{\tau(1+n)^2}\cdot\left[\frac{1}{2}+\frac{9\tau^2}{4}\cdot(2(1+n)^3-b(1+n)^2-b)\right]\\ &\geq \frac{1}{\tau(1+n)^2}\cdot\left[\frac{1}{2}-\frac{9\tau^2}{4}\cdot(b^3+b)\right]\\ &> 0.\end{aligned}$$

If there is a point $\bar{n}\in(0,b-1)$ on the trajectory such that $F(\bar{n})>\frac{3}{2}\Xi(\bar{n})$, then noting $\Xi(n)>0$ and $I>0$ on $(\bar{n},b-1)$, we get from (87) that $F(n)>\frac{3}{2}\Xi(n)$ on $(\bar{n},b-1)$. Because $(b-1,0)$ is a saddle point, this trajectory will go to infinity. We hence get (89). ∎

LEMMA 5.4. *When* $0 < \tau < \frac{1}{2\sqrt{b^3+b}}$, *all interior positive trajectories to system* (80) *with* $F \geq 0$ *are Lipschitz continuous on a neighborhood of* $n = 0$.

Proof. We first present a lower bound of $\frac{dF}{dn}$. Notice that all critical points of trajectories lie on the curve $\Xi = \Xi(n)$. We claim that an interior positive trajectory to system (80) must have at least one critical point on $(0, b-1)$. Otherwise, the trajectory has no critical point on $(0, b-1)$; then

$$F'(n) > 0 \text{ on } (0, b-1) \quad \text{or} \quad F'(n) < 0 \text{ on } (0, b-1).$$

If the former case holds, by (81) and (82),

$$(F - \Xi)'(n) > 0 \text{ on } (0, b-1).$$

By Lemma 5.2, when $\tau < \frac{1}{2\sqrt{b^3+b}}$, it holds that $F(0) = 0 = \Xi(0)$; then it follows that $F(n) > \Xi(n)$ on $(0, b-1)$. Since $(b-1, 0)$ is a saddle point, this indicates that the trajectory cannot go back to the line $n = 0$, but goes to infinity. If the latter case holds, since $F(0) = 0$, we get

$$F(n) < 0 \quad \text{for any } n \in (0, b-1).$$

Using (81) again, noting $n + 1 - b < 0$ for $n \in (0, b-1)$, we derive

$$F'(n) > \frac{1}{\tau(1+n)^2} > 0 \text{ on } (0, b-1),$$

which is a contradiction. Thus, an interior positive trajectory to system (80) has at least one critical point over $(0, b-1)$.

We next claim that an interior positive trajectory has at most one critical point. Denote by n_0 a critical point of this trajectory. Taking $\beta = 1$ in (87), and using (83), we have

$$\begin{aligned} \left(F^2 - \Xi^2\right)' &= \frac{2(F-\Xi)}{\tau(1+n)^2} + 2\Xi\tau\left(2 - b + 2n - \frac{b}{(1+n)^2}\right) \\ &= \left(F^2 - \Xi^2\right) \cdot \frac{2}{\tau(1+n)^2(F+\Xi)} - 2\Xi\Xi'. \end{aligned} \tag{90}$$

Recall that n^* is the maximal point of the function $\Xi(n)$ on $(0, b-1)$. If $n_0 \geq n^*$, noting $F(n_0) = \Xi(n_0)$ and $\Xi(n) > 0$, $\Xi'(n) < 0$ on $(n^*, b-1)$, it follows from (90) that

$$F(n) > \Xi(n) \text{ over } (n^*, b-1).$$

Because $(b-1, 0)$ is a saddle point, this trajectory will go to infinity. Thus, $n_0 \in (0, n^*)$.

Now since $\Xi(n) > 0$, $\Xi'(n) > 0$ on (n_0, n^*) and $F(n_0) = \Xi(n_0)$, by (90) again, we have

$$F(n) < \Xi(n) \text{ over } (n_0, n^*]. \tag{91}$$

Since all critical points of the trajectory are on the curve $\Xi(n)$, (91) indicates that there is no other critical point on $(n_0, n^*]$ for this trajectory. On the other hand, suppose that there is a critical point $n_1 \in (0, n_0)$ for this trajectory; then

$$F(n_1) = \Xi(n_1), \quad \Xi(n) > 0, \quad \text{and} \quad \Xi'(n) > 0 \text{ on } (n_1, n^*].$$

Applying (90) repeatedly, we get

$$F(n) < \Xi(n) \quad \text{for } n \in (n_1, n^*].$$

This contradicts the fact that $F(n_0) = \Xi(n_0)$ because $n_0 \in (n_1, n^*)$. Thus, there is no critical point on $(0, n_0)$ for this trajectory, and n_0 is the unique critical point of this interior positive trajectory. As a consequence, we conclude that

$$\frac{dF(n)}{dn} > 0 \quad \text{on } (0, n_0). \tag{92}$$

We next derive an upper bound of $\frac{dF}{dn}$. By (80), we get

$$F_x = n + 1 - b + \frac{n_x}{\tau(1+n)^2} = n + 1 - b - \frac{1}{\tau} \cdot \left(\frac{1}{1+n}\right)_x. \tag{93}$$

Noting $n(0) = 0$, by the continuity of the trajectory, $0 \le n < b - 1$ on $[0, z]$ for some $z > 0$. Noting $F(0) = 0$, hence, for $x \in [0, z]$,

$$F(x) < -\frac{1}{\tau} \int_0^x \left(\frac{1}{1+n}\right)_x dx = \frac{1}{\tau} - \frac{1}{\tau(1+n)}.$$

