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Asymptotic behavior of solutions to bipolar Euler–Poisson equations with time-dependent damping

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ABSTRACT

In this paper, we study the one-dimensional Euler–Poisson equations of bipolar hydrodynamic model for semiconductor device with time-dependent damping effect $-\frac{J}{(1+t)^{\lambda}}$ for $-1 < \lambda < 1$, where the damping effect is time-gradually-degenerate for $\lambda > 0$, and time-gradually-enhancing for $\lambda < 0$. Such a damping effect makes the hydrodynamic system possess the nonlinear diffusion phenomena time-asymptotic-weakly or strongly. Based on technical observation, and by using the time-weighted energy method, where the weights are artfully chosen, we prove that the system admits a unique global smooth solution, which time-asymptotically converges to the corresponding diffusion wave, when the initial perturbation around the diffusion wave is small enough. The convergence rates are specified in the algebraic forms $O(t^{-\frac{3}{4}(1+\lambda)})$ and $O(t^{-(1-\lambda)})$ according to different values of λ in $(-1, \frac{1}{7})$ and $(\frac{1}{7}, 1)$, respectively, where $\lambda = \frac{1}{7}$ is the critical point, and the convergence rate at the critical point is $O(t^{-\frac{6}{7}} \ln t)$. All these convergence rates obtained in different cases are optimal in the sense when the initial perturbations are L^2-integrable. Particularly, when $\lambda = \frac{1}{7}$, the convergence rate is the fastest, namely, the asymptotic profile of the original system at the critical point is the best.

1. Introduction

Models and mathematical equations. Proposed first by Baccarani and Wordeman [1] in 1985, the bipolar hydrodynamic models, generally used in description of the charged fluid particles such as electrons and holes in bipolar semiconductor devices [18] or positively and negatively charged ions in a plasma [30], are the Euler–Poisson equations with damping as follows:

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$$\begin{cases} n_{1t} + J_{1x} = 0, \\ J_{1t} + \left(\dfrac{J_1^2}{n_1} + p(n_1) \right)_x = n_1 E - \dfrac{J_1}{\tau}, \\ n_{2t} + J_{2x} = 0, \\ J_{2t} + \left(\dfrac{J_2^2}{n_2} + p(n_2) \right)_x = -n_2 E - \dfrac{J_2}{\tau}, \\ E_x = n_1 - n_2 - D(x). \end{cases} \tag{1.1}$$

Here $n_1(x,t)$ and $n_2(x,t)$ represent the densities of electrons and holes for bipolar semiconductor devices, $J_1(x,t)$ and $J_2(x,t)$ denote the current densities for electrons and holes, respectively. $E(x,t)$ is the electric field. $D(x) > 0$ is the doping profile standing for the density of impurities in the semiconductor device. $p(s)$ is the pressure function for both of electrons and holes. $\tau > 0$ is the relaxation-time.

In the system (1.1), the terms $-\frac{J_i}{\tau}$ for $i = 1,2$ are called the damping effects, which play the key role for the regularity of the solutions as well as the time-asymptotic behavior. In fact, if we differentiate $(1.1)_2$ and $(1.1)_4$ with respect to x, and notice $J_{ix} = -n_{it}$ $(i = 1,2)$ from $(1.1)_1$ and $(1.1)_3$, respectively, then we formally have the following damped wave equations:

$$n_{1tt} + \frac{1}{\tau} n_{1t} - \left(\frac{J_1^2}{n_1} + p(n_1) \right)_{xx} = -(n_1 E)_x,$$

$$n_{2tt} + \frac{1}{\tau} n_{2t} - \left(\frac{J_2^2}{n_2} + p(n_2) \right)_{xx} = (n_2 E)_x.$$

When $\tau \to \infty$, the damping effects to the system (1.1) (see also the above damped wave equations) will be vanishing, and the system becomes the pure Euler–Poisson system. In this case, the solutions will loss their regularity as we know. While, when $\tau \to 0^+$, the damping effects to the system will extremely enlarge, and the solutions are expected to converge to their profiles much faster. It is very interesting but also challenging to study the asymptotic behavior of the solutions as $\tau \to \infty$ and $\tau \to 0^+$ respectively. In order to see how the solutions change time-asymptotically as $\tau \to \infty$ and $\tau \to 0^+$, respectively, let us mathematically take $\tau = (1+t)^\lambda$ for some constant λ, where, when $t \to \infty$, then $\tau = (1+t)^\lambda \to \infty$ for $\lambda > 0$ and $\tau = (1+t)^\lambda \to 0$ for $\lambda < 0$. Thus, the damping effects in Euler–Poisson system (1.1) become time-gradually-degenerate for $\lambda > 0$ and time-gradually-enhancing for $\lambda < 0$. To understand the structure of solutions in these cases is important and difficult from the mathematical point of view, and it is also the first attempt to the Euler–Poisson system for bipolar semiconductors as we know.

Therefore, in this paper, we are mainly interested in the following one dimensional Euler–Poisson system with time-dependent damping

$$\begin{cases} n_{1t} + J_{1x} = 0, \\ J_{1t} + \left(\dfrac{J_1^2}{n_1} + p(n_1) \right)_x = n_1 E - \dfrac{J_1}{(1+t)^\lambda}, \\ n_{2t} + J_{2x} = 0, \\ J_{2t} + \left(\dfrac{J_2^2}{n_2} + p(n_2) \right)_x = -n_2 E - \dfrac{J_2}{(1+t)^\lambda}, \\ E_x = n_1 - n_2, \end{cases} \tag{1.2}$$

subjected to the initial value conditions

$$(n_1, n_2, J_1, J_2)\big|_{t=0} = (n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_{1\pm}, J_{2\pm}) \quad \text{as } x \to \pm\infty, \tag{1.3}$$

where $n_\pm > 0$, $J_{1\pm}$ and $J_{2\pm}$ are constants. Throughout the paper, we assume that

$$p(s) \geq 0,\ p'(s) > 0 \text{ for } s > 0,\ -1 < \lambda < 1,\ D(x) \equiv 0. \tag{1.4}$$

The condition $p'(s) > 0$ is physical, and the example is $p(s) = Ts^\gamma$ for $\gamma \geq 1$, where $\gamma = 1$ is the isothermal case, and $\gamma > 1$ is the isentropic case. The doping profile $D(x) \equiv 0$ is not a physical case but an important example in mathematical point of view, and this case has been intensively studied in [5,7,9,10]. While, when $D(x) > 0$, the expected asymptotic profiles (stationary waves) for the system will be totally different from what we study here. This will be another issue for us to investigate in future.

Expected asymptotic profiles. The target in this paper is to study the large time behaviors of smooth solutions to (1.2)–(1.3). Notice that, except the hyperbolic properties for the solutions of the hydrodynamical system (1.2), the damping effects make that the hydrodynamical system possesses diffusion phenomena like nonlinear diffusion equations, so we are mainly concerned with the nonlinear diffusive phenomena of the model, in the sense that both the densities of electrons and holes tend time-asymptotically to the same shifted nonlinear diffusion waves.

In order to see what will be the properly asymptotic profile for (1.2) for $-1 < \lambda < 1$, let us take the following variable scalings for arbitrary small $\varepsilon > 0$,

$$t \to (1+\bar{t})/\varepsilon^{\theta_1}, \quad x \to \bar{x}/\varepsilon^{\theta_2}, \quad n_i \to \bar{n}_i, \quad J_i \to \varepsilon^{\theta_3}\bar{J}_i, \quad E \to \bar{E}(=0), \quad i = 1, 2, \tag{1.5}$$

where θ_1, θ_2, θ_3 are positive constants and will be determined later. We first have from $(1.2)_1$ and $(1.2)_3$ that

$$\varepsilon^{\theta_1}\bar{n}_{i\bar{t}} + \varepsilon^{\theta_2+\theta_3}\bar{J}_{i\bar{x}} = 0, \quad i = 1, 2.$$

In order to balance it, let

$$\theta_1 = \theta_2 + \theta_3, \tag{1.6}$$

then

$$\bar{n}_{i\bar{t}} + \bar{J}_{i\bar{x}} = 0, \quad i = 1, 2. \tag{1.7}$$

On the other hand, it follows from $(1.2)_2$ and $(1.2)_4$ that

$$\varepsilon^{\theta_1-\theta_2+\theta_3}\bar{J}_{i\bar{t}} + \varepsilon^{2\theta_3}\left(\frac{\bar{J}_i^2}{\bar{n}_i}\right)_{\bar{x}} + p(\bar{n}_i)_{\bar{x}} = -\frac{\varepsilon^{\theta_1\lambda-\theta_2+\theta_3}\bar{J}_i}{(\varepsilon^{\theta_1}+1+\bar{t})^\lambda}, \quad i = 1, 2. \tag{1.8}$$

Noting (1.6) also gives $\theta_1 - \theta_2 + \theta_3 = 2\theta_3$, and further

$$\theta_1\lambda - \theta_2 + \theta_3 = 2\theta_3 + (\lambda - 1)\theta_1 < 2\theta_3 \text{ for } -1 < \lambda < 1,$$

one can see that the leading order equation of (1.8) is

$$p(\bar{n}_i)_{\bar{x}} = -\frac{\varepsilon^{\theta_1\lambda-\theta_2+\theta_3}\bar{J}_i}{(1+\bar{t})^\lambda}, \quad i = 1, 2.$$

By setting

$$\theta_1\lambda - \theta_2 + \theta_3 = 0 \text{ for } -1 < \lambda < 1, \tag{1.9}$$

we have

$$p(\bar{n}_i)_{\bar{x}} = -\frac{\bar{J}_i}{(1+\bar{t})^{\lambda}}, \quad i = 1, 2. \tag{1.10}$$

So, the asymptotic profiles of (n_1, n_2) for (1.2) are expected to be the solutions of the coupled system of equations

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ \bar{J} := \bar{n}\bar{u} = -(1+t)^{\lambda} p(\bar{n})_x, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad -1 < \lambda < 1, \tag{1.11}$$

with

$$(\bar{n}, \bar{J})(x,t) \to (n_{\pm}, 0) \quad \text{as } x \to \pm\infty. \tag{1.12}$$

Substituting $(1.11)_2$ into $(1.11)_1$, we get the following nonlinear diffusion equation with time-dependent diffusion coefficient

$$\bar{n}_t = (1+t)^{\lambda} p(\bar{n})_{xx} \quad \text{with } \bar{n} \to n_{\pm} \text{ as } x \to \pm\infty.$$

Such a nonlinear diffusion equation with $-1 < \lambda < 1$ possesses the self-similar solutions in the form of

$$(\bar{n}, \bar{u})(x,t) = (\bar{n}, \bar{u}) \left(\frac{x}{\sqrt{(1+t)^{1+\lambda}}} \right),$$

which are usually called *nonlinear diffusion waves*.

Under this scaling argument, one may observe that $\lambda = \pm 1$ are the critical cases in which system (1.2) may exhibit different mathematical structures, and hence the asymptotic behaviors of the system may be different from what we study here. We will settle this project in a subsequent paper.

Background of relevant research. Regarding the diffusion phenomena, let us first draw a background picture. For the scalar damped wave equation

$$u_{tt} + u_t - \Delta u = F(u, u_x),$$

the pioneering work on the optimal decay rates for the solutions affected by the damping was given by Matsumura [19] in 1977. This result has been then extensively generalized in many different cases. In 2004, Wirth [35] proposed the wave equation with time-gradually-degenerate damping

$$u_{tt} + (1+t)^{-\lambda} u_t - \Delta u = 0, \quad \lambda > 0,$$

and significantly studied the $L^p - L^q$ decay rates of the solutions in [36,37]. See also the recent development in more general cases [26,33].

For the p-system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{1.13}$$

where $\alpha > 0$ is a constant, Hsiao and Liu [6] first proved that the solutions (v, u) of damped p-system will converge to the diffusion waves $(\bar{v}, \bar{u})$ of the system

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}, \end{cases}$$

in the sense $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$. Then Nishihara [23] improved the convergence rates as $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}})$ for the initial perturbations in L^2. Subsequently, by constructing an approximate Green function, Wang and Yang [34] further improved the rates as $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-\frac{3}{2}})$ for the initial perturbations in L^1. Furthermore, Mei [22] observed that, the best asymptotic profile is the solution for the corresponding nonlinear diffusion equation with some particularly selected initial data, and the convergence to the best asymptotic profile is in the form of $\|(v - \bar{v}, u - \bar{u})(t)\|_{L^\infty} = O(1)\big((1+t)^{-\frac{3}{2}} \ln(2+t), (1+t)^{-2} \ln(2+t)\big)$, when the initial perturbations are in L^1. For other results related to (1.13) with nonlinear damping or vacuum, we refer to the interesting works [12,8,21,16,17,24,25,27]. See also the boundary cases studied in [3,4,14–17].

When the damping is time-gradually-degenerate, Pan [28,29] considered the following 1-d isentropic Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\dfrac{\mu}{(1+t)^\lambda} \rho u, \\ \rho|_{t=0} = 1 + \varepsilon \rho_0(x), \quad u|_{t=0} = \varepsilon u_0(x), \end{cases} \tag{1.14}$$

where $\rho_0(x)$, $u_0(x) \in C_0^\infty(\mathbb{R})$, $|x| \le R$ and $\varepsilon > 0$ is a sufficiently small constant, and proved that $\lambda = 1$, $\mu = 2$ is the critical threshold of (1.14) to separate the global existence and finite-time blow up of solutions. More precisely, when $\lambda = 1$, $\mu > 2$ or $0 \le \lambda < 1$, $\mu > 0$, (1.14) has global smooth solutions; while, when $\lambda = 1$, $0 \le \mu \le 2$ or $\lambda > 1$, $\mu \ge 0$, the C^1 solutions of (1.14) will blow up in finite time.

For the p-system with time-gradually-degenerate damping,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\dfrac{\alpha}{(1+t)^\lambda} u, \end{cases} \tag{1.15}$$

very recently, Li–Li–Mei–Zhang (2017) [13] and Cui–Yin–Zhang–Zhu (2018) [2] independently studied the same problem for the time-algebraical convergence to diffusion waves at almost the same time, where the convergence rates obtained in [2] are better than that in [13]. When $\lambda \ge 1$, the solutions of the corresponding initial–boundary-value problem are proved to be bounded, but their derivatives will blow up in finite time [31].

For bipolar Euler–Poisson equations with regular damping (i.e. $\lambda = 0$), Gasser, Hsiao and Li [5] first studied the nonlinear diffusive phenomena, and showed that, the smooth solutions of (1.2)–(1.3) tend to the diffusion waves with algebraic convergence rate, under the assumptions of

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)] dx = 0 \text{ or equivalently, } E(+\infty, t) - E(-\infty, t) = 0.$$

In physics, this condition represents the switch-off situation of semiconductor device. Later, Huang and Li [7] generalize the results of [5] to the weak entropy solutions on the base of compensated compactness.

The more interesting and challenging case of switch-on semiconductor device has been recently studied by Huang, Mei and Wang [9]. Here the switch-on case means that

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]dx \neq 0 \quad (\text{or} \quad J_{i+} \neq J_{i-} \text{ for } i = 1, 2).$$

By artfully constructing correction functions, the authors prove that in this case, the smooth solutions of (1.2)–(1.3) with $\lambda = 0$ also converge to the diffusion waves with optimal algebraic convergence rate.

Our main contributions and developed techniques. When the damping is time-dependent for Euler–Poisson system, the story is totally different. To show the specific diffusion phenomena of the solutions for (1.2)–(1.3) will be our mission in the present paper. By the time-weighted energy method with artful selection of time-weight-functions, we shall prove the convergence of the original solutions $(n_i, J_i)(x, t)$ to the corresponding diffusion waves $(\bar{n}_i, \bar{J}_i)\big(\frac{x}{\sqrt{(1+t)^{1+\lambda}}}\big)$ for $i = 1, 2$ as follows:

$$\|(n_i - \bar{n})(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} O(1)(1+t)^{-\frac{3}{4}(1+\lambda)}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{6}{7}} \ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{\lambda - 1}, & \frac{1}{7} < \lambda < 1, \end{cases}$$

and

$$\|(J_i - \bar{J})(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} O(1)(1+t)^{-\frac{\lambda+5}{4}}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{9}{7}} \ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{-\frac{3(1-\lambda)}{2}}, & \frac{1}{7} < \lambda < 1. \end{cases}$$

Here, all convergence rates are optimal in the sense when the initial perturbations are in L^2. See Remark 2.1 below for details. Particularly, $\lambda = \frac{1}{7}$ is the critical point for the convergence involving $\ln(2+t)$, which is the first result obtained for the studies related to time-dependent damping phenomena. And to our knowledge, it is also noticed that the case for time-gradually-enhancing damping (i.e. $\lambda < 0$) is first proposed in this paper.

Through the analysis of decay rate of solutions to the linearized equations around the diffusion waves (for the details, see Remark 2.1 below), one can observe that it is necessary to decompose the range of λ into three parts, $-1 < \lambda < \frac{1}{7}$, $\lambda = \frac{1}{7}$ and $\frac{1}{7} < \lambda < 1$, to obtain the optimal convergence rate. Similar decomposition was also used by Cui et al. [2]. However the convergence rate at the critical point $\lambda = \frac{1}{7}$ was not clarified in [2], because the energy estimates in that paper relied on Gronwall's inequalities on two regions, which may hide some structures of the equations especially in the case of $\lambda = \frac{1}{7}$. In this paper, we employ a different strategy to derive the optimal convergence rates for all $\lambda \in (-1, 1)$. Actually, we take the time-weight-functions $(\beta + t)^\alpha$ and then $\gamma(\beta + t)^{\alpha - 1}$ to establish the essential L^2 estimate, where α, β and γ are specific parameters to be technically determined according to different λ. There are two advantages of this approach: (1) One only needs to simply chose β suitably large rather than establishing the finite time estimates as that in Proposition 3.1 of [2] to close the energy estimates; (2) It avoids the use of Gronwall's inequality and hence captures the full dissipative structures of the equations.

