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Asymptotic behaviour of solutions of the hydrodynamic model of semiconductors

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Degond and Markowich discussed the existence and uniqueness of a steady-state solution in the subsonic case for the one-dimensional hydrodynamic model of semiconductors. In the present paper, we reconsider the existence and uniqueness of a globally smooth subsonic steady-state solution, and prove its stability for small perturbation. The proof method we adopt in this paper is based on elementary energy estimates.

1. Introduction

Since its introduction by Bløtekjær [2], the hydrodynamic model for semiconductors has recently attracted a lot of attention because of its ability to model hot electron effects that are not accounted for in the classical drift-diffusion model. For more discussion on these models in physics and engineering, and their derivation from kinetic transport equation, we refer to [17, 23, 26–29, 31] for details.

Recently, many papers were written on the hydrodynamic model of semiconductors. For the steady-state system, Degond and Markowich [5, 6] investigated the existence and uniqueness of subsonic solutions in one dimension and, for irrational flow, in three dimensions (see also [32]), respectively; Markowich [24] discussed the existence of subsonic solutions in two dimensions. The corresponding investigations on transonic solutions in one dimension were done by Ascher *et al.* [1] and by Markowich and Pietra [25] via phase plane analysis and the representation of discontinuous solutions, and by Gamba [8] and Gamba and Morawetz [9] via artificial viscosity. For the Cauchy problem of unipolar time-dependent hydrodynamic systems, Luo *et al.* [19] investigated the global existence and the asymptotic behaviour of smooth solutions of the hydrodynamic model for semiconductors and discussed their convergence to the stationary solution of the drift-diffusion equation in $\mathbb{R}^1$, but they had to assume the stationary current density J to be zero due to a technical difficulty in the analysis. The stability and, respectively, instability of the steady-state solutions of the Cauchy problem for the semiconductor model was analysed by Hattori and Zhu [12] and by Hattori [11]. Stability with convergence rates of global solutions of the hydrodynamic model to the corresponding steady-state solution on a quarter plane was analysed by Marcati and Mei [20]. The assumption of zero current density was removed therein, too. Furthermore, for the initial boundary-value problem in a bounded domain, under the assumption of zero current density on the boundary, Hsiao and Yang [14] discussed the time-asymptotic convergence of the smooth solutions of the hydrodynamic model and those of the drift-diffusion model to the unique steady-state solution. Subsequent to [14,19,20], in this paper we study the asymptotic stability of steady-state solutions for the non-zero current density case in a bounded domain, which improves the previous works [14,19,20].

Regarding other topics on such hydrodynamic models for semiconductor devices, we note the following. In [30], Poupaud *et al.* showed the global existence of the solutions with arbitrarily large data by using a trick concerning charge conservation. By finite-difference schemes and compensated compactness, Marcati and Natalini [21,22] proved the existence of weak solutions and discussed the relaxation limit to the drift-diffusion model for $1 < \gamma \leqslant \frac{5}{3}$. Zhang [35,36] discussed the existence of weak solutions and the relaxation limits for $\gamma > \frac{5}{3}$. Gasser and Natalini [10] discussed the relaxation limit for the non-isentropic hydrodynamic model. On the strip domain, the local existence of smooth solutions was proved by Zhang [34]; its large-time behaviour was analysed by Chen *et al.* [4]. The existence of weak solutions was obtained by Zhang [33] via Godunov schemes and by Fang and Ito [7] via vanishing viscosity. Hsiao and Zhang [15,16] discussed the relaxation limit and verified the boundary condition for weak solutions in the sense of trace. By shock-capturing schemes, Chen and Wang [3] investigated the existence of weak solutions in a high-dimensional compact domain with geometry symmetry. Also, there are lots of works on numerical analysis and simulation, for instance, by Jerome and Shu [18] and references therein.

After an appropriate scaling, the one-dimensional time-dependent system in the case of one carrier type, i.e. electrons, reads

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = \rho \phi_x - \frac{\rho u}{\tau}, \tag{1.2}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.3}$$

where $\rho > 0$, u and ϕ denote the electron density, velocity and the electrostatic potential, respectively. $j = \rho u$ is called the current density. $p = p(\rho)$ is the pressure-density relation, which satisfies

$$\rho^2 p'(\rho) \text{ is strictly monotonically increasing from } (0,\infty) \text{ into } (0,\infty). \tag{1.4}$$

In the present paper, we assume that

$$p \in C^3(0, +\infty). \tag{1.5}$$

And $\tau = \tau(\rho, \rho u) > 0$ is the momentum relaxation time. The device domain is the x-interval $(0, 1)$. $\mathcal{C} = \mathcal{C}(x) > 0$ is the doping profile, which stands for the given background density of changed ions. We assume that there is a function $\mathcal{A} = \mathcal{A}(x) \in C^2(0, 1)$ such that

$$\mathcal{A}(x) > 0, \quad \mathcal{A}(0) = \rho_1, \quad \mathcal{A}(1) = \rho_2, \quad \mathcal{A}(x) - \mathcal{C}(x) \in C(0, 1). \tag{1.6}$$

In the present paper, we first consider the initial boundary-value problems (IBVP for simplicity) for (1.1)–(1.3) with the following initial data,

$$(\rho, j)(x, 0) = (\tilde{\rho}, \tilde{j})(x), \quad x \in (0, 1), \tag{1.7}$$

and the density and potential Dirichlet boundary conditions

$$\rho(0, t) = \rho_1, \quad \rho(1, t) = \rho_2, \quad t \geqslant 0, \tag{1.8}$$

$$\phi(0, t) = 0, \quad \phi(1, t) = \phi_1, \quad t \geqslant 0. \tag{1.9}$$

These kind of boundary conditions are of importance in physics of semiconductor devices [23].

Our interest is to investigate the existence and stability of the smooth steady-state solutions of the hydrodynamic model of semiconductors, namely, the solutions of the boundary-value problem (BVP) for the following system

$$j = \text{const.}, \tag{1.10}$$

$$\left(\frac{j^2}{\rho} + p(\rho)\right)_x = \rho\phi_x - \frac{j}{\tau}, \tag{1.11}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.12}$$

with boundary conditions (1.8) and (1.9).

The main results in the present paper show that, for any $J_0 \neq 0$ satisfying condition (2.5) with $|\rho_2 - \rho_1| \ll 1$, there is a $\Phi_0 > 0$ such that, for any $0 < \phi_1 \leqslant \Phi_0$, the BVP (1.10)–(1.12) and (1.8)–(1.9) has a unique regular solution $(\rho_0, j_0, \phi_0)(x)$, with $|j_0| \leqslant |J_0|$, and, for any small initial perturbation of (ρ_0, j_0), the global solutions (ρ, j, ϕ) of the IBVP (1.1)–(1.3) and (1.7)–(1.9) exists and tends exponentially to the solution (ρ_0, j_0, ϕ_0) as $t \to +\infty$.

This paper is arranged as follows. In § 2, the existence, uniqueness and properties on the solutions to the BVP (1.10)–(1.12) and (1.8)–(1.9) are shown. The global existence and the asymptotic behaviour of the solution of the IBVP (1.1)–(1.3) and (1.7)–(1.9) are introduced and proved in § 3.

NOTATION. We make some notation for simplicity. C always denotes a positive constant. $L^2(0, 1)$ is the space of square-integrable real-valued functions defined on $[0, 1]$ with the norm $\|\cdot\|$, and $H^k(0, 1)$ denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially, $\|\cdot\|_0 = \|\cdot\|$. Let T and B be a positive constant and a Banach space, respectively. $C^k(0, T; B)$ $(k \geqslant 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0, T]$, and $L^2(0, T; B)$ denotes the space of B-valued L^2-functions on $(0, T)$. The corresponding spaces of B-valued functions on $[0, \infty)$ are defined analogously.

2. Steady-state system

In this section, we consider the properties of the steady-state solution of the BVP (1.10)–(1.12), (1.8)–(1.9) for the hydrodynamic model of semiconductors. For simplicity, we assume $\tau = 1$ from now on.

According to those shown in [5] for subsonic solutions, the boundary data of the BVP (1.10)–(1.12) and (1.8)–(1.9) should satisfy the current–voltage relationship

$$\phi_1 = F(\rho_2, j) - F(\rho_1, j) + j \int_0^1 \frac{\mathrm{d}x}{\rho(x)}, \tag{2.1}$$

where

$$F(\rho, j) = \frac{j^2}{2\rho^2} + h(\rho), \qquad h'(\rho) = \frac{1}{\rho} p'(\rho). \tag{2.2}$$

Since, by (2.1), the case $j = 0$ yields $\phi_1 = 0$, we consider, in the present paper, the physically more interesting case $j \neq 0$.