It then follows that for $n \in [0, b-1]$ and $F \ge 0$,

$$\frac{dF(n)}{dn} < \frac{\tau(n+1-b)(2+n)}{(1+n)^2} + \frac{1}{\tau(1+n)^2} \le \frac{1}{\tau}.$$

This estimate together with (92) implies the trajectory is Lipschitz continuous on $(0, n_0)$. ∎

LEMMA 5.5. *When* $0 < \tau < \min\{\frac{1}{3\sqrt{b^3+b}}, \frac{1}{4\sqrt{b-1}}\}$, *all interior positive trajectories to system* (80) *with* $F \ge 0$ *are* C^1*-smooth on a neighborhood of* $n = 0$ *and*

$$\frac{dF}{dn}(0) = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right). \tag{94}$$

Proof. By Lemma 5.4, we only need to show that the second order derivative of the trajectory does not change sign on a neighborhood of $n = 0$, i.e.,

$$\frac{d^2F}{dn^2} \text{ does not change sign if } 0 < n \ll 1. \tag{95}$$

Step 1. We first compute $\frac{d^2F}{dn^2}$. By (81) and (82),

$$(1+n)^2 F F' = \frac{1}{\tau}(F - \Xi). \tag{96}$$

Notice that $F(n)$ is C^∞ over $(0, b-1)$. Differentiating (96) in n and using the first equality of (85), a direct calculation yields

$$\begin{aligned}
&(1+n)^2 F F'' \\
&= -2(1+n)FF' - (1+n)^2(F')^2 + \frac{1}{\tau}(F' - \Xi') \\
&= -\frac{1}{\tau(1+n)F^2}\left[2F^3 - (2\Xi - (1+n)\Xi')F^2 - \frac{\Xi F}{\tau(1+n)} + \frac{\Xi^2}{\tau(1+n)}\right].
\end{aligned} \tag{97}$$

By (82) and (83), it is easy to see that

$$2\Xi-(1+n)\Xi' = -\frac{2\tau(n+1-b)(2+n)n}{1+n}+\tau(1+n)\left(2-b+2n-\frac{b}{(1+n)^2}\right)$$
$$=\frac{\tau}{1+n}\left[2(n+1-b)+bn(2+n)\right].$$

It then follows that

$$\begin{aligned}F'' &= -\frac{2}{\tau(1+n)^3F^3}\Big\{F^3-\frac{\tau\left[2(n+1-b)+bn(2+n)\right]}{2(1+n)}\cdot F^2\\ &\quad+\frac{(2+n)n(n+1-b)}{2(1+n)^2}\cdot F+\frac{\tau(2+n)^2n^2(n+1-b)^2}{2(1+n)^3}\Big\}\\ &\triangleq -\frac{2}{\tau(1+n)^3F^3}\cdot H_2(n,F).\end{aligned}\tag{98}$$

Step 2. We next solve the equation $H_2(n,F)=0$, which is a third order algebraic equation in the form

$$F^3+kF^2+mF+\ell=0,\tag{99}$$

where

$$\begin{aligned}k&=-\frac{\tau\left[2(n+1-b)+bn(2+n)\right]}{2(1+n)},\quad m=\frac{(2+n)n(n+1-b)}{2(1+n)^2},\\ \ell&=\frac{\tau(2+n)^2n^2(n+1-b)^2}{2(1+n)^3}.\end{aligned}\tag{100}$$

Denote $p=-\frac{k^2}{3}+m$, $q=2(\frac{k}{3})^3-\frac{km}{3}+\ell$; by Cardan's formula, (99) has three roots:

$$F_1=A^{\frac13}+B^{\frac13},\quad F_2=\varpi A^{\frac13}+\varpi^2B^{\frac13},\quad F_3=\varpi^2A^{\frac13}+\varpi B^{\frac13},$$

where $\varpi=\frac{-1+\sqrt{3}i}{2}$, $A=-\frac{q}{2}+[(\frac{q}{2})^2+(\frac{p}{3})^3]^{\frac12}$, and $B=-\frac{q}{2}-[(\frac{q}{2})^2+(\frac{p}{3})^3]^{\frac12}$. Furthermore, if $(\frac{q}{2})^2+(\frac{p}{3})^3\le 0$, then all roots are real valued. We claim that when $\tau<\frac{1}{4\sqrt{b-1}}$ and $0<n\ll 1$, then $(\frac{q}{2})^2+(\frac{p}{3})^3\le 0$. Actually, a simple calculation gives

$$\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=\frac{1}{4\cdot 3^4}[(km-9\ell)^2-4(k^2-3m)(m^2-3k\ell)].\tag{101}$$

When $0<n\ll 1$, by (100),

$$k=\tau(b-1)+O(n),\quad m=(1-b)n+O(n^2),\quad \ell=2\tau(b-1)^2n^2+O(n^3).$$

It then follows that

$$\begin{aligned}km-9\ell&=-\tau(b-1)^2n+O(n^2),\quad k^2-3m=(b-1)^2\tau^2+O(n),\\ m^2-3k\ell&=(b-1)^2n^2-6\tau^2(b-1)^3n^2+O(n^3).\end{aligned}$$

Substituting these three estimates into (101) yields

$$\begin{aligned}&\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3\\ &=\frac{1}{4\cdot 3^4}\cdot[(b-1)^4\tau^2n^2-4(b-1)^2\tau^2((b-1)^2-6\tau^2(b-1)^3)n^2+O(n^3)]\\ &=\frac{1}{4\cdot 3^4}\cdot[3(b-1)^4\tau^2(-1+8\tau^2(b-1))n^2+O(n^3)].\end{aligned}$$

Thus, when $\tau < \frac{1}{4\sqrt{b-1}}$ and $0 < n \ll 1$, we have $(\frac{q}{2})^2 + (\frac{p}{3})^3 < 0$.