The rest of this paper is organized as follows. In Section 2, we shall recall some fundamental properties of the nonlinear diffusion waves, and present the estimates for correction functions. After such settings, we state our main results. Section 3 is devoted to the proofs of our main theorems. We leave the details of construction of correction functions in the Appendix.

2. Preliminaries and main results

Before proceeding, we clarify some notions used throughout the paper. $C > 0$ denotes a generic constant which may change from line to line. $H^m(\mathbb{R})(m \geq 0)$ is the usual Sobolev space whose norm is abbreviated as $\|f\|_m := \sum_{k=0}^{m} \|\partial_x^k f\|$ with $\|f\| := \|f\|_{L^2(\mathbb{R})}$. For convenience, we also write $\|(f,g)\|^2 := \|f\|^2 + \|g\|^2$ and

$$\delta := |n_+ - n_-| + |J_{1+}| + |J_{1-}| + |J_{2+}| + |J_{2-}| + |E_+|. \tag{2.1}$$

2.1. Diffusion waves

We first state some fundamental properties of the nonlinear diffusion waves. The nonlinear diffusion equation (1.11) is equivalent to

$$\bar{n}_t = (1+t)^\lambda p(\bar{n})_{xx}, \quad p'(\bar{n}) > 0. \tag{2.2}$$

Thanks to the works of van Duyn and Peletier [32] (see also [6,9,13]), one can see that (2.2) has a unique (up to a shift) monotone self-similar solution called nonlinear diffusion waves in the form

$$\bar{n}(x,t) = \varphi\left(\frac{x}{\sqrt{(1+t)^{1+\lambda}}}\right) \triangleq \varphi(z), \quad z \in \mathbb{R}, \text{ with } \varphi(\pm\infty) = n_\pm. \tag{2.3}$$

As in [13], it is easy to prove that $\varphi(z)$ satisfies

$$\sum_{k=1}^{4} \left|\partial_z^k \varphi(z)\right| + \left|\varphi(z) - n_+\right|_{z>0} + \left|\varphi(z) - n_-\right|_{z<0} \leq C\left|n_+ - n_-\right| e^{-C(1+\lambda)z^2} \tag{2.4}$$

for some constant $C > 0$. A direct calculation from (2.4) yields the following estimates for the diffusion waves.

Lemma 2.1. *For the self-similar solution of* (2.2)*, it holds that*

$$\left|\partial_x^k \partial_t^l \bar{n}\right| \leq C\left|n_+ - n_-\right|(1+t)^{-\frac{k(1+\lambda)+2l}{2}}, \quad k+l \geq 1, \quad k,l \geq 0, \tag{2.5}$$

$$\int_{\mathbb{R}} \left|\partial_x^k \partial_t^l \bar{n}\right|^2 dx \leq C\left|n_+ - n_-\right|^2 (1+t)^{-\frac{(2k-1)(1+\lambda)}{2} - 2l}, \quad k+l \geq 1. \tag{2.6}$$

2.2. Correction functions

Because we are interested in the physical switch-on situation for the device, that is the initial data satisfy

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]dx \neq 0 \quad (\text{or} \quad J_{i+} \neq J_{i-} \text{ for } i = 1,2),$$

or, equivalently

$$E(+\infty,t) - E(-\infty,t) \neq 0,$$

there will be a gap between the original solutions and the diffusion waves at far fields because of the difference of voltage. In order to fill such gap, as in [9,11], we need to construct the correction functions $(\hat{n}_1, \hat{J}_1, \hat{n}_2, \hat{J}_2, \hat{E})(x,t)$.

As the analysis in the Appendix for the behaviors of the solutions to (1.2)–(1.3) at the far fields $x = \pm\infty$, for $0 \le \lambda < 1$

$$\begin{cases} |n_i(\pm\infty,t) - \bar{n}(\pm\infty,t)| = 0, & i = 1,2, \\ |J_i(+\infty,t) - \bar{J}(+\infty,t)| = O(\delta)e^{-Ct^{1-\lambda}} \neq 0, & i = 1,2, \\ |J_i(-\infty,t) - \bar{J}(-\infty,t)| = |J_{i-}|e^{-Ct^{1-\lambda}} \neq 0, & i = 1,2, \\ |E(+\infty,t) - 0| = O(\delta)e^{-Ct^{1-\lambda}} \neq 0, \\ E(-\infty,t) = E_- = 0, \end{cases} \tag{2.7}$$

and for $-1 < \lambda < 0$

$$\begin{cases} |n_i(\pm\infty,t) - \bar{n}(\pm\infty,t)| = 0, & i = 1,2, \\ |J_i(+\infty,t) - \bar{J}(+\infty,t)| = O(\delta)e^{-Ct^{1+\lambda}} \neq 0, & i = 1,2, \\ |J_i(-\infty,t) - \bar{J}(-\infty,t)| = |J_{i-}|e^{-Ct^{1+\lambda}} \neq 0, & i = 1,2, \\ |E(+\infty,t) - 0| = O(\delta)e^{-Ct^{1+\lambda}} \neq 0, \\ E(-\infty,t) = E_- = 0, \end{cases} \tag{2.8}$$

one can observe from (2.7) and (2.8) that there are some gaps between $J_i(\pm\infty,t)$ and $\bar{J}(\pm\infty,t)$. In other words,

$$J_i(x,t) - \bar{J}(x,t) \notin L^2(\mathbb{R}) \quad \text{and} \quad E(x,t) \notin L^2(\mathbb{R}).$$

To overcome such difficulty, as shown in Appendix later, we can construct some correction functions $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(x,t)$, which satisfy

$$\begin{cases} \hat{n}_{1t} + \hat{J}_{1x} = 0, \\ \hat{J}_{1t} = \check{n}\hat{E} - \dfrac{\hat{J}_1}{(1+t)^\lambda}, \\ \hat{n}_{2t} + \hat{J}_{2x} = 0, \\ \hat{J}_{2t} = -\check{n}\hat{E} - \dfrac{\hat{J}_2}{(1+t)^\lambda}, \\ \hat{E}_x = \hat{n}_1 - \hat{n}_2, \end{cases} \quad \text{with} \quad \begin{cases} \hat{J}_i(x,t) \to J_i^\pm(t) & \text{as} \quad x \to \pm\infty, \\ \hat{E}(x,t) \to 0 & \text{as} \quad x \to -\infty, \\ \hat{E}(x,t) \to E^+(t) & \text{as} \quad x \to +\infty. \end{cases} \tag{2.9}$$

Here $\check{n} = \check{n}(x)$, $\hat{J}_i(x,0)$ and $\hat{E}(x,0)$ are selected such that

$$\check{n}(x) \to n_\pm, \quad \hat{J}_i(x,0) \to J_{i\pm} \quad \text{and} \quad \hat{E}(x,0) \to E_0^\pm \quad \text{as} \quad x \to \pm\infty.$$

We summarize the estimates for $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})$ in the following lemma. The detailed construction for these correction functions can be found in the Appendix of the paper.

Lemma 2.2. *For $-1 < \lambda < 1$, there exist some correction functions $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(x,t)$ satisfying*

$$\|(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(t)\|_{L^\infty(\mathbb{R})} \le C\delta e^{-Ct^{\sigma_0}} \quad \textit{for some constant } \sigma_0 > 0, \tag{2.10}$$

$$|J_i(\pm\infty,t) - \bar{J}(\pm\infty,t) - \hat{J}_i(\pm\infty,t)| = 0, \tag{2.11}$$

$$|E(\pm\infty,t) - \hat{E}(\pm\infty,t)| = 0, \tag{2.12}$$

$$\text{supp } \hat{n}_1 \subset \text{supp } m_0, \ \text{supp } \hat{n}_2 \subset \text{supp } m_0. \tag{2.13}$$

2.3. Main theorems

Next, we consider the perturbation equations of (1.2) with respect to the diffusion waves (1.11). By (1.2), (1.11) and (2.9), we have

$$\begin{cases}(n_1-\hat{n}_1-\bar{n})_t+(J_1-\hat{J}_1-\bar{J})_x=0,\\ (J_1-\hat{J}_1-\bar{J})_t+\left(\dfrac{J_1^2}{n_1}+p(n_1)-p(\bar{n})\right)_x=n_1E-\check{n}\hat{E}-\dfrac{J_1-\hat{J}_1-\bar{J}}{(1+t)^\lambda}+f_0,\\ (n_2-\hat{n}_2-\bar{n})_t+(J_2-\hat{J}_2-\bar{J})_x=0,\\ (J_2-\hat{J}_2-\bar{J})_t+\left(\dfrac{J_2^2}{n_2}+p(n_2)-p(\bar{n})\right)_x=-n_2E+\check{n}\hat{E}-\dfrac{J_2-\hat{J}_2-\bar{J}}{(1+t)^\lambda}+f_0,\\ (E-\hat{E})_x=(n_1-\hat{n}_1-\bar{n})-(n_2-\hat{n}_2-\bar{n}).\end{cases} \tag{2.14}$$

Here $f_0:=(1+t)^\lambda p(\bar{n})_{xt}+\lambda(1+t)^{\lambda-1}p(\bar{n})_x$, and $(\bar{n},\bar{J})=(\bar{n},\bar{J})(x+x_0,t)$ are the shifted diffusion waves, where the shift x_0 can be determined as follows. Integrating $(2.14)_1$ and $(2.14)_3$ over $\mathbb{R}$ with respect to x, and noting (2.9), (1.12) and (2.11), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}[n_i(x,t)-\hat{n}_i(x,t)-\bar{n}(x+x_0,t)]dx\\ &=-[J_i(+\infty,t)-\hat{J}_i(+\infty,t)-\bar{J}(+\infty,t)]+[J_i(-\infty,t)-\hat{J}_i(-\infty,t)-\bar{J}(-\infty,t)]\\ &=0,\quad i=1,2.\end{aligned}$$

Integrating this equation with respect to t yields

$$\begin{aligned}\int_{\mathbb{R}}[n_i(x,t)-\hat{n}_i(x,t)-\bar{n}(x+x_0,t)]dx&=\int_{\mathbb{R}}[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x+x_0,0)]dx\\ &\triangleq I_i(x_0),\quad i=1,2.\end{aligned}$$

Let us determine x_0 such that $I_i(x_0)=0$ for $i=1,2$. It is easy to see that

$$I_i'(x_0)=\frac{d}{dx_0}\left(\int_{\mathbb{R}}\big[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x+x_0,0)\big]dx\right)=-(n_+-n_-),\ i=1,2.$$

It then follows that

$$I_i(x_0)=I_i(0)+\int_0^{x_0}I_i'(y)dy=I_i(0)-(n_+-n_-)x_0.$$

Recalling that $I_i(x_0)=0$, we hence have

$$x_0=\frac{1}{n_+-n_-}I_i(0)=\frac{1}{n_+-n_-}\int_{\mathbb{R}}\big[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x,0)\big]dx,\ i=1,2.$$

We claim

$$\int_{\mathbb{R}}\big[n_1(x,0)-\hat{n}_1(x,0)-\bar{n}(x,0)\big]dx=\int_{\mathbb{R}}\big[n_2(x,0)-\hat{n}_2(x,0)-\bar{n}(x,0)\big]dx, \tag{2.15}$$

and hence x_0 is well-defined. In fact, integrating (1.2)$_5$ with respect to x over $\mathbb{R}$ and taking $t=0$, we obtain

$$\int_{\mathbb{R}} \big[n_1(x,0)-n_2(x,0)\big]dx = E_+ - E_- = E_+. \tag{2.16}$$

On the other hand, integrating (4.24) in the Appendix over $\mathbb{R}$, noting that $\hat{E}(+\infty,t)=E_+$, we get

$$\int_{\mathbb{R}} \hat{n}_1(x,t)dx = \frac{1}{2}(J_{1+}-J_{1-}+J_{2+}-J_{2-})\int_t^{+\infty} e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]}ds + \frac{1}{2}E_+.$$

Similarly, integrating (4.25) in the Appendix over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} \hat{n}_2(x,t)dx = \frac{1}{2}(J_{1+}-J_{1-}+J_{2+}-J_{2-})\int_t^{+\infty} e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]}ds - \frac{1}{2}E_+.$$

Thus,

$$\int_{\mathbb{R}} \big[\hat{n}_1(x,0)-\hat{n}_2(x,0)\big]dx = \int_{\mathbb{R}} \big[\hat{n}_1(x,t)-\hat{n}_2(x,t)\big]dx = E_+,$$

which, in combination with (2.16), leads to

$$\int_{\mathbb{R}} \big[n_1(x,0)-n_2(x,0)\big]dx = \int_{\mathbb{R}} [\hat{n}_1(x,0)-\hat{n}_2(x,0)]dx.$$

Hence, (2.15) holds.

Because the shift x_0 is selected such that $I_i(x_0)=0$, we could use the approach of anti-derivative to study the long time behavior of the perturbation equation (2.14). In other words, we define

$$\begin{cases} \phi_i(x,t) := \displaystyle\int_{-\infty}^{x} \big[n_i(\xi,t)-\hat{n}_i(\xi,t)-\bar{n}(\xi+x_0,t)\big]d\xi, \\ \psi_i(x,t) := J_i(x,t)-\hat{J}_i(x,t)-\bar{J}(x+x_0,t), \quad i=1,2, \\ \mathcal{H}(x,t) := E(x,t)-\hat{E}(x,t). \end{cases} \tag{2.17}$$

Then ϕ_i, ψ_i and $\mathcal{H}$ satisfy

$$\begin{cases} \phi_{1t}+\psi_1 = 0, \\ \psi_{1t} + \left(\dfrac{(-\phi_{1t}+\hat{J}_1+\bar{J})^2}{\phi_{1x}+\hat{n}_1+\bar{n}} + p(\phi_{1x}+\hat{n}_1+\bar{n}) - p(\bar{n})\right)_x \\ \qquad = f_0 + (\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H} + (\phi_{1x}+\hat{n}_1+\bar{n}-\check{n})\hat{E} - \dfrac{\psi_1}{(1+t)^\lambda}, \\ \phi_{2t}+\psi_2 = 0, \\ \psi_{2t} + \left(\dfrac{(-\phi_{2t}+\hat{J}_2+\bar{J})^2}{\phi_{2x}+\hat{n}_2+\bar{n}} + p(\phi_{2x}+\hat{n}_2+\bar{n}) - p(\bar{n})\right)_x \\ \qquad = f_0 - (\phi_{2x}+\hat{n}_2+\bar{n})\mathcal{H} - (\phi_{2x}+\hat{n}_2+\bar{n}-\check{n})\hat{E} - \dfrac{\psi_2}{(1+t)^\lambda}, \\ \mathcal{H} = \phi_1-\phi_2, \end{cases} \tag{2.18}$$

with the initial data

$$\begin{cases} \phi_{i0}(x) := \phi_i(x,0) = \int\limits_{-\infty}^{x} \big[n_{i0}(\xi) - \hat{n}_i(\xi,0) - \bar{n}(\xi + x_0, 0)\big] d\xi, \\ \psi_{i0}(x) := \psi_i(x,0) = J_{i0}(x) - \hat{J}_i(x,0) - \bar{J}(x + x_0, 0), \quad i = 1,2, \\ \mathcal{H}_0(x) := \phi_{10}(x) - \phi_{20}(x). \end{cases} \tag{2.19}$$

We are now ready to state our main results.

Theorem 2.1 *(Case of $-1 < \lambda < \frac{1}{7}$). Let $(\phi_{10}, \phi_{20}, \psi_{10}, \psi_{20})(x) \in H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R})$, and $\Phi_0 := \|(\phi_{10}, \phi_{20})\|_3 + \|(\psi_{10}, \psi_{20})\|_2$. Then, there is a $\delta_0 > 0$ such that if $\delta + \Phi_0 \le \delta_0$, then the IVP (1.2) and (1.3) admits a unique global smooth solution, which satisfies*

$$\sum_{k=0}^{2} (1+t)^{(k+1)(1+\lambda)} \|\partial_x^k (n_1 - \bar{n}, n_2 - \bar{n})(t)\|^2 + \sum_{k=0}^{2} (1+t)^{k(1+\lambda)+2} \|\partial_x^k (J_1 - \bar{J}, J_2 - \bar{J})(t)\|^2 \le C\big(\delta + \Phi_0^2\big).$$

Theorem 2.2 *(Case of $\lambda = \frac{1}{7}$). Under the same assumptions of Theorem 2.1, there is a $\delta_0 > 0$ such that if $\delta + \Phi_0 \le \delta_0$, then the IVP (1.2) and (1.3) admits a unique global smooth solution, which satisfies*

$$\sum_{k=0}^{2} (1+t)^{\frac{8}{7}(k+1)} \ln^{-2}(2+t) \|\partial_x^k (n_1 - \bar{n}, n_2 - \bar{n})(t)\|^2 + \sum_{k=0}^{2} (1+t)^{\frac{8k}{7}+2} \ln^{-2}(2+t) \|\partial_x^k (J_1 - \bar{J}, J_2 - \bar{J})(t)\|^2 \le C\big(\delta + \Phi_0^2\big).$$

Theorem 2.3 *(Case of $\frac{1}{7} < \lambda < 1$). Under the same assumptions of Theorem 2.1, there is a $\delta_0 > 0$ such that if $\delta + \Phi_0 \le \delta_0$, then the IVP (1.2) and (1.3) admits a unique global smooth solution, which satisfies*

$$\sum_{k=0}^{2} (1+t)^{(k+1)(1+\lambda)+\frac{1}{2}-\frac{7\lambda}{2}} \|\partial_x^k (n_1 - \bar{n}, n_2 - \bar{n})(t)\|^2 + \sum_{k=0}^{2} (1+t)^{k(1+\lambda)+\frac{5}{2}-\frac{7\lambda}{2}} \|\partial_x^k (J_1 - \bar{J}, J_2 - \bar{J})(t)\|^2 \le C\big(\delta + \Phi_0^2\big).$$

Owing to the cancellation structure of bipolar model, we show that the difference of the densities of electrons and holes decays faster.

