J. Math. Anal. Appl. 456 (2017) 849–871

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Convergence to nonlinear diffusion waves for solutions of p-system with time-dependent damping

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ARTICLE INFO

Article history:
Received 24 May 2017
Available online 19 July 2017
Submitted by D. Wang

Keywords:
Time-dependent damping
Nonlinear diffusion waves
Convergence rates
Time-weighted energy estimates

ABSTRACT

In this paper, we study the Cauchy problem for the hyperbolic p-system with time-gradually-degenerate damping term $-\frac{1}{(1+t)^\lambda}u$ for $0 \leqslant \lambda < 1$, and show that the damped p-system has a couple of global solutions uniquely, and such solutions tend time-asymptotically to the shifted nonlinear diffusion waves, which are the solutions of the corresponding nonlinear parabolic equation governed by the Darcy's law. We further derive the convergence rates when the initial perturbations are in L^2. The approach adopted is the technical time-weighted energy method.

1. Introduction

In order to investigate how the weak dissipation affects the solutions to the damped wave equations, Wirth [39–41] first studied the linear wave equation with time-dependent damping

$$v_{tt} + b(t)v_t - \Delta v = 0,$$

and derived the standard $L^p - L^q$ estimates for the solutions, where the damping coefficient $b(t)$ is time-gradually vanishing, for example, $b(t) = (1+t)^{-\lambda}$ with $\lambda > 0$. Later then, Nishihara and his collaborators [18,27,31] intensively studied the global existence or blow up of the solutions to the time-dependently damped wave equations with nonlinear source

$$v_{tt} + b(t)v_t - \Delta v + |v|^{\rho-1}v = 0,$$

see also some recent developments [4,28,37] and the references cited therein.

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http://dx.doi.org/10.1016/j.jmaa.2017.07.025
0022-247X/

Instead of nonlinear source term, the other typical models are the visco-elasticity dynamics or the porous media flow with nonlinear diffusion

$$v_{tt} + b(t)v_t - \Delta\sigma(v) = 0, \quad \text{or} \quad v_{tt} + b(t)v_t + \Delta p(v) = 0,$$

with $\sigma'(v) > 0$ or $p'(v) < 0$. Here, $\sigma(v)$ is the stress function for visco-elastic material, and $p(v)$ is the pressure function for porous media flow. If we set $u_x = v_t$ for the above 1-d equation, then we derive the following damped p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -b(t)u. \end{cases}$$

In this paper, we are interested in the following Cauchy problem for the p-system of hyperbolic conservation laws with time-gradually-degenerate damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\frac{1}{(1+t)^\lambda}u, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.1}$$

Physically, system (1.1) models a compressible flow through porous media in Lagrangian coordinates, where, $v > 0$ is the specific volume, u is the velocity, and the pressure $p(v)$ is a smooth function of v with $p(v) > 0$ and $p'(v) < 0$. A typical example is the polytropic gas, where $p(v) = v^{-\gamma}$ with $\gamma \geqslant 1$. The damping term $\frac{1}{(1+t)^\lambda}u$ represents the time-gradually-vanishing friction effect. Here and hereafter, we assume that $0 \leqslant \lambda < 1$.

Subjected to (1.1), the initial value is proposed as

$$(v,u)(x,0) = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty, \tag{1.2}$$

where $v_\pm > 0$ and $u_\pm$ are the constant states.

From Darcy's law, the asymptotic profiles of the solutions to system (1.1)–(1.2) are expected to be the solutions of the following nonlinear diffusion equations with time-dependent damping:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\frac{1}{(1+t)^\lambda}\bar{u}, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad 0 \leqslant \lambda < 1, \tag{1.3}$$

or

$$\begin{cases} \bar{v}_t = -(1+t)^\lambda p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\frac{1}{(1+t)^\lambda}\bar{u}, \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad 0 \leqslant \lambda < 1, \tag{1.4}$$

with

$$(\bar{v},\bar{u})(x,t) \to (v_\pm, 0) \quad \text{as } x \to \pm\infty. \tag{1.5}$$

This system possesses some self-similar solutions in the form of

$$(\bar{v},\bar{u})(x,t) = (\bar{v},\bar{u})\Big(\frac{x}{\sqrt{(1+t)^{1+\lambda}}}\Big),$$

which are usually called nonlinear diffusion waves. When $\lambda = 0$, C.T. van Duyn and L.A. Peletier [3] first showed the existence of the self-similar solutions in the form of $(\bar{v},\bar{u})\Big(\frac{x}{\sqrt{1+t}}\Big)$.

The p-system without time-dependent damping (i.e., the case of $\lambda = 0$) has been drawn a lot of attention:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{1.6}$$

where $\alpha > 0$ is the damping constant. In their pioneering work, Hsiao and Liu [9] first proved that the solutions (v, u) of (1.6) converge to the diffusion waves $(\bar{v}, \bar{u})$ of the system

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases}$$

in the sense $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$. Then Nishihara [24] significantly improved the convergence rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}})$ for the initial perturbations in L^2-sense. Subsequently, Nishihara, Wang and Yang [29], Wang and Yang [38] further improved the rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-\frac{3}{2}})$ by constructing an approximate Green function for the initial perturbations in L^1-sense. Furthermore, Mei [23] observed that the best asymptotic profiles are the solutions for the corresponding nonlinear diffusion equation with some particularly selected initial data, and showed that the convergence to the best asymptotic profile is in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)\big((1+t)^{-\frac{3}{2}}\log(2+t), (1+t)^{-2}\log(2+t)\big)$, when the initial perturbations are in L^1. The new rates are much better than the existing convergence rates to the diffusion waves, which was further improved to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)\big((1+t)^{-\frac{3}{2}}, (1+t)^{-2}\big)$ by Geng and Wang [5]. For other results related to (1.6) with nonlinear damping or vacuum or bounded domain, and so on, we refer to these interesting works [1,2,6,10–17,19,20,22,25,26,30,42–47] and the references therein.

Regarding the time-dependent damping to the p-system, the following 1-d isentropic Euler equations have been paid more attention in recent years:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\frac{\mu}{(1+t)^\lambda}\rho u, \\ \rho|_{t=0} = 1 + \varepsilon \rho_0(x), \quad u|_{t=0} = \varepsilon u_0(x), \end{cases} \tag{1.7}$$

where $\rho_0(x)$, $u_0(x) \in C_0^\infty(\mathbb{R})$, $|x| \leqslant R$ and $\varepsilon > 0$ is sufficiently small. Here $\rho(x)$ is the density, $u(x)$ denotes the fluid velocity, $P(\rho)$ stands for the pressure. $\lambda \geqslant 0$, $\mu \geqslant 0$ are two constants to describe the scale of the damping. Pan [34] showed that $\lambda = 1$, $\mu = 2$ are the critical values of (1.7), which separate the global existence and finite-time blow up of solutions. More precisely, when $\lambda = 1$, $\mu > 2$ or $0 \leqslant \lambda < 1$, $\mu > 0$, (1.7) has global smooth solutions; while, when $\lambda = 1$, $0 \leqslant \mu \leqslant 2$ or $\lambda > 1$, $\mu \geqslant 0$, the C^1 solutions of (1.7) will blow up in finite time. The case dealt with $\lambda = 1$, $\mu > 2$ has been reported in [33], and the case of $\lambda = 1$, $0 \leqslant \mu \leqslant 2$ is treated in [32]. Furthermore, Pan [34] gave a finite upper bound for the lifespan of classical solutions when $\lambda = 1$, $0 \leqslant \mu \leqslant 2$ or $\lambda > 1$, $\mu \geqslant 0$. For more results about 2-d and 3-d Euler equations with time-dependent damping, we refer to [7,8,35,36].

Motivated by the above results, in the present paper, we study the case of $0 \leqslant \lambda < 1$ corresponding to (1.1). We shall prove the existence of global smooth solutions, and derive the convergence of the solutions (v, u) to the diffusion waves $(\bar{v}, \bar{u})$, which are the solutions of the corresponding nonlinear parabolic equation given by the Darcy's law. Here the diffusion waves have the form $(\bar{v}, \bar{u})(x/\sqrt{(1+t)^{1+\lambda}})$, which are different from the existing studies in the form of $(\bar{v}, \bar{u})(x/\sqrt{1+t})$, due to the gradual vanishing of the damping. See Section 2 for more details.