Now all roots of the equation $H_2(n, F) = 0$ are real valued functions. And clearly, they are analytic in n on $(0, b-1)$. We then take an expansion of the roots denoted by $F_0(n)$ as

$$F_0(n) = \theta_0 + \theta_1 n + O(n^2),$$

and substitute this formula into $H_2(n, F) = 0$ to get

$$\theta_0 = 0 \quad \text{or} \quad \theta_0 = -\tau(b-1) < 0,$$

and

$$\theta_1 = \frac{1}{2}\left(\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) \quad \text{or} \quad \theta_1 = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right).$$

Notice that when $\tau \ll 1$, $\frac{1}{2}\big(\frac{1}{\tau}+\sqrt{\frac{1}{\tau^2} - 8(b-1)}\big) = O(\frac{1}{\tau})$ and $\frac{1}{2}\big(\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2} - 8(b-1)}\big) = \frac{4(b-1)\tau}{1+\sqrt{1-8(b-1)\tau^2}} = O(\tau)$. Because we are interested in the interior positive trajectories with $F \geq 0$, by Lemma 5.3, it holds that

$$\theta_0 = 0 \quad \text{and} \quad \theta_1 = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right). \tag{102}$$

Thus, the solution curve of the equation $H_2(n, F) = 0$ satisfies

$$F_0(n) = \theta_1 n + O(n^2) = \frac{4(b-1)\tau}{1 + \sqrt{1 - 8(b-1)\tau^2}} \cdot n + O(n^2). \tag{103}$$

Step 3. We proceed to show that when $0 < n \ll 1$, the function $\frac{dF}{dn}(n)$ is monotone.

Assume that $\hat{n}_0 > 0$ is a critical point of the function $\frac{dF}{dn}(n)$; then $\frac{d^2F}{dn^2}(\hat{n}_0) = 0$. We claim that when $\hat{n}_0$ is small enough, it holds that $\frac{d^3F}{dn^3}(\hat{n}_0) > 0$. Differentiating (97) in n, we have

$$\begin{aligned} &2(1+n)FF'' + (1+n)^2F'F'' + (1+n)^2FF''' \\ &= -2FF' - 4(1+n)(F')^2 - 2(1+n)FF'' - 2(1+n)^2F'F'' + \frac{1}{\tau}(F'' - \Xi''). \end{aligned}$$

Noting $F''(\hat{n}_0) = 0$, it then follows from (84) that

$$\begin{aligned} (1+\hat{n}_0)^2FF'''(\hat{n}_0) &= -2FF'(\hat{n}_0) - 4(1+\hat{n}_0)(F'(\hat{n}_0))^2 - \frac{\Xi''(\hat{n}_0)}{\tau} \\ &= -2FF'(\hat{n}_0) - 4(1+\hat{n}_0)(F'(\hat{n}_0))^2 + 2 + \frac{2b}{(1+\hat{n}_0)^3}. \end{aligned} \tag{104}$$

Using (97) again, since $F''(\hat{n}_0) = 0$, it holds that

$$(1+\hat{n}_0)^2(F'(\hat{n}_0))^2 = -2(1+\hat{n}_0)FF'(\hat{n}_0) + \frac{1}{\tau}(F'(\hat{n}_0) - \Xi'(\hat{n}_0)).$$

Substituting this equality into (104) and using (85) leads to

(105)
$$
\begin{aligned}
&(1+\hat{n}_0)^3 FF'''(\hat{n}_0)\\
&= -2(1+\hat{n}_0)FF'(\hat{n}_0) - 4(1+\hat{n}_0)^2(F'(\hat{n}_0))^2 + 2(1+\hat{n}_0) + \frac{2b}{(1+\hat{n}_0)^2}\\
&= 6(1+\hat{n}_0)FF'(\hat{n}_0) - \frac{4F'(\hat{n}_0)}{\tau} + \frac{4\Xi'(\hat{n}_0)}{\tau} + 2(1+\hat{n}_0) + \frac{2b}{(1+\hat{n}_0)^2}\\
&= \frac{6(F(\hat{n}_0)-\Xi(\hat{n}_0))}{\tau(1+\hat{n}_0)} - \frac{4(F(\hat{n}_0)-\Xi(\hat{n}_0))}{\tau^2(1+\hat{n}_0)^2F(\hat{n}_0)} + \frac{4\Xi'(\hat{n}_0)}{\tau} + 2(1+\hat{n}_0) + \frac{2b}{(1+\hat{n}_0)^2}\\
&= \frac{2}{(1+\hat{n}_0)^2F}\cdot\left[\frac{3F(F-\Xi)(1+\hat{n}_0)}{\tau} - \frac{2(F-\Xi)}{\tau^2} + \frac{2(1+\hat{n}_0)^2F\Xi'}{\tau} + (1+\hat{n}_0)^3F + bF\right]\\
&\triangleq \frac{2}{(1+\hat{n}_0)^2F}\cdot J(\hat{n}_0).
\end{aligned}
$$