Theorem 2.4. *If $\delta + \Phi_0 \ll 1$ and $-1 < \lambda < 1$, then it holds that*

$$\|(n_1 - n_2)(t)\|_1^2 + \|(J_1 - J_2)(t)\|_1^2 + \|(E - \hat{E})(t)\|_2^2 \le C(\delta^2 + \Phi_0^2) e^{-Ct^\sigma},$$

for some constant $\sigma > 0$.

Using the Sobolev inequality

$$\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}, \tag{2.20}$$

one can further derive the following estimates.

Corollary 2.1 *(Convergence to diffusion waves). Under the assumptions of Theorem 2.1, one has for $i = 1, 2$,*

$$\|(n_i - \bar{n})(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} O(1)(1+t)^{-\frac{3}{4}(1+\lambda)}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{6}{7}}\ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{\lambda-1}, & \frac{1}{7} < \lambda < 1, \end{cases} \tag{2.21}$$

and

$$\|(J_i - \bar{J})(t)\|_{L^\infty(\mathbb{R})} \leq \begin{cases} O(1)(1+t)^{-\frac{\lambda+5}{4}}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{9}{7}}\ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{-\frac{3(1-\lambda)}{2}}, & \frac{1}{7} < \lambda < 1. \end{cases} \tag{2.22}$$

Remark 2.1. We now show that the convergence rates presented in (2.21) and (2.22) are optimal in the sense of L^2 initial perturbations, and explain why there appears $\ln(2+t)$ in the critical case $\lambda = \frac{1}{7}$. In view of (3.1), the main working equations with source terms involving diffusion waves $\bar{n}$, let us consider the following linear nonhomogeneous heat equation

$$\begin{cases} \dfrac{u_t}{(1+t)^\lambda} - u_{xx} = -F, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.23}$$

where $F = (1+t)^\lambda p(\bar{n})_{xt} + \lambda(1+t)^{\lambda-1}p(\bar{n})_x - \left(\frac{\bar{J}^2}{\bar{n}}\right)_x$. Multiplying (2.23) by $(1+t)^\lambda$,

$$\begin{cases} u_t - (1+t)^\lambda u_{xx} = -(1+t)^\lambda F, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.24}$$

By the principle of superposition, $u = v + w$, where v and w satisfy

$$\begin{cases} v_t - (1+t)^\lambda v_{xx} = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ v(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.25}$$

and

$$\begin{cases} w_t - (1+t)^\lambda w_{xx} = -(1+t)^\lambda F(x,t), & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ w(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{2.26}$$

A direct calculation by Fourier transformation for (2.25) yields

$$v(x,t) = \frac{1}{2\sqrt{\pi A(t)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4A(t)}} u_0(y)dy, \tag{2.27}$$

where $A(t) := \frac{(1+t)^{1+\lambda}-1}{1+\lambda}$. It is then easy to see that

$$
\begin{aligned}
|v_x(x,t)| &= \frac{1}{2\sqrt{\pi A}}\left|\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{4A}}\left(-\frac{x-y}{2A}\right)u_0(y)dy\right| \\
&= \frac{O(1)}{A^{3/2}}\left(\int\limits_{\mathbb{R}} e^{-\frac{y^2}{2A}}y^2dy\right)^{1/2}\|u_0\|_{L^2} \\
&= O(1)A^{-3/4}\|u_0\|_{L^2} \\
&= O(1)(1+t)^{-\frac{3(1+\lambda)}{4}}.
\end{aligned}
$$

Here we have used the fact that $A(t) \sim \frac{(1+t)^{1+\lambda}}{1+\lambda}$ as $t \to \infty$. We next estimate $|w_x(x,t)|$. Let $\bar{w}(s;x,t)$ be the solution of

$$
\begin{cases}
\bar{w}_t - (1+t+s)^\lambda \bar{w}_{xx} = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}_+, \\
\bar{w}(s;x,0) = -(1+s)^\lambda F(x,s), & x \in \mathbb{R}.
\end{cases}
$$

It is easy to see that $w(x,t) = \int_0^t \bar{w}(s;x,t-s)ds$. Similar to (2.27), we have

$$
\bar{w}(s;x,t) = \frac{1}{2\sqrt{\pi B(s,t)}}\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{4B(s,t)}}[-(1+s)^\lambda F(y,s)]dy,
$$

where $B(s,t) := \frac{(1+t+s)^{1+\lambda}-(1+s)^{1+\lambda}}{1+\lambda}$. Then,

$$
w(x,t) = \int\limits_0^t \frac{1}{2\sqrt{\pi B(s,t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{4B(s,t-s)}}(1+s)^\lambda(-F(y,s))dyds.
$$

Thus, by (2.6)

$$
\begin{aligned}
|w_x| &= \left|\int\limits_0^t \frac{1}{2\sqrt{\pi B(s,t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{4B(s,t-s)}}\frac{-(x-y)}{2B(s,t-s)}(1+s)^\lambda F(y,s)dyds\right| \\
&\le \int\limits_0^t \frac{(1+s)^\lambda}{4\sqrt{\pi}B^{3/2}(s,t-s)}\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{4B(s,t-s)}}|x-y||F|dyds \\
&\le O(1)\int\limits_0^t \frac{(1+s)^\lambda}{B^{3/2}(s,t-s)}\left(\int\limits_{\mathbb{R}} e^{-\frac{(x-y)^2}{2B(s,t-s)}}|x-y|^2\right)^{1/2}\left(\int\limits_{\mathbb{R}} F^2\right)^{1/2} ds \\
&\le O(1)\int\limits_0^t B^{-3/4}(s,t-s)(1+s)^{\frac{7\lambda-5}{4}}ds \\
&= O(1)\int\limits_0^{\frac{t}{2}} B^{-3/4}(s,t-s)(1+s)^{\frac{7\lambda-5}{4}}ds + O(1)\int\limits_{\frac{t}{2}}^t B^{-3/4}(s,t-s)(1+s)^{\frac{7\lambda-5}{4}}ds \\
&\triangleq I + II.
\end{aligned}
$$

When $s \in (0, \frac{t}{2})$, $B(s,t-s) = \frac{(1+t)^{1+\lambda}-(1+s)^{1+\lambda}}{1+\lambda} \sim t^{1+\lambda}$ as $t \to \infty$, hence

$$I = O(1)t^{-\frac{3}{4}(1+\lambda)} \int_0^{\frac{t}{2}} (1+s)^{\frac{7\lambda-5}{4}} ds = \begin{cases} O(1)(1+t)^{-\frac{3}{4}(1+\lambda)}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{6}{7}} \ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{\lambda-1}, & \frac{1}{7} < \lambda < 1. \end{cases}$$

When $s \in (\frac{t}{2}, t)$, $B(s,t-s) \sim (1+t)^{1+\lambda} - (1+s)^{1+\lambda}$ as $t \to \infty$. Then by changing $\bar{s} := (1+t)^{1+\lambda} - (1+s)^{1+\lambda}$, a simple calculation gives

$$\begin{aligned} II &= O(1)(1+t)^{\frac{7\lambda-5}{4}} \int_0^{(1+t)^{1+\lambda}-(1+\frac{t}{2})^{1+\lambda}} \bar{s}^{-\frac{3}{4}} \left[(1+t)^{1+\lambda} - \bar{s}\right]^{\frac{1}{1+\lambda}-1} d\bar{s} \\ &= O(1)(1+t)^{\frac{7\lambda-5}{4}} (1+t)^{(1+\lambda)(\frac{1}{1+\lambda}-1)} \int_0^{(1+t)^{1+\lambda}-(1+\frac{t}{2})^{1+\lambda}} \bar{s}^{-\frac{3}{4}} d\bar{s} \\ &= O(1)(1+t)^{\lambda-1}. \end{aligned}$$

Noting that $\lambda - 1 < -\frac{3}{4}(1+\lambda)$ for $\lambda < \frac{1}{7}$ and $\lambda - 1 > -\frac{3}{4}(1+\lambda)$ for $\lambda > \frac{1}{7}$, we obtain

$$|u_x| \le |v_x| + |w_x| = \begin{cases} O(1)(1+t)^{-\frac{3}{4}(1+\lambda)}, & -1 < \lambda < \frac{1}{7}, \\ O(1)(1+t)^{-\frac{6}{7}} \ln(2+t), & \lambda = \frac{1}{7}, \\ O(1)(1+t)^{\lambda-1}, & \frac{1}{7} < \lambda < 1. \end{cases}$$

This indicates that the convergence rates obtained in (2.21) and (2.22) are optimal.

Remark 2.2. The argument above motivates us to divide the range of λ into three parts, $-1 < \lambda < \frac{1}{7}$, $\lambda = \frac{1}{7}$ and $\frac{1}{7} < \lambda < 1$. This decomposition was also adopted in [2], where the diffusive phenomenon of damped p-system was studied for $\lambda \in (0,1)$. However, in this paper, by artful construction of various time-weights, we show that in the critical case $\lambda = \frac{1}{7}$ the growth factor is exactly $\ln(2+t)$.

3. Convergence to nonlinear diffusion waves

In this section, we will prove the global existence of smooth solutions to equations (2.18)–(2.19). Substituting $(2.18)_1$ into $(2.18)_2$, and $(2.18)_3$ into $(2.18)_4$, respectively, we obtain a system of damped wave equations

$$\begin{cases} \phi_{1tt} + \dfrac{1}{(1+t)^\lambda}\phi_{1t} - \left(p'(\bar{n})\phi_{1x}\right)_x = -F - (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} - f_1 + g_{1x} + G_{1x}, \\ \phi_{2tt} + \dfrac{1}{(1+t)^\lambda}\phi_{2t} - \left(p'(\bar{n})\phi_{2x}\right)_x = -F + (\phi_{2x} + \hat{n}_2 + \bar{n})\mathcal{H} + f_2 + g_{2x} + G_{2x}, \end{cases} \tag{3.1}$$

with initial data

$$\phi_i(x,0) = \phi_{i0}(x), \quad \phi_{it}(x,0) = -\psi_{i0}(x). \tag{3.2}$$

Here

$$\begin{cases} F := (1+t)^{\lambda} p(\bar{n})_{xt} + \lambda(1+t)^{\lambda-1} p(\bar{n})_x - \left(\dfrac{\bar{J}^2}{\bar{n}}\right)_x \\ f_i := (\phi_{ix} + \hat{n}_i + \bar{n} - \check{n})\hat{E}, \\ g_i := \dfrac{(-\phi_{it} + \hat{J}_i + \bar{J})^2}{\phi_{ix} + \hat{n}_i + \bar{n}} - \dfrac{\bar{J}^2}{\bar{n}}, \\ G_i := p(\phi_{ix} + \hat{n}_i + \bar{n}) - p(\bar{n}) - p'(\bar{n})\phi_{ix}, \quad i = 1, 2. \end{cases}$$

Equivalently, we only need to study the global existence of smooth solutions to (3.1)–(3.2). As in [20], the local existence of solutions can be obtained by the iteration method. Thus, we only need to establish the a priori estimates for the solutions by technical time-weighted energy method.

Let $T \in (0, +\infty]$, we seek for the solution of (3.1)–(3.2) in the following space

$$X(T) = \left\{ \phi_i(x,t) \middle| \partial_t^j \phi_i \in C([0,T]; H^{3-j}(\mathbb{R})), i = 1, 2,\ j = 0, 1, 2 \right\}$$

where its norm is defined in the following sense: if $-1 < \lambda < \frac{1}{7}$, then

$$\begin{aligned} N(T)^2 := \sup_{0 \le t \le T} \Bigg\{ &\sum_{k=0}^{3} (1+t)^{k(1+\lambda)} \|\partial_x^k \phi_i(t)\|^2 + \sum_{k=0}^{2} (1+t)^{k(1+\lambda)+2} \|\partial_x^k \phi_{it}(t)\|^2 \\ &+ \sum_{k=0}^{1} (1+t)^{k(1+\lambda)+3+\lambda} \|\partial_x^k \phi_{itt}(t)\|^2 \Bigg\}; \end{aligned} \tag{3.3}$$

if $\lambda = \frac{1}{7}$, then

$$\begin{aligned} N(T)^2 := \sup_{0 \le t \le T} \Bigg\{ &\sum_{k=0}^{3} (1+t)^{\frac{8k}{7}} \ln^{-2}(2+t) \|\partial_x^k \phi_i(t)\|^2 + \sum_{k=0}^{2} (1+t)^{\frac{8k}{7}+2} \ln^{-2}(2+t) \|\partial_x^k \phi_{it}(t)\|^2 \\ &+ \sum_{k=0}^{1} (1+t)^{\frac{8k}{7}+\frac{22}{7}} \ln^{-2}(2+t) \|\partial_x^k \phi_{itt}(t)\|^2 \Bigg\}; \end{aligned} \tag{3.4}$$

and if $\frac{1}{7} < \lambda < 1$, then

$$\begin{aligned} N(T)^2 := \sup_{0 \le t \le T} \Bigg\{ &\sum_{k=0}^{3} (1+t)^{k(1+\lambda)+\frac{1}{2}-\frac{7\lambda}{2}} \|\partial_x^k \phi_i(t)\|^2 + \sum_{k=0}^{2} (1+t)^{k(1+\lambda)+\frac{5}{2}-\frac{7\lambda}{2}} \|\partial_x^k \phi_{it}(t)\|^2 \\ &+ \sum_{k=0}^{1} (1+t)^{k(1+\lambda)+\frac{7}{2}-\frac{5\lambda}{2}} \|\partial_x^k \phi_{itt}(t)\|^2 \Bigg\}. \end{aligned} \tag{3.5}$$

It is easy to see that owing to the Sobolev inequality (2.20), if $N(T) \ll 1$, then there exists a constant c_1 such that

$$0 < \frac{1}{c_1} \le n_i = \phi_{ix} + \hat{n}_i + \bar{n} \le c_1, \quad i = 1, 2.$$

Before the estimation of solutions of (3.1), we first show the decay rate of $\mathcal{H}$, which also proves Theorem 2.4. Subtracting $(3.1)_2$ from $(3.1)_1$, we get the following wave equation with time-dependent damping

$$\mathcal{H}_{tt}+\frac{1}{(1+t)^{\lambda}}\mathcal{H}_t-(p'(\bar{n})\mathcal{H}_x)_x+2\bar{n}\mathcal{H}=(g_1-g_2)_x+(G_1-G_2)_x-h_1-h_2, \tag{3.6}$$

where $h_1=(\phi_{1x}+\phi_{2x}+\hat{n}_1+\hat{n}_2)\mathcal{H}$ and $h_2=[\phi_{1x}+\phi_{2x}+\hat{n}_1+\hat{n}_2+2(\bar{n}-\check{n})]\hat{E}$.

Lemma 3.1. *It holds that for $-1<\lambda<1$*

$$\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t,\mathcal{H}_{xx},\mathcal{H}_{xt},\mathcal{H}_{tt})(t)\|^2\le C(\delta^2+\Phi_0^2)e^{-Ct^{\sigma}}\ \textit{for some constant}\ \sigma>0, \tag{3.7}$$

provided that $N(T)+\delta\ll 1$.