Dividing (1.11) by ρ, differentiating it again and using (1.10) and (1.12), we finally obtain

$$\left(\frac{\partial F}{\partial \rho}(\rho, j)\rho_x\right)_x + j\left(\frac{1}{\rho}\right)_x - \rho = -\mathcal{C}(x), \quad 0 < x < 1. \tag{2.3}$$

Thus, to make sure the existence of regular solutions, the subsonic condition is required to be satisfied, i.e.

$$\frac{\partial F}{\partial \rho}(\rho, j) = -\frac{j^2}{\rho^3} + \frac{1}{\rho} p'(\rho) > 0 \quad \Leftrightarrow \quad \rho^2 p'(\rho) > j^2. \tag{2.4}$$

Due to (1.4), we conclude that there is a unique $\rho_m = \rho_m(j)$ such that

$$\frac{\partial F}{\partial \rho}(\rho, j) > 0 \quad \text{for } \rho > \rho_m.$$

Also, by (2.4), we know that the minimal point ρ_m of $\rho \to F(\rho, j)$ is a strictly increasing function of j with $\rho_m(j = 0) = 0$. This implies that the equation (2.3) is uniformly elliptic for $\rho \geqslant \rho^* > \rho_m$, which, by (2.4), means a fully subsonic flow $|u| < c(\rho)$. Here, $c(\rho) = \sqrt{p'(\rho)}$ denotes the speed of sound.

The main result in this section is the following theorem.

THEOREM 2.1. *Assume that (1.4), (1.5) and (1.6) hold. Let $J_0 \neq 0$ be such that*

$$\rho_1,\ \rho_2,\ \inf_{x \in (0,1)} \mathcal{C}(x) > \rho_m(J_0), \tag{2.5}$$

and assume that $|\rho_2 - \rho_1| \ll 1$. Then there is a constant $\Phi_0 > 0$ such that, for all $0 < \phi_1 \leqslant \Phi_0$, the BVP (1.10)–(1.12) and (1.8)–(1.9) has a unique solution $(\rho_0, j_0, \phi_0)(x)$, which satisfies $|j_0| \leqslant |J_0|$ and

$$\mathcal{C}_- \triangleq \min\{\rho_1, \rho_2, \inf_{x \in (0,1)} \mathcal{C}(x)\} \leqslant \rho_0(x) \leqslant \max\{\rho_1, \rho_2, \sup_{x \in (0,1)} \mathcal{C}(x)\} \triangleq \mathcal{C}_+, \tag{2.6}$$

$$\|\rho_0 - \mathcal{A}\|_2^2 + \|\rho_{0x}\|_1^2 \leqslant C_0 \delta_0, \tag{2.7}$$

$$\|\phi_{0x}\|_1^2 \leqslant C_0 \delta_0, \tag{2.8}$$

where C_0 is a positive constant related to $\mathcal{C}_\pm$ and $|j_0|$, and

$$\delta_0 = \max_{x\in(0,1)} \{|\mathcal{A}'(x)|+|\mathcal{A}''(x)|+|\mathcal{A}(x)-\mathcal{C}(x)|\}+(|\phi_1|+|\rho_2-\rho_1|)(1+|\ln\mathcal{C}_+ - \ln\mathcal{C}_+|).$$

REMARK 2.2. The choice of function $\mathcal{A} = \mathcal{A}(x)$ will be perfect if it approximates $\mathcal{C}(x)$ sufficiently with small enough oscillations. A simple choice is

$$\mathcal{A}(x) = \rho_1 + x(\rho_2 - \rho_1), \quad x \in [0, 1].$$

A careful analysis will show that the constant C_0 increases if the given doping profile is near the transonic region. For the given doping profile, the bounds of the right-hand side terms in (2.7)–(2.8) is dependent of the choice of $\mathcal{A}(x)$ and the oscillation of $\mathcal{C}(x)$.

Proof. We are going to prove theorem 2.1 in the following two steps.

STEP 1 (The *a priori* estimates). Let $(\rho_0, j_0, \phi_0)(x)$ be a regular solution of the BVP (1.10)–(1.12) and (1.8)–(1.9), which is bounded and satisfies the subsonic condition (2.4). We prove that (2.7) and (2.8) hold for $(\rho_0, j_0, \phi_0)(x)$.

Set

$$\chi = \rho_0 - \mathcal{A}(x). \tag{2.9}$$

By (1.11) and (1.12), we obtain the equation for χ,

$$\left(\frac{p'(\rho_0) - j_0^2/\rho_0^2}{\rho_0}(\mathcal{A}'(x) + \chi_x)\right)_x + \left(\frac{j_0}{\rho_0}\right)_x = \chi + \mathcal{A}(x) - \mathcal{C}(x). \tag{2.10}$$

Multiplying (2.10) with χ, integrating it over $(0, 1)$, using $\chi(0) = \chi(1) = 0$ and integration by parts, one has

$$\begin{aligned}\int_0^1 \chi^2 \,\mathrm{d}x &+ \int_0^1 \frac{p'(\rho_0) - j_0^2/\rho_0^2}{\rho_0}\chi_x^2 \,\mathrm{d}x \\ &= -\int_0^1 \chi(\mathcal{A} - \mathcal{C})(x) \,\mathrm{d}x - \int_0^1 \frac{j_0}{\rho_0}\chi_x \,\mathrm{d}x - \int_0^1 \frac{p'(\rho_0) - j_0^2/\rho_0^2}{\rho_0}\mathcal{A}'(x)\chi_x \,\mathrm{d}x.\end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}\left|\int_0^1 \frac{j_0}{\rho_0}\chi_x \,\mathrm{d}x\right| &\leqslant \left|\int_0^1 \frac{j_0}{\rho_0}\rho_{0x} \,\mathrm{d}x\right| + \left|\int_0^1 \frac{j_0}{\rho_0}\mathcal{A}'(x) \,\mathrm{d}x\right| \\ &\leqslant (|\phi_1| + |\rho_2 - \rho_1|)\left(|\ln\mathcal{C}_+ - \ln\mathcal{C}_-| + \frac{1}{\mathcal{C}_-}\max_{x\in(0,1)}|\mathcal{A}'(x)|\right),\end{aligned} \tag{2.12}$$

it follows from (2.1), (2.11) and (2.12) that

$$\begin{aligned}\int_0^1 \chi^2 \,\mathrm{d}x &+ \int_0^1 \chi_x^2 \,\mathrm{d}x \\ &\leqslant C_0 \max_{x\in(0,1)}(|(\mathcal{A} - \mathcal{C})(x)| + |\mathcal{A}'(x)| + (|\phi_1| + |\rho_2 - \rho_1|)\ln\mathcal{C}_+ - \ln\mathcal{C}_-|),\end{aligned} \tag{2.13}$$

where $C_0 > 0$ is a constant related to $|j_0|$ and $\mathcal{C}_\pm$.

We multiply (2.10) with χ_{xx} and integrate by parts over $(0,1)$. With $\chi(0) = \chi(1) = 0$ and (2.13), we have, similarly to (2.13),

$$\begin{aligned}
&\int_0^1 \chi_x^2 \,\mathrm{d}x + \int_0^1 \chi_{xx}^2 \,\mathrm{d}x \\
&\quad \leqslant C_0 \max_{x\in(0,1)} (|(\mathcal{A}-\mathcal{C})(x)| + |\mathcal{A}'(x)| + |\mathcal{A}''(x)| + (|\phi_1| + |\rho_2-\rho_1) \ln \mathcal{C}_+ - \ln \mathcal{C}_-|).
\end{aligned} \tag{2.14}$$

The combination of (2.13) and (2.14), in view of (2.9), yields (2.7).

Multiplying (1.12) with $[\phi_0(x) - x\phi_1]$ and integrating by parts over $(0,1)$, one has

$$\int_0^1 \phi_{0x}^2 \,\mathrm{d}x \leqslant O(1)\biggl(|\phi_1| + |\mathcal{A}(x) - \mathcal{C}(x)| + \int_0^1 \chi^2 \,\mathrm{d}x\biggr), \tag{2.15}$$

which implies (2.8) in view of (2.1), (1.12), (2.9) and (2.7).