In order to see what of kind gaps between the original solutions and the target profiles (diffusion waves) at far fields is, let us make the following heuristic analysis. For a diffusion wave $(\bar{v}, \bar{u})(x + x_0, t)$ with some constant x_0, which will be determined later, we first get from $(1.1)_1$ and $(1.3)_1$ that

$$(v-\bar{v})_t-(u-\bar{u})_x=0. \tag{1.8}$$

Integrating (1.8) with respect to x over $(-\infty,+\infty)$, and noting (1.5), we have

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{+\infty}[v(x,t)-\bar{v}(x+x_0,t)]dx&=[u(+\infty,t)-\bar{u}(+\infty,t)]-[u(-\infty,t)-\bar{u}(-\infty,t)]\\&=u(+\infty,t)-u(-\infty,t).\end{aligned} \tag{1.9}$$

In order to overcome the difficulty caused by $u(+\infty,t)\neq u(-\infty,t)$, we need to construct a pair of correction functions $(\hat{v},\hat{u})(x,t)$. As in [22], we define

$$\begin{cases}\hat{u}(x,t)=m(x)e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]},\\ \hat{v}(x,t)=-m'(x)\int_t^{+\infty}e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]}ds,\end{cases}$$

where

$$m(x)=u_-+(u_+-u_-)\int_{-\infty}^{x}m_0(y)dy.$$

Here $m_0(x)>0$, $m_0(x)\in C_0^\infty(\mathbb{R})$ and $\int_{-\infty}^{+\infty}m_0(x)dx=1$. Thus

$$\hat{v}(x,t)=(u_--u_+)m_0(x)\int_t^{+\infty}e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]}ds, \tag{1.10}$$

and it holds that

$$\begin{cases}\hat{v}_t-\hat{u}_x=0,\\ \hat{u}_t=-\frac{1}{(1+t)^\lambda}\hat{u}.\end{cases} \tag{1.11}$$

Therefore, by (1.1), (1.3) and (1.11), we have

$$\begin{cases}(v-\bar{v}-\hat{v})_t-(u-\bar{u}-\hat{u})_x=0,\\ (u-\bar{u}-\hat{u})_t+[p(v)-p(\bar{v})]_x=-\left(\frac{u}{(1+t)^\lambda}-\frac{\bar{u}}{(1+t)^\lambda}-\frac{\hat{u}}{(1+t)^\lambda}\right)-\bar{u}_t,\end{cases} \tag{1.12}$$

where $(\bar{v},\bar{u})$ is the shifted diffusion wave $(\bar{v},\bar{u})(x+x_0,t)$.

Next, we are going to determine the shift x_0. Integrating the first equation of (1.12) with respect to x over $(-\infty,+\infty)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{+\infty}[v(x,t)-\bar{v}(x+x_0,t)-\hat{v}(x,t)]dx\\&=[u(+\infty,t)-\bar{u}(+\infty,t)-\hat{u}(+\infty,t)]-[u(-\infty,t)-\bar{u}(-\infty,t)-\hat{u}(-\infty,t)]\\&=0.\end{aligned} \tag{1.13}$$

Integrating (1.13) with respect to t, we have

$$\int_{-\infty}^{+\infty}[v(x,t)-\bar{v}(x+x_0,t)-\hat{v}(x,t)]dx=\int_{-\infty}^{+\infty}[v_0(x)-\bar{v}(x+x_0,0)-\hat{v}(x,0)]dx$$
$$\triangleq I(x_0).$$

Now, we determine x_0 such that $I(x_0)=0$. Since

$$\begin{aligned}I'(x_0)&=\frac{\partial}{\partial x_0}\left(\int_{-\infty}^{+\infty}\big[v_0(x)-\bar{v}(x+x_0,0)-\hat{v}(x,0)\big]dx\right)\\&=-\int_{-\infty}^{+\infty}\bar{v}'(x+x_0,0)dx\\&=-\big[\bar{v}(+\infty,0)-\bar{v}(-\infty,0)\big]\\&=-(v_+-v_-),\end{aligned}$$

we get

$$I(x_0)-I(0)=\int_0^{x_0}I'(y)dy=-(v_+-v_-)x_0,$$

with $I(x_0)=0$, we further have

$$x_0=\frac{1}{v_+-v_-}I(0)=\frac{1}{v_+-v_-}\int_{-\infty}^{+\infty}\big[v_0(x)-\bar{v}(x,0)-\hat{v}(x,0)\big]dx.$$

Define

$$\begin{cases}V(x,t)\triangleq\int_{-\infty}^{x}\big[v(y,t)-\bar{v}(y+x_0,t)-\hat{v}(y,t)\big]dy,\\ z(x,t)\triangleq u(x,t)-\bar{u}(x+x_0,t)-\hat{u}(x,t),\end{cases}\tag{1.14}$$

and

$$\begin{cases}V_0(x,t)\triangleq\int_{-\infty}^{x}\big[v_0(y)-\bar{v}(y+x_0,0)-\hat{v}(y,0)\big]dy,\\ z_0(x)\triangleq u_0(x)-\bar{u}(x+x_0,0)-\hat{u}(x,0).\end{cases}\tag{1.15}$$

Thus, (1.12) can be transformed into

$$\begin{cases}V_t-z=0,\\ z_t+\big(p'(\bar{v})V_x\big)_x=-\frac{1}{(1+t)^\lambda}z+F-G_x,\\ (V,z)(x,0)=(V_0,z_0)(x),\end{cases}\tag{1.16}$$

where

$$F\triangleq(1+t)^\lambda p(\bar{v})_{xt}+\lambda(1+t)^{\lambda-1}p(\bar{v})_x\quad\text{and}\quad G\triangleq p(V_x+\bar{v}+\hat{v})-p(\bar{v})-p'(\bar{v})V_x.$$

Notations. Throughout this paper, C denotes a generic positive constant which may change from one line to another. $H^m(\mathbb{R})$ $(m\geqslant 0)$ is the usual Sobolev space whose norm is abbreviated as

$$\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\| \quad \text{with } \|f\| := \|f\|_{L^2(\mathbb{R})}.$$

For convenience, we also write

$$\|(f,g)\|^2 := \|f\|^2 + \|g\|^2.$$

Our main result is as follows.

Theorem 1.1. *Suppose that both $\|V_0\|_3 + \|z_0\|_2$ and $\delta =: |v_+ - v_-| + |u_+ - u_-|$ are sufficiently small. Then, there exists a unique couple of time-global solutions $(V,z)(x,t)$ of (1.16), which satisfies*

$$V(x,t) \in C^k\big(0,\infty; H^{3-k}(\mathbb{R})\big),\ k = 0,1,2,3, \quad z(x,t) \in C^k\big(0,\infty; H^{2-k}(\mathbb{R})\big),\ k = 0,1,2,$$

and

$$\begin{aligned}
&\sum_{k=0}^{3}(1+t)^{k+(k-3)\lambda}\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+t)^{2+k+(k-3)\lambda}\|\partial_x^k z(\cdot,t)\|^2 \\
&+ \int_0^t \Bigg[\sum_{k=0}^{3}(1+s)^{k-1+(k-3)\lambda}\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2}(1+s)^{k+1+(k-3)\lambda}\|\partial_x^k z(\cdot,t)\|^2\Bigg] ds \\
&\le C\big(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta\big).
\end{aligned}$$

By the Sobolev inequality

$$\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|^{\frac12}\|f_x\|^{\frac12},$$

we further derive the following estimates.

Corollary 1.1. *Under the previous hypotheses, one has*

$$\|V_x(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{3(1-\lambda)}{4}}, \tag{1.17}$$

and

$$\|V_t(\cdot,t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{5(1-\lambda)}{4}}. \tag{1.18}$$

In view of (1.14), (1.15) and (1.10), the estimates (1.17) and (1.18) with $0 \le \lambda < 1$ imply the convergence of $(v,u)(x,t)$ to $(\bar v,\bar u)(x+x_0,t)$, namely, we obtain the following corollary.

Corollary 1.2 *(Convergence to diffusion waves). Under the conditions in Theorem 1.1, the system (1.1), (1.2) possesses a unique couple of global solutions $(v,u)(x,t)$, which converges to its nonlinear diffusion wave $(\bar v,\bar u)(x+x_0,t)$ in the form of*

$$\|(v-\bar v)(\cdot,t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3(1-\lambda)}{4}}, \tag{1.19}$$

$$\|(u-\bar u)(\cdot,t)\|_{L^\infty} = O(1)(1+t)^{-\frac{5(1-\lambda)}{4}}. \tag{1.20}$$

Remark 1.1. If the initial perturbations are in L^1, we first need to derive the optimal decays for the linear equations with time-gradually vanishing damping

$$v_{tt} + \frac{1}{(1+t)^\lambda} v_t - a(x,t) v_{xx} = 0, \quad a(x,t) > 0,$$

by the method of approximate Green function, then we show the optimal convergence of the original solutions $(v,u)(x,t)$ with time-dependent damping to the corresponding diffusion waves. This will be our next target in future.

2. Diffusion waves

In this section, we firstly state some fundamental properties of the nonlinear diffusion waves. Then we will consider the dissipative nature of the nonlinear diffusion waves.