By (82), (83), and (103), when $\hat{n}_0 \ll 1$,

$$
\begin{aligned}
&F(\hat{n}_0) = \theta_1\hat{n}_0 + O(\hat{n}_0^2), \quad F(\hat{n}_0) - \Xi(\hat{n}_0) = (\theta_1 - 2\tau(b-1))\hat{n}_0 + O(\hat{n}_0^2),\\
&\hat{\Xi}'(\hat{n}_0) = 2\tau(b-1) + O(\hat{n}_0).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
J(\hat{n}_0) &= -\frac{2(\theta_1 - 2\tau(b-1))}{\tau^2}\hat{n}_0 + 4(b-1)\theta_1\hat{n}_0 + \theta_1\hat{n}_0 + b\theta_1\hat{n}_0 + O(\hat{n}_0^2)\\
&= \left[\frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} + (5b-3)\theta_1\right]\hat{n}_0 + O(\hat{n}_0^2).
\end{aligned}
\tag{106}
$$

By (102),
$$
\frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} = -\theta_1\cdot\frac{8(b-1)}{1+\sqrt{1-8(b-1)\tau^2}}.
$$

It hence follows that if $\tau \ll 1$ such that $\tau^2 < \frac{1}{16(b-1)} < \frac{2}{25(b-1)}$, then

$$
\begin{aligned}
\frac{4(b-1)}{\tau} - \frac{2\theta_1}{\tau^2} + (5b-3)\theta_1 &= \theta_1\left[5b-3-\frac{8(b-1)}{1+\sqrt{1-8(b-1)\tau^2}}\right]\\
&> \theta_1(5b-3-5(b-1))\\
&= 2\theta_1 > 0.
\end{aligned}
$$

Substituting this inequality into (106) and then (105), we conclude that

$$
F'''(\hat{n}_0) > 0 \ \text{ if } \hat{n}_0 \ll 1 \quad \text{and} \quad \tau < \frac{1}{4\sqrt{b-1}}.
$$

Thus, the critical point $\hat{n}_0$ must be the local minimal point of $\frac{dF}{dn}(n)$, and hence there exists $n_2 > 0$ such that the function $\frac{dF}{dn}(n)$ has at most one critical point over $(0, n_2)$. This implies $\frac{d^2F}{dn^2}$ could change sign at most once on $(0, n_2)$. As a consequence, there exists $n_3 \in (0, n_2)$ such that the function $\frac{dF}{dn}(n)$ is monotone on $(0, n_3)$.

Step 4. Now by Lemma 5.4 and the monotonicity of $\frac{dF}{dn}$, one can easily see that

$$
\lim_{n\to 0^+} F'(n) \text{ exists } \triangleq F'(0).
$$

Then $F'(n)$ is continuous on $[0, n_2]$. It remains to show (94). Applying L'Hôpital's principle to (81) at $n = 0$, it holds that

$$F'(0) = \frac{2(1-b)}{F'(0)} + \frac{1}{\tau}.$$

Thus,

$$F'(0) = \frac{1}{2}\left(\frac{1}{\tau} + \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) = O\left(\frac{1}{\tau}\right)$$

or

$$F'(0) = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) = O(\tau).$$

By Lemma 5.3, $F'(0) = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right)$. ☐

THEOREM 5.6. *Assume that $b(x) = b > 1$ is a constant. There exists a constant $\tau_0 = \tau_0(b)$ depending only on b, such that for any $0 < \tau < \tau_0$ the interior subsonic solution to system* (5) *satisfies*

$$\rho \in C^1[0,\epsilon], \quad \rho(0) = 1, \quad E(0) = \frac{1}{\tau}, \quad \textit{and} \quad \rho_x(0) = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) \tag{107}$$

for some $\epsilon > 0$.

Proof. Recalling the transformation (79), $\rho = n + 1$ and $E = F + \frac{1}{\tau(n+1)}$. By Lemma 5.2, one can find that all interior subsonic trajectories to system (5) must start from $(1, \frac{1}{\tau})$. In other words, all interior subsonic solutions to system (5) must satisfy $\rho(0) = 1$ and $E(0) = \frac{1}{\tau}$. By L'Hôpital's principle and (94),

$$\lim_{n\to 0^+} \frac{n}{F(n)} = \lim_{n\to 0^+} \frac{1}{F'(n)} = \frac{1}{\theta_1},$$

which together with the first equation of (80) gives

$$n_x(0) = \lim_{x\to 0^+} \frac{F(x)}{2n(x)} = \frac{\theta_1}{2}.$$

Thus, $n \in C^1[0,\epsilon]$ for some $\epsilon > 0$. Recalling $n = \rho - 1$, we have $\rho \in C^1[0,\epsilon]$ for some $\epsilon > 0$, and

$$\rho_x(0) = \frac{\theta_1}{2} = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right),$$

where we have used $\theta_1 = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right)$. ☐

We next study the structure of the interior supersonic solutions. To do so, we still study the transformed equations (80) and (81) but with $n \in (-1, 0]$.

LEMMA 5.7. *When $0 < \tau < \frac{1}{3\sqrt{b}}$, all interior negative trajectories to system* (80) *end at the point* $(0,0)$.