Step 2 (Existence of regular solutions). Define

$$\varPhi_0 = F(\rho_2, J_0) - F(\rho_1, J_0) + \frac{|J_0|}{\mathcal{C}_+}, \tag{2.16}$$

which implies, in terms of (2.5) and $|\rho_2 - \rho_1| \ll 1$, that

$$\varPhi_0 > 0. \tag{2.17}$$

In addition, one can verify that

$$\frac{\mathrm{d}\varPhi}{\mathrm{d}j} =: \frac{\partial F}{\partial j}(\rho_2, j) - \frac{\partial F}{\partial j}(\rho_1, j) + \frac{1}{\mathcal{C}_+} > 0 \tag{2.18}$$

for $|j| < |J_0|$ and $|\rho_2 - \rho_1| \ll 1$.

Without loss of generality, we assume that

$$\rho_2 \geqslant \rho_1. \tag{2.19}$$

For any $0 < \phi_1 \leqslant \varPhi_0$, define the operator $T : p \to P$ by solving the following linear equation with the Dirichlet boundary:

$$\left(\frac{\partial F}{\partial p}(p, J)P_x\right)_x - JP_x - P = \mathcal{C}, \quad 0 < x < 1, \tag{2.20}$$

$$P(0) = \rho_1, \quad P(1) = \rho_2, \tag{2.21}$$

where $J = J[p]$ satisfies

$$\phi_1 = F(\rho_2, J) - F(\rho_1, J) + J\int_0^1 \frac{\mathrm{d}x}{p(x)}, \tag{2.22}$$

and

$$J < \frac{\rho_1^2\rho_2^2}{\mathcal{C}_+(\rho_2^2 - \rho_1^2)} \quad \text{for } \rho_2 > \rho_1. \tag{2.23}$$

Suppose

$$\mathcal{C}_- \leqslant p(x) \leqslant \mathcal{C}_+. \tag{2.24}$$

By (2.22), (2.18), (2.16) and (2.5), one finds that

$$|J| \leqslant |J_0|, \qquad \frac{\partial F}{\partial \rho}(p, J) > \alpha_0 > 0, \tag{2.25}$$

which implies that the equation (2.20) is elliptic. For the linear elliptic BVP (2.20)–(2.21), applying the maximum principle and using energy estimates similar to (2.13)–(2.15), one obtains

$$\mathcal{C}_- \leqslant P = T(p) \leqslant \mathcal{C}_+, \quad P = T(p) \in C^2(0,1), \quad \|P - \mathcal{A}\|_2 = \|T(p) - \mathcal{A}\|_2 \leqslant C \tag{2.26}$$

and

$$|J[P]| \leqslant |J_0|, \qquad \frac{\partial F}{\partial \rho}(P, J[P]) > \alpha_0 > 0, \tag{2.27}$$

with α_0 a constant.

Thus it is easily shown that the operator T is continuous and bounded in $H^2(0,1)$. Applying Schauder's fixed-point theorem and the compact embedding of $H^2(0,1)$ into $C^1(0,1)$, one obtains the existence of a fixed point, say $\rho_0(x)$, of the operator T such that the fixed point $\rho_0 = T(\rho_0)$ satisfies (2.20) and (2.21), with $p = P = \rho_0$. Let j_0 and $\phi_0(x)$ be determined by

$$\phi_1 = F(\rho_2, j_0) - F(\rho_1, j_0) + j_0 \int_0^1 \frac{\mathrm{d}x}{\rho_0(x)}, \tag{2.28}$$

$$j_0 < \frac{\rho_1^2 \rho_2^2}{\mathcal{C}_+(\rho_2^2 - \rho_1^2)} \quad \text{for } \rho_2 > \rho_1 \tag{2.29}$$

and

$$\phi_{0xx} = \rho_0 - \mathcal{C}, \quad 0 < x < 1, \tag{2.30}$$

$$\phi_0(0) = 0, \quad \phi_0(1) = \phi_1. \tag{2.31}$$

Thus (ρ_0, j_0, ϕ_0) is a solution to the BVP (1.10)–(1.12) and (1.8)–(1.9) satisfying $|j_0| \leqslant |J_0|$, and (2.6)–(2.8). Applying the maximum principle and a similar argument to [5, 19], one can easily prove the uniqueness of the solution. □

3. Hydrodynamic model

In this section, we consider the stability of the steady-state solution obtained in § 2.

Set

$$\psi_0 = \tilde{\rho} - \rho_0, \qquad \eta_0 = \tilde{j} - j_0. \tag{3.1}$$

The main result in this section is the following one.

THEOREM 3.1. *Let $(\rho_0, j_0, \phi_0)(x)$ be the regular solution of the BVP (1.10)–(1.12) and (1.8)–(1.9) given by theorem 2.1. Assume $(\psi_0, \eta_0) \in H^2$. Then, there exists $\varepsilon_0 > 0$ such that if $\|(\psi_0, \eta_0)\|_2 + \delta_0 \leqslant \varepsilon_0$, then the global smooth solution $(\rho, j, \phi)(x, t)$ to the IBVP (1.1)–(1.3) and (1.7)–(1.9) exists and satisfies*

$$\|(\rho - \rho_0, j - j_0, \phi - \phi_0)(\cdot, t)\|_2^2 \leqslant O(1)\|(\psi_0, \eta_0)\|_2^2 \exp\{-\beta t\}, \quad t \geqslant 0, \tag{3.2}$$

with a positive constant β.

Proof. By standard methods, we can prove the local existence of a solution of the IBVP (1.1)–(1.3) and (1.7)–(1.9), and show its regularity due to the effect of relaxation as in [13]. To show the global existence of a smooth solution, we shall establish uniform *a priori* estimates, i.e. lemmas 3.2–3.4.

Let $(\rho_0, j_0, \phi_0)(x)$ be the steady-state solution to the BVP (1.10)–(1.12) and (1.8)–(1.9). For any $T > 0$, assume that $(\rho, j, \phi)(x, t)$ is the solution of the IBVP (1.1)–(1.3) and (1.7)–(1.9).

Set

$$\psi = \rho - \rho_0, \qquad \eta = j - j_0, \qquad e = \phi - \phi_0. \tag{3.3}$$

Then the corresponding IBVP for (ϕ, η, e) on $(0, 1) \times [0, +\infty)$ is

$$\psi_t + \eta_x = 0, \tag{3.4}$$

$$\eta_t + \left[\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right]_x = \psi\phi_{0x} + (\rho_0+\psi)e_x - \eta, \tag{3.5}$$

$$e_{xx} = \psi, \tag{3.6}$$

$$\psi(0, t) = \psi(1, t) = 0, \quad t \geqslant 0, \tag{3.7}$$

$$e(0, t) = e(1, t) = 0, \quad t \geqslant 0, \tag{3.8}$$

$$\psi(x, 0) = \psi_0(x), \quad \eta(x, 0) = \eta_0(x), \quad x \in (0, 1). \tag{3.9}$$

For $T > 0$, denote the basic space for the IBVP (3.4)–(3.9) as

$$X(T) = \{(\psi, \eta, e) \in H^2,\ 0 \leqslant t \leqslant T\}, \tag{3.10}$$

with norm given by

$$M(0, T) = \max_{0 \leqslant t \leqslant T} \|(\psi, \eta, e)(t)\|_2,$$

and assume that the following assumption holds:

$$N(T) = \max_{0 \leqslant t \leqslant T} \|(\psi, \eta)(t)\|_2 \ll 1. \tag{3.11}$$

It is easy to verify that, under the assumption (3.11), it holds that

$$0 < \rho_- \leqslant \rho_0 + \phi \leqslant \rho_+, \qquad j_- \leqslant j_0 + \eta \leqslant j_+,$$

with ρ_-, ρ_+, j_- and j_+ constants, and the subsonic condition (2.4) holds for (ρ, j, ϕ).