Inspired by the classical works of van Duyn and Peletier [3], we confirm that the nonlinear diffusion equation

$$\bar{v}_t = -(1+t)^\lambda p(\bar{v})_{xx}, \quad p'(\bar{v}) < 0, \tag{2.1}$$

has a unique (up to a shift) monotone self-similar solution called nonlinear diffusion wave in the form

$$\bar{v}(x,t) = \varphi\left(\frac{x}{\sqrt{(1+t)^{\lambda+1}}}\right) \triangleq \varphi(\xi), \quad \xi \in \mathbb{R}, \tag{2.2}$$

with

$$\varphi(\pm\infty) = v_\pm.$$

Substituting (2.2) into (2.1), it follows that

$$\varphi''(\xi) + \frac{p''(\varphi(\xi))\varphi'(\xi) - \frac{\lambda+1}{2}\xi}{p'(\varphi(\xi))} \cdot \varphi'(\xi) = 0.$$

As in [9], it is easy to see that the self-similar solution $\varphi(\xi)$ satisfies

$$\sum_{k=1}^{4} \left|\partial_\xi^k \varphi(\xi)\right| + \left|\varphi(\xi) - v_+\right|_{\xi>0} + \left|\varphi(\xi) - v_-\right|_{\xi<0} \leqslant C\left|v_+ - v_-\right| e^{-C(\lambda+1)\xi^2}, \tag{2.3}$$

for some constant $C > 0$. A direct calculation from (2.2) gives

$$\begin{cases} \bar{v}_x(x,t) = \frac{\varphi'(\xi)}{\sqrt{(1+t)^{\lambda+1}}}, \quad \bar{v}_t(x,t) = -\frac{(\lambda+1)\xi\varphi'(\xi)}{2(1+t)}, \quad \bar{v}_{xx}(x,t) = \frac{\varphi''(\xi)}{(1+t)^{\lambda+1}}, \\ \bar{v}_{xt}(x,t) = -\frac{(\lambda+1)\left[\varphi'(\xi)+\xi\varphi''(\xi)\right]}{2(1+t)\sqrt{(1+t)^{\lambda+1}}}, \quad \bar{v}_{tt}(x,t) = \frac{(\lambda+1)(\lambda+3)\xi\varphi'(\xi)+(\lambda+1)^2\xi^2\varphi''(\xi)}{4(1+t)^2}, \\ \bar{v}_{xxx}(x,t) = \frac{\varphi'''(\xi)}{(1+t)^{\lambda+1}\sqrt{(1+t)^{\lambda+1}}}, \quad \bar{v}_{xxt}(x,t) = -\frac{(\lambda+1)\left[2\varphi''(\xi)+\xi\varphi'''(\xi)\right]}{2(1+t)^{\lambda+2}}, \\ \bar{v}_{xtt}(x,t) = \frac{(\lambda+1)(\lambda+3)\varphi'(\xi)+(\lambda+1)(3\lambda+5)\xi\varphi''(\xi)+(\lambda+1)^2\xi^2\varphi'''(\xi)}{4(1+t)^2\sqrt{(1+t)^{\lambda+1}}}, \\ \bar{v}_{xxxt}(x,t) = -\frac{(\lambda+1)\left[3\varphi'''(\xi)+\xi\varphi^{(4)}(\xi)\right]}{2(1+t)^{\lambda+2}\sqrt{(1+t)^{\lambda+1}}}, \\ \bar{v}_{xxtt}(x,t) = \frac{4(\lambda+1)(\lambda+2)\varphi''(\xi)+(\lambda+1)(5\lambda+7)\xi\varphi'''(\xi)+(\lambda+1)^2\xi^2\varphi^{(4)}(\xi)}{4(1+t)^{3+\lambda}}. \end{cases} \tag{2.4}$$

Thanks to (2.3) and (2.4), we have the following decay estimates for the diffusion waves.

Lemma 2.1. *Assume that* $0 \le \lambda < 1$, *it holds that*

$$\int_{-\infty}^{+\infty} \left|\partial_x^k \bar{v}(x,t)\right|^2 dx = O(1)\left|v_+ - v_-\right|^2 (1+t)^{-\frac{(2k-1)(\lambda+1)}{2}}, \quad k = 1,2,3,$$

$$\int_{-\infty}^{+\infty} \left|\partial_x^k \bar{v}_t(x,t)\right|^2 dx = O(1)\left|v_+ - v_-\right|^2 (1+t)^{-\frac{(3-\lambda)+2k(\lambda+1)}{2}}, \quad k = 0,1,2,3,$$

$$\int_{-\infty}^{+\infty} \left|\bar{v}_{tt}(x,t)\right|^2 dx = O(1)\left|v_+ - v_-\right|^2 (1+t)^{-\frac{7-\lambda}{2}},$$

$$\int_{-\infty}^{+\infty} \left|\bar{v}_{xtt}(x,t)\right|^2 dx = O(1)\left|v_+ - v_-\right|^2 (1+t)^{-\frac{9+\lambda}{2}},$$

$$\int_{-\infty}^{+\infty} \left|\bar{v}_{xxtt}(x,t)\right|^2 dx = O(1)\left|v_+ - v_-\right|^2 (1+t)^{-\frac{11+3\lambda}{2}}.$$

3. Proof of Theorem 1.1

By (1.16), we obtain

$$\begin{cases} V_{tt} + \dfrac{1}{(1+t)^\lambda} V_t + \left(p'(\bar{v})V_x\right)_x = F - G_x, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ (V, V_t)|_{t=0} = (V_0, z_0)(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where

$$F = (1+t)^\lambda p(\bar{v})_{xt} + \lambda(1+t)^{\lambda-1} p(\bar{v})_x,\ G = p(V_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})V_x.$$

As is well known, Theorem 1.1 can be proved by the classical continuation method based on the local existence and the a priori estimates. The local existence of the solution for (3.1) can be obtained by the iteration method which can be found in [21] for details. Next, we shall devote ourselves to establishing the a priori estimates for the solution by the time-weighted energy method. It is technical and will be the main effort in this section.

Let $T \in [0, \infty]$, we define

$$N(T)^2 := \sup_{0 \leqslant t \leqslant T} \left\{ \sum_{k=0}^{3} (1+t)^{k+(k-3)\lambda} \|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{2} (1+t)^{2+k+(k-3)\lambda} \|\partial_x^k V_t(\cdot,t)\|^2 \right\}. \tag{3.2}$$

Lemma 3.1. *Assume that* $N(T) + \delta \ll 1$, *then*

$$\begin{aligned} &(1+t)^{-3\lambda}\|V(\cdot,t)\|^2 + (1+t)^{-\lambda}\|(V_x, V_t)(\cdot,t)\|^2 \\ &\quad + \int_0^t (1+s)^{-2\lambda}\|(V_x, V_t)(\cdot,s)\|^2 ds \le C\left(\|V_0\|_1^2 + \|z_0\|^2 + \delta\right). \end{aligned} \tag{3.3}$$

Proof. Multiplying (3.1) by $2(1+t)^{-\lambda}V_t$, we get

$$\begin{aligned}&\left[(1+t)^{-\lambda}\left(V_t^2-p'(\bar v)V_x^2\right)\right]_t+\left[2(1+t)^{-2\lambda}+\lambda(1+t)^{-1-\lambda}\right]V_t^2\\&\quad+\left[\lambda(1+t)^{-\lambda-1}|p'(\bar v)|+(1+t)^{-\lambda}p''(\bar v)\bar v_t\right]V_x^2+2\big((1+t)^{-\lambda}p'(\bar v)V_xV_t\big)_x\\&=2(1+t)^{-\lambda}V_tF-2(1+t)^{-\lambda}V_tG_x.\end{aligned}\tag{3.4}$$

In view of (2.3) and the second equality of (2.4), we have

$$(1+t)^{-\lambda}|p''(\bar v)\bar v_t|V_x^2\le C\delta(1+t)^{-\lambda-1}V_x^2.\tag{3.5}$$

To estimate the right hand side of (3.4), we set

$$\begin{aligned}(1+t)^{-\lambda}V_tF&=p(\bar v)_{xt}V_t+\lambda(1+t)^{-1}p(\bar v)_xV_t\\&\triangleq I_1+I_2.\end{aligned}\tag{3.6}$$

By Young's inequality and the second equality of (2.4),

$$\begin{aligned}I_1&\le\frac18(1+t)^{-2\lambda}V_t^2+2(1+t)^{2\lambda}\left(p''(\bar v)\bar v_x\bar v_t+p'(\bar v)\bar v_{xt}\right)^2\\&\le\frac18(1+t)^{-2\lambda}V_t^2+C(1+t)^{2\lambda}\left(|\bar v_x\bar v_t|^2+|\bar v_{xt}|^2\right)\\&\le\frac18(1+t)^{-2\lambda}V_t^2+C\delta(1+t)^{2\lambda-2}|\bar v_x|^2+C(1+t)^{2\lambda}|\bar v_{xt}|^2.\end{aligned}\tag{3.7}$$

Analogous to (3.7),

$$I_2=\lambda(1+t)^{-1}p'(\bar v)\bar v_xV_t\le\frac18(1+t)^{-2\lambda}V_t^2+C(1+t)^{2\lambda-2}|\bar v_x|^2.\tag{3.8}$$

The second term on the right hand side of (3.4) can be estimated as follows.