Proof. By (82)–(84),

$$\Xi(n) \le 0, \quad \Xi'(n) > 2\tau(b-1) > 0, \quad \Xi''(n) < 0 \quad \text{for } n \in (-1, 0], \tag{108}$$

$$\lim_{n\to -1} \Xi(n) = -\infty. \tag{109}$$

We next focus on the region $F \le 0$. Notice that (87) still holds. If $F(n) = -h < 0$, then $F^2(0) - \beta^2\Xi^2(0) = h^2 > 0$ for any $\beta > 0$. To ensure $F^2(n) > \beta^2\Xi^2(n)$ for any $n \in (-1, 0]$, we also need to determine β such that $I > 0$ for $n \in (-1, 0]$. Setting $\beta = \frac{c_1}{\tau^2}$ with $c_1 = \frac{1}{3b}$, when $\tau < \frac{1}{3\sqrt{b}}$, we have for $n \in (-1, 0]$

$$\begin{aligned}
I &= \frac{1}{\tau(1+n)^2} \cdot \left[\frac{c_1}{\tau^2} - 1 + \frac{c_1^2}{\tau^2} \cdot (2(1+n)^3 - b(1+n)^2 - b)\right] \\
&\ge \frac{1}{\tau(1+n)^2} \cdot \left[\frac{c_1}{\tau^2} - 1 - \frac{2bc_1^2}{\tau^2}\right] \\
&= \frac{1}{\tau(1+n)^2} \cdot \left(\frac{1}{9b\tau^2} - 1\right) \\
&> 0.
\end{aligned}$$

It then follows from (87) that $F^2(n) > \frac{\Xi^2(n)}{3b\tau^2}$ for $n \in (-1, 0)$. Noting $F(0) < 0$ and $\Xi(n) < 0$ on $(-1, 0)$, we get

$$F(n) < \Xi(n) \quad \text{for } n \in (-1, 0).$$

It hence follows from (109) that

$$\lim_{n\to -1} F(n) = -\infty,$$

and the trajectory ending at $(0, -h)$ with $h > 0$ does not start from a point of the line $n = 0$. Thus, when $\tau < \frac{1}{3\sqrt{b}}$, all interior negative trajectories should end at the point $(0, 0)$. ∎

LEMMA 5.8. *When $\tau < \frac{1}{3\sqrt{b}}$, all interior negative trajectories to system* (80) *satisfy*

$$F(n) \ge \frac{3}{2} \cdot \Xi(n) \quad \textit{for } n \in (-1, 0]. \tag{110}$$

Proof. Taking $\beta = \frac{3}{2}$ in (87), when $\tau < \frac{1}{3\sqrt{b}}$, we have for $n \in (-1, 0]$

$$\begin{aligned}
I &= \frac{1}{\tau(1+n)^2} \cdot \left[\frac{1}{2} + \frac{9\tau^2}{4} \cdot (2(1+n)^3 - b(1+n)^2 - b)\right] \\
&\ge \frac{1}{\tau(1+n)^2} \cdot \left[\frac{1}{2} - \frac{9\tau^2}{4} \cdot 2b\right] \\
&> 0.
\end{aligned}$$

If there is a point $\bar{n} \in (-1, 0)$ on the trajectory such that $F(\bar{n}) < \frac{3}{2} \cdot \Xi(\bar{n}) < 0$, then noting $\Xi(n) < 0$ and $I > 0$ on $(-1, \bar{n})$, by (87), we have

$$F^2(n) > \frac{9}{4} \cdot \Xi^2(n) \quad \text{on } (-1, \bar{n}).$$

Because $\Xi(\bar{n}) < 0$ and $F(\bar{n}) < 0$ on $(-1, \bar{n})$, we have $F(n) < \frac{3}{2} \cdot \Xi(n)$ for $n \in (-1, \bar{n})$. Thus, by (109), $\lim_{n\to -1} F(n) = -\infty$, and this trajectory starts from infinity and cannot be an interior negative trajectory to system (80). We hence get (110). ∎

LEMMA 5.9. *When* $\tau < \frac{1}{3\sqrt{b}}$, *all interior negative trajectories to system* (80) *with* $F \le 0$ *are Lipschitz continuous on a neighborhood of* $n = 0$.

Proof. We first show that an interior negative trajectory must have at least one critical point on $(-1, 0)$. Otherwise, the trajectory has no critical point over $(-1, 0)$; then

$$F'(n) > 0 \text{ on } (-1,0) \quad \text{or} \quad F'(n) < 0 \text{ on } (-1,0).$$

If $F'(n) > 0$ on $(-1, 0)$, then

$$(F - \Xi)'(n) > 0 \text{ on } (-1, 0).$$

By Lemma 5.7, when $\tau < \frac{1}{3\sqrt{b}}$, $F(0) = \Xi(0) = 0$, we then have $F(n) < \Xi(n) < 0$ on $(-1, 0)$. Thus, by (109),

$$\lim_{n\to-1} F(n) < \lim_{n\to-1} \Xi(n) = -\infty.$$

This implies the trajectory cannot start from a point of the line $n = 0$. If $F'(n) < 0$ on $(-1, 0)$, noting $F(0) = 0$, it holds that $F(n) > 0$ for $n \in (-1, 0)$. By (81), since $\frac{(n+1-b)(2-n)n}{(1+n)^3 F} > 0$, we have

$$F'(n) > \frac{1}{\tau(1+n)^2} > 0 \text{ on } (-1, 0),$$

which is a contradiction. Thus, an interior negative trajectory to system (80) has at least one critical point on $(-1, 0)$.