Lemma 3.2. *It holds for $(\psi, \eta, e) \in X(T)$, provided that $N(T) + \delta_0$ is small enough, that*

$$\int_0^1 e_x^2 \,\mathrm{d}x \leqslant O(1) \int_0^1 \psi^2 \,\mathrm{d}x, \qquad e_x^2 \leqslant O(1) \int_0^1 \psi^2 \,\mathrm{d}x, \tag{3.12}$$

$$\int_0^1 e_{xt}^2 \,\mathrm{d}x \leqslant O(1) \int_0^1 \psi_t^2 \,\mathrm{d}x, \qquad e_{xt}^2 \leqslant O(1) \int_0^1 \psi_t^2 \,\mathrm{d}x, \qquad e_t^2 \leqslant O(1) \int_0^1 \psi^2 \,\mathrm{d}x, \tag{3.13}$$

$$\int_0^1 \eta^2 \,\mathrm{d}x \leqslant O(1)\left(\exp\{-c_0 t\} \int_0^1 \eta_0^2 \,\mathrm{d}x + \int_0^1 (\psi_t^2 + \psi^2) \,\mathrm{d}x\right), \tag{3.14}$$

$$\eta^2 \leqslant O(1)\left(\exp\{-c_0 t\} \int_0^1 \eta_0^2 \,\mathrm{d}x + \int_0^1 (\psi_t^2 + \psi^2) \,\mathrm{d}x\right), \tag{3.15}$$

$$\int_0^1 \eta_t^2 \,\mathrm{d}x \leqslant O(1)\Big(\exp\{-c_0 t\}\int_0^1 \eta_0^2 \,\mathrm{d}x + \int_0^1 (\psi_t^2 + \psi_x^2 + \psi^2)\,\mathrm{d}x\Big), \tag{3.16}$$

with $c_0 > 0$ *a constant.*

We first show (3.12). Multiplying (3.6) with e and integrating over $(0,1)$ yields, by (3.7) and integration by parts,

$$\int_0^1 e_x^2 \,\mathrm{d}x + \int_0^1 e\psi \,\mathrm{d}x = 0, \tag{3.17}$$

which implies, in terms of Hölder's inequality, that

$$\int_0^1 e_x^2 \,\mathrm{d}x \leqslant \Big(\int_0^1 e^2 \,\mathrm{d}x\Big)^{1/2}\Big(\int_0^1 \psi^2 \,\mathrm{d}x\Big)^{1/2}. \tag{3.18}$$

Then it follows that

$$\int_0^1 e_x^2 \,\mathrm{d}x \leqslant 2\int_0^1 \psi^2 \,\mathrm{d}x, \tag{3.19}$$

from (3.8), (3.18) and the following Poincaré inequality (noting $e(0,t) = e(1,t) = 0$):

$$\Big(\int_0^1 e^2 \,\mathrm{d}x\Big)^{1/2} \leqslant 2\Big(\int_0^1 e_x^2 \,\mathrm{d}x\Big)^{1/2}. \tag{3.20}$$

Integrating the above inequality gives (3.20).

On the other hand, by the integral mean-value theorem, there exists a curve $x_1(t)$ satisfying $0 < x_1(t) < 1$ such that

$$e_x^2(x_1(t),t) = \int_0^1 e_x^2(x,t)\,\mathrm{d}x.$$

Thanks to (3.6) and (3.19), we have

$$\begin{aligned} e_x^2(x,t) &= e_x^2(x_1(t),t) + 2\Big|\int_{x_1(t)}^x e_x e_{xx} \,\mathrm{d}x\Big| \\ &\leqslant \int_0^1 e_x^2 \,\mathrm{d}x + 2\int_0^1 |e_x e_{xx}|\,\mathrm{d}x \\ &\leqslant 2\int_0^1 e_x^2 \,\mathrm{d}x + \int_0^1 e_{xx}^2 \,\mathrm{d}x \\ &\leqslant O(1)\int_0^1 \psi^2 \,\mathrm{d}x. \end{aligned}$$

Now we deal with (3.13). Differentiating (3.6) with respect to t leads to

$$e_{xxt} = \psi_t. \tag{3.21}$$

Multiplying (3.21) with e_t, integrating over $(0,1)$ and noticing $e_t(0,t) = e_t(1,t) = 0$, we have, after integration by parts,

$$\int_0^1 e_{xt}^2 \,\mathrm{d}x + \int_0^1 e_t\psi_t \,\mathrm{d}x = 0. \tag{3.22}$$

Similarly to (3.19), one has

$$\int_0^1 e_{xt}^2\,\mathrm{d}x \leqslant O(1)\int_0^1 \psi_t^2\,\mathrm{d}x. \tag{3.23}$$

The other estimates in (3.13) follow from (3.23), (3.21) and the following inequalities:

$$e_{xt}^2 \leqslant 2\int_0^1 e_{xt}^2\,\mathrm{d}x + \int_0^1 e_{xxt}^2\,\mathrm{d}x, \qquad e_t^2 \leqslant \int_0^1 e_t^2\,\mathrm{d}x + \int_0^1 e_{xt}^2\,\mathrm{d}x.$$

Finally, we estimate (3.14)–(3.16). Since it holds by (3.4) that

$$\eta^2 \leqslant \int_0^1 \eta^2\,\mathrm{d}x + 2\int_0^1 |\eta_x\eta|\,\mathrm{d}x \leqslant 2\int_0^1 \eta^2\,\mathrm{d}x + \int_0^1 \psi_t^2\,\mathrm{d}x \tag{3.24}$$

and, in view of (3.5), (3.12), (2.7), and (2.8), that

$$\begin{aligned}\eta_t^2 &\leqslant O(1)\left\{\left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)_x\right\}^2\\ &\qquad + O(1)(\eta^2 + (\psi_x\phi_{0x})^2 + (\rho_{0x}+\psi_x)^2 e_x^2)\\ &\leqslant O(1)(\eta^2+\psi_x^2+\psi_t^2+\psi^2),\end{aligned} \tag{3.25}$$

it is sufficient to prove (3.14). Multiplying (3.5) with η and integrating it over $(0,1)$, one has, by (3.8) and integration by parts, that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \eta^2\,\mathrm{d}x\right) + \int_0^1 \eta^2\,\mathrm{d}x\\ &\quad = -\left[\eta\frac{(j_0+\eta)^2 - j_0^2}{\rho_0}\right]\Bigg|_0^1 + \int_0^1 \eta(\psi\phi_{0x} + (\rho_0+\psi)e_x)\,\mathrm{d}x\\ &\qquad + \int_0^1 \eta_x\left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)\mathrm{d}x\\ &\quad \triangleq I_1 + I_2 + I_3.\end{aligned} \tag{3.26}$$

The I_1, I_2 and I_3 can be estimated as follows,

$$\begin{aligned}I_1 &\leqslant \int_0^1 \left|\eta_x\frac{(j_0+\eta)^2-j_0^2}{\rho_0} + 2\eta\eta_x\frac{j_0+\eta}{\rho_0} - \eta\rho_{0x}\frac{(j_0+\eta)^2-j_0^2}{\rho_0^2}\right|\mathrm{d}x\\ &\leqslant O(1)\int_0^1 |\psi_t\eta|\,\mathrm{d}x + \delta_0\int_0^1 \eta^2\,\mathrm{d}x\\ &\leqslant (O(1)\delta_0 + \tfrac{1}{12})\int_0^1 \eta^2\,\mathrm{d}x + O(1)\int_0^1 \psi_t^2\,\mathrm{d}x,\end{aligned} \tag{3.27}$$

$$\begin{aligned}I_2 &\leqslant \frac{1}{12}\int_0^1 \eta^2\,\mathrm{d}x + O(1)\int_0^1 (\psi^2 + e_x^2)\,\mathrm{d}x\\ &\leqslant \frac{1}{12}\int_0^1 \eta^2\,\mathrm{d}x + O(1)\int_0^1 \psi^2\,\mathrm{d}x,\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
I_3 &\leqslant \int_0^1 \left|\psi_t\left(2\frac{j_0+\theta_1\eta}{\rho_0+\theta_2\psi}\eta - \frac{(j_0+\theta_3\eta)^2}{(\rho_0+\theta_4\psi)^2}\psi\right)\right| \mathrm{d}x + \int_0^1 |\psi_t\psi p'(\rho_0+\theta_5\psi)| \,\mathrm{d}x \\
&\leqslant \frac{1}{12}\int_0^1 \eta^2 \,\mathrm{d}x + O(1)\int_0^1 (\psi_t^2+\psi^2)\,\mathrm{d}x,
\end{aligned} \tag{3.29}
$$

with $0<\theta_i<1$ $(i=1,2,3,4,5)$. Here we used (3.4).