Proof. *Step 1.* Multiplying (3.6) by $2\mathcal{H}_t$, we get

$$\begin{aligned}&\left(\mathcal{H}_t^2+2\bar{n}\mathcal{H}^2+p'(\bar{n})\mathcal{H}_x^2\right)_t+2(1+t)^{-\lambda}\mathcal{H}_t^2-2\bar{n}_t\mathcal{H}^2-p''(\bar{n})\bar{n}_t\mathcal{H}_x^2\\&\quad-[2p'(\bar{n})\mathcal{H}_t\mathcal{H}_x]_x=2\mathcal{H}_t[(g_1-g_2)_x+(G_1-G_2)_x-h_1-h_2].\end{aligned} \tag{3.8}$$

The right hand side of (3.8) can be estimated as follows. Noting

$$\begin{aligned}(g_1-g_2)_x=&-\frac{J_1^2}{n_1^2}\mathcal{H}_{xx}-\frac{2J_1}{n_1}\mathcal{H}_{xt}+O(1)(\hat{n}_{1x}+\hat{n}_{2x}+\hat{J}_{1x}+\hat{J}_{2x})\\&+O(1)(\phi_{2xx}+\phi_{2xt}+\bar{n}_x+\bar{J}_x+\hat{n}_{2x}+\hat{J}_{2x})(\mathcal{H}_x+\mathcal{H}_t+\hat{n}_1+\hat{n}_2+\hat{J}_1+\hat{J}_2),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&2\mathcal{H}_t\left(-\frac{J_1^2}{n_1^2}\mathcal{H}_{xx}-\frac{2J_1}{n_1}\mathcal{H}_{xt}\right)\\&=-\left(\frac{2J_1^2}{n_1^2}\mathcal{H}_t\mathcal{H}_x+\frac{2J_1}{n_1}\mathcal{H}_t^2\right)_x+\left(\frac{J_1^2}{n_1^2}\mathcal{H}_x^2\right)_t-\left(\frac{J_1^2}{n_1^2}\right)_t\mathcal{H}_x^2+\left(\frac{2J_1^2}{n_1^2}\right)_x\mathcal{H}_t\mathcal{H}_x+\left(\frac{2J_1}{n_1}\right)_x\mathcal{H}_t^2,\end{aligned}$$

by Young's inequality and (2.5), we obtain

$$\begin{aligned}2\mathcal{H}_t(g_1-g_2)_x\le&-\left(\frac{2J_1^2}{n_1^2}\mathcal{H}_t\mathcal{H}_x+\frac{2J_1}{n_1}\mathcal{H}_t^2\right)_x+\left(\frac{J_1^2}{n_1^2}\mathcal{H}_x^2\right)_t+C(\delta+N(t))(1+t)^{-\lambda}\mathcal{H}_t^2\\&+C(\delta+N(t))(1+t)^{-1}\mathcal{H}_x^2+C(1+t)^{-1}(\hat{n}_1^2+\hat{n}_2^2+\hat{J}_1^2+\hat{J}_2^2)\\&+C(1+t)^{\lambda}(\hat{n}_{1x}^2+\hat{n}_{2x}^2+\hat{J}_{1x}^2+\hat{J}_{2x}^2).\end{aligned}$$

A direct computation yields

$$\begin{aligned}&(G_1-G_2)_x\\&=p'(n_1)(\phi_{1xx}+\hat{n}_{1x}+\bar{n}_x)-p'(n_2)(\phi_{2xx}+\hat{n}_{2x}+\bar{n}_x)-p''(\bar{n})\bar{n}_x\mathcal{H}_x-p'(\bar{n})\mathcal{H}_{xx}\\&=[p'(n_1)-p'(n_2)](\phi_{2xx}+\hat{n}_{2x}+\bar{n}_x)+(p'(n_1)-p'(\bar{n}))\mathcal{H}_{xx}+p'(n_1)(\hat{n}_{1x}-\hat{n}_{2x})\\&\quad-p''(\bar{n})\bar{n}_x\mathcal{H}_x.\end{aligned} \tag{3.10}$$

Then, by Taylor's formula, we get

$$\begin{aligned}2\mathcal{H}_t(G_1-G_2)_x \le{}& [2(p'(n_1)-p'(\bar n))\mathcal{H}_t\mathcal{H}_x]_x - \left[p''(\xi_1)(\phi_{1x}+\hat n_1)\mathcal{H}_x^2\right]_t\\ &+\left(\frac18+N(t)\right)(1+t)^{-\lambda}\mathcal{H}_t^2 + C(\delta+N(t))(1+t)^{-1}\mathcal{H}_x^2\\ &+C(1+t)^{\lambda}(\hat n_{1x}^2+\hat n_{2x}^2)+C(1+t)^{-1}(\hat n_1^2+\hat n_2^2).\end{aligned}$$

Similarly,

$$\begin{aligned}&2|\mathcal{H}_t h_1| \le C(\delta+N(t))(1+t)^{-\lambda}\mathcal{H}_t^2 + C(\delta+N(t))(1+t)^{-1}\mathcal{H}^2,\\ &2|\mathcal{H}_t h_2| \le \left(\frac18+N(t)\right)(1+t)^{-\lambda}\mathcal{H}_t^2 + C(1+t)^{\lambda}\hat E^2 + C(1+t)^{\lambda}(\bar n-\check n)^2\hat E^2.\end{aligned}$$

Thus, integrating (3.8) over $(0,t)\times\mathbb{R}$, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[\mathcal{H}_t^2+2\bar n\mathcal{H}^2+\left(p'(\bar n)-\frac{J_1^2}{n_1^2}+p''(\xi_1)(\phi_{1x}+\hat n_1)\right)\mathcal{H}_x^2\right]\\ &+\left(\frac74-C\delta-CN(t)\right)\int_{\mathbb{R}}(1+t)^{-\lambda}\mathcal{H}_t^2\\ &\le C(\delta+N(t))\int_{\mathbb{R}}(1+t)^{-1}(\mathcal{H}^2+\mathcal{H}_x^2)+C\delta^2e^{-Ct^{\sigma_0}}.\end{aligned} \tag{3.11}$$

Step 2. If $0\le\lambda<1$, multiplying (3.6) by $(1+t)^{-\lambda}\mathcal{H}$, we obtain

$$\begin{aligned}&\left[(1+t)^{-\lambda}\mathcal{H}\mathcal{H}_t+\frac{\lambda}{2}(1+t)^{-\lambda-1}\mathcal{H}^2+\frac12(1+t)^{-2\lambda}\mathcal{H}^2\right]_t-(1+t)^{-\lambda}\mathcal{H}_t^2\\ &+\left[\frac{\lambda}{2}(\lambda+1)(1+t)^{-\lambda-2}+\lambda(1+t)^{-2\lambda-1}+2\bar n(1+t)^{-\lambda}\right]\mathcal{H}^2+p'(\bar n)(1+t)^{-\lambda}\mathcal{H}_x^2\\ &=(p'(\bar n)(1+t)^{-\lambda}\mathcal{H}_x\mathcal{H})_x+(1+t)^{-\lambda}\mathcal{H}[(g_1-g_2)_x+(G_1-G_2)_x-h_1-h_2].\end{aligned} \tag{3.12}$$

The right-hand side of (3.12) can be estimated as below. By (3.9) and Young's inequality,

$$\begin{aligned}&(1+t)^{-\lambda}\mathcal{H}(g_1-g_2)_x\\ &\le-\left[(1+t)^{-\lambda}\mathcal{H}\left(\frac{J_1^2}{n_1^2}\mathcal{H}_x+\frac{2J_1}{n_1}\mathcal{H}_t\right)\right]_x+C(\delta+N(t))(1+t)^{-\lambda}(\mathcal{H}^2+\mathcal{H}_t^2)\\ &\quad+\left(\frac{p'(\bar n)}{8}+N(t)\right)(1+t)^{-\lambda}\mathcal{H}_x^2+C(1+t)^{-2\lambda-1}(\hat n_1^2+\hat n_2^2+\hat J_1^2+\hat J_2^2)\\ &\quad+C(1+t)^{-\lambda}(\hat n_{1x}^2+\hat n_{2x}^2+\hat J_{1x}^2+\hat J_{2x}^2).\end{aligned}$$

By (3.10),

$$\begin{aligned}&(1+t)^{-\lambda}\mathcal{H}(G_1-G_2)_x\\ &\le\left[(1+t)^{-\lambda}(p'(n_1)-p'(\bar n))\mathcal{H}\mathcal{H}_x\right]_x+\left(\frac{p'(\bar n)}{8}+N(t)\right)(1+t)^{-2\lambda-1}\mathcal{H}_x^2\\ &\quad+C(\delta+N(t))(1+t)^{-\lambda}\mathcal{H}^2+C(1+t)^{-\lambda}(\hat n_{1x}^2+\hat n_{2x}^2)+C(1+t)^{-2\lambda-1}(\hat n_1^2+\hat n_2^2).\end{aligned}$$

Similarly,

$$(1+t)^{-\lambda}|\mathcal{H}(h_1+h_2)| \le \left(\frac{\bar{n}}{8} + C(\delta+N(t))\right)(1+t)^{-\lambda}\mathcal{H}^2 + C(1+t)^{-\lambda}\hat{E}^2 + C(1+t)^{-\lambda}(\bar{n}-\check{n})^2\hat{E}^2.$$

Integrating (3.12) over $\mathbb{R}$, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{-\lambda}\mathcal{H}\mathcal{H}_t + \frac{\lambda}{2}(1+t)^{-\lambda-1}\mathcal{H}^2 + \frac{1}{2}(1+t)^{-2\lambda}\mathcal{H}^2\right] + \frac{3}{4}\int_{\mathbb{R}} p'(\bar{n})(1+t)^{-\lambda}\mathcal{H}_x^2 \\
&\quad + \int_{\mathbb{R}}\left[\frac{\lambda}{2}(\lambda+1)(1+t)^{-\lambda-2} + \lambda(1+t)^{-2\lambda-1} + \frac{15}{8}\bar{n}(1+t)^{-\lambda}\right]\mathcal{H}^2 \\
&\le \big(1+C(\delta+N(t))\big)\int_{\mathbb{R}}(1+t)^{-\lambda}\mathcal{H}_t^2 + C(\delta+N(t))\int_{\mathbb{R}}(1+t)^{-\lambda}(\mathcal{H}^2+\mathcal{H}_x^2) \\
&\quad + C\delta^2 e^{-Ct^{\sigma_0}}.
\end{aligned} \tag{3.13}$$

Adding (3.13) to (3.11), noting that

$$|J_1| = |\psi_1 + \hat{J}_1 + \bar{J}| = |-\phi_{1t} + \hat{J}_1 - (1+t)^{\lambda}p'(\bar{n})\bar{n}_x| \le C(\delta+N(t)),$$

and for some positive constants C_0 and C_1,

$$\mathcal{H}_t^2 + (1+t)^{-\lambda}\mathcal{H}\mathcal{H}_t + \frac{1}{2}(1+t)^{-2\lambda}\mathcal{H}^2 \ge C_0\mathcal{H}_t^2 + C_1(1+t)^{-2\lambda}\mathcal{H}^2,$$

when $\delta + N(t) \ll 1$, by Gronwall's inequality, we get

$$\int_{\mathbb{R}}[\mathcal{H}_t^2 + \mathcal{H}^2 + \mathcal{H}_x^2] \le C(\delta^2+\Phi_0^2)e^{-Ct^{\sigma}} \text{ for some constant } \sigma > 0. \tag{3.14}$$

If $-1<\lambda<0$, multiplying (3.6) by $(1+t)^{\lambda}\mathcal{H}$, and integrating the resulting equation over $\mathbb{R}$, as in (3.13), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{\lambda}\mathcal{H}\mathcal{H}_t - \frac{\lambda}{2}(1+t)^{\lambda-1}\mathcal{H}^2 + \frac{1}{2}\mathcal{H}^2\right] \\
&\quad + \int_{\mathbb{R}} p'(\bar{n})(1+t)^{\lambda}\mathcal{H}_x^2 + \int_{\mathbb{R}}\left[\frac{\lambda}{2}(\lambda-1)(1+t)^{\lambda-2} + 2\bar{n}(1+t)^{\lambda}\right]\mathcal{H}^2 \\
&\le \big(1+C(\delta+N(t))\big)\int_{\mathbb{R}}(1+t)^{\lambda}\mathcal{H}_t^2 + C(\delta+N(t))\int_{\mathbb{R}}(1+t)^{\lambda}\mathcal{H}^2 \\
&\quad + \left(\frac{1}{4} + C(\delta+N(t))\right)\int_{\mathbb{R}}(1+t)^{\lambda}\mathcal{H}_x^2 + C\delta^2 e^{-Ct^{\sigma_0}}.
\end{aligned} \tag{3.15}$$

Adding (3.15) to (3.11), when $\delta + N(t) \ll 1$, by Gronwall's inequality, we get

$$\int_{\mathbb{R}}[\mathcal{H}_t^2 + \mathcal{H}^2 + \mathcal{H}_x^2] \le C(\delta^2+\Phi_0^2)e^{-Ct^{\sigma}}. \tag{3.16}$$

Step 3. Differentiating (3.6) in x gives

$$\begin{aligned}&\mathcal{H}_{xtt}+\frac{1}{(1+t)^{\lambda}}\mathcal{H}_{xt}-(p'(\bar n)\mathcal{H}_{xx})_x+2\bar n\mathcal{H}_x\\&=(p''(\bar n)\bar n_x\mathcal{H}_x)_x-2\bar n_x\mathcal{H}+(g_1-g_2)_{xx}+(G_1-G_2)_{xx}-h_{1x}-h_{2x}.\end{aligned}\tag{3.17}$$

Multiplying (3.17) by $2\mathcal{H}_{xt}$, and integrating the resultant equation on $\mathbb{R}$, noting that by (2.5),

$$\begin{aligned}(p''(\bar n)\bar n_x\mathcal{H}_x)_x\mathcal{H}_{xt}&=(p'''(\bar n)\bar n_x^2+p''(\bar n)\bar n_{xx})\mathcal{H}_x\mathcal{H}_{xt}+p''(\bar n)\bar n_x\mathcal{H}_{xx}\mathcal{H}_{xt}\\&\le\delta(1+t)^{-\lambda}\mathcal{H}_{xt}^2+\frac{C}{\delta}(1+t)^{\lambda}(\bar n_x^4+\bar n_{xx}^2)\mathcal{H}_x^2+\frac{C}{\delta}(1+t)^{\lambda}\bar n_x^2\mathcal{H}_{xx}^2\\&\le\delta(1+t)^{-\lambda}\mathcal{H}_{xt}^2+C\delta(1+t)^{-2-\lambda}\mathcal{H}_x^2+C\delta(1+t)^{-1}\mathcal{H}_{xx}^2,\end{aligned}$$

as in (3.11), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[\mathcal{H}_{xt}^2+2\bar n\mathcal{H}_x^2+\left(p'(\bar n)-\frac{J_1^2}{n_1^2}+p''(\xi_1)(\phi_{1x}+\hat n_1)\right)\mathcal{H}_{xx}^2\right]\\&\quad+\left(\frac{7}{4}-C\delta-CN(t)\right)\int_{\mathbb{R}}(1+t)^{-\lambda}\mathcal{H}_{xt}^2\\&\le C(\delta+N(t))\int_{\mathbb{R}}(1+t)^{-\lambda}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2)+C\delta^2e^{-Ct^{\sigma_0}}.\end{aligned}\tag{3.18}$$

On the other hand, if $0\le\lambda<1$, multiplying (3.17) by $(1+t)^{-\lambda}\mathcal{H}_x$, we get

$$\begin{aligned}&\left[(1+t)^{-\lambda}\mathcal{H}_x\mathcal{H}_{xt}+\frac{\lambda}{2}(1+t)^{-\lambda-1}\mathcal{H}_x^2+\frac{1}{2}(1+t)^{-2\lambda}\mathcal{H}_x^2\right]_t+p'(\bar n)(1+t)^{-\lambda}\mathcal{H}_{xx}^2\\&\quad-(1+t)^{-\lambda}\mathcal{H}_{xt}^2+\left[\frac{\lambda}{2}(\lambda+1)(1+t)^{-\lambda-2}+\lambda(1+t)^{-2\lambda-1}+2\bar n(1+t)^{-\lambda}\right]\mathcal{H}_x^2\\&=\left[p'(\bar n)(1+t)^{-\lambda}\mathcal{H}_{xx}\mathcal{H}_x+p''(\bar n)(1+t)^{-\lambda}\bar n_x\mathcal{H}_x^2\right]_x-2\bar n_x(1+t)^{-\lambda}\mathcal{H}\mathcal{H}_x\\&\quad-p''(\bar n)(1+t)^{-\lambda}\bar n_x\mathcal{H}_x\mathcal{H}_{xx}+(1+t)^{-\lambda}\mathcal{H}_x[(g_1-g_2)_{xx}+(G_1-G_2)_{xx}-h_{1x}-h_{2x}].\end{aligned}\tag{3.19}$$

Noting that by Young's inequality and (2.5), we get

$$\begin{aligned}|p''(\bar n)(1+t)^{-\lambda}\bar n_x\mathcal{H}_x\mathcal{H}_{xx}|&\le\delta(1+t)^{-\lambda}\mathcal{H}_{xx}^2+C\delta(1+t)^{-1}\mathcal{H}_x^2,\\|(1+t)^{-\lambda}\bar n_x\mathcal{H}\mathcal{H}_x|&\le\delta(1+t)^{-\lambda}\mathcal{H}_x^2+C\delta(1+t)^{-\lambda}\mathcal{H}^2.\end{aligned}$$

Then integrating (3.19) on $\mathbb{R}$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{-\lambda}\mathcal{H}_x\mathcal{H}_{xt}+\frac{\lambda}{2}(1+t)^{-\lambda-1}\mathcal{H}_x^2+\frac{1}{2}(1+t)^{-2\lambda}\mathcal{H}_x^2\right]+\int_{\mathbb{R}}p'(\bar n)(1+t)^{-\lambda}\mathcal{H}_{xx}^2\\&\quad-\int_{\mathbb{R}}(1+t)^{-\lambda}\mathcal{H}_{xt}^2+\int_{\mathbb{R}}\left[\frac{\lambda}{2}(\lambda+1)(1+t)^{-\lambda-2}+\lambda(1+t)^{-2\lambda-1}+2\bar n(1+t)^{-\lambda}\right]\mathcal{H}_x^2\\&\le C(\delta+N(t))\int_{\mathbb{R}}(1+t)^{-\lambda}(\mathcal{H}^2+\mathcal{H}_x^2+\mathcal{H}_t^2+\mathcal{H}_{xx}^2)+C\delta^2e^{-Ct^{\sigma_0}}.\end{aligned}\tag{3.20}$$