We next claim that this interior negative trajectory has at most one critical point. Denote by $\widetilde{n_0} \in (-1, 0)$ a critical point of this trajectory. Then by (108),

$$F(\widetilde{n_0}) = \Xi(\widetilde{n_0}) < 0, \quad \Xi(n) < 0, \quad \Xi'(n) > 0 \text{ on } (-1, \widetilde{n_0}). \tag{111}$$

Noting (90) still holds, it follows from (111) and (90) that when $F \le 0$,

$$0 \ge F(n) > \Xi(n) \text{ for } n \in (n_*, \widetilde{n_0}),$$

where n_* is the point such that $F(n_*) = 0$. In other words, there is no critical point on $(-1, \widetilde{n_0})$. On the other hand, if there is a critical point $\widetilde{n_1} \in (\widetilde{n_0}, 0)$, then

$$F(\widetilde{n_1}) = \Xi(\widetilde{n_1}) < 0, \quad \Xi(n) < 0, \quad \text{and } \Xi'(n) > 0 \text{ on } (\widetilde{n_0}, \widetilde{n_1}).$$

Applying (90) again, we have

$$F(n) > \Xi(n) \text{ for } n \in (n_*, \widetilde{n_1}),$$

which contradicts the fact that $F(\widetilde{n_0}) = \Xi(\widetilde{n_0})$. Thus, there is no critical point on $(\widetilde{n_0}, 0)$ for this trajectory, and $\widetilde{n_0}$ is the unique critical point of this trajectory. As a consequence, we obtain

$$\frac{dF(n)}{dn} > 0 \text{ on } (\widetilde{n_0}, 0). \tag{112}$$

We next derive an upper bound of $\frac{dF}{dn}$. Integrating (93) on $(x, 1)$, noting $F(1) = 0$, we have

$$F(x) > \frac{1}{\tau} \int_x^1 \left(\frac{1}{1+n} \right)_x dx = \frac{1}{\tau} - \frac{1}{\tau(1+n)} = \frac{n}{\tau(1+n)}.$$

Noting $F < 0$ on $(x, 1)$, it follows that $\frac{n}{F} > \tau(1+n)$. By (81), we obtain

$$\frac{dF(n)}{dn} < \frac{\tau(n+1-b)(2+n)}{(1+n)^2} + \frac{1}{\tau(1+n)^2} \leq \frac{1}{\tau(1+n)^2} < \frac{1}{\tau(1+\widetilde{n_0})^2} \quad \text{for } n \in (\widetilde{n_0}, 0).$$

This estimate together with (112) implies the trajectory is Lipschitz continuous on $(\widetilde{n_0}, 0)$. ∎

LEMMA 5.10. *When $\tau < \min\{\frac{1}{3\sqrt{b}}, \frac{1}{4\sqrt{b-1}}\}$, all interior negative trajectories to system (80) with $F \leq 0$ are C^1-smooth on a neighborhood of $n = 0$ and*

$$\frac{dF}{dn}(0) = \frac{1}{2}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right).$$

Proof. The proof is quite similar to that of Lemma 5.5. The main difference is that now the unique critical point of the function $\frac{dF}{dn}$ is the maximal point of $\frac{dF}{dn}$. Other changes are obvious. ∎

On the basis of Lemma 5.10, analogously to Theorem 5.6, one can obtain the refined structure of the interior supersonic solution established in Theorem 3.1.

THEOREM 5.11. *Assume that $b(x) = b > 1$ is a constant. There exists a constant $\tau_0 = \tau_0(b)$ such that for any $0 < \tau < \tau_0$ the interior supersonic solution (ρ, E) on an interval $[0, L]$ satisfies*

(113)
$$\rho \in C^1[L-\epsilon, L],\ \rho(0) = \rho(L) = 1,\ E(L) = \frac{1}{\tau},\ \text{ and }\ \rho_x(L) = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right)$$

for some $\epsilon > 0$.

On the basis of Theorems 5.6 and 5.11, we are able to construct interior C^1-smooth transonic solutions to system (5)–(6).

THEOREM 5.12. *Assume that $b(x) = b > 1$ is a constant. There exists a constant $\tau_0 = \tau_0(b)$ such that for any $0 < \tau < \tau_0$, there exist infinitely many interior C^1-smooth transonic solutions to system (5)–(6) in the form*

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where $x_0 \in (0, 1)$ is the location of transition, and $0 < \rho_{sup}(x) \leq 1$ and $\rho_{sub}(x) \geq 1$ satisfy

$$\rho_{sup}(x_0) = \rho_{sub}(x_0) = 1, \tag{114}$$

$$\begin{aligned} (\rho_{sup})_x(x_0) = (\rho_{sub})_x(x_0) &= \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right), \\ E_{sup}(x_0) = E_{sub}(x_0) &= \frac{1}{\tau}. \end{aligned} \tag{115}$$

Proof. For any $x_0 \in (0, 1)$, by Theorem 3.1, system (5) admits an interior supersonic solution ρ_{sup} on $[0, x_0]$ satisfying

$$\rho_{sup}(0) = \rho_{sup}(x_0) = 1.$$

By Theorem 5.11, there exists a constant $\tau_0 = \tau_0(b)$ such that for any $0 < \tau < \tau_0$

$$\begin{cases} \rho_{sup} \in C^1[x_0 - \epsilon_0, x_0], \ E_{sup}(x_0) = \frac{1}{\tau}, \\ (\rho_{sup})_x(x_0) = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) \end{cases} \tag{116}$$

for some $\epsilon_0 > 0$.