Substituting (3.27)–(3.28) into (3.26) leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \eta^2 \,\mathrm{d}x\right) + (\tfrac{3}{2} - O(1)\delta_0)\int_0^1 \eta^2 \,\mathrm{d}x \leqslant O(1)\int_0^1 (\psi^2+\psi^2)\,\mathrm{d}x. \tag{3.30}
$$

Integrating (3.30) over $[0,t]$ gives

$$
\int_0^1 \eta^2 \,\mathrm{d}x \leqslant \exp\{-c_0 t\}\int_0^1 \eta_0^2 \,\mathrm{d}x + O(1)(1-\exp\{-c_0 t\})\int_0^1 (\psi^2+\psi^2)\,\mathrm{d}x, \tag{3.31}
$$

with a constant $0<c_0<\frac{3}{2}-O(1)\delta_0$, which implies (3.14). □

LEMMA 3.3. *It holds, for $(\psi,\eta,e)\in X(T)$, that*

$$
\int_0^1 (\psi_t^2+\psi_x^2+\psi^2)\,\mathrm{d}x \leqslant O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_1 t\}, \tag{3.32}
$$

$$
\int_0^1 (e^2+e_x^2+e_{xx}^2)\,\mathrm{d}x \leqslant O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_1 t\}, \tag{3.33}
$$

with $\beta_1>0$ a constant, provided that $N(T)+\delta_0$ is small enough.

Proof. Differentiating (3.5) with respect to x and using (3.4) and (3.6), we obtain the 'wave equation with friction'

$$
\psi_{tt}+\psi_t+(\rho_0+\phi_{0xx}+\psi)\psi+(\rho_0+\psi)_x e_x - \left[\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right]_{xx} = 0. \tag{3.34}
$$

Multiplying (3.34) with ψ and integrating over $(0,1)$, one has, by (3.8) and integration by parts,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tfrac{1}{2}\psi^2+\psi\psi_t \,\mathrm{d}x\right) - \int_0^1 \psi_t^2 \,\mathrm{d}x + \int_0^1 (\rho_0+\phi_{0xx}+\psi)\psi^2 \,\mathrm{d}x \\
&\quad = -\int_0^1 (\rho_0+\psi)_x e_x\psi \,\mathrm{d}x - \int_0^1 \left(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\right)_x \psi_x \,\mathrm{d}x \\
&\quad \triangleq I_4 + I_5.
\end{aligned} \tag{3.35}
$$

By the Hölder inequality and (3.12), I_4 can be estimated as

$$
\begin{aligned}
|I_4| &\leqslant (\delta_0+N(T))\left(\int_0^1 e_x^2 \,\mathrm{d}x\right)^{1/2}\left(\int_0^1 \psi^2 \,\mathrm{d}x\right)^{1/2} \\
&\leqslant O(1)(\delta_0+N(T))\int_0^1 \psi^2 \,\mathrm{d}x.
\end{aligned} \tag{3.36}
$$

By (3.4) and (3.14), we can estimate I_5 as

$$
\begin{aligned}
I_5 = &-\int_0^1 \psi_x\left\{\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi\right\}_x \mathrm{d}x\\
&-\int_0^1 \psi_x\left\{\left(\tfrac{1}{2}p''(\rho_0+\theta_6\psi)+\frac{j_0^2}{(\rho_0+\theta_7\psi)^2}\right)\psi^2\right\}_x \mathrm{d}x\\
&-\int_0^1 \psi_x\left\{\frac{2j_0\eta}{\rho_0}+\frac{\eta^2}{\rho_0+\theta_8\psi}-\frac{2(j_0+\theta_{10}\eta)}{(\rho_0+\theta_9\psi)^2}\psi\eta\right\}_x \mathrm{d}x\\
\leqslant &-\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\,\mathrm{d}x\\
&+O(1)(\delta_0+N(T))\int_0^1(|\psi\psi_x|+|\eta\psi_x|)\,\mathrm{d}x+\int_0^1\left(\left|\frac{2j_0}{\rho_0}\right|+N(T)\right)|\psi_t\psi_x|\,\mathrm{d}x\\
\leqslant &-\frac{1}{2}\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\,\mathrm{d}x\\
&+O(1)(\delta_0+N(T))\int_0^1(\psi_x^2+\psi^2+\eta^2)\,\mathrm{d}x+2\int_0^1\frac{j_0^2}{\rho_0^2p'(\rho_0)-j_0^2}\psi_t^2\,\mathrm{d}x\\
\leqslant &-\frac{1}{2}\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x 2\,\mathrm{d}x+O(1)(\delta_0+N(T))\int_0^1(\psi_x^2+\psi^2+\psi_t^2)\,\mathrm{d}x\\
&+2a\int_0^1\psi_t^2+\exp\{-c_0t\}\int_0^1\eta_0^2\,\mathrm{d}x,
\end{aligned}
\tag{3.37}
$$

where $0<\theta_i<1$ $(i=6,7,8,9,10)$ and

$$
0<a<\max_{x\in(0,1)}\frac{j_0(x)^2}{\rho_0(x)^2p'(\rho_0(x))-j_0(x)^2}. \tag{3.38}
$$

Substituting (3.36) and (3.37) into (3.35), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\tfrac{1}{2}\psi_t^2+\psi_t\psi\,\mathrm{d}x\right)\\
&\quad-(1+2a)\int_0^1\psi_t^2\,\mathrm{d}x\int_0^1(\rho_0+\phi_{0xx}+\psi)\psi^2\,\mathrm{d}x+\frac{1}{2}\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\,\mathrm{d}x\\
&\qquad\leqslant O(1)(\delta_0+N(T))\int_0^1(\psi_x^2+\psi^2+\psi_t^2)\,\mathrm{d}x+\exp\{-c_0t\}\int_0^1\eta_0^2\,\mathrm{d}x.
\end{aligned}
\tag{3.39}
$$

We multiply (3.34) with ψ_t and integrate over $(0,1)$. Noticing that $\psi_t(0,t)=\psi_t(1,t)=0$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2\,\mathrm{d}x\right)+\int_0^1(\psi_t^2-\tfrac{1}{2}\psi_t\psi^2)\,\mathrm{d}x\\
&\quad=-\int_0^1(\rho_0+\psi)_x e_x\psi_t\,\mathrm{d}x-\int_0^1\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_x\psi_{xt}\,\mathrm{d}x\\
&\quad\triangleq I_6+I_7.
\end{aligned}
\tag{3.40}
$$

We estimate I_6 and I_7 as follows. Due to (3.12), it holds that

$$|I_6| \leqslant O(1)(\delta_0 + N(T)) \int_0^1 (\psi^2 + \psi_t^2)\,\mathrm{d}x. \tag{3.41}$$

By (3.4), (3.14) and (3.16), I_7 can be estimated as

$$\begin{aligned}
I_7 = &-\int_0^1 \psi_{xt}\biggl(\biggl(p'(\rho_0+\psi) - \frac{j_0^2+\eta}{\rho_0^2+\psi}\biggr)\psi_x \\
&\quad - 2\frac{j_0+\eta}{\rho_0+\psi}\psi_t + \rho_{0x}\biggl(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\biggr)\biggr)\,\mathrm{d}x \\
= &-\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 \biggl(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\biggr)\psi_x^2\,\mathrm{d}x\biggr) \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 (\tfrac{1}{2}\psi_x^2 + \psi_x\rho_{0x})\biggl(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\biggr)\,\mathrm{d}x\biggr) \\
&\quad + \int_0^1 (\tfrac{1}{2}\psi_x^2 + \psi_x\rho_{0x})\biggl(p''(\rho_0+\psi)\psi_t + \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - \frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t\biggr)\,\mathrm{d}x \\
&\quad - \int_0^1 \psi_t^2\biggl(\frac{\eta_x}{\rho_0+\psi} - \frac{j_0+\eta}{(\rho_0+\psi)^2}(\rho_0+\psi)_x\biggr)\,\mathrm{d}x \\
\leqslant &-\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{2}\int_0^1 \biggl(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\biggr)\psi_x^2\,\mathrm{d}x\biggr) \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 (\tfrac{1}{2}\psi_x^2 + \psi_x\rho_{0x})\biggl(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\biggr)\,\mathrm{d}x\biggr) \\
&\quad + O(1)(N(T)+\delta_0)\int_0^1 (\psi_x^2 + \psi_t^2 + \eta_t^2 + \eta^2)\,\mathrm{d}x \\
\leqslant &-\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\tfrac{1}{2}\int_0^1 \biggl(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\biggr)\psi_x^2\,\mathrm{d}x\biggr) \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 (\tfrac{1}{2}\psi_x^2 + \psi_x\rho_{0x})\biggl(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\biggr)\,\mathrm{d}x\biggr) \\
&\quad + O(1)(N(T)+\delta_0)\int_0^1 (\psi_x^2 + \psi_t^2 + \psi^2)\,\mathrm{d}x + O(1)\exp\{-c_0 t\}\int_0^1 \eta_0^2\,\mathrm{d}x.
\end{aligned} \tag{3.42}$$