$$\begin{aligned}&-(1+t)^{-\lambda}V_tG_x\\&=-(1+t)^{-\lambda}\left[p'(V_x+\bar v+\hat v)(V_{xx}+\bar v_x+\hat v_x)-p'(\bar v)\bar v_x-p''(\bar v)\bar v_xV_x-p'(\bar v)V_{xx}\right]V_t\\&=-(1+t)^{-\lambda}\Big[\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xx}+\left(p'(V_x+\bar v+\hat v)-p'(\bar v)-p''(\bar v)V_x\right)\bar v_x\\&\quad+p'(V_x+\bar v+\hat v)\hat v_x\Big]V_t\\&\le C(1+t)^{-\lambda}\left(|V_x+\hat v||V_{xx}|+(|\hat v|+|V_x+\hat v|^2)|\bar v_x|+|\hat v_x|\right)|V_t|,\end{aligned}\tag{3.9}$$

where we have used

$$\begin{aligned}&|p'(V_x+\bar v+\hat v)-p'(\bar v)|=O(1)|V_x+\hat v|\\&|p'(V_x+\bar v+\hat v)-p'(\bar v)-p''(\bar v)V_x|=p''(\bar v)\hat v+O(1)|V_x+\hat v|^2.\end{aligned}\tag{3.10}$$

By Gagliardo–Nirenberg's inequality,

$$\begin{aligned}\|V_x(\cdot,t)\|_{L^\infty}&\le C\|V_x(\cdot,t)\|_{L^2}^{\frac12}\|V_{xx}(\cdot,t)\|_{L^2}^{\frac12}\le CN(t)(1+t)^{\frac{3(\lambda-1)}{4}},\\\|V_{xx}(\cdot,t)\|_{L^\infty}&\le C\|V_{xx}(\cdot,t)\|_{L^2}^{\frac12}\|V_{xxx}(\cdot,t)\|_{L^2}^{\frac12}\le CN(t)(1+t)^{\frac{\lambda-5}{4}}.\end{aligned}\tag{3.11}$$

Thus, it follows from (3.9) and the first equality of (2.4) that

$$(1+t)^{-\lambda}|V_tG_x| \le \frac{1}{8}(1+t)^{-2\lambda}V_t^2 + CN(t)(1+t)^{\frac{\lambda-5}{2}}V_x^2 + C(|\hat{v}|+|\hat{v}_x|). \tag{3.12}$$

In view of (1.10), we get

$$|\hat{v}(x,t)| + |\hat{v}_x(x,t)| \le C\delta e^{-C(1+t)^{1-\lambda}}(m_0(x)+|m_0'(x)|). \tag{3.13}$$

Now, integrating (3.4) over $\mathbb{R}\times[0,t]$, and noting that by Lemma 2.1 and (3.13),

$$\int_0^t\int \left[(1+s)^{2\lambda}|\bar{v}_{xt}|^2 + (1+s)^{2\lambda-2}|\bar{v}_x|^2 + |\hat{v}| + |\hat{v}_x|\right] \le C\delta^2, \tag{3.14}$$

it follows from (3.5)–(3.12) that

$$\begin{aligned} &\int (1+t)^{-\lambda}\left(V_t^2 - p'(\bar{v})V_x^2\right) + \int_0^t\int \left[\frac{5}{4}(1+s)^{-2\lambda} + \lambda(1+s)^{-1-\lambda}\right]V_t^2 \\ &+ \int_0^t\int \left(\lambda|p'(\bar{v})| - C\delta\right)(1+s)^{-\lambda-1}V_x^2 \\ &\le CN(t)\int_0^t\int (1+s)^{\frac{\lambda-5}{2}}V_x^2 + C\int\left(z_0^2+V_{0x}^2\right) + C\delta. \end{aligned} \tag{3.15}$$

On the other hand, multiplying (3.1) by $(1+t)^{-2\lambda}V$, we get

$$\begin{aligned} &\left[(1+t)^{-2\lambda}VV_t + \frac{1}{2}(1+t)^{-3\lambda}V^2 + \lambda(1+t)^{-2\lambda-1}V^2\right]_t - (1+t)^{-2\lambda}V_t^2 \\ &+ \left[\lambda(2\lambda+1)(1+t)^{-2\lambda-2} + \frac{3\lambda}{2}(1+t)^{-3\lambda-1}\right]V^2 + (1+t)^{-2\lambda}(-p'(\bar{v}))V_x^2 \\ &= \left[-(1+t)^{-2\lambda}p'(\bar{v})V_xV + (1+t)^{-\lambda}Vp(\bar{v})_t - (1+t)^{-2\lambda}VG\right]_x \\ &\quad - (1+t)^{-\lambda}V_xp'(\bar{v})\bar{v}_t + \lambda(1+t)^{-\lambda-1}Vp'(\bar{v})\bar{v}_x + (1+t)^{-2\lambda}V_xG. \end{aligned} \tag{3.16}$$

By Young's inequality,

$$\begin{aligned} (1+t)^{-\lambda}|V_xp'(\bar{v})\bar{v}_t| &\le \frac{1}{2}(1+t)^{-2\lambda}|p'(\bar{v})|V_x^2 + \frac{1}{2}|p'(\bar{v})|\bar{v}_t^2, \\ \lambda(1+t)^{-\lambda-1}|Vp'(\bar{v})\bar{v}_x| &\le \frac{\lambda}{2}(1+t)^{-3\lambda-1}V^2 + \frac{\lambda}{2}(1+t)^{\lambda-1}|p'(\bar{v})|^2\bar{v}_x^2. \end{aligned}$$

By Taylor's formula,

$$\begin{aligned} (1+t)^{-2\lambda}|V_xG| &\le C(1+t)^{-2\lambda}(|\hat{v}| + |V_x+\hat{v}|^2)|V_x| \\ &\le \left(\frac{1}{4}+CN(t)\right)(1+t)^{-2\lambda}|p'(\bar{v})|V_x^2 + C(1+t)^{-2\lambda}\hat{v}^2. \end{aligned}$$

Hence, integrating (3.16) over $\mathbb{R} \times [0,t]$ gives

$$\begin{aligned} &\int \left[(1+t)^{-2\lambda} V V_t + \frac{1}{2}(1+t)^{-3\lambda} V^2 + \lambda(1+t)^{-2\lambda-1} V^2\right] - \int_0^t \int (1+s)^{-2\lambda} V_t^2 \\ &\quad + \lambda \int_0^t \int (1+s)^{-3\lambda-1} V^2 + \left(\frac{1}{4} - CN(t)\right) \int_0^t \int (1+s)^{-2\lambda} |p'(\bar{v})| V_x^2 \\ &\leq C \int \left(z_0^2 + V_0^2\right) + C\delta. \end{aligned} \tag{3.17}$$

Combining (3.15) with (3.17), and noting that

$$(1+t)^{-\lambda} V_t^2 + (1+t)^{-2\lambda} V V_t + \frac{1}{2}(1+t)^{-3\lambda} V^2 \geq C_0 (1+t)^{-\lambda} V_t^2 + C_1 (1+t)^{-3\lambda} V^2,$$

for some positive constants C_0 and C_1, and that $-2\lambda > -\lambda - 1 > \frac{\lambda-5}{2}$, it follows that

$$\begin{aligned} &\int \left[C_0(1+t)^{-\lambda} V_t^2 + C_1 (1+t)^{-3\lambda} V^2 + (1+t)^{-\lambda} V_x^2\right] + \lambda \int_0^t \int (1+s)^{-3\lambda-1} V^2 \\ &\quad + \frac{1}{4} \int_0^t \int (1+s)^{-2\lambda} V_t^2 + \left(\frac{1}{4} - CN(t) - C\delta\right) \int_0^t \int (1+s)^{-2\lambda} V_x^2 \\ &\leq C \int \left(z_0^2 + V_{0x}^2 + V_0^2\right) + C\delta. \end{aligned} \tag{3.18}$$

Therefore, when $N(T) + \delta \ll 1$, we get the desired estimate (3.3). □

Lemma 3.2. *If $N(T) + \delta \ll 1$, then it holds that*

$$(1+t)^{1-2\lambda} \|(V_x, V_t)(\cdot,t)\|^2 + \int_0^t (1+s)^{1-3\lambda} \|V_t(\cdot,s)\|^2 ds \leqslant C \left(\|V_0\|_1^2 + \|z_0\|^2 + \delta\right). \tag{3.19}$$

Proof. Multiplying (3.1) by $2(1+t)^{1-2\lambda} V_t$, we get

$$\begin{aligned} &\left[(1+t)^{1-2\lambda} \left(V_t^2 - p'(\bar{v}) V_x^2\right)\right]_t + \left[2(1+t)^{1-3\lambda} - (1-2\lambda)(1+t)^{-2\lambda}\right] V_t^2 \\ &= -\left[(1-2\lambda)(1+t)^{-2\lambda} p'(\bar{v}) + (1+t)^{1-2\lambda} p''(\bar{v}) \bar{v}_t\right] V_x^2 - 2\big((1+t)^{1-2\lambda} p'(\bar{v}) V_x V_t\big)_x \\ &\quad + 2(1+t)^{1-2\lambda} V_t F - 2(1+t)^{1-2\lambda} V_t G_x. \end{aligned} \tag{3.20}$$

The right hand side of (3.20) can be estimated as follows. Analogous to (3.5), we get

$$(1+t)^{1-2\lambda} |p''(\bar{v}) \bar{v}_t| V_x^2 \leq C\delta (1+t)^{-2\lambda} V_x^2. \tag{3.21}$$

As in (3.6),

$$\begin{aligned} (1+t)^{1-2\lambda} V_t F &= (1+t)^{1-\lambda} p(\bar{v})_{xt} V_t + \lambda (1+t)^{-\lambda} p(\bar{v})_x V_t \\ &\triangleq I_3 + I_4. \end{aligned} \tag{3.22}$$