Adding (3.20) to (3.18), when $\delta + N(t) \ll 1$, by Gronwall's inequality, we get

$$\int_{\mathbb{R}} [\mathcal{H}_{xx}^2 + \mathcal{H}_{xt}^2 + \mathcal{H}_x^2] \leq C(\delta^2 + \Phi_0^2)e^{-Ct^\sigma}. \tag{3.21}$$

If $-1 < \lambda < 0$, multiplying (3.17) by $(1+t)^\lambda \mathcal{H}_x$, adding the resulting equation to (3.18), as in (3.20), we get

$$\int_{\mathbb{R}} [\mathcal{H}_{xx}^2 + \mathcal{H}_{xt}^2 + \mathcal{H}_x^2] \leq C(\delta^2 + \Phi_0^2)e^{-Ct^\sigma}. \tag{3.22}$$

In view of equation (3.6), it is easy to see that, for $-1 < \lambda < 1$

$$\|\mathcal{H}_{tt}(t)\|^2 \leq C(\delta^2 + \Phi_0^2)e^{-Ct^\sigma}. \tag{3.23}$$

The desired estimate (3.7) follows directly from (3.14), (3.16), (3.21), (3.22) and (3.23). $\square$

Lemma 3.2. *Assume that $N(T) + \delta \ll 1$, then for $-1 < \lambda < \frac{1}{7}$*

$$\begin{aligned} &\|\phi_i(t)\|^2 + (1+t)^{1+\lambda}\|(\phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t \left[(1+s)^\lambda \|\phi_{ix}(s)\|^2 + (1+s)\|\phi_{it}(s)\|^2\right] ds \\ &\leq C(\delta + \Phi_0^2); \end{aligned} \tag{3.24}$$

for $\lambda = \frac{1}{7}$

$$\begin{aligned} &\|\phi_i(t)\|^2 + (1+t)^{\frac{8}{7}}\|(\phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t \left[(1+s)^{\frac{1}{7}} \|\phi_{ix}(s)\|^2 + (1+s)\|\phi_{it}(s)\|^2\right] ds \\ &\leq C(\delta + \Phi_0^2)\ln^2(2+t); \end{aligned} \tag{3.25}$$

and for $\frac{1}{7} < \lambda < 1$

$$\begin{aligned} &\|\phi_i(t)\|^2 + (1+t)^{1+\lambda}\|(\phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t \left[(1+s)^\lambda \|\phi_{ix}(s)\|^2 + (1+s)\|\phi_{it}(s)\|^2\right] ds \\ &\leq C(\delta + \Phi_0^2)(1+t)^{\frac{7\lambda}{2} - \frac{1}{2}}. \end{aligned} \tag{3.26}$$

Proof. Multiplying $(3.1)_1$ by $2(\beta+t)^\alpha \phi_{1t}$, where $\beta \geq 1$ and $\alpha \geq 0$ are constants to be determined later, and integrating the resultant equation over $\mathbb{R}$, we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \left[(\beta+t)^\alpha \left(\phi_{1t}^2 + p'(\bar{n})\phi_{1x}^2\right)\right] + \int_{\mathbb{R}} \left[\frac{2(\beta+t)^\alpha}{(1+t)^\lambda} - \alpha(\beta+t)^{\alpha-1}\right]\phi_{1t}^2 \\ &- \int_{\mathbb{R}} \left[\alpha(\beta+t)^{\alpha-1}p'(\bar{n}) + (\beta+t)^\alpha p''(\bar{n})\bar{n}_t\right]\phi_{1x}^2 \\ &= 2\int_{\mathbb{R}} (\beta+t)^\alpha \phi_{1t}\big[-F - (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} - f_1 + g_{1x} + G_{1x}\big]. \end{aligned} \tag{3.27}$$

The right hand side of (3.27) can be estimated as follows. By (1.11), (2.5) and (2.6),

$$
\begin{aligned}
-\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1t}F &\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + \frac{C}{\delta}\int_{\mathbb{R}} \left[(1+t)^{\alpha+3\lambda-2}|\bar{n}_x|^2 + (1+t)^{\alpha+3\lambda}|\bar{n}_{xt}|^2\right] \\
&\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + C\delta(1+t)^{\frac{5\lambda-5+2\alpha}{2}}.
\end{aligned}
\tag{3.28}
$$

By Lemmas 2.2 and 3.1,

$$
\begin{aligned}
&-\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1t}(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H} \\
&\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + \frac{C}{\delta}\int_{\mathbb{R}} (\beta+t)^{\alpha+\lambda}(\phi_{1x}^2+\hat{n}_1^2+\bar{n}^2)\mathcal{H}^2 \\
&\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + C\delta e^{-Ct^{\sigma}},
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1t}f_1 &\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + \frac{C}{\delta}\int_{\mathbb{R}} (\beta+t)^{\alpha+\lambda}\left[\phi_{1x}^2+\hat{n}_1^2+(\bar{n}-\check{n})^2\right]\hat{E}^2 \\
&\le \delta\int_{\mathbb{R}} (\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + C\delta e^{-Ct^{\sigma}},
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1t}g_{1x} \\
&= -\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1tx}\left[\frac{(\phi_{1t}-\hat{J}_1-2\bar{J})(\phi_{1t}-\hat{J}_1)}{n_1} - \frac{\bar{J}^2(\phi_{1x}+\hat{n}_1)}{n_1\bar{n}}\right] \\
&\le \int_{\mathbb{R}} (\beta+t)^{\alpha}\left(-\frac{\phi_{1t}^2\phi_{1tx}}{n_1} + \frac{2\bar{J}\phi_{1t}\phi_{1tx}}{n_1} + \frac{\bar{J}^2\phi_{1x}\phi_{1tx}}{n_1\bar{n}}\right) + C\delta e^{-Ct^{\sigma}} \\
&= \int_{\mathbb{R}} (\beta+t)^{\alpha}\left[\frac{\phi_{1t}^3}{3}\left(\frac{1}{n_1}\right)_x - \phi_{1t}^2\left(\frac{\bar{J}}{n_1}\right)_x\right] + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1x}^2\frac{\bar{J}^2}{n_1\bar{n}} \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} \left[(\beta+t)^{\alpha}\frac{\bar{J}^2}{n_1\bar{n}}\right]_t \phi_{1x}^2 + C\delta e^{-Ct^{\sigma}} \\
&\le \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (\beta+t)^{\alpha}\phi_{1x}^2\frac{\bar{J}^2}{n_1\bar{n}} + C\delta\int_{\mathbb{R}} [(\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + (\beta+t)^{\alpha-1}\phi_{1x}^2] + C\delta e^{-Ct^{\sigma}},
\end{aligned}
\tag{3.31}
$$

where we have used the fact that

$$
\|\phi_{1xx}(\cdot,t)\|_{L^\infty} \le C\|\phi_{1xx}(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|\phi_{1xxx}(\cdot,t)\|_{L^2}^{\frac{1}{2}} \le CN(t)(1+t)^{-\frac{1+\lambda}{2}} \quad \text{for } -1<\lambda<1. \tag{3.32}
$$

By Taylor's formula,

$$\begin{aligned} &p'(\phi_{1x}+\hat{n}_1+\bar{n})-p'(\bar{n})=O(1)|\phi_{1x}+\hat{n}_1|,\\ &p'(\phi_{1x}+\hat{n}_1+\bar{n})-p'(\bar{n})-p''(\bar{n})\phi_{1x}=p''(\bar{n})\hat{n}_1+O(1)|\phi_{1x}+\hat{n}_1|^2. \end{aligned} \tag{3.33}$$

It then follows that,

$$\begin{aligned} &\int_{\mathbb{R}}(\beta+t)^\alpha\phi_{1t}G_{1x}\\ &=\int_{\mathbb{R}}(\beta+t)^\alpha\big[p'(\phi_{1x}+\hat{n}_1+\bar{n})(\phi_{1xx}+\hat{n}_{1x}+\bar{n}_x)-p'(\bar{n})\bar{n}_x\\ &\quad-p''(\bar{n})\bar{n}_x\phi_{1x}-p'(\bar{n})\phi_{1xx}\big]\phi_{1t}\\ &=\int_{\mathbb{R}}(\beta+t)^\alpha\Big[\left(p'(\phi_{1x}+\hat{n}_1+\bar{n})-p'(\bar{n})\right)\phi_{1xx}+p'(\phi_{1x}+\hat{n}_1+\bar{n})\hat{n}_{1x}\\ &\quad+\left(p'(\phi_{1x}+\hat{n}_1+\bar{n})-p'(\bar{n})-p''(\bar{n})\phi_{1x}\right)\bar{n}_x\Big]\phi_{1t}\\ &\le C\int_{\mathbb{R}}(\beta+t)^\alpha\left(|\phi_{1x}+\hat{n}_1||\phi_{1xx}|+(|\hat{n}_1|+|\phi_{1x}+\hat{n}_1|^2)|\bar{n}_x|+|\hat{n}_{1x}|\right)|\phi_{1t}|\\ &\le C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1t}^2+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1x}^2+C\delta e^{-Ct^\sigma}. \end{aligned} \tag{3.34}$$

Here we have used (3.32), (2.5) and $\|\phi_{1x}(\cdot,t)\|_{L^\infty}\le N(t)$. Thus,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^\alpha\left(\phi_{1t}^2+\Big(p'(\bar{n})-\frac{\bar{J}^2}{n_1\bar{n}}\Big)\phi_{1x}^2\right)\right]+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^\alpha}{(1+t)^\lambda}-\alpha(\beta+t)^{\alpha-1}\right]\phi_{1t}^2\\ &\le\int_{\mathbb{R}}\left(\alpha p'(\bar{n})+C(\delta+N(t))\right)(\beta+t)^{\alpha-1}\phi_{1x}^2+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1t}^2\\ &\quad+C\delta(1+t)^{\frac{5\lambda-5+2\alpha}{2}}+C\delta e^{-Ct^\sigma}. \end{aligned} \tag{3.35}$$

On the other hand, multiplying $(3.1)_1$ by $\gamma(\beta+t)^{\alpha-1}\phi_1$, where $\gamma>0$ is another constant to be determined later, and integrating the resultant equation over $\mathbb{R}$, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}}\left[\gamma(\beta+t)^{\alpha-1}\phi_1\phi_{1t}-\frac{\gamma}{2}(\alpha-1)(\beta+t)^{\alpha-2}\phi_1^2+\frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\phi_1^2\right]\\ &\quad+\gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}p'(\bar{n})\phi_{1x}^2+\int_{\mathbb{R}}\left[\frac{\gamma}{2}(\alpha-1)(\alpha-2)(\beta+t)^{\alpha-3}-\left(\frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\right)_t\right]\phi_1^2\\ &=\gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1t}^2+\gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_1\big[-F-(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}-f_1+g_{1x}+G_{1x}\big]. \end{aligned}$$

By Young's inequality, (2.5) and (2.6),

$$
\begin{aligned}
-\gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_1 F &\le O(1)\int_{\mathbb{R}}(1+t)^{\alpha+\lambda-2}|\phi_1\bar n_x| + O(1)\int_{\mathbb{R}}(1+t)^{\alpha+\lambda-1}|\phi_1\bar n_{xt}| \\
&\le C(1+t)^{\alpha+\lambda-2}\left(\int_{\mathbb{R}}\phi_1^2\right)^{1/2}\left(\int_{\mathbb{R}}\bar n_x^2\right)^{1/2} \\
&\quad + C(1+t)^{\alpha+\lambda-1}\left(\int_{\mathbb{R}}\phi_1^2\right)^{1/2}\left(\int_{\mathbb{R}}\bar n_{xt}^2\right)^{1/2} \\
&\le C\delta(1+t)^{\frac{4\alpha+3\lambda-9}{4}}\|\phi_1\|_{L^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\gamma(\beta+t)^{\alpha-1}\phi_1\phi_{1t} - \frac{\gamma}{2}(\alpha-1)(\beta+t)^{\alpha-2}\phi_1^2 + \frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\phi_1^2\right] \\
&\quad + \gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}p'(\bar n)\phi_{1x}^2 + \int_{\mathbb{R}}\left[\frac{\gamma}{2}(\alpha-1)(\alpha-2)(\beta+t)^{\alpha-3} - \left(\frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\right)_t\right]\phi_1^2 \\
&\le \gamma\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1t}^2 + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1t}^2 + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1x}^2 \\
&\quad + C\delta(1+t)^{\frac{4\alpha+3\lambda-9}{4}}\|\phi_1\|_{L^2} + C\delta e^{-Ct^\sigma}.
\end{aligned}
\tag{3.36}
$$

Adding (3.36) to (3.35), it then follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^\alpha\left(\phi_{1t}^2 + \left(p'(\bar n) - \frac{\bar J^2}{n_1\bar n}\right)\phi_{1x}^2\right) - \frac{\gamma}{2}(\alpha-1)(\beta+t)^{\alpha-2}\phi_1^2\right. \\
&\quad \left. + \gamma(\beta+t)^{\alpha-1}\phi_1\phi_{1t} + \frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\phi_1^2\right] + \int_{\mathbb{R}}\left[\frac{2(\beta+t)^\alpha}{(1+t)^\lambda} - (\alpha+\gamma)(\beta+t)^{\alpha-1}\right]\phi_{1t}^2 \\
&\quad + \int_{\mathbb{R}}\left[\frac{\gamma}{2}(\alpha-1)(\alpha-2)(\beta+t)^{\alpha-3} - \left(\frac{\gamma(\beta+t)^{\alpha-1}}{2(1+t)^\lambda}\right)_t\right]\phi_1^2 \\
&\quad + (\gamma-\alpha)\int_{\mathbb{R}}(\beta+t)^{\alpha-1}p'(\bar n)\phi_{1x}^2 \\
&\le C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1x}^2 + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1t}^2 \\
&\quad + C\delta(1+t)^{\frac{5\lambda-5+2\alpha}{2}} + C\delta(1+t)^{\frac{4\alpha+3\lambda-9}{4}}\|\phi_1\|_{L^2} + C\delta e^{-Ct^\sigma}.
\end{aligned}
\tag{3.37}
$$

When $-1<\lambda<0$, taking $\gamma=1$, $\alpha=1+\lambda$, $\beta=1$, and noting by (3.3), $\|\phi_1\|_{L^2}\le N(t)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{1+\lambda}\left(\phi_{1t}^2 + \left(p'(\bar n) - \frac{\bar J^2}{n_1\bar n}\right)\phi_{1x}^2\right) + (1+t)^\lambda\phi_1\phi_{1t} - \frac{\lambda}{2}(1+t)^{\lambda-1}\phi_1^2 + \frac{1}{2}\phi_1^2\right] \\
&\quad + \int_{\mathbb{R}}\left[2(1+t) - (2+\lambda)(1+t)^\lambda\right]\phi_{1t}^2 - \lambda\int_{\mathbb{R}}(1+t)^\lambda p'(\bar n)\phi_{1x}^2 + \frac{\lambda}{2}(\lambda-1)\int_{\mathbb{R}}(1+t)^{\lambda-2}\phi_1^2 \\
&\le C(\delta+N(t))\int_{\mathbb{R}}(1+t)^\lambda\phi_{1x}^2 + C(\delta+N(t))\int_{\mathbb{R}}(1+t)\phi_{1t}^2 + C\delta(1+t)^{\frac{7\lambda-5}{4}} + C\delta e^{-Ct^\sigma}.
\end{aligned}
$$

Thus, integrating this inequality over $(0,t)$, and noting that for some positive constants C_2, C_3,

$$(1+t)^{1+\lambda}\phi_{1t}^2 + (1+t)^{\lambda}\phi_1\phi_{1t} + \frac{1}{2}\phi_1^2 \geq C_2(1+t)^{1+\lambda}\phi_{1t}^2 + C_3\phi_1^2,$$
$$\frac{\bar{J}^2}{n_1\bar{n}} \leq C(1+t)^{2\lambda}\bar{n}_x^2 \leq C\delta^2(1+t)^{\lambda-1} \leq C\delta^2,$$

and that $2(1+t) - (2+\lambda)(1+t)^{\lambda} \geq -\lambda > 0$, when $\delta + N(t) \ll 1$, we then have

$$\int_{\mathbb{R}} \left[(1+t)^{1+\lambda}\left(\phi_{1t}^2 + \phi_{1x}^2\right) + \phi_1^2\right] + \int_0^t\int_{\mathbb{R}} (1+s)\phi_{1t}^2 + \int_0^t\int_{\mathbb{R}} (1+s)^{\lambda}\phi_{1x}^2 \leq C\delta + \Phi_0^2. \tag{3.38}$$