Substituting (3.41) and (3.42) into (3.40), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 \tfrac{1}{2}\psi_t^2 + \tfrac{1}{2}\biggl(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\biggr)\psi_x^2 + (\rho_0 + \phi_{0xx} + \psi)\psi^2\,\mathrm{d}x\biggr) \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 (\tfrac{1}{2}\psi_x^2 + \psi_x\rho_{0x})\biggl(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\biggr)\,\mathrm{d}x\biggr) \\
&\qquad\quad + \int_0^1 \psi_t^2\,\mathrm{d}x \\
&\quad \leqslant O(1)(N(T)+\delta_0)\int_0^1 (\psi_x^2 + \psi_t^2 + \psi^2)\,\mathrm{d}x + O(1)\exp\{-c_0 t\}\int_0^1 \eta_0^2\,\mathrm{d}x.
\end{aligned} \tag{3.43}$$

By $[(3.39)+2(1+2a)\times(3.43)]$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 (1+2a)\psi_t^2+\tfrac{1}{2}\psi^2+\psi\psi_t+(1+2a)\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\,\mathrm{d}x\right)\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 2(1+2a)(\rho_0+\phi_{0xx}+\psi)\psi^2\,\mathrm{d}x\right)\\
&\quad\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 (1+2a)(\psi_x^2+2\psi_x\rho_{0x})\left(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2}\right)\mathrm{d}x\right)\\
&\quad\quad\quad+\int_0^1\left((1+2a)\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2+\frac{1}{2}\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\right)\mathrm{d}x\\
&\quad\quad\quad\quad\leqslant O(1)(N(T)+\delta_0)\int_0^1(\psi_x^2+\psi_t^2+\psi^2)\,\mathrm{d}x+O(1)\exp\{-c_0t\}\int_0^1\eta_0^2\,\mathrm{d}x.
\end{aligned}
\tag{3.44}
$$

Noticing, for positive constants c_1, c_2, c_3, that

$$
\begin{aligned}
&c_1(\psi_t^2+\psi_x^2+\psi^2)\\
&\qquad\leqslant(1+2a)\psi_t^2+\tfrac{1}{2}\psi^2+\psi\psi_t+(1+2a)\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\\
&\qquad\qquad+2(1+2a)(\rho_0+\phi_{0xx}+\psi)\psi^2\\
&\qquad\qquad+(1+2a)(\psi_x^2+2\psi_x\rho_{0x})\left(p'(\rho_0+\psi)-p'(\rho_0)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}+\frac{j_0^2}{\rho_0^2}\right)\\
&\qquad\leqslant c_2^{-1}\left((1+2a)\psi_t^2+(\rho_0+\phi_{0xx}+\psi)\psi^2+\frac{1}{2}\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_x^2\right)\\
&\qquad\leqslant c_3(\psi_t^2+\psi_x^2+\psi^2),
\end{aligned}
\tag{3.45}
$$

and integrating (3.44) over $[0,t]$, we obtain (3.32) for a constant $\beta_1>0$ from the Sobolev embedding theorem, provided that $N(T)+\delta_0$ is small enough.

Thus (3.33) follows from (3.32), (3.20), (3.12) and (3.6). □

LEMMA 3.4. *For $(\psi,\eta,e)\in X(T)$, we have*

$$
\int_0^1(\psi_t^2+\psi_{xt}^2+\psi_{tt}^2+\psi_{xx}^2)\,\mathrm{d}x\leqslant O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_2t\},\tag{3.46}
$$

$$
\int_0^1(e_t^2+e_{xt}^2+e_{xxt}^2)\,\mathrm{d}x\leqslant O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_2t\},\tag{3.47}
$$

with $\beta_2>0$, provided that $N(T)+\delta_0+\|(\psi_0,\eta_0)\|_2$ is small enough.

Proof. Differentiating (3.34) with respect to t leads to

$$
\begin{aligned}
&\psi_{ttt}+\psi_{tt}+(\rho_0+\phi_{0xx}+2\psi)\psi_t+(\rho_0+\psi)_xe_{xt}+e_x\psi_{xt}\\
&\qquad-\left[\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}-\frac{j_0^2}{\rho_0^2}+p(\rho_0+\psi)-p(\rho_0)\right]_{xxt}=0.
\end{aligned}
\tag{3.48}
$$

Multiplying (3.48) with ψ_t, integrating it over $(0,1)$ and using $\psi_t(0,t)=\psi_t(1,t)=0$ and (3.6), we have, after integration by parts,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tfrac{1}{2}\psi_t^2+\psi_t\psi_{tt}\,\mathrm{d}x\right)-\int_0^1 \psi_{tt}^2\,\mathrm{d}x+\int_0^1(\rho_0+\phi_{0xx}+\tfrac{3}{2}\psi)\psi_t^2\,\mathrm{d}x\\
&\quad=-\int_0^1(\rho_0+\psi)_x e_{xt}\psi_t\,\mathrm{d}x-\int_0^1\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_{xt}\psi_{xt}\,\mathrm{d}x\\
&\quad\triangleq I_8+I_9. \qquad (3.49)
\end{aligned}$$

By the Hölder inequality and (3.13), it holds that

$$\begin{aligned}
|I_8|&\leqslant(\delta_0+N(T))\left(\int_0^1 e_{xt}^2\,\mathrm{d}x\right)^{1/2}\left(\int_0^1\psi_t^2\,\mathrm{d}x\right)^{1/2}\\
&\leqslant O(1)(\delta_0+N(T))\int_0^1\psi_t^2\,\mathrm{d}x. \qquad (3.50)
\end{aligned}$$

By (3.4), (3.15), (3.16) and (3.32), we estimate I_9 as

$$\begin{aligned}
I_9=&-\int_0^1\Bigg[\left(p'(\rho_0+\psi)-\frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}\right)\psi_{xt}-2\frac{j_0+\eta}{(\rho_0+\psi)^2}\eta_t(\rho_0+\psi)_x\\
&\quad+2\frac{(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t(\rho_0+\psi)_x+p''(\rho_0+\psi)\psi_t(\rho_0+\psi)_x\\
&\quad+2\frac{j_0+\eta}{\rho_0+\psi}\eta_{xt}+2\frac{\eta_t\psi_t}{\rho_0+\psi}-2\frac{j_0+\eta}{(\rho_0+\psi)^2}\eta_x\psi_t\Bigg]\psi_{xt}\,\mathrm{d}x\\
\leqslant&-\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_{xt}^2\,\mathrm{d}x+2\int_0^1\left|\frac{j_0}{\rho_0}\right||\psi_{tt}\psi_{xt}|\,\mathrm{d}x\\
&+O(1)\int_0^1(|\eta|+N(T))(|\psi_{tt}\psi_{xt}|+\psi_{xt}^2)\,\mathrm{d}x\\
&+O(1)(\delta_0+N(T))\int_0^1|\psi_{xt}|(|\eta_t|+|\psi_t|+|\psi_x|)\,\mathrm{d}x\\
\leqslant&-\frac{1}{2}\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_{xt}^2\,\mathrm{d}x+2\int_0^1\frac{j_0^2}{\rho_0^2p'(\rho_0)-j_0^2}\psi_{tt}^2\,\mathrm{d}x\\
&+O(1)(\|(\psi_0,\eta_0)\|_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)\,\mathrm{d}x\\
&+O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0t\}+\exp\{-\beta_1t\}). \qquad (3.51)
\end{aligned}$$

Substituting (3.50) and (3.51) into (3.49) leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\tfrac{1}{2}\psi_t^2+\psi_t\psi_{tt}\,\mathrm{d}x\right)-(1+2a)\int_0^1\psi_{tt}^2\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int_0^1\left(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\right)\psi_{xt}^2\,\mathrm{d}x+\int_0^1(\rho_0+\phi_{0xx}+\tfrac{3}{2}\psi)\psi_t^2\,\mathrm{d}x
\end{aligned}$$

$$\leqslant O(1)(\|(\psi_0,\eta_0)\|_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x + O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\}). \tag{3.52}$$