By Young's inequality and (2.4),

$$\begin{aligned} I_3 &\le \frac{1}{8}(1+t)^{1-3\lambda}V_t^2 + 2(1+t)^{1+\lambda}\left(p''(\bar v)\bar v_x \bar v_t + p'(\bar v)\bar v_{xt}\right)^2 \\ &\le \frac{1}{8}(1+t)^{1-3\lambda}V_t^2 + C\delta(1+t)^{\lambda-1}|\bar v_x|^2 + C(1+t)^{1+\lambda}|\bar v_{xt}|^2. \end{aligned} \tag{3.23}$$

Similarly,

$$I_4 \le \frac{1}{8}(1+t)^{1-3\lambda}V_t^2 + C(1+t)^{\lambda-1}|\bar v_x|^2. \tag{3.24}$$

Analogous to (3.9), by (3.10), we have

$$\begin{aligned} (1+t)^{1-2\lambda}|V_t G_x| &\le C(1+t)^{1-2\lambda}\left(|V_x+\hat v||V_{xx}| + (|\hat v| + |V_x+\hat v|^2)|\bar v_x| + |\hat v_x|\right)|V_t| \\ &\le \frac{1}{8}(1+t)^{1-3\lambda}V_t^2 + CN(t)(1+t)^{-\frac{\lambda+3}{2}}V_x^2 + C(|\hat v| + |\hat v_x|). \end{aligned} \tag{3.25}$$

Now, integrating (3.20) over $\mathbb{R}\times[0,t]$, by (3.21)–(3.25) and Lemma 2.1, we obtain

$$\begin{aligned} &\int (1+t)^{1-2\lambda}\left(V_t^2 - p'(\bar v)V_x^2\right) + \int_0^t\int \left(\frac{5}{4}(1+s)^{1-3\lambda} - (1-2\lambda)(1+s)^{-2\lambda}\right)V_t^2 \\ &\le C\int_0^t\int (1+s)^{-2\lambda}V_x^2 + CN(t)\int_0^t\int (1+s)^{-\frac{\lambda+3}{2}}V_x^2 + C\int\left(z_0^2 + V_{0x}^2\right) + C\delta. \end{aligned}$$

Since $-2\lambda > -\frac{\lambda+3}{2}$, we further get

$$\begin{aligned} &\int (1+t)^{1-2\lambda}\left(V_t^2 - p'(\bar v)V_x^2\right) + \frac{5}{4}\int_0^t\int (1+s)^{1-3\lambda}V_t^2 \\ &\le C\int_0^t\int (1+s)^{-2\lambda}(V_x^2 + V_t^2) + C\int\left(z_0^2 + V_{0x}^2\right) + C\delta. \end{aligned} \tag{3.26}$$

Therefore, owing to Lemma 3.1, we obtain (3.19). □

Lemma 3.3. *When $N(T)+\delta \ll 1$, it holds that*

$$\begin{aligned} &(1+t)^{2-\lambda}\|(V_{xx},V_{xt})(\cdot,t)\|^2 + \int_0^t \left[(1+s)^{1-\lambda}\|V_{xx}(\cdot,s)\|^2 + (1+s)^{2-2\lambda}\|V_{xt}(\cdot,s)\|^2\right]ds \\ &\leqslant C\left(\|V_0\|_2^2 + \|z_0\|_1^2 + \delta\right). \end{aligned} \tag{3.27}$$

Proof. Differentiating (3.1) in x yields

$$V_{xtt} + \frac{1}{(1+t)^\lambda}V_{xt} + \left(p'(\bar v)V_{xx}\right)_x = -\left(p''(\bar v)\bar v_x V_x\right)_x + F_x - G_{xx}. \tag{3.28}$$

Multiplying (3.28) by $2(\beta+t)V_{xt}$ with β being a constant to be determined later, and integrating the resultant equation, we have

$$\begin{aligned}&\frac{d}{dt}\int\left[(\beta+t)\left(V_{xt}^2-p'(\bar v)V_{xx}^2\right)\right]+\int\left[\frac{2(\beta+t)}{(1+t)^\lambda}-1\right]V_{xt}^2\\&=-\int\left[p'(\bar v)+(\beta+t)p''(\bar v)\bar v_t\right]V_{xx}^2+2(\beta+t)\int V_{xt}F_x\\&\quad-2(\beta+t)\int V_{xt}\left(G_{xx}+\left(p''(\bar v)\bar v_xV_x\right)_x\right).\end{aligned}\tag{3.29}$$

In view of (2.4), we get

$$(\beta+t)\int|p''(\bar v)\bar v_t|V_{xx}^2\le C\delta\int V_{xx}^2.\tag{3.30}$$

By Young's inequality, (2.4) and Lemma 2.1,

$$\begin{aligned}&2(\beta+t)\int V_{xt}F_x\\&\le C(\beta+t)\int\left[(1+t)^\lambda(|\bar v_t|\bar v_x^2+|\bar v_x\bar v_{xt}|+|\bar v_t\bar v_{xx}|+|\bar v_{xxt}|)+(1+t)^{\lambda-1}(\bar v_x^2+|\bar v_{xx}|)\right]|V_{xt}|\\&\le\delta(\beta+t)^{1-\lambda}\int V_{xt}^2+\frac{C}{\delta}\left[(1+t)^{2\lambda-2}\int\bar v_x^2+(1+t)^{3\lambda-1}\int\bar v_{xx}^2+(1+t)^{3\lambda+1}\int\bar v_{xxt}^2\right]\\&\le\delta(\beta+t)^{1-\lambda}\int V_{xt}^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.\end{aligned}\tag{3.31}$$

We estimate the last term on the right hand side of (3.29) as follows.

$$\begin{aligned}&-2(\beta+t)\int V_{xt}\left(G_{xx}+\left(p''(\bar v)\bar v_xV_x\right)_x\right)\\&=-2(\beta+t)\int\left[p'(V_x+\bar v+\hat v)(V_{xx}+\bar v_x+\hat v_x)-p'(\bar v)\bar v_x-p'(\bar v)V_{xx}\right]_xV_{xt}\\&\triangleq I_5+I_6,\end{aligned}\tag{3.32}$$

where

$$\begin{aligned}I_5&=-2(\beta+t)\int\left[\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xx}\right]_xV_{xt}\\&=\frac{d}{dt}\left[(\beta+t)\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xx}^2\right]\\&\quad-\int\left[p'(V_x+\bar v+\hat v)-p'(\bar v)+(\beta+t)\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)_t\right]V_{xx}^2,\end{aligned}$$

and

$$\begin{aligned}I_6&\le C(\beta+t)\int\left[|V_{xx}||\bar v_x|+(|\bar v_{xx}|+|\bar v_x|^2)(|\hat v|+|V_x|)+|\hat v_x|+|\hat v_{xx}|\right]|V_{xt}|\\&\le\delta(\beta+t)^{1-\lambda}\int V_{xt}^2+C\delta\int V_{xx}^2+C\delta(1+t)^{-1-\lambda}\int V_x^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.\end{aligned}$$

By Gagliardo–Nirenberg's inequality,

$$\|V_{xt}(\cdot,t)\|_{L^\infty} \le C\|V_{xt}(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|V_{xxt}(\cdot,t)\|_{L^2}^{\frac{1}{2}} \le CN(t)(1+t)^{\frac{3\lambda-7}{4}}.$$

It then follows that

$$\begin{aligned}(\beta+t)\int |(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v}))_t|\,V_{xx}^2 &\le C(\beta+t)\int (|V_{xt}|+|\bar{v}_t|+|\hat{v}_t|)V_{xx}^2\\ &\le C(N(t)+\delta)\int V_{xx}^2.\end{aligned} \tag{3.33}$$

Hence, by (3.29)–(3.33), we obtain that

$$\begin{aligned}&\frac{d}{dt}\int \left[(\beta+t)\left(V_{xt}^2-p'(\bar{v})V_{xx}^2-\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)V_{xx}^2\right)\right]\\ &\quad+\int \left[2(\beta+t)^{1-\lambda}-1-3\delta\right]V_{xt}^2\\ &\le \int \left(|p'(\bar{v})|+C(N(t)+\delta)\right)V_{xx}^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.\end{aligned} \tag{3.34}$$

Integrating (3.34) over $(0,t)$ gives

$$\begin{aligned}&\int(\beta+t)V_{xt}^2+\int(\beta+t)[-p'(\bar{v})-C\delta-CN(t)]V_{xx}^2+\int_0^t\int\left[2(\beta+s)^{1-\lambda}-1-3\delta\right]V_{xt}^2\\ &\le \int_0^t\int\left(|p'(\bar{v})|+C(N(t)+\delta)\right)V_{xx}^2+C\int\left(z_{0x}^2+V_{0xx}^2\right)+C\delta.\end{aligned} \tag{3.35}$$