When $0 \leq \lambda < 1$, taking $\gamma = 2\alpha$ and $\alpha = 1+\lambda$ in (3.37), and integrating the resulting equation over $(0,t)$, but noting that

$$-(1+\lambda)\left(\frac{(\beta+t)^{\lambda}}{(1+t)^{\lambda}}\right)_t = \lambda(1+\lambda)(\beta-1)(\beta+t)^{\lambda-1}(1+t)^{-\lambda-1} \geq 0,$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left[(\beta+t)^{1+\lambda}\left(\phi_{1t}^2 + \frac{p'(\bar{n})}{2}\phi_{1x}^2\right) + 2(1+\lambda)(\beta+t)^{\lambda}\phi_1\phi_{1t} - \lambda(1+\lambda)(\beta+t)^{\lambda-1}\phi_1^2\right.\\
&\quad \left. + \frac{(1+\lambda)(\beta+t)^{\lambda}}{(1+t)^{\lambda}}\phi_1^2\right] + \int_0^t\int_{\mathbb{R}} \left[2(\beta+s) - 3(1+\lambda)(\beta+s)^{\lambda}\right]\phi_{1t}^2\\
&\quad + (1+\lambda)\int_0^t\int_{\mathbb{R}} (\beta+s)^{\lambda}p'(\bar{n})\phi_{1x}^2\\
&\leq \lambda(1-\lambda)(1+\lambda)\int_0^t\int_{\mathbb{R}} (\beta+s)^{\lambda-2}\phi_1^2 + C(\delta+N(t))\int_0^t\int_{\mathbb{R}} (\beta+s)^{\lambda}\phi_{1x}^2\\
&\quad + C(\delta+N(t))\int_0^t\int_{\mathbb{R}} (\beta+s)\phi_{1t}^2 + C\delta\int_0^t (1+s)^{\frac{7\lambda-3}{2}}\\
&\quad + C\delta\int_0^t (1+s)^{\frac{7\lambda-5}{4}}\|\phi_1\|_{L^2} + C\delta + \Phi_0^2.
\end{aligned} \tag{3.39}$$

We next estimate $\int_0^t\int_{\mathbb{R}}(\beta+s)^{\lambda-2}\phi_1^2$ with $\lambda \geq 0$. To do so, we take $\gamma = 2\alpha$ and $\alpha = 2\lambda \geq 0$ in (3.37) to have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left[(\beta+t)^{2\lambda}\left(\phi_{1t}^2 + \left(p'(\bar{n}) - \frac{\bar{J}^2}{n_1\bar{n}}\right)\phi_{1x}^2\right) - 2\lambda(2\lambda-1)(\beta+t)^{2\lambda-2}\phi_1^2\right.\\
&\left. +4\lambda(\beta+t)^{2\lambda-1}\phi_1\phi_{1t} + \frac{2\lambda(\beta+t)^{2\lambda-1}}{(1+t)^{\lambda}}\phi_1^2\right] + \int_{\mathbb{R}} \left[\frac{2(\beta+t)^{2\lambda}}{(1+t)^{\lambda}} - 6\lambda(\beta+t)^{2\lambda-1}\right]\phi_{1t}^2\\
&+2\lambda\int_{\mathbb{R}} \left[(2\lambda-1)(2\lambda-2)(\beta+t)^{2\lambda-3} - \left(\frac{(\beta+t)^{2\lambda-1}}{(1+t)^{\lambda}}\right)_t\right]\phi_1^2 + 2\lambda\int_{\mathbb{R}} (\beta+t)^{2\lambda-1}p'(\bar{n})\phi_{1x}^2
\end{aligned}$$

$$\leq C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{2\lambda-1}\phi_{1x}^2+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\lambda}\phi_{1t}^2$$
$$+C\delta(1+t)^{\frac{9\lambda-5}{2}}+C\delta(1+t)^{\frac{11\lambda-9}{4}}\|\phi_1\|_{L^2}+C\delta e^{-Ct^{\sigma}}. \tag{3.40}$$

Noting

$$\begin{aligned}-\left(\frac{(\beta+t)^{2\lambda-1}}{(1+t)^{\lambda}}\right)_t&=\frac{(\beta+t)^{2\lambda-2}}{(1+t)^{\lambda}}\left(-2\lambda+1+\lambda\frac{\beta+t}{1+t}\right)\\&\geq(1-\lambda)\frac{(\beta+t)^{2\lambda-2}}{(1+t)^{\lambda}}\\&\geq(1-\lambda)(\beta+t)^{\lambda-2},\end{aligned}$$

we have

$$\begin{aligned}&(2\lambda-1)(2\lambda-2)(\beta+t)^{2\lambda-3}-\left(\frac{(\beta+t)^{2\lambda-1}}{(1+t)^{\lambda}}\right)_t\\&\geq(1-\lambda)(\beta+t)^{\lambda-2}\big[1-2(2\lambda-1)(\beta+t)^{\lambda-1}\big].\end{aligned}$$

Now choosing $\beta=10^{\frac{1}{1-\lambda}}$, it holds that

$$\begin{aligned}(2\lambda-1)(2\lambda-2)(\beta+t)^{2\lambda-3}-\left(\frac{(\beta+t)^{2\lambda-1}}{(1+t)^{\lambda}}\right)_t&\geq\frac{1}{2}(1-\lambda)(\beta+t)^{\lambda-2},\\\frac{2(\beta+t)^{2\lambda}}{(1+t)^{\lambda}}-6\lambda(\beta+t)^{2\lambda-1}&\geq(\beta+t)^{\lambda}.\end{aligned}$$

Hence, integrating (3.40) over $(0,t)$, it follows that when $\delta+N(t)\ll1$,

$$\int_0^t\int_{\mathbb{R}}(\beta+s)^{\lambda-2}\phi_1^2\leq C\delta\int_0^t(1+s)^{\frac{9\lambda-5}{2}}+C\delta\int_0^t(1+s)^{\frac{11\lambda-9}{4}}\|\phi_1\|_{L^2}+C\delta+\Phi_0^2. \tag{3.41}$$

Substituting (3.41) into (3.39) and noting that

$$2(\beta+s)-3(1+\lambda)(\beta+s)^{\lambda}\geq(\beta+s),$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\Big[(\beta+t)^{1+\lambda}\left(\phi_{1t}^2+\phi_{1x}^2\right)+\phi_1^2\Big]+\int_0^t\int_{\mathbb{R}}(\beta+s)\phi_{1t}^2+\int_0^t\int_{\mathbb{R}}(\beta+s)^{\lambda}\phi_{1x}^2\\&\leq C\delta\int_0^t(1+s)^{\frac{7\lambda-3}{2}}+C\delta\int_0^t(1+s)^{\frac{9\lambda-5}{2}}+C\delta\int_0^t(1+s)^{\frac{7\lambda-5}{4}}\|\phi_1\|_{L^2}\\&\quad+C\delta\int_0^t(1+s)^{\frac{11\lambda-9}{4}}\|\phi_1\|_{L^2}+C\delta+\Phi_0^2.\end{aligned} \tag{3.42}$$

When $0 < \lambda < \frac{1}{7}$, by (3.3), $\|\phi_1\|_{L^2} \le N(t)$, then it is easy to see that

$$\int_0^t (1+s)^{\frac{7\lambda-3}{2}} + \int_0^t (1+s)^{\frac{9\lambda-5}{2}} + \int_0^t (1+s)^{\frac{7\lambda-5}{4}} \|\phi_1\|_{L^2} + \int_0^t (1+s)^{\frac{11\lambda-9}{4}} \|\phi_1\|_{L^2} \le C.$$

It then follows from (3.42) that when $\delta + N(t) \ll 1$,

$$\int_{\mathbb{R}} \left[(1+t)^{1+\lambda}\left(\phi_{1t}^2 + \phi_{1x}^2\right) + \phi_1^2\right] + \int_0^t \int_{\mathbb{R}} (1+s)\phi_{1t}^2 + \int_0^t \int_{\mathbb{R}} (1+s)^{\lambda}\phi_{1x}^2 \le C\delta + \Phi_0^2.$$

When $\lambda = \frac{1}{7}$, by (3.4), $\|\phi_1\|_{L^2} \le N(t)\ln(2+t)$, it is easy to see that

$$\begin{aligned}&\int_0^t (1+s)^{\frac{7\lambda-3}{2}} + \int_0^t (1+s)^{\frac{9\lambda-5}{2}} + \int_0^t (1+s)^{\frac{7\lambda-5}{4}} \|\phi_1\|_{L^2} + \int_0^t (1+s)^{\frac{11\lambda-9}{4}} \|\phi_1\|_{L^2} \\ &\le 2\ln^2(2+t) + \ln(2+t) + \frac{7}{6}.\end{aligned}$$

It then follows from (3.42) that

$$\int_{\mathbb{R}} \left[(1+t)^{\frac{8}{7}}\left(\phi_{1t}^2 + \phi_{1x}^2\right) + \phi_1^2\right] + \int_0^t \int_{\mathbb{R}} (1+s)\phi_{1t}^2 + \int_0^t \int_{\mathbb{R}} (1+s)^{\frac{1}{7}}\phi_{1x}^2 \le C\delta\ln^2(2+t) + \Phi_0^2.$$

When $\frac{1}{7} < \lambda < 1$, by (3.5), $\|\phi_1\|_{L^2} \le N(t)(1+t)^{\frac{7\lambda-1}{4}}$, then

$$\int_0^t (1+s)^{\frac{7\lambda-3}{2}} + \int_0^t (1+s)^{\frac{9\lambda-5}{2}} + \int_0^t (1+s)^{\frac{7\lambda-5}{4}} \|\phi_1\|_{L^2} + \int_0^t (1+s)^{\frac{11\lambda-9}{4}} \|\phi_1\|_{L^2} \le C(1+t)^{\frac{7\lambda-1}{2}}.$$

It hence follows from (3.42) that

$$\int_{\mathbb{R}} \left[(1+t)^{1+\lambda}\left(\phi_{1t}^2 + \phi_{1x}^2\right) + \phi_1^2\right] + \int_0^t \int_{\mathbb{R}} (1+s)\phi_{1t}^2 + \int_0^t \int_{\mathbb{R}} (1+s)^{\lambda}\phi_{1x}^2 \le C\delta(1+t)^{\frac{7\lambda-1}{2}} + \Phi_0^2.$$

Calculating similarly to $i = 2$, we also have the desired estimates when $N(T) + \delta \ll 1$. $\square$

Lemma 3.3. *When $N(T) + \delta \ll 1$, then for $-1 < \lambda < \frac{1}{7}$*

$$\begin{aligned}&(1+t)^{2+2\lambda}\|(\phi_{ixx}, \phi_{ixt})(t)\|^2 + \int_0^t \left[(1+s)^{1+2\lambda}\|\phi_{ixx}(s)\|^2 + (1+s)^{2+\lambda}\|\phi_{ixt}(s)\|^2\right] ds \\ &\le C(\delta + \Phi_0^2);\end{aligned} \tag{3.43}$$

for $\lambda = \frac{1}{7}$

$$\begin{aligned}&(1+t)^{\frac{16}{7}}\|(\phi_{ixx},\phi_{ixt})(t)\|^2+\int_0^t\left[(1+s)^{\frac{9}{7}}\|\phi_{ixx}(s)\|^2+(1+s)^{\frac{15}{7}}\|\phi_{ixt}(s)\|^2\right]ds\\&\le C(\delta+\Phi_0^2)\ln^2(2+t);\end{aligned}\tag{3.44}$$

and for $\frac{1}{7}<\lambda<1$

$$\begin{aligned}&(1+t)^{2+2\lambda}\|(\phi_{ixx},\phi_{ixt})(t)\|^2+\int_0^t\left[(1+s)^{1+2\lambda}\|\phi_{ixx}(s)\|^2+(1+s)^{2+\lambda}\|\phi_{ixt}(s)\|^2\right]ds\\&\le C(\delta+\Phi_0^2)(1+t)^{\frac{7\lambda}{2}-\frac{1}{2}}.\end{aligned}\tag{3.45}$$

Proof. Differentiating $(3.1)_1$ in x yields

$$\begin{aligned}&\phi_{1xtt}+\frac{1}{(1+t)^\lambda}\phi_{1xt}-\big(p'(\bar n)\phi_{1xx}\big)_x\\&=(p''(\bar n)\bar n_x\phi_{1x})_x-F_x-[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}]_x-f_{1x}+g_{1xx}+G_{1xx}.\end{aligned}\tag{3.46}$$

Multiplying (3.46) by $2(\beta+t)^{2+2\lambda}\phi_{1xt}$ with $\beta=10^{\frac{1}{1-\lambda}}$ and integrating the resultant equation over $\mathbb{R}$ gives

$$\begin{aligned}&\frac{d}{dt}\int\left[(\beta+t)^{2+2\lambda}\left(\phi_{1xt}^2+p'(\bar n)\phi_{1xx}^2\right)\right]+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^{2+2\lambda}}{(1+t)^\lambda}-(2+2\lambda)(\beta+t)^{1+2\lambda}\right]\phi_{1xt}^2\\&-\int_{\mathbb{R}}\left[(2+2\lambda)(\beta+t)^{1+2\lambda}p'(\bar n)+(\beta+t)^{2+2\lambda}p''(\bar n)\bar n_t\right]\phi_{1xx}^2\\&=2\int_{\mathbb{R}}(\beta+t)^{2+2\lambda}\phi_{1xt}\Big[(p''(\bar n)\bar n_x\phi_{1x})_x-F_x-[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}+f_1]_x+g_{1xx}+G_{1xx}\Big].\end{aligned}\tag{3.47}$$

We estimate the right hand side of (3.47) as follows. By Young's inequality and (2.6),

$$-2\int_{\mathbb{R}}(\beta+t)^{2+2\lambda}\phi_{1xt}F_x\le\delta\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}},\tag{3.48}$$

$$-2\int_{\mathbb{R}}(\beta+t)^{2+2\lambda}\phi_{1xt}[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}+f_1]_x\le\delta\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2+C\delta e^{-Ct^\sigma}.\tag{3.49}$$

As in (3.31),

$$\begin{aligned}&2\int_{\mathbb{R}}(\beta+t)^{2+2\lambda}\phi_{1xt}g_{1xx}\\&\le\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{2+2\lambda}\phi_{1xx}^2\frac{J_1^2}{n_1^2}\right]+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2\\&+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xx}^2+C\int_{\mathbb{R}}(\beta+t)\phi_{1t}^2+C\int_{\mathbb{R}}(\beta+t)^\lambda\phi_{1x}^2\\&+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^\sigma},\end{aligned}\tag{3.50}$$

where we have used $\|\phi_{1tt}\|_{L^\infty} \le C\|\phi_{1tt}\|_{L^2}^{\frac{1}{2}}\|\phi_{1xtt}\|_{L^2}^{\frac{1}{2}} \le CN(t)(1+t)^{-1}$ for $-1<\lambda<1$, and $J_1 = -\phi_{1t} + \hat{J}_1 + \bar{J}$. As in (3.34),

$$
\begin{aligned}
&2\int_{\mathbb{R}} (\beta+t)^{2+2\lambda}\phi_{1xt}\left((p''(\bar{n})\bar{n}_x\phi_{1x})_x + G_{xx}\right)\\
&= 2\int_{\mathbb{R}} (\beta+t)^{2+2\lambda}\left[p'(\phi_{1x}+\hat{n}_1+\bar{n})(\phi_{1xx}+\hat{n}_{1x}+\bar{n}_x) - p'(\bar{n})\bar{n}_x - p'(\bar{n})\phi_{1xx}\right]_x\phi_{1xt}\\
&\triangleq I_1 + I_2,
\end{aligned}
\tag{3.51}
$$

where

$$
\begin{aligned}
I_1 &= 2\int_{\mathbb{R}} (\beta+t)^{2+2\lambda}\left[\left(p'(\phi_{1x}+\hat{n}_1+\bar{n}) - p'(\bar{n})\right)\phi_{1xx}\right]_x\phi_{1xt}\\
&\le -\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{2+2\lambda}\left(p'(\phi_{1x}+\hat{n}_1+\bar{n}) - p'(\bar{n})\right)\phi_{1xx}^2\right]\\
&\quad + C\int_{\mathbb{R}}(\beta+t)^{1+2\lambda}|\phi_{1x}+\hat{n}_1|\phi_{1xx}^2 + C\int_{\mathbb{R}}(\beta+t)^{2+2\lambda}(|\phi_{1xt}|+|\bar{n}_t|+|\hat{n}_{1t}|)\phi_{1xx}^2\\
&\le -\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{2+2\lambda}\left(p'(\phi_{1x}+\hat{n}_1+\bar{n}) - p'(\bar{n})\right)\phi_{1xx}^2\right]\\
&\quad + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xx}^2,
\end{aligned}
\tag{3.52}
$$

and

$$
\begin{aligned}
I_2 &\le C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2 + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xx}^2\\
&\quad + C\int_{\mathbb{R}}(\beta+t)^{\lambda}\phi_{1x}^2 + C\delta e^{-Ct^\sigma}.
\end{aligned}
\tag{3.53}
$$

Hence, by (3.48)–(3.53), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\left[(\beta+t)^{2+2\lambda}\Big(\phi_{1xt}^2 + \big(p'(\bar{n}) + O(1)(\delta+N(t))\big)\phi_{1xx}^2\Big)\right]\\
&\quad + \int_{\mathbb{R}}\left[\frac{2(\beta+t)^{2+2\lambda}}{(1+t)^\lambda} - (2+2\lambda)(\beta+t)^{1+2\lambda}\right]\phi_{1xt}^2\\
&\le \int_{\mathbb{R}}\left[(2+2\lambda)p'(\bar{n}) + C(\delta+N(t))\right](\beta+t)^{1+2\lambda}\phi_{1xx}^2 + C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2\\
&\quad + C\int_{\mathbb{R}}(\beta+t)\phi_{1t}^2 + C\int_{\mathbb{R}}(\beta+t)^\lambda\phi_{1x}^2 + C\delta(1+t)^{\frac{7\lambda-3}{2}} + C\delta e^{-Ct^\sigma}.
\end{aligned}
\tag{3.54}
$$