We multiply (3.48) by ψ_{tt} and integrate the resulting equation over $(0,1)$. In terms of $\psi_t(0,t) = \psi_t(1,t) = 0$ and integration by parts, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 \tfrac{1}{2}\psi_{tt}^2 + \tfrac{1}{2}(\rho_0 + \phi_{0xx} + 2\psi)\psi_t^2\,\mathrm{d}x\biggr) + \int_0^1 \psi_{tt}^2\,\mathrm{d}x - \int_0^1 \psi_t^3\,\mathrm{d}x \\
= -\int_0^1 ((\rho_0+\psi)_x e_{xt} + e_x\psi_{xt})\psi_{tt}\,\mathrm{d}x \\
- \int_0^1 \biggl(\frac{(j_0+\eta)^2}{\rho_0+\psi} - \frac{j_0^2}{\rho_0} + p(\rho_0+\psi) - p(\rho_0)\biggr)_{xt}\psi_{xtt}\,\mathrm{d}x \\
\triangleq I_{10} + I_{11}.
\end{aligned} \tag{3.53}$$

By (3.12), (3.13) and the Cauchy inequality, it is easy to verify that

$$\begin{aligned}
|I_{10}| &\leqslant \tfrac{1}{2}(\delta_0 + N(T))\int_0^1 (e_{xt}^2 + \psi_{tt}^2)\,\mathrm{d}x + \frac{1}{2}\int_0^1 |e_x|(\psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x \\
&\leqslant O(1)(\delta_0 + N(T))\int_0^1 (\psi_t^2 + \psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x.
\end{aligned} \tag{3.54}$$

By (3.4), we have

$$\begin{aligned}
I_{11} = &-\int_0^1 \biggl(p'(\rho_0+\psi) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}\biggr)\psi_{xt}\psi_{xtt}\,\mathrm{d}x \\
&+ \int_0^1 \biggl(\frac{2(j_0+\eta)}{\rho_0+\psi}\psi_{tt} + \frac{2\eta_t\psi_t}{\rho_0+\psi} - \frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\psi_t^2\biggr)\psi_{xtt}\,\mathrm{d}x \\
&+ \int_0^1 \biggl(\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t - \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - p''(\rho_0+\psi)\psi_t\biggr)(\rho_0+\psi)_x\psi_{xtt}\,\mathrm{d}x \\
\triangleq\; & K_1 + K_2 + K_3.
\end{aligned} \tag{3.55}$$

By (3.25), (3.15) and (3.32), we have

$$\begin{aligned}
K_1 = &-\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 \frac{1}{2}\biggl(p'(\rho_0+\psi) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}\biggr)\psi_{xt}^2\,\mathrm{d}x\biggr) \\
&+ \frac{1}{2}\int_0^1 \psi_{xt}^2\biggl(p''(\rho_0+\psi)\psi_t + \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - \frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t\biggr)\,\mathrm{d}x \\
\leqslant &-\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_0^1 \frac{1}{2}\biggl(p'(\rho_0+\psi) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2}\biggr)\psi_{xt}^2\,\mathrm{d}x\biggr) \\
&+ O(1)(N(T) + \|(\psi_0,\eta_0)\|_2)\int_0^1 \psi_{xt}^2\,\mathrm{d}x.
\end{aligned} \tag{3.56}$$

Noticing $\psi_{tt}(0,t)=\psi_{tt}(1,t)=0$, we have, by (3.25), (3.15), (3.12) and (3.32),

$$\begin{aligned}
|K_2| &= \left|\int_0^1 \psi_{tt}^2\left(\frac{2(j_0+\eta)}{\rho_0+\psi}\right)_x \mathrm{d}x\right| + \left|\int_0^1 \psi_{tt}\left(\frac{2\eta_t\psi_t}{\rho_0+\psi}-\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\psi_t^2\right)_x \mathrm{d}x\right| \\
&\leqslant O(1)\int_0^1 \psi_{tt}^2(|\psi_t|+|\psi_x|+|\rho_{0x}|)\,\mathrm{d}x \\
&\quad + O(1)\int_0^1 |\psi_{tt}\psi_{xt}|(|\psi_t|+|\eta_t|)\,\mathrm{d}x \\
&\quad + O(1)\int_0^1 |\psi_{tt}|(\psi_t^2+|\eta_t\psi_t|+|\psi_x\psi_t|)\,\mathrm{d}x \\
&\leqslant O(1)(N(T)+\delta_0+\|(\psi_0,\eta_0)\|_2)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)\,\mathrm{d}x \\
&\quad + O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_1 t\}.
\end{aligned} \tag{3.57}$$

At last, we estimate K_3 as

$$\begin{aligned}
K_3 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]\mathrm{d}x\right) \\
&\quad -\int_0^1 \psi_{xt}^2\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]\mathrm{d}x \\
&\quad -\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_{tt}-4\eta_t\psi_t\frac{(j_0+\eta)}{(\rho_0+\psi)^3}+\frac{2\eta_t^2}{(\rho_0+\psi)^2}\right]\mathrm{d}x \\
&\quad +\int_0^1 \psi_{xt}(\rho_0+\psi)_x\psi_{tt}\left[\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}+p''(\rho_0+\psi)\right]\mathrm{d}x \\
&\quad +\int_0^1 \psi_{xt}(\rho_0+\psi)_x\psi_t\left[\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}+p''(\rho_0+\psi)\right]_t\mathrm{d}x \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\left[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\right]\mathrm{d}x\right) \\
&\quad + O(1)(\|(\psi_0,\eta_0)\|_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)\,\mathrm{d}x \\
&\quad + O(1)\|(\psi_0,\eta_0)\|_2^2\exp\{-\beta_1 t\}+O(1)(N(T)+\delta_0)\int_0^1 \eta_{tt}^2\,\mathrm{d}x
\end{aligned} \tag{3.58}$$

Differentiating (3.5) with respect to t, and using (3.13), (3.16) and (3.32), we can estimate the last term in the right-hand side of (3.58) as

$$\begin{aligned}
\int_0^1 \eta_{tt}^2\,\mathrm{d}x &\leqslant O(1)\int_0^1\left|\left(\frac{(j_0+\eta)^2}{\rho_0+\psi}-\frac{j_0^2}{\rho_0}+p(\rho_0+\psi)-p(\rho_0)\right)_{xt}\right|^2\mathrm{d}x \\
&\qquad + O(1)\int_0^1(\eta_t^2+\psi_t^2+e_{xt}^2)\,\mathrm{d}x \\
&\leqslant O(1)\int_0^1(\psi_{xt}^2+\psi_{tt}^2)\,\mathrm{d}x+O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0 t\}+\exp\{-\beta_1 t\}).
\end{aligned} \tag{3.59}$$

Substituting (3.59) into (3.58) leads to

$$\begin{aligned} K_3 \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\bigg[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t - \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - p''(\rho_0+\psi)\psi_t\bigg]\,\mathrm{d}x\bigg) \\ + O(1)(\|(\psi_0,\eta_0)\|_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x \\ + O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\}). \end{aligned} \tag{3.60}$$

Combining (3.56), (3.57), and (3.60) with (3.55), we have

$$\begin{aligned} I_{11} \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \frac{1}{2}\bigg(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \\ - \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \frac{1}{2}\bigg(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \\ + \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\bigg[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t - \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - p''(\rho_0+\psi)\psi_t\bigg]\,\mathrm{d}x\bigg) \\ + O(1)(\|(\psi_0,\eta_0)\|_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x \\ + O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\}). \end{aligned} \tag{3.61}$$

Then it follows from (3.53), (3.54), (3.32) and (3.61) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \tfrac{1}{2}\psi_{tt}^2 + \tfrac{1}{2}(\rho_0+\phi_{0xx}+2\psi)\psi_t^2 + \frac{1}{2}\bigg(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \\ - \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \tfrac{1}{2}\bigg(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \\ + \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \psi_{xt}(\rho_0+\psi)_x\bigg[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t - \frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t - p''(\rho_0+\psi)\psi_t\bigg]\,\mathrm{d}x\bigg) \\ + \int_0^1 \psi_{tt}^2\,\mathrm{d}x \\ \leqslant O(1)(\|(\psi_0,\eta_0)\|_2 + N(T) + \delta_0)\int_0^1 (\psi_{tt}^2 + \psi_{xt}^2)\,\mathrm{d}x \\ + O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0 t\} + \exp\{-\beta_1 t\}). \end{aligned} \tag{3.62}$$