On the other hand, multiplying (3.28) by $(\beta+t)^{1-\lambda}V_x$, and integrating the equation, we get

$$\begin{aligned}&\frac{d}{dt}\int\left[(\beta+t)^{1-\lambda}V_{xt}V_x+\frac{1}{2}(\beta+t)^{1-\lambda}(1+t)^{-\lambda}V_x^2-\frac{1-\lambda}{2}(\beta+t)^{-\lambda}V_x^2\right]\\ &\quad-\int(\beta+t)^{1-\lambda}V_{xt}^2-(\beta+t)^{1-\lambda}\int p'(\bar{v})V_{xx}^2\\ &=\frac{1}{2}\left(\lambda(1-\lambda)(\beta+t)^{-1-\lambda}+[(\beta+t)^{1-\lambda}(1+t)^{-\lambda}]_t\right)\int V_x^2\\ &\quad+(\beta+t)^{1-\lambda}\int V_xF_x+(\beta+t)^{1-\lambda}\int V_{xx}\left(G_x+p''(\bar{v})\bar{v}_xV_x\right).\end{aligned} \tag{3.36}$$

As in (3.31),

$$\begin{aligned}&(\beta+t)^{1-\lambda}\int V_xF_x\\ &\le\delta(\beta+t)^{-2\lambda}\int V_x^2+\frac{C}{\delta}\left[(1+t)^{\lambda-1}\int\bar{v}_x^2+(1+t)^{2\lambda}\int\bar{v}_{xx}^2+(1+t)^{2+2\lambda}\int\bar{v}_{xxt}^2\right]\\ &\le\delta(\beta+t)^{-2\lambda}\int V_x^2+C\delta(1+t)^{\frac{\lambda-3}{2}}.\end{aligned}$$

As in (3.32),

$$
\begin{aligned}
&(\beta+t)^{1-\lambda}\int V_{xx}\left(G_x+p''(\bar v)\bar v_x V_x\right)\\
&=(\beta+t)^{1-\lambda}\int (p'(V_x+\bar v+\hat v)-p'(\bar v))(V_{xx}^2+\bar v_x V_{xx})+(\beta+t)^{1-\lambda}\int p'(V_x+\bar v+\hat v)\hat v_x V_{xx}\\
&\le C(N(t)+\delta)(\beta+t)^{1-\lambda}\int V_{xx}^2+C\delta(1+t)^{-2\lambda}\int V_x^2+C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}
$$

Integrating (3.36) over $(0,t)$, by Lemma 3.1, we get

$$
\begin{aligned}
&\int\left[(\beta+t)^{1-\lambda}V_{xt}V_x+\frac{1}{2}(\beta+t)^{1-2\lambda}V_x^2\right]-\int_0^t\int(\beta+s)^{1-\lambda}V_{xt}^2\\
&+\int_0^t\int(\beta+s)^{1-\lambda}(|p'(\bar v)|-CN(t)-C\delta)V_{xx}^2\\
&\le C\delta\int_0^t\int(1+s)^{-2\lambda}V_x^2+C\int(1+t)^{-\lambda}V_x^2+C\int(V_{0x}^2+z_{0x}^2)+C\delta\\
&\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}
\tag{3.37}
$$

Adding (3.37) to (3.35) leads to

$$
\begin{aligned}
&\int\left[C_2(\beta+t)(V_{xt}^2+V_{xx}^2)+C_3(\beta+t)^{1-2\lambda}V_x^2\right]+\int_0^t\int[(\beta+s)^{1-\lambda}-1-3\delta]V_{xt}^2\\
&+\int_0^t\int[((\beta+s)^{1-\lambda}-1)|p'(\bar v)|-CN(t)-C\delta]V_{xx}^2\\
&\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}
$$

Now choosing $\beta=10^{\frac{1}{1-\lambda}}$, it is easy to see that if $N(T)+\delta\ll 1$, then

$$
\begin{aligned}
&\int\left[(\beta+t)(V_{xt}^2+V_{xx}^2)+(\beta+t)^{1-2\lambda}V_x^2\right]+\int_0^t\int(\beta+s)^{1-\lambda}(V_{xt}^2+V_{xx}^2)\\
&\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}
\tag{3.38}
$$

Multiplying (3.34) by $(\beta+t)^{1-\lambda}$, integrating the resultant equation over $(0,t)$, by (3.38), we then have

$$
\int(\beta+t)^{2-\lambda}(V_{xt}^2+V_{xx}^2)+\int_0^t\int(\beta+s)^{2-2\lambda}V_{xt}^2\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).
$$

This inequality in combination with (3.38) gives (3.27). □

Lemma 3.4. *If $N(T)+\delta \ll 1$, then*

$$(1+t)^3\|(V_{xxx},V_{xxt})(\cdot,t)\|^2+\int_0^t\left[(1+s)^2\|V_{xxx}(\cdot,s)\|^2+(1+s)^{3-\lambda}\|V_{xxt}(\cdot,s)\|^2\right]ds \leqslant C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big). \tag{3.39}$$

Proof. Differentiating (3.28) in x leads to

$$V_{xxtt}+\frac{1}{(1+t)^\lambda}V_{xxt}+\big(p'(\bar v)V_{xxx}\big)_x=F_{xx}-G_{xxx}-H_x, \tag{3.40}$$

where $H=p'''(\bar v)\bar v_x^2V_x+p''(\bar v)\bar v_{xx}V_x+2p''(\bar v)\bar v_xV_{xx}$. Multiplying (3.40) by $2(\beta+t)^3V_{xxt}$ with $\beta=10^{\frac{1}{1-\lambda}}$, and integrating the resultant equation, we have

$$\begin{aligned}&\frac{d}{dt}\int(\beta+t)^3\left(V_{xxt}^2-p'(\bar v)V_{xxx}^2\right)+\int\left[2(\beta+t)^3/(1+t)^\lambda-3(\beta+t)^2\right]V_{xxt}^2\\&+\int\left[3(\beta+t)^2p'(\bar v)+(\beta+t)^3p''(\bar v)\bar v_t\right]V_{xxx}^2=2(\beta+t)^3\int V_{xxt}(F_{xx}-G_{xxx}-H_x).\end{aligned} \tag{3.41}$$

As in (3.31), by (2.4) and Lemma 2.1,

$$\begin{aligned}&2(\beta+t)^3\int V_{xxt}F_{xx}\\&\le\delta(\beta+t)^{3-\lambda}\int V_{xxt}^2+\frac{C}{\delta}(1+t)^{3+3\lambda}\int\big(|\bar v_t\bar v_x^3|^2+|\bar v_{xt}\bar v_x^2|^2+|\bar v_t\bar v_x\bar v_{xx}|^2+|\bar v_{xt}\bar v_{xx}|^2\\&\quad+|\bar v_x\bar v_{xxt}|^2+|\bar v_t\bar v_{xxx}|^2+|\bar v_{xxxt}|^2\big)+\frac{C}{\delta}(1+t)^{1+3\lambda}\int(\bar v_x^6+|\bar v_x\bar v_{xx}|^2+\bar v_{xxx}^2)\\&\le\delta(\beta+t)^{3-\lambda}\int V_{xxt}^2+C\delta(1+t)^{\frac{\lambda-3}{2}}.\end{aligned}$$

As in (3.32),

$$-2(\beta+t)^3\int V_{xxt}(G_{xxx}+H_x)=I_7+I_8,$$

where

$$\begin{aligned}I_7&=-2(\beta+t)^3\int\left[\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xxx}\right]_xV_{xxt}\\&=\frac{d}{dt}\left[\int(\beta+t)^3\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xxx}^2\right]\\&\quad-3(\beta+t)^2\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xxx}^2-(\beta+t)^3\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)_tV_{xxx}^2\\&\le\frac{d}{dt}\left[\int(\beta+t)^3\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xxx}^2\right]+C(N(t)+\delta)(\beta+t)^2\int V_{xxx}^2,\end{aligned}$$

and

$$
\begin{aligned}
I_8 &= O(1)(\beta+t)^3 \int |V_x+\hat{v}|(|\bar{v}_x|^3+|\bar{v}_x\bar{v}_{xx}|+|\bar{v}_{xxx}|)|V_{xxt}| \\
&\quad + O(1)(\beta+t)^3 \int \Big(|V_{xx}|^3+|V_{xx}|^2|\bar{v}_x|+|V_{xx}||\bar{v}_x|^2+|V_{xxx}||V_{xx}| \\
&\quad + |V_{xxx}||\bar{v}_x|+|V_{xx}||\bar{v}_{xx}|+|\hat{v}_x||V_{xxx}|+|\hat{v}_x|+|\hat{v}_{xx}|+|\hat{v}_{xxx}|\Big)|V_{xxt}| \\
&\le C(N(t)+\delta)(\beta+t)^{3-\lambda}\int V_{xxt}^2 + C(N(t)+\delta)(\beta+t)^2\int V_{xxx}^2 \\
&\quad + C(1+t)^{1-\lambda}\int V_{xx}^2 + (1+t)^{-2\lambda}\int V_x^2 + C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}
$$