On the other hand, multiplying (3.46) by $4(\beta+t)^{1+2\lambda}\phi_{1x}$ with $\beta = 10^{\frac{1}{1-\lambda}}$ and integrating the resultant equation over $\mathbb{R}$, by Young's inequality and Lemma 2.1, we get

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}}\left[4(\beta+t)^{1+2\lambda}\phi_{1x}\phi_{1xt}-2(1+2\lambda)(\beta+t)^{2\lambda}\phi_{1x}^2+\frac{2(\beta+t)^{1+2\lambda}}{(1+t)^{\lambda}}\phi_{1x}^2\right]\\
&\quad+4\int\limits_{\mathbb{R}}(\beta+t)^{1+2\lambda}p'(\bar n)\phi_{1xx}^2\\
&\le 4\int\limits_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xt}^2+C(\delta+N(t))\int\limits_{\mathbb{R}}\left[(\beta+t)^{2+\lambda}\phi_{1xt}^2+(\beta+t)^{1+2\lambda}\phi_{1xx}^2\right]\\
&\quad+C\int\limits_{\mathbb{R}}(\beta+t)^{\lambda}\phi_{1x}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}
\tag{3.55}
$$

Adding (3.55) to (3.54) gives

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}}\Big[(\beta+t)^{2+2\lambda}\Big(\phi_{1xt}^2+\big(p'(\bar n)+O(1)(\delta+N(t))\big)\phi_{1xx}^2\Big)+4(\beta+t)^{1+2\lambda}\phi_{1x}\phi_{1xt}\\
&\quad-2(1+2\lambda)(\beta+t)^{2\lambda}\phi_{1x}^2+\frac{2(\beta+t)^{1+2\lambda}}{(1+t)^{\lambda}}\phi_{1x}^2\Big]\\
&\quad+\int\limits_{\mathbb{R}}\left[\frac{2(\beta+t)^{2+2\lambda}}{(1+t)^{\lambda}}-2(3+\lambda)(\beta+t)^{1+2\lambda}\right]\phi_{1xt}^2+2(1-\lambda)\int\limits_{\mathbb{R}}p'(\bar n)(\beta+t)^{1+2\lambda}\phi_{1xx}^2\\
&\le C(\delta+N(t))\int\limits_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2+C(\delta+N(t))\int\limits_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xx}^2+C\int\limits_{\mathbb{R}}(\beta+t)\phi_{1t}^2\\
&\quad+C\int\limits_{\mathbb{R}}(\beta+t)^{\lambda}\phi_{1x}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}
\tag{3.56}
$$

Integrating (3.56) over $(0,t)$, by Lemma 3.2 and noting that

$$
\frac{2(\beta+t)^{2+2\lambda}}{(1+t)^{\lambda}}-2(3+\lambda)(\beta+t)^{1+2\lambda}\ge(\beta+t)^{2+\lambda},
$$

we have the desired estimates (3.43)–(3.45) provided $N(T)+\delta\ll 1$. □

Lemma 3.4. *Assume that $N(T)+\delta\ll 1$, then for $-1<\lambda<\frac{1}{7}$*

$$
\begin{aligned}
&(1+t)^{3+3\lambda}\|(\phi_{ixxx},\phi_{ixxt})(t)\|^2+\int\limits_0^t\left[(1+s)^{2+3\lambda}\|\phi_{ixxx}(s)\|^2+(1+s)^{3+2\lambda}\|\phi_{ixxt}(s)\|^2\right]ds\\
&\le C(\delta+\Phi_0^2),
\end{aligned}
$$

for $\lambda=\frac{1}{7}$

$$
\begin{aligned}
&(1+t)^{\frac{24}{7}}\|(\phi_{ixxx},\phi_{ixxt})(t)\|^2+\int\limits_0^t\left[(1+s)^{\frac{17}{7}}\|\phi_{ixxx}(s)\|^2+(1+s)^{\frac{23}{7}}\|\phi_{ixxt}(s)\|^2\right]ds\\
&\le C(\delta+\Phi_0^2)\ln^2(2+t),
\end{aligned}
$$

and for $\frac{1}{7}<\lambda<1$

$$(1+t)^{3+3\lambda}\|(\phi_{ixxx},\phi_{ixxt})(t)\|^2+\int_0^t\left[(1+s)^{2+3\lambda}\|\phi_{ixxx}(s)\|^2+(1+s)^{3+2\lambda}\|\phi_{ixxt}(s)\|^2\right]ds$$
$$\le C(\delta+\Phi_0^2)(1+t)^{\frac{7\lambda}{2}-\frac{1}{2}}.$$

Proof. Differentiating (3.46) in x leads to

$$\begin{aligned}&\phi_{1xxtt}+\frac{1}{(1+t)^\lambda}\phi_{1xxt}-\left(p'(\bar n)\phi_{1xxx}\right)_x\\&=-F_{xx}-[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}]_{xx}-f_{1xx}+g_{1xxx}+G_{1xxx}+K_x,\end{aligned}\tag{3.57}$$

where $K=p'''(\bar n)\bar n_x^2\phi_{1x}+p''(\bar n)\bar n_{xx}\phi_{1x}+2p''(\bar n)\bar n_x\phi_{1xx}$. Multiplying (3.57) by $2(\beta+t)^{3+3\lambda}\phi_{1xxt}$ with $\beta=10^{\frac{1}{1-\lambda}}$, and integrating it over $\mathbb{R}$ yields

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{3+3\lambda}\left(\phi_{1xxt}^2+p'(\bar n)\phi_{1xxx}^2\right)\right]\\&+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^{3+3\lambda}}{(1+t)^\lambda}-(3+3\lambda)(\beta+t)^{2+3\lambda}\right]\phi_{1xxt}^2\\&-\int_{\mathbb{R}}\left[(3+3\lambda)(\beta+t)^{2+3\lambda}p'(\bar n)+(\beta+t)^{3+3\lambda}p''(\bar n)\bar n_t\right]\phi_{1xxx}^2\\&=2\int_{\mathbb{R}}(\beta+t)^{3+3\lambda}\phi_{1xxt}\big[-[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}]_{xx}-f_{1xx}+g_{1xxx}-F_{xx}+G_{1xxx}+K_x\big].\end{aligned}\tag{3.58}$$

We next estimate the right hand side of (3.58). As in (3.48) and (3.49),

$$\begin{aligned}&-2\int_{\mathbb{R}}(\beta+t)^{3+3\lambda}\phi_{1xxt}\big[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}+f_1+F\big]_{xx}\\&\le C\delta\int_{\mathbb{R}}(\beta+t)^{3+2\lambda}\phi_{1xxt}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}}+\delta\int_{\mathbb{R}}(\beta+t)^{2+3\lambda}\phi_{1xxx}^2+C\delta e^{-Ct^\sigma}.\end{aligned}\tag{3.59}$$

As in (3.50),

$$\begin{aligned}&2\int_{\mathbb{R}}(\beta+t)^{3+3\lambda}\phi_{1xxt}\left[g_{1xxx}+K_x+G_{1xxx}\right]\\&\le\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{3+3\lambda}\phi_{1xxx}^2\frac{J_1^2}{n_1^2}\right]-\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{3+3\lambda}\left(p'(\phi_{1x}+\hat n_1+\bar n)-p'(\bar n)\right)\phi_{1xxx}^2\right]\\&+C(\delta+N(t))\int_{\mathbb{R}}\left[(\beta+t)^{3+2\lambda}\phi_{1xxt}^2+(\beta+t)^{2+3\lambda}\phi_{1xxx}^2\right]\\&+C\int_{\mathbb{R}}\left[(\beta+t)^{2+\lambda}\phi_{1xt}^2+(\beta+t)^{1+2\lambda}\phi_{1xx}^2\right]+C\int_{\mathbb{R}}\left[(\beta+t)\phi_{1t}^2+(\beta+t)^\lambda\phi_{1x}^2\right]\\&+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^\sigma},\end{aligned}\tag{3.60}$$

where we have used $\|\phi_{1tt}\|_{L^\infty}\le C\|\phi_{1tt}\|_{L^2}^{\frac12}\|\phi_{1xtt}\|_{L^2}^{\frac12}\le CN(t)(1+t)^{-1}$ for $-1<\lambda<1$, and $J_1=-\phi_{1t}+\hat J_1+\bar J$. Substituting (3.59)–(3.60) to (3.58),

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{3+3\lambda}\Big(\phi_{1xxt}^2+\big(p'(\bar n)+O(1)(\delta+N(t))\big)\phi_{1xxx}^2\Big)\right]\\
&\quad+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^{3+3\lambda}}{(1+t)^{\lambda}}-(3+3\lambda)(\beta+t)^{2+3\lambda}\right]\phi_{1xxt}^2\\
&\le\int_{\mathbb{R}}\left[(3+3\lambda)p'(\bar n)+C(\delta+N(t))\right](\beta+t)^{2+3\lambda}\phi_{1xxx}^2+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{3+2\lambda}\phi_{1xxt}^2\\
&\quad+C\int_{\mathbb{R}}\left[(\beta+t)^{2+\lambda}\phi_{1xt}^2+(\beta+t)^{1+2\lambda}\phi_{1xx}^2+(\beta+t)\phi_{1t}^2+(\beta+t)^{\lambda}\phi_{1x}^2\right]\\
&\quad+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}\tag{3.61}$$

On the other hand, multiplying (3.57) by $6(\beta+t)^{2+3\lambda}\phi_{1xx}$, integrating it over $\mathbb{R}$, similar to (3.59)–(3.60), and noting that

$$\left(\frac{3(\beta+t)^{2+3\lambda}}{(1+t)^{\lambda}}\right)_t-3(2+3\lambda)(1+3\lambda)(\beta+t)^{3\lambda}\le C(\beta+t)^{1+2\lambda},$$

we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[6(\beta+t)^{2+3\lambda}\phi_{1xx}\phi_{1xxt}-3(2+3\lambda)(\beta+t)^{1+3\lambda}\phi_{1xx}^2+\frac{3(\beta+t)^{2+3\lambda}}{(1+t)^{\lambda}}\phi_{1xx}^2\right]\\
&\quad+6\int_{\mathbb{R}}(\beta+t)^{2+3\lambda}p'(\bar n)\phi_{1xxx}^2\\
&\le 6\int_{\mathbb{R}}(\beta+t)^{2+3\lambda}\phi_{1xxt}^2+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{3+2\lambda}\phi_{1xxt}^2\\
&\quad+C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{2+3\lambda}\phi_{1xxx}^2+C\int_{\mathbb{R}}(\beta+t)^{2+\lambda}\phi_{1xt}^2+C\int_{\mathbb{R}}(\beta+t)^{1+2\lambda}\phi_{1xx}^2\\
&\quad+C\int_{\mathbb{R}}(\beta+t)\phi_{1t}^2+C\int_{\mathbb{R}}(\beta+t)^{\lambda}\phi_{1x}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}\tag{3.62}$$

Adding (3.62) to (3.61), integrating the resultant equation over $(0,t)$, by Lemmas 3.2, 3.3 and

$$\frac{2(\beta+t)^{3+3\lambda}}{(1+t)^{\lambda}}-3(3+\lambda)(\beta+t)^{2+3\lambda}\ge(\beta+t)^{3+2\lambda}\quad\text{for }\beta=10^{\frac{1}{1-\lambda}},$$

we get Lemma 3.4. □

Lemma 3.5. *When $N(T)+\delta\ll 1$, then for $-1<\lambda<\frac{1}{7}$*

$$(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{3+\lambda}\|(\phi_{ixt},\phi_{itt})(t)\|^2+\int_0^t(1+s)^3\|\phi_{itt}(s)\|^2ds\le C(\delta+\Phi_0^2),\tag{3.63}$$

for $\lambda=\frac{1}{7}$

$$(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{\frac{22}{7}}\|(\phi_{ixt},\phi_{itt})(t)\|^2+\int_0^t(1+s)^3\|\phi_{itt}(s)\|^2ds \\ \le C(\delta+\Phi_0^2)\ln^2(2+t), \tag{3.64}$$

and for $\frac{1}{7}<\lambda<1$

$$(1+t)^2\|\phi_{it}(t)\|^2+(1+t)^{3+\lambda}\|(\phi_{ixt},\phi_{itt})(t)\|^2+\int_0^t(1+s)^3\|\phi_{itt}(s)\|^2ds \\ \le C(\delta+\Phi_0^2)(1+t)^{\frac{7\lambda}{2}-\frac{1}{2}}. \tag{3.65}$$

Proof. Differentiating $(3.1)_1$ in t yields

$$\phi_{1ttt}-\lambda(1+t)^{-\lambda-1}\phi_{1t}+\frac{1}{(1+t)^\lambda}\phi_{1tt}-\big(p'(\bar{n})\phi_{1xt}\big)_x \\ =(p''(\bar{n})\bar{n}_t\phi_{1x})_x-F_t-[(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}]_t-f_{1t}+g_{1xt}+G_{1xt}. \tag{3.66}$$

Multiplying (3.66) by $2(\beta+t)^\alpha\phi_{1tt}$ and integrating it over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^\alpha\left(\phi_{1tt}^2+p'(\bar{n})\phi_{1xt}^2\right)-\frac{\lambda(\beta+t)^\alpha}{(1+t)^{1+\lambda}}\phi_{1t}^2\right]+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^\alpha}{(1+t)^\lambda}-\alpha(\beta+t)^{\alpha-1}\right]\phi_{1tt}^2 \\ -\int_{\mathbb{R}}\left[\alpha(\beta+t)^{\alpha-1}p'(\bar{n})+(\beta+t)^\alpha p''(\bar{n})\bar{n}_t\right]\phi_{1xt}^2+\int_{\mathbb{R}}\left[\frac{\lambda(\beta+t)^\alpha}{(1+t)^{1+\lambda}}\right]_t\phi_{1t}^2 \\ =2\int_{\mathbb{R}}(\beta+t)^\alpha\phi_{1tt}\Big[(p''(\bar{n})\bar{n}_t\phi_{1x})_x-F_t-[(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}]_t-f_{1t}+g_{1xt}+G_{1xt}\Big]. \tag{3.67}$$

By Young's inequality and (2.5),

$$\int_{\mathbb{R}}(\beta+t)^\alpha\left|\phi_{1tt}[F+(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}+f_1]_t\right| \\ \le\delta\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1tt}^2+C\delta(1+t)^{\frac{2\alpha+5\lambda-9}{2}}+C\delta e^{-Ct^\sigma}. \tag{3.68}$$

As in (3.50),

$$2(\beta+t)^\alpha\int\phi_{1tt}\left[g_{1xt}+(p''(\bar{n})\bar{n}_t\phi_{1x})_x+G_{1xt}\right] \\ \le\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^\alpha\phi_{1xt}^2\frac{J_1^2}{n_1^2}\right]-\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^\alpha\left(p'(\phi_{1x}+\hat{n}_1+\bar{n})-p'(\bar{n})\right)\phi_{1xt}^2\right] \\ +C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1tt}^2+C\int_{\mathbb{R}}\left[(\beta+t)^{\alpha-1}\phi_{1xt}^2+(\beta+t)^{\alpha-2+\lambda}\phi_{1xx}^2\right] \\ +C\int_{\mathbb{R}}(\beta+t)^{\alpha-2-\lambda}\phi_{1t}^2+C\int_{\mathbb{R}}(\beta+t)^{\alpha-3}\phi_{1x}^2+C\delta(1+t)^{\frac{2\alpha+5\lambda-9}{2}}+C\delta e^{-Ct^\sigma}.$$

Thus,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(\beta+t)^{\alpha}\Big(\phi_{1tt}^2+\big(p'(\bar n)+O(1)(\delta+N(t))\big)\phi_{1xt}^2\Big)-\frac{\lambda(\beta+t)^{\alpha}}{(1+t)^{1+\lambda}}\phi_{1t}^2\right]\\
&\quad+\int_{\mathbb{R}}\left[\frac{2(\beta+t)^{\alpha}}{(1+t)^{\lambda}}-\alpha(\beta+t)^{\alpha-1}\right]\phi_{1tt}^2+\int_{\mathbb{R}}\left[\frac{\lambda(\beta+t)^{\alpha}}{(1+t)^{1+\lambda}}\right]_t\phi_{1t}^2\\
&\le C(\delta+N(t))\int_{\mathbb{R}}(\beta+t)^{\alpha-\lambda}\phi_{1tt}^2+C\int_{\mathbb{R}}(\beta+t)^{\alpha-1}\phi_{1xt}^2+C\int_{\mathbb{R}}(\beta+t)^{\alpha-2+\lambda}\phi_{1xx}^2\\
&\quad+C\int_{\mathbb{R}}(\beta+t)^{\alpha-2-\lambda}\phi_{1t}^2+C\int_{\mathbb{R}}(\beta+t)^{\alpha-3}\phi_{1x}^2+C\delta(1+t)^{\frac{2\alpha+5\lambda-9}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}
\tag{3.69}
$$

Taking $\alpha=2+2\lambda$, $\beta=10^{\frac{1}{1-\lambda}}$ in (3.69), integrating it over $(0,t)$, and noting that

$$
\frac{2(\beta+t)^{2+2\lambda}}{(1+t)^{\lambda}}-(2+2\lambda)(\beta+t)^{1+2\lambda}\ge(\beta+t)^{2+\lambda},
$$

by Lemmas 3.2 and 3.3, we have,

$$
\int_0^t\int_{\mathbb{R}}(\beta+s)^{2+\lambda}\phi_{1tt}^2\le C\delta\int_0^t(1+s)^{\frac{9\lambda-5}{2}}+C\delta+\Phi_0^2. \tag{3.70}
$$