On the other hand, by Theorem 2.1, system (5) has a unique interior subsonic solution ρ_{sub} on $[x_0, 1]$ satisfying

$$\rho_{sub}(x_0) = \rho_{sub}(1) = 1.$$

By Theorem 5.6, there exists a constant $\tau_1 = \tau_1(b)$ such that for any $0 < \tau < \tau_1$

$$\begin{cases} \rho_{sub} \in C^1[x_0, x_0 + \epsilon_1], \ E_{sub}(x_0) = \frac{1}{\tau}, \\ (\rho_{sub})_x(x_0) = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(b-1)}\right) \end{cases} \tag{117}$$

for some $\epsilon_1 > 0$. We can now construct an interior C^1-smooth transonic solution by

$$\rho(x) = \begin{cases} \rho_{sup}(x), \ x \in [0, x_0], \\ \rho_{sub}(x), \ x \in [x_0, 1]. \end{cases}$$

Furthermore, (114) and (115) follow from (116) and (117). Because $x_0 \in (0, 1)$ is arbitrary, the C^1-smooth transonic solutions are infinitely many. ◻

As a by-product, one can easily see that when $0 < \tau \ll 1$, there is no transonic solution with shock. In other words, when τ is small, system (5) admits transonic solution of C^1-smooth type only.

Theorem 5.13. *Assume that $b(x) = b > 1$ is a constant. There exists a constant $\tau_0 = \tau_0(b)$ such that for any $0 < \tau < \tau_0$, system (5)–(6) has no transonic shock solution.*

Proof. We argue by contradiction. Assume that there is a transonic solution with shock. Denote by x_0 the jump location. By the Rankine–Hugoniot condition (11) and (12),

$$E_l = E_r \quad \text{and} \quad \rho_l \rho_r = 1. \tag{118}$$

Because the solution is discontinuous, it holds that $0 < \rho_l < 1 < \rho_r$. Clearly, there are two cases for the value of E_l:

$$E_l \le \frac{1}{\tau} \quad \text{or} \quad E_l > \frac{1}{\tau}.$$

If the former case holds, observing that at x_0, $\rho_{sup}(x_0)E_{sup}(x_0) - \frac{1}{\tau} = \rho_l E_l - \frac{1}{\tau} < 0$, it follows from the first equation of (5) that

$$(\rho_{sup})_x(x_0) = \frac{\rho_l E_l - \frac{1}{\tau}}{1 - \frac{1}{\rho_l^2}} > 0.$$

Thus, we can extend this supersonic solution to an interval $[0, L]$ such that

$$\rho_{sup}(L) = 1 \quad \text{and} \quad E_{sup}(L) < E_{sup}(x_0) = E_l < \frac{1}{\tau}.$$

Here we have used the fact that E_{sup} is monotone decreasing. Recalling the transformation (79), this implies

$$F_{sup}(L) = E_{sup}(L) - \frac{1}{\tau} < 0.$$

In view of the proof of Lemma 5.7, we find that the corresponding trajectory satisfies

$$\lim_{n\to -1^+} F_{sup}(n) = -\infty.$$

Thus, this supersonic solution cannot satisfy the left boundary condition $\rho_{sup}(0) = 1$, which is a contradiction.

If the latter case happens, by (118), we get

$$E_r > \frac{1}{\tau} \quad \text{and} \quad \rho_r > 1.$$

Thus, we can extend backward this subsonic part to an interior subsonic solution of system (5), still denoted by (ρ_{sub}, E_{sub}) such that for some $x_{-1} \in \mathbb{R}$,

$$\rho_{sub}(x_{-1}) = 1, \quad E_{sub}(x_{-1}) > E_r > \frac{1}{\tau},$$

where we have used the fact that E_{sub} is monotone decreasing. Recalling the transformation again, we have

$$F_{sub}(x_{-1}) = E_{sub}(x_{-1}) - \frac{1}{\tau} > E_r - \frac{1}{\tau} > 0.$$

In view of the proof of Lemma 5.2, one can see that the corresponding trajectory will go to infinity, which contradicts the right boundary condition $\rho_{sub}(1) = 1$. Therefore, there is no transonic solution with shock. ∎

Proof of Theorem 1.3. Combining Theorems 2.1, 3.1, 4.2, 5.12, and 5.13, we immediately obtain Theorem 1.3. ∎

6. Examples. Let us test a special case when $b(x) \equiv b > 1$ (constant). We may clearly observe the structure of stationary solutions to system (5)–(6) from the phase-plane analysis, which also further confirms our main theorem mentioned above. Notice that, when $b > 1$, the critical point of system (5) is $A = \left(b, \frac{1}{\tau b}\right)$, and the Jacobian matrix of system (5) at A is

$$J(A) = \begin{bmatrix} \dfrac{b}{\tau(b^2-1)} & \dfrac{b^3}{b^2-1} \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the eigenvalues λ of matrix $J(A)$ satisfy the following characteristic equation:

$$\lambda^2 - \frac{b\lambda}{\tau(b^2-1)} - \frac{b^3}{b^2-1} = 0. \tag{119}$$

Notice that $\lambda_1\lambda_2 = -\dfrac{b^3}{b^2-1} < 0$, where λ_1 and λ_2 are the roots of (119). Thus, A is a saddle point. On the other hand, it follows from system (5) that

$$\frac{dE}{d\rho} = \frac{(\rho - b)(1 - \frac{1}{\rho^2})}{\rho E - \frac{1}{\tau}}, \tag{120}$$

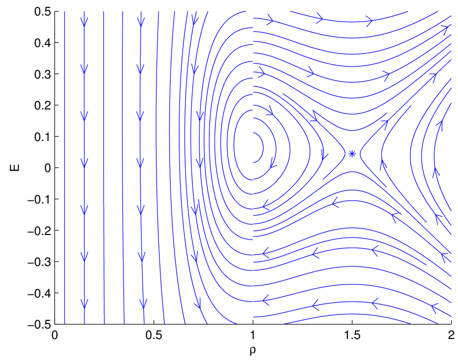

FIG. 2. *Phase-plane of* (ρ, E) *with* $\tau = 15$ *and* $b = 1.5$. $*$ *is the saddle point* $A = (1.5, 2/45)$.