By $[(3.52) + 2(1+2a)\times(3.62) + (3.16) + (3.32)]$, we have, due to (3.32),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 \tfrac{1}{2}\psi_t^2 + \psi_t\psi_{tt} + (1+2a)\psi_{tt}^2\,\mathrm{d}x\bigg) \\ + \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 (1+2a)(\rho_0+\phi_{0xx}+2\psi)\psi_t^2 + (1+2a)\bigg(p'(\rho_0) - \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \\ - \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 (1+2a)\bigg(p'(\rho_0+\psi) - p'(\rho_0) - \frac{(j_0+\eta)^2}{(\rho_0+\psi)^2} + \frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x\bigg) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^1 2(1+2a)\psi_{xt}(\rho_0+\psi)_x \\
&\qquad\qquad \times\bigg[\frac{2(j_0+\eta)}{(\rho_0+\psi)^2}\eta_t-\frac{2(j_0+\eta)^2}{(\rho_0+\psi)^3}\psi_t-p''(\rho_0+\psi)\psi_t\bigg]\,\mathrm{d}x\bigg) \\
&\quad+(1+2a)\int_0^1\psi_{tt}^2\,\mathrm{d}x+\frac{1}{2}\int_0^1\bigg(p'(\rho_0)-\frac{j_0^2}{\rho_0^2}\bigg)\psi_{xt}^2\,\mathrm{d}x+\int_0^1(\psi_t^2+\eta_t^2)\,\mathrm{d}x \\
&\qquad\qquad\leqslant O(1)(\|(\psi_0,\eta_0)\|_2+N(T)+\delta_0)\int_0^1(\psi_{tt}^2+\psi_{xt}^2)\,\mathrm{d}x \\
&\qquad\qquad\quad+O(1)\|(\psi_0,\eta_0)\|_2^2(\exp\{-c_0t\}+\exp\{-\beta_1t\}). \qquad (3.63)
\end{aligned}
$$

By (3.34), we have

$$
\int_0^1\psi_{xx}^2\,\mathrm{d}x\leqslant O(1)\int_0^1(\psi_{tt}^2+\psi_t^2+\psi_x^2+\psi_{xt}^2+e_x^2+\eta^2)\,\mathrm{d}x. \qquad (3.64)
$$

Then, integration of (3.63) over $(0,t)$, in terms of (3.16), (3.32), similar inequalities to (3.45), and the Sobolev embedding theorem, lead to (3.46).

The estimate (3.47) follows from (3.46) and (3.13). □

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References

1 U. Ascher, P. A. Markowich and C. Schmeiser. A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model. *Math. Models Meth. Appl. Sci.* **1** (1991), 347–376.

2 K. Bløtekjær. Transport equations for electrons in two-valley semiconductors. *IEEE Trans. Electron Devices* **17** (1970), 38–47.

3 G. Chen and D. Wang. Convergence of shock schemes for the compressible Euler–Poisson equations. *Commun. Math. Phys.* **179** (1996), 333–364.

4 G. Chen, J. Jerome and B. Zhang. Particle hydrodynamic moment models in biology and microelectronics: singular relaxation limits. *Nonlin. Analysis* **30** (1997), 233–244.

5 P. Degond and P. A. Markowich. On a one-dimensional steady-state hydrodynamic model. *Appl. Math. Lett.* **3** (1990), 25–29.

6 P. Degond and P. A. Markowich. A steady-state potential flow model for semiconductors. *Ann. Math. Pure Appl.* **IV** (1993), 87–98.

7 W. Fang and K. Ito. Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors. *J. Diff. Eqns* **133** (1997), 224–244.

8 I. Gamba. Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor *Commun. PDEs* **17** (1992), 553–577.

9 I. Gamba and C. S. Morawetz. A viscous approximation for a two-dimensional steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow. *Commun. Pure Appl. Math.* **49** (1996), 999–1049.

10 I. Gasser and R. Natalini. The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors. *Q. Appl. Math.* **57** (1996), 269–282.

11 H. Hattori. Stability and instability of steady-state solutions for a hydrodynamic model of semiconductors. *Proc. R. Soc. Edinb.* A **127** (1997), 781–796.

12 H. Hattori and C. Zhu. Asymptotic behavior of the solutions to a non-isentropic hydrodynamic model of semiconductors. *J. Diff. Eqns* **144** (1998), 353–389.

13 L. Hsiao and H. L. Li. Shock reflection for the damped p-system. *Q. Appl. Math.* (In the press.)

14 L. Hsiao and T. Yang. Asymptotic of initial boundary value problems for hydrodynamic and drift diffusion models for semiconductors. *J. Diff. Eqns* **170** (2001), 472–493.

15 L. Hsiao and K. Zhang. The relaxation of the hydrodynamic model for semiconductors to drift diffusion equations. *J. Diff. Eqns* **165** (2000), 315–354.

16 L. Hsiao and K. Zhang. The global weak solution and relaxation limits of the initial boundary value problem to the bipolar hydrodynamic model for semiconductors. *Math. Models Meth. Appl. Sci.* **10** (2000), 1333–1361.

17 J. Jerome. *Analysis of charge transport: a mathematical study of semiconductor devices* (Springer, 1996).

18 J. Jerome and C. Shu. Energy transport systems for semiconductors: analysis and simulation. In *World Congress of Nonlinear Analysis '92, Tampa, FL*, vols I–IV, pp. 3835–3846 (Berlin: de Gruyter, 1996).

19 T. Luo, R. Natalini and Z. Xin. Large time behavior of the solutions to a hydrodynamic model for semiconductors. *SIAM J. Appl. Math.* **59** (1998), 810–830.

20 P. Marcati and M. Mei. Asymptotic convergence to steady-state solutions of the initial boundary value problem to a hydrodynamic model for semiconductors. *Q. Appl. Math.* **58** (2000), 763–784.

21 P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem. *Proc. R. Soc. Edinb.* A **125** (1995), 115–131.

22 P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation. *Arch. Ration. Mech. Analysis* **129** (1995), 129–145.

23 P. A. Markowich. *The steady-state semiconductor device equations* (Springer, 1986).

24 P. A. Markowich. On steady-state Euler–Poisson model for semiconductors. *Z. Angew. Math. Phys.* **62** (1991), 389–407.

25 P. A. Markowich and P. Pietra. A non-isentropic Euler–Poisson model for a collisionless plasma. *Math. Meth. Appl. Sci.* **16** (1993), 409–442.

26 P. A. Markowich and C. Schmeiser. The drift-diffusion limit for electron–phonon interaction in semiconductors. *Math. Models Meth. Appl. Sci.* **7** (1997), 707–729.

27 P. A. Markowich, C. Ringhofer and C. Schmeiser. *Semiconductor equations* (Springer, 1989).

28 F. Poupaud. On a system of nonlinear Boltzmann equations of semiconductor physics. *SIAM J. Appl. Math.* **50** (1990), 1593–1606.

29 F. Poupaud. Diffusion approximation of the linear semiconductor Boltzmann equation: analysis of boundary layer. *Asymp. Analysis* **4** (1991), 293–317.

30 F. Poupaud, M. Rascle and J.-P. Vila. Global solutions to the isothermal Euler–Poisson system with arbitrarily large data. *J. Diff. Eqns* **123** (1995), 93–121.

31 S. Selberherr. *Analysis and simulation of semiconductor device equations* (Springer, 1984).

32 L. Yeh. Subsonic solutions of hydrodynamic model for semiconductors. *Math. Meth. Appl. Sci.* **20** (1997), 1389–1410.

33 B. Zhang. Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices. *Commun. Math. Phys.* **157** (1993), 1–22.

34 B. Zhang. On a local existence theorem for a simplified one-dimensional hydrodynamic model for semiconductor devices. *SIAM J. Math. Analysis* **25** (1994), 941–947.

35 K. Zhang. Global weak solutions of the Cauchy problem to a hydrodynamic model for semiconductors. *J. PDEs* **12** (1999) 369–383.

36 K. Zhang. Zero relaxation limit of global weak solutions of the Cauchy problem to a hydrodynamic model for semiconductors. *J. Diff. Eqns* **171** (2001), 251–293.

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