Taking $\alpha=3+\lambda$ and $\beta=1$ in (3.69) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[(1+t)^{3+\lambda}\Big(\phi_{1tt}^2+\big(p'(\bar n)+O(1)(\delta+N(t))\big)\phi_{1xt}^2\Big)-\lambda(1+t)^2\phi_{1t}^2\right]+2\int_{\mathbb{R}}(1+t)^3\phi_{1tt}^2\\
&\le C(\delta+N(t))\int_{\mathbb{R}}(1+t)^3\phi_{1tt}^2+C\int_{\mathbb{R}}(1+t)^{2+\lambda}\phi_{1tt}^2+C\int_{\mathbb{R}}(1+t)^{2+\lambda}\phi_{1xt}^2\\
&\quad+C\int_{\mathbb{R}}(1+t)^{1+2\lambda}\phi_{1xx}^2+C\int_{\mathbb{R}}(1+t)\phi_{1t}^2+C\int_{\mathbb{R}}(1+t)^{\lambda}\phi_{1x}^2\\
&\quad+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}
\tag{3.71}
$$

On the other hand, multiplying (3.66) by $2(1+t)^{2+\lambda}\phi_{1t}$ and integrating the resultant equation over $\mathbb{R}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[2(1+t)^{2+\lambda}\phi_{1t}\phi_{1tt}-(2+\lambda)(1+t)^{1+\lambda}\phi_{1t}^2+(1+t)^2\phi_{1t}^2\right]\\
&\le 2\int_{\mathbb{R}}(1+t)^{2+\lambda}\phi_{1tt}^2+C(\delta+N(t))\int_{\mathbb{R}}(1+t)^3\phi_{1tt}^2+C\int_{\mathbb{R}}(1+t)^{2+\lambda}\phi_{1xt}^2\\
&\quad+C\int_{\mathbb{R}}(1+t)\phi_{1t}^2+C\int_{\mathbb{R}}(1+t)^{\lambda}\phi_{1x}^2+C\delta(1+t)^{\frac{7\lambda-3}{2}}+C\delta e^{-Ct^{\sigma}}.
\end{aligned}
\tag{3.72}
$$

Combining (3.72) and (3.71), and integrating the equation over $(0,t)$, by Lemmas 3.2, 3.3 and (3.70), we get the desired inequalities if $N(T)+\delta\ll 1$. $\square$

Lemma 3.6. *When $N(T)+\delta \ll 1$, then for $-1<\lambda<\frac{1}{7}$*

$$(1+t)^{4+2\lambda}\|(\phi_{ixxt},\phi_{ixtt})(t)\|^2+\int_0^t(1+s)^{4+\lambda}\|\phi_{ixtt}(s)\|^2ds \le C(\delta+\Phi_0^2),$$

for $\lambda=\frac{1}{7}$

$$(1+t)^{\frac{30}{7}}\|(\phi_{ixxt},\phi_{ixtt})(t)\|^2+\int_0^t(1+s)^{\frac{29}{7}}\|\phi_{ixtt}(s)\|^2ds \le C(\delta+\Phi_0^2)\ln^2(2+t),$$

and for $\frac{1}{7}<\lambda<1$

$$(1+t)^{4+2\lambda}\|(\phi_{ixxt},\phi_{ixtt})(t)\|^2+\int_0^t(1+s)^{4+\lambda}\|\phi_{ixtt}(s)\|^2ds \le C(\delta+\Phi_0^2)(1+t)^{\frac{7\lambda}{2}-\frac{1}{2}}.$$

Proof. Differentiating (3.66) in x yields

$$\begin{aligned}&\phi_{1xttt}-\lambda(1+t)^{-\lambda-1}\phi_{1xt}+\frac{1}{(1+t)^{\lambda}}\phi_{1xtt}-\big(p'(\bar n)\phi_{1xxt}\big)_x\\&=-F_{xt}-[(\phi_{1x}+\hat n_1+\bar n)\mathcal{H}]_{xt}-f_{1xt}+g_{1xxt}+G_{1xxt}+W_x,\end{aligned}\tag{3.73}$$

where $W=p''(\bar n)\bar n_x\phi_{1xt}+(p''(\bar n)\bar n_t\phi_{1x})_x$. Multiplying (3.73) by $2(\beta+t)^{4+2\lambda}\phi_{1xtt}$ with $\beta=10^{\frac{1}{1-\lambda}}$, and applying similar argument as above, we obtain the desired estimates. □

Theorem 2.1–Theorem 2.3 are direct consequences of Lemmas 2.2, 3.2–3.6, and the transformation (2.17).

4. Appendix

This section is devoted to the construction of correction functions $(\hat n_1,\hat J_1,\hat n_2,\hat J_2,\hat E)(x,t)$ satisfying the equations (2.9). First of all, let us investigate the behaviors of the solutions to (1.2)–(1.3) at the far fields $x=\pm\infty$. Set

$$n_i^{\pm}(t):=n_i(\pm\infty,t),\ J_i^{\pm}(t):=J_i(\pm\infty,t),\ E^{\pm}(t):=E(\pm\infty,t),\quad i=1,2.\tag{4.1}$$

From $(1.2)_1$ and $(1.2)_3$, since $\partial_x J_i|_{x=\pm\infty}=0$ for $i=1,2$, we have

$$n_i^{\pm}(t)=n_i(\pm\infty,t)\equiv n_{\pm}.\tag{4.2}$$

Differentiating $(1.2)_5$ with respect to t and by $(1.2)_1$ and $(1.2)_3$, we get

$$E_{xt}=(n_1-n_2)_t=-(J_1-J_2)_x.$$

Integrating this equation with respect to x over $\mathbb{R}$ yields

$$\frac{d}{dt}E^+(t)-\frac{d}{dt}E^-(t)=-[J_1^+(t)-J_2^+(t)]+[J_1^-(t)-J_2^-(t)].\tag{4.3}$$

In view of $(1.2)_2$, $(1.2)_4$, and (4.2) we obtain

$$\frac{d}{dt}J_1^{\pm}(t) = n_{\pm}E^{\pm}(t) - \frac{J_1^{\pm}(t)}{(1+t)^{\lambda}}, \quad \frac{d}{dt}J_2^{\pm}(t) = -n_{\pm}E^{\pm}(t) - \frac{J_2^{\pm}(t)}{(1+t)^{\lambda}}. \tag{4.4}$$

Noting that there are six unknown functions $J_i^{\pm}(t)(i = 1, 2)$ and $E^{\pm}(t)$, to determine these six functions, we need one more boundary condition at far field

$$E(-\infty, t) = E_-. \tag{4.5}$$

Without loss of generality, we assume that

$$E_- = 0. \tag{4.6}$$

Then, by (4.4), we get

$$\frac{d}{dt}J_i^-(t) = -\frac{J_i^-(t)}{(1+t)^{\lambda}}, \; i = 1, 2.$$

Hence it is easy to compute that

$$J_i^-(t) = J_{i-}e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}, \quad i = 1, 2. \tag{4.7}$$

We next study the behaviors of $J_i^+(t)$, $i = 1, 2$ and $E^+(t)$. Notice that $J_i^+(t)$ satisfy

$$\begin{cases} \dfrac{d}{dt}J_1^+(t) = n_+E^+(t) - \dfrac{J_1^+(t)}{(1+t)^{\lambda}}, \\ \dfrac{d}{dt}J_2^+(t) = -n_+E^+(t) - \dfrac{J_2^+(t)}{(1+t)^{\lambda}}, \\ J_i^+(0) = J_{i+}, \quad i = 1, 2. \end{cases} \tag{4.8}$$

Because $E^-(t) = E(-\infty, t) = E_- = 0$, we get from (4.3) and (4.7) that

$$\frac{d}{dt}E^+(t) = -[J_1^+(t) - J_2^+(t)] + (J_{1-} - J_{2-})e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}. \tag{4.9}$$

By $(1.2)_5$, the initial data of $E^+(t)$ can be determined by

$$E^+(0) = \int\limits_{\mathbb{R}} (n_{10}(y) - n_{20}(y))dy \triangleq E_+.$$

Adding the two equations of (4.8) gives

$$\begin{cases} \dfrac{d}{dt}\big[J_1^+(t) + J_2^+(t)\big] = -\dfrac{1}{(1+t)^{\lambda}}\big[J_1^+(t) + J_2^+(t)\big], \\ \big[J_1^+(t) + J_2^+(t)\big]\big|_{t=0} = J_{1+} + J_{2+}, \end{cases} \tag{4.10}$$

from which, we get

$$J_1^+(t) + J_2^+(t) = (J_{1+} + J_{2+})e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}. \tag{4.11}$$

Similarly, subtracting $(4.8)_2$ from $(4.8)_1$, we obtain

$$\begin{cases} \dfrac{d}{dt}\big[J_1^+(t)-J_2^+(t)\big] = 2n_+E^+(t) - \dfrac{1}{(1+t)^\lambda}\big[J_1^+(t)-J_2^+(t)\big], \\ \big[J_1^+(t)-J_2^+(t)\big]\big|_{t=0} = J_{1+}-J_{2+}, \end{cases} \tag{4.12}$$

which, in combination with (4.9), leads to

$$\begin{cases} \dfrac{d^2E^+(t)}{dt^2} + \dfrac{1}{(1+t)^\lambda}\cdot\dfrac{dE^+(t)}{dt} + 2n_+E^+(t) = 0, \\ \dfrac{dE^+}{dt}\Big|_{t=0} = (J_{2+}-J_{2-})-(J_{1+}-J_{1-}), \quad E^+(0)=E_+. \end{cases} \tag{4.13}$$

It is easy to see that (4.13) is locally well-posed. To have a global solution for (4.13) we only need to derive proper decay estimates when $-1<\lambda<1$. To achieve this, we divide the estimates into two cases.

Case 1: $0 \le \lambda < 1$. Multiplying (4.13) by $2E_t^+ + (1+t)^{-\lambda}E^+$, we get

$$\begin{aligned} &\left[(E_t^+)^2 + (1+t)^{-\lambda}E^+E_t^+ + \frac{\lambda}{2}(1+t)^{-\lambda-1}(E^+)^2 + \frac{1}{2}(1+t)^{-2\lambda}(E^+)^2 + 2n_+(E^+)^2\right]_t \\ &+ (1+t)^{-\lambda}(E_t^+)^2 + \left[\frac{\lambda}{2}(\lambda+1)(1+t)^{-\lambda-2} + \lambda(1+t)^{-2\lambda-1} + 2n_+(1+t)^{-\lambda}\right](E^+)^2 = 0. \end{aligned}$$

Set $F_1 := (E_t^+)^2 + (1+t)^{-\lambda}E^+E_t^+ + \frac{\lambda}{2}(1+t)^{-\lambda-1}(E^+)^2 + \frac{1}{2}(1+t)^{-2\lambda}(E^+)^2 + 2n_+(E^+)^2$. Noting

$$\frac{1}{2}(E_t^+)^2 + (1+t)^{-\lambda}E^+E_t^+ + \frac{1}{2}(1+t)^{-2\lambda}(E^+)^2 = \left(\frac{E_t^+}{\sqrt{2}} + \frac{1}{\sqrt{2}}(1+t)^{-\lambda}E^+\right)^2 \ge 0, \tag{4.14}$$

we have

$$\frac{dF_1(t)}{dt} + \sigma(1+t)^{-\lambda}F_1(t) \le 0, \quad \text{for some constant } \sigma > 0.$$

It then follows from Gronwall's inequality that

$$F_1(t) \le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]},$$

where δ is given by (2.1). Thus, owing to (4.14), we have

$$|E_t^+(t)| + |E^+(t)| \le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]} \le C\delta e^{-Ct^{1-\lambda}}. \tag{4.15}$$

Notice that (4.9) also gives

$$J_1^+(t) - J_2^+(t) = -\frac{d}{dt}E^+(t) + (J_{1-}-J_{2-})e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

This identity, together with (4.11), implies

$$J_1^+(t) = -\frac{1}{2}E_t^+ + \frac{1}{2}(J_{1+}+J_{2+}+J_{1-}-J_{2-})e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

Thus, by (4.15),

$$J_1^+(t) \le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]} \le C\delta e^{-Ct^{1-\lambda}}. \tag{4.16}$$

Similarly,

$$J_2^+(t) \le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]} \le C\delta e^{-Ct^{1-\lambda}}. \tag{4.17}$$

In summary, it follows from (4.2), (4.6), (4.7), (4.15)–(4.17) that

$$\begin{cases} n_i(\pm\infty, t) = n_\pm, & i = 1, 2, \\ |J_i(+\infty, t)| = O(\delta)e^{-Ct^{1-\lambda}}, \ J_i(-\infty, t) = J_{i-}e^{-Ct^{1-\lambda}}, & i = 1, 2, \\ |E(+\infty, t)| = O(\delta)e^{-Ct^{1-\lambda}}, \ |E(-\infty, t)| = E_- = 0. \end{cases}$$

Because the diffusion waves $(\bar{n}, \bar{J})$ satisfy $(\bar{n}, \bar{J})(\pm\infty, t) = (n_\pm, 0)$, it holds that for $0 \le \lambda < 1$

$$\begin{cases} |n_i(\pm\infty, t) - \bar{n}(\pm\infty, t)| = 0, & i = 1, 2, \\ |J_i(+\infty, t) - \bar{J}(+\infty, t)| = O(\delta)e^{-Ct^{1-\lambda}} \ne 0, & i = 1, 2, \\ |J_i(-\infty, t) - \bar{J}(-\infty, t)| = |J_{i-}|e^{-Ct^{1-\lambda}} \ne 0, & i = 1, 2, \\ |E(+\infty, t) - 0| = O(\delta)e^{-Ct^{1-\lambda}} \ne 0, \ E(-\infty, t) = E_- = 0. \end{cases} \tag{4.18}$$

Case 2: $-1 < \lambda < 0$. Multiplying (4.13) by $2E_t^+ + (1+t)^\lambda E^+$, we get

$$\left[(E_t^+)^2 + (1+t)^\lambda E^+ E_t^+ - \frac{\lambda}{2}(1+t)^{\lambda-1}(E^+)^2 + \frac{1}{2}(E^+)^2 + 2n_+(E^+)^2\right]_t$$
$$+ [2(1+t)^{-\lambda} - (1+t)^\lambda](E_t^+)^2 + \left[\frac{\lambda}{2}(\lambda-1)(1+t)^{\lambda-2} + 2n_+(1+t)^\lambda\right](E^+)^2 = 0.$$

As in (4.15)–(4.17), we have

$$|E_t^+| + |E^+| \le C\delta e^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]} \le C\delta e^{-Ct^{1+\lambda}},$$
$$|J_1^+(t)| + |J_2^+(t)| \le C\delta e^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]} \le C\delta e^{-Ct^{1+\lambda}}.$$

And then, as in (4.18), we derive for $-1 < \lambda < 0$

$$\begin{cases} |n_i(\pm\infty, t) - \bar{n}(\pm\infty, t)| = 0, & i = 1, 2, \\ |J_i(+\infty, t) - \bar{J}(+\infty, t)| = O(\delta)e^{-Ct^{1+\lambda}} \ne 0, & i = 1, 2, \\ |J_i(-\infty, t) - \bar{J}(-\infty, t)| = |J_{i-}|e^{-Ct^{1+\lambda}} \ne 0, & i = 1, 2, \\ |E(+\infty, t) - 0| = O(\delta)e^{-Ct^{1+\lambda}} \ne 0, \ E(-\infty, t) = E_- = 0. \end{cases} \tag{4.19}$$

Thus, one can observe from (4.18) and (4.19) that there are some gaps between $J_i(\pm\infty, t)$ and $\bar{J}(\pm\infty, t)$. In other words,

$$J_i(x, t) - \bar{J}(x, t) \notin L^2(\mathbb{R}) \quad \text{and} \quad E(x, t) \notin L^2(\mathbb{R}).$$

To overcome such difficulty, inspired by the works of [9], we have to construct some correction functions $(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(x, t)$, which satisfy

$$\begin{cases} \hat{n}_{1t} + \hat{J}_{1x} = 0, \\ \hat{J}_{1t} = \check{n}\hat{E} - \dfrac{\hat{J}_1}{(1+t)^\lambda}, \\ \hat{n}_{2t} + \hat{J}_{2x} = 0, \\ \hat{J}_{2t} = -\check{n}\hat{E} - \dfrac{\hat{J}_2}{(1+t)^\lambda}, \\ \hat{E}_x = \hat{n}_1 - \hat{n}_2, \end{cases} \quad \text{with} \quad \begin{cases} \hat{J}_i(x,t) \to J_i^{\pm}(t) & \text{as} \quad x \to \pm\infty, \\ \hat{E}(x,t) \to 0 & \text{as} \quad x \to -\infty, \\ \hat{E}(x,t) \to E^+(t) & \text{as} \quad x \to +\infty. \end{cases} \tag{4.20}$$

Here $\check{n} = \check{n}(x)$, $\hat{J}_i(x,0)$ and $\hat{E}(x,0)$ are selected such that

$$\check{n}(x) \to n_\pm, \quad \hat{J}_i(x,0) \to J_{i\pm} \quad \text{and} \quad \hat{E}(x,0) \to E_0^\pm \quad \text{as} \quad x \to \pm\infty.$$

In fact, we choose

$$\begin{cases} \check{n}(x) = n_- + (n_+ - n_-) \displaystyle\int_{-\infty}^{x} m_0(y)dy, \ \