Now, integrating (3.41) over $(0,t)$, by Lemmas 3.1 and 3.3, we obtain

$$
\begin{aligned}
&\int (\beta+t)^3\left(V_{xxt}^2 - p'(\bar{v})V_{xxx}^2\right) + \int_0^t\int \left[(2-C\delta-CN(t))(\beta+s)^{3-\lambda}-3(\beta+s)^2\right]V_{xxt}^2 \\
&\le \int_0^t\int \left(3|p'(\bar{v})|+C\delta+CN(t)\right)(\beta+s)^2V_{xxx}^2 + C\big(\|V_0\|_3^2+\|z_0\|_2^2\big)+C\delta.
\end{aligned}
\tag{3.42}
$$

On the other hand, multiplying (3.40) by $4(\beta+t)^2V_{xx}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\int\left[4(\beta+t)^2V_{xx}V_{xxt}+\frac{2(\beta+t)^2}{(1+t)^\lambda}V_{xx}^2-4(\beta+t)V_{xx}^2\right] \\
&\quad -4(\beta+t)^2\int V_{xxt}^2+4(\beta+t)^2\int(-p'(\bar{v}))V_{xxx}^2+4\int V_{xx}^2 \\
&=2\left[\frac{(\beta+t)^2}{(1+t)^\lambda}\right]_t\int V_{xx}^2+4(\beta+t)^2\int V_{xx}F_{xx}+4(\beta+t)^2\int V_{xxx}(G_{xx}+H).
\end{aligned}
\tag{3.43}
$$

By Young's inequality,

$$
\begin{aligned}
4(\beta+t)^2\int V_{xx}F_{xx} &= -4(\beta+t)^2(1+t)^\lambda\int p(\bar{v})_{xxt}V_{xxx}+\frac{4\lambda(\beta+t)^2}{(1+t)^{1-\lambda}}\int p(\bar{v})_{xxx}V_{xx} \\
&\le C\delta(\beta+t)^2\int V_{xxx}^2+C\delta(\beta+t)^{1-\lambda}\int V_{xx}^2+C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(\beta+t)^2\int V_{xxx}(G_{xx}+H) \\
&=(\beta+t)^2\int\Big[\left(p'(V_x+\bar{v}+\hat{v})-p'(\bar{v})\right)(V_{xxx}+\bar{v}_{xx})+\left(p''(V_x+\bar{v}+\hat{v})-p''(\bar{v})\right)\bar{v}_x^2\Big]V_{xxx} \\
&\quad+O(1)(\beta+t)^2\int\left(V_{xx}^2+|V_{xx}||\bar{v}_x|+|\hat{v}_x|(|V_{xx}|+1)+|\hat{v}_{xx}|\right)|V_{xxx}| \\
&\le C(N(t)+\delta)(\beta+t)^2\int V_{xxx}^2+C(1+t)^{1-\lambda}\int V_{xx}^2+(1+t)^{-2\lambda}\int V_x^2+C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}
\tag{3.44}
$$

Hence, integrating (3.43) over $(0,t)$ leads to

$$
\begin{aligned}
&\int \left[4(\beta+t)^2 V_{xx}V_{xxt} + 10(\beta+t)^{2-\lambda}V_{xx}^2\right] - 4\int_0^t\int(\beta+s)^2 V_{xxt}^2 \\
&\quad + (4 - CN(t) - C\delta)\int_0^t\int(\beta+s)^2(-p'(\bar{v}))V_{xxx}^2 \\
&\le \left[4(\beta+t) + 8(\beta+t)^{2-\lambda}\right]\int V_{xx}^2 + C\int_0^t\int(1+s)^{1-\lambda}V_{xx}^2 \\
&\quad + C\int_0^t\int(1+s)^{-2\lambda}V_x^2 + C\int\left(z_{0xx}^2 + V_{0xx}^2\right) + C\delta.
\end{aligned}
\tag{3.45}
$$

Therefore, combining (3.42) with (3.45), and noting $\beta = 10^{1/(1-\lambda)}$, the desired estimate (3.39) follows from Lemmas 3.1 and 3.3 provided that $N(T) + \delta \ll 1$. □

Lemma 3.5. *If $N(T) + \delta \ll 1$, then*

$$
\begin{aligned}
&(1+t)^{3-2\lambda}\|(V_{tt}, V_{xt})(\cdot,t)\|^2 + (1+t)^{2-3\lambda}\|V_t(\cdot,t)\|^2 \\
&\quad + \int_0^t\left[(1+s)^{3-3\lambda}\|V_{tt}(\cdot,s)\|^2 + (1+s)^{2-2\lambda}\|V_{xt}(\cdot,s)\|^2\right]ds \\
&\leqslant C\left(\|V_0\|_2^2 + \|z_0\|_1^2 + \delta\right).
\end{aligned}
\tag{3.46}
$$

Proof. It follows from (3.1) that

$$
V_{tt} = -\frac{1}{(1+t)^{\lambda}}V_t - \left(p'(\bar{v})V_x\right)_x + F - G_x.
$$

By Lemmas 3.1–3.3, we get

$$
\int_0^t\int(1+s)^{1-\lambda}V_{tt}^2 \le C\left(\|V_0\|_2^2 + \|z_0\|_1^2 + \delta\right). \tag{3.47}
$$

Differentiating (3.1) in t yields

$$
V_{ttt} + \frac{1}{(1+t)^{\lambda}}V_{tt} - \frac{\lambda}{(1+t)^{\lambda+1}}V_t + \left(p'(\bar{v})V_{tx}\right)_x = F_t - G_{xt} - \left(p''(\bar{v})\bar{v}_t V_x\right)_x. \tag{3.48}
$$

Multiplying (3.48) by $2(1+t)^{2-\lambda}V_{tt}$, and integrating the resultant equation, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\left[(1+t)^{2-\lambda}\left(V_{tt}^2 - p'(\bar{v})V_{xt}^2\right) - \lambda(1+t)^{1-2\lambda}V_t^2\right] + 2(1+t)^{2-2\lambda}\int V_{tt}^2 \\
&= (2-\lambda)(1+t)^{1-\lambda}\int V_{tt}^2 - \lambda(1-2\lambda)(1+t)^{-2\lambda}\int V_t^2 \\
&\quad - \int\left[(2-\lambda)(1+t)^{1-\lambda}p'(\bar{v}) + (1+t)^{2-\lambda}p''(\bar{v})\bar{v}_t\right]V_{xt}^2 \\
&\quad + 2(1+t)^{2-\lambda}\int V_{tt}F_t - 2(1+t)^{2-\lambda}\int V_{tt}\left[G_{xt} + \left(p''(\bar{v})\bar{v}_t V_x\right)_x\right].
\end{aligned}
\tag{3.49}
$$

Analogous to (3.31),

$$\begin{aligned}
&2(1+t)^{2-\lambda}\int V_{tt}F_t\\
&\le \delta(1+t)^{2-2\lambda}\int V_{tt}^2+\frac{C}{\delta}\int\left[(1+t)^{2\lambda-2}|\bar v_x|^2+(1+t)^{2\lambda}|\bar v_{xt}|^2+C(1+t)^{2+2\lambda}|\bar v_{xtt}|^2\right]\\
&\le \delta(1+t)^{2-2\lambda}\int V_{tt}^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.
\end{aligned}$$

Analogous to (3.32),

$$\begin{aligned}
&-2(1+t)^{2-\lambda}\int V_{tt}\left[G_{xt}+\left(p''(\bar v)\bar v_tV_x\right)_x\right]\\
&=\frac{d}{dt}\left[(1+t)^{2-\lambda}\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xt}^2\right]\\
&-\int\left[(2-\lambda)(1+t)^{1-\lambda}\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)+(1+t)^{2-\lambda}\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)_t\right]V_{xt}^2\\
&+O(1)(1+t)^{2-\lambda}\int\Big[|V_x+\hat v|(|\bar v_x||\bar v_t|+|\bar v_{xt}|)+|V_{xx}||\bar v_t|+|\hat v_t|(|V_{xx}|+1)+|\hat v_x|+|\hat v_{xt}|\Big]|V_{tt}|\\
&\le\frac{d}{dt}\left[(1+t)^{2-\lambda}\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xt}^2\right]+C(N(t)+\delta)(1+t)^{1-\lambda}\int V_{xt}^2\\
&+\delta(1+t)^{2-2\lambda}\int V_{tt}^2+C\delta\int V_{xx}^2+C\delta(1+t)^{-\lambda-1}\int V_x^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.
\end{aligned}$$

Thus, it follows from (3.49) that

$$\begin{aligned}
&\frac{d}{dt}\left[\int(1+t)^{2-\lambda}\left(V_{tt}^2-p'(\bar v)V_{xt}^2\right)\right]+(2-2\delta)(1+t)^{2-2\lambda}\int V_{tt}^2\\
&\le\frac{d}{dt}\left[\lambda\int(1+t)^{1-2\lambda}V_t^2+(1+t)^{2-\lambda}\int\left(p'(V_x+\bar v+\hat v)-p'(\bar v)\right)V_{xt}^2\right]\\
&+(1+t)^{1-\lambda}\left[(2-\lambda)|p'(\bar v)|+C\delta+CN(t)\right]\int V_{xt}^2+C(1+t)^{1-\lambda}\int V_{tt}^2\\
&+C\int V_{xx}^2+C(1+t)^{-2\lambda}\int V_t^2+C(1+t)^{-\lambda-1}\int V_x^2+C\delta(1+t)^{\frac{3\lambda-5}{2}}.
\end{aligned}\tag{3.50}$$