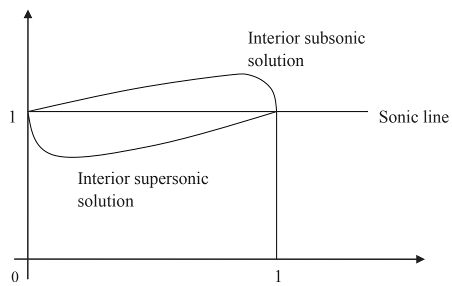

FIG. 3. *Interior subsonic solution and interior supersonic solution.*

which helps to determine the directions of all trajectories. Here and in what follows, to avoid confusion, we denote by $E = E(\rho)$ the function of the trajectory.

Example 1: *Subsonic/supersonic solutions.* Let $\tau = 15$ and $b = 1.5$. We draw the phase-plane of (ρ, E) for (5) in Figure 2, from which we observe that there exist at least one interior subsonic solution and one interior supersonic solution. These interior subsonic/supersonic solutions can be roughly sketched in Figure 3.

Example 2: *Transonic shock solutions.* In this example, we show in Figure 4 how to construct an interior transonic shock solution when τ is large: the discontinuous trajectory in blue stands for a transonic shock solution with smaller length (e.g., $\frac{1}{2}$) and is structured by a stationary shock at x_0 with the Rankine–Hugoniot jump condition (11) linking the other two solutions. One is a supersonic solution $\rho_{sup}(x)$ with $\rho_{sup}(0) = 1$ and $\rho_{sup}(x_0^-) = \rho_l < 1$, and the other is a subsonic solution $\rho_{sub}(x)$ with $\rho_{sub}(x_0^+) = \rho_r > 1$ and $\rho_{sub}(\frac{1}{2}) = 1$. The discontinuous trajectory in red (color available online) represents a similar transonic shock solution with larger length (e.g., $\frac{3}{2}$) satisfying the entropy condition and the Rankine–Hugoniot condition at some jump location. By continuity, there is an interior transonic shock solution to (5) on $[0, 1]$. Since the choice of $\rho_l = \rho_{sup}(x_0^+)$ can be infinitely many when $\rho_r - \rho_l \ll 1$, there are infinitely many transonic shock solutions. In Figure 5, we draw two transonic shock solutions to system (5) with different ρ_l.

Example 3: *Smooth transonic solutions.* When τ is small, for example, by taking $\tau = 0.5$, we see in Figure 6 that the phase-plane changes dramatically: many subsonic trajectories start from the same point $(1, \frac{1}{\tau})$, and many supersonic trajectories end at the same point $(1, \frac{1}{\tau})$. As a result, one can see that there are possibly smooth

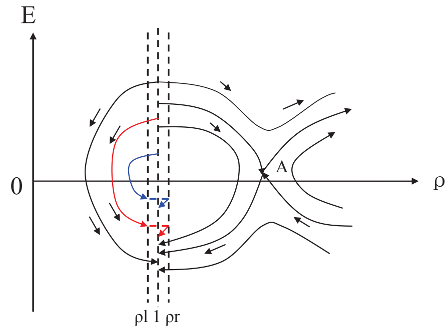

FIG. 4. *Transonic shock trajectories in the phase-plane of* (ρ, E) *when* τ *is large.*

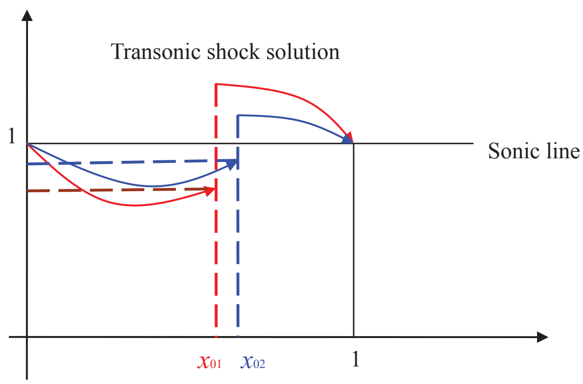

FIG. 5. *Transonic shock solutions when* τ *is large.*

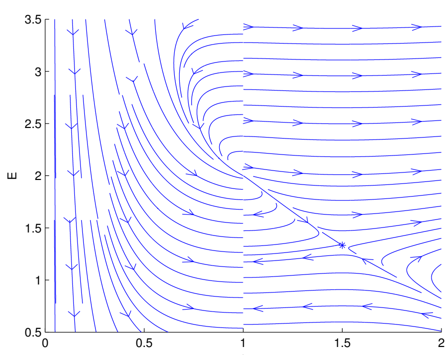

FIG. 6. *Phase-plane of* (ρ, E) *with* $\tau = 0.5$ *and* $b = 1.5$. $*$ *is the saddle point* $A = (1.5, 4/3)$.

transonic solutions, which are constructed by two solutions at some location x_0: one is an interior supersonic solution with $\rho_{sup}(0) = 1 = \rho_{sup}(x_0)$, and the other is an interior subsonic solution with $\rho_{sub}(x_0) = 1 = \rho_{sub}(1)$. Since the transition location x_0 can be chosen arbitrarily in $(0, 1)$, these smooth transonic solutions are infinitely many.

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