Integrating (3.50) over $(0,t)$, by (3.47) and Lemmas 3.1–3.3, it is easy to see that, if $N(T)+\delta\ll 1$, then

$$\int(1+t)^{2-\lambda}\left(V_{tt}^2+V_{xt}^2\right)+\int_0^t\int(1+s)^{2-2\lambda}V_{tt}^2\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{3.51}$$

On the other hand, multiplying (3.48) by $2(1+t)^{1-\lambda}V_t$ and integrating the equation, we get

$$\begin{aligned}
&\frac{d}{dt}\int\left[2(1+t)^{1-\lambda}V_tV_{tt}+(1+t)^{1-2\lambda}V_t^2-(1-\lambda)(1+t)^{-\lambda}V_t^2\right]-2(1+t)^{1-\lambda}\int p'(\bar v)V_{xt}^2\\
&=2(1+t)^{1-\lambda}\int V_{tt}^2+\left(\lambda(1-\lambda)(1+t)^{-1-\lambda}+(1+t)^{-2\lambda}\right)\int V_t^2\\
&+2(1+t)^{1-\lambda}\int V_tF_t+2(1+t)^{1-\lambda}\int V_{tx}(G_t+p''(\bar v)\bar v_tV_x).
\end{aligned}\tag{3.52}$$

By Lemma 2.1, a simple calculation yields

$$2(1+t)^{1-\lambda}\int V_tF_t \le \delta(1+t)^{-2\lambda}\int V_t^2 + C\delta(1+t)^{\frac{3\lambda-5}{2}},$$

$$\begin{aligned}&2(1+t)^{1-\lambda}\int V_{tx}(G_t+p''(\bar v)\bar v_t V_x)\\ &\le C(1+t)^{1-\lambda}\int \big(|V_x+\hat v|(|V_{xt}|+|\bar v_t|)+|\hat v_t|\big)|V_{xt}|\\ &\le C(\delta+N(t))(1+t)^{1-\lambda}\int V_{xt}^2 + C\delta(1+t)^{-1-\lambda}\int V_x^2 + C\delta(1+t)^{\frac{3\lambda-5}{2}}.\end{aligned}$$

It then follows from (3.52) that, if $N(T)+\delta \ll 1$, then

$$\begin{aligned}&\frac{d}{dt}\int \big[2(1+t)^{1-\lambda}V_tV_{tt}+(1+t)^{1-2\lambda}V_t^2\big] + (1+t)^{1-\lambda}\int |p'(\bar v)|V_{xt}^2\\ &\le \frac{d}{dt}\left[\int (1-\lambda)(1+t)^{-\lambda}V_t^2\right] + C(1+t)^{1-\lambda}\int V_{tt}^2 + C(1+t)^{-1-\lambda}\int V_x^2\\ &\quad + C(1+t)^{-2\lambda}\int V_t^2 + C\delta(1+t)^{\frac{3\lambda-5}{2}}.\end{aligned} \tag{3.53}$$

Multiplying (3.53) by $(1+t)^{1-\lambda}$, and integrating the resulting equation over $(0,t)$, noting that

$$2\int (1+t)^{2-2\lambda}V_tV_{tt} \le \frac{1}{2}\int (1+t)^{2-3\lambda}V_t^2 + 2\int (1+t)^{2-\lambda}V_{tt}^2,$$

$$2\int_0^t\int (1+s)^{1-2\lambda}V_tV_{tt} \le \int_0^t\int (1+s)^{-2\lambda}V_t^2 + \int_0^t\int (1+s)^{2-2\lambda}V_{tt}^2,$$

by (3.51) and Lemmas 3.1 and 3.2, we get

$$\int (1+t)^{2-3\lambda}V_t^2 + \int_0^t\int (1+s)^{2-2\lambda}V_{xt}^2 \le C\big(\|V_0\|_2^2+\|z_0\|_1^2+\delta\big). \tag{3.54}$$

Finally, multiplying (3.50) by $(1+t)^{1-\lambda}$, and integrating the resulting equation over $(0,t)$, by (3.51), (3.54) and Lemmas 3.1–3.3, we obtain the desired estimate (3.46). □

Lemma 3.6. *If $N(T)+\delta \ll 1$, then*

$$\begin{aligned}&(1+t)^{4-\lambda}\|(V_{xtt},V_{xxt})(\cdot,t)\|^2 + \int_0^t (1+s)^{4-2\lambda}\|V_{xtt}(\cdot,s)\|^2 ds\\ &\leqslant C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big).\end{aligned} \tag{3.55}$$

Proof. Differentiating (3.48) in x gives

$$V_{xttt} + \frac{1}{(1+t)^\lambda}V_{xtt} - \frac{\lambda}{(1+t)^{\lambda+1}}V_{xt} + \big(p'(\bar v)V_{txx}\big)_x = F_{xt} - G_{xxt} - J_x, \tag{3.56}$$

where $J = \big(p''(\bar{v})\bar{v}_t V_x\big)_x + p''(\bar{v})\bar{v}_x V_{tx}$. Multiplying (3.56) by $2(\beta+t)^{4-\lambda}V_{xtt}$ with $\beta = 10^{\frac{1}{1-\lambda}}$, and integrating the equation, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\int (\beta+t)^{4-\lambda}\left(V_{xtt}^2 - p'(\bar{v})V_{xxt}^2\right)\right] + \left[\frac{2(\beta+t)^{4-\lambda}}{(1+t)^{\lambda}} - (4-\lambda)(\beta+t)^{3-\lambda}\right]\int V_{xtt}^2 \\
&= \lambda\frac{d}{dt}\left[\int \frac{(\beta+t)^{4-\lambda}}{(1+t)^{1+\lambda}}V_{xt}^2\right] - \int \left[(4-\lambda)(\beta+t)^{3-\lambda}p'(\bar{v}) + (\beta+t)^{4-\lambda}p''(\bar{v})\bar{v}_t\right]V_{xxt}^2 \\
&\quad - \lambda\left(\frac{(\beta+t)^{4-\lambda}}{(1+t)^{1+\lambda}}\right)_t \int V_{xt}^2 + 2(\beta+t)^{4-\lambda}\int V_{xtt}(F_{xt} - G_{xxt} - J_x).
\end{aligned} \tag{3.57}$$

As in (3.31),

$$\begin{aligned}
&2(\beta+t)^{4-\lambda}\int V_{xtt}F_{xt} \\
&\le \delta\int(\beta+t)^{4-2\lambda}V_{xtt}^2 + \frac{C}{\delta}\int\Big[(1+t)^{\lambda-1}|\bar{v}_x|^2 + (1+t)^{1+\lambda}|\bar{v}_{xt}|^2 \\
&\quad + (1+t)^{2\lambda}|\bar{v}_{xx}|^2 + (1+t)^{2+2\lambda}|\bar{v}_{xxt}|^2 + (1+t)^{3+\lambda}|\bar{v}_{xtt}|^2 + (1+t)^{4+2\lambda}|\bar{v}_{xxtt}|^2\Big] \\
&\le \delta(\beta+t)^{4-2\lambda}\int V_{xtt}^2 + C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}$$

As in (3.32),

$$\begin{aligned}
&-2(\beta+t)^{4-\lambda}\int V_{xtt}(G_{xxt}+J_x) \\
&\le \frac{d}{dt}\Big[(\beta+t)^{4-\lambda}\int\left(p'(V_x+\bar{v}+\hat{v}) - p'(\bar{v})\right)V_{xxt}^2\Big] + C(\delta+N(t))(\beta+t)^{4-2\lambda}\int V_{xtt}^2 \\
&\quad + C(\delta+N(t))\int\Big[(1+t)^{3-\lambda}V_{xxt}^2 + (1+t)^2V_{xxx}^2 + (1+t)^{2-2\lambda}V_{xt}^2 + (1+t)^{1-\lambda}V_{xx}^2 \\
&\quad + (1+t)^{-2\lambda}V_x^2\Big] + C\delta(1+t)^{\frac{\lambda-3}{2}}.
\end{aligned}$$

Integrating (3.57) over $(0,t)$, by Lemmas 3.1, 3.3–3.5, noting $\beta = 10^{\frac{1}{1-\lambda}}$, it follows that

$$\begin{aligned}
&\int(\beta+t)^{4-\lambda}(V_{xtt}^2 - p'(\bar{v})V_{xxt}^2) + \int_0^t\int(\beta+s)^{4-2\lambda}V_{xtt}^2 \\
&\le C(\delta+N(t))\int_0^t\int(\beta+s)^{4-2\lambda}V_{xtt}^2 + C\big(\|V_0\|_3^2 + \|z_0\|_2^2 + \delta\big).
\end{aligned}$$

We thus derive the desired estimate (3.55) provided that $N(T)+\delta \ll 1$. □

Acknowledgments

J. Li's work was partially supported by the National Science Foundation of China (No. 11571066), M. Mei's work was partially supported by NSERC grant RGPIN 354724-2016 and FRQNT grant 192571, K. Zhang's work was partially supported by the National Science Foundation of China (No. 11371082) and the Fundamental Research Funds for the Central Universities (No. 111065201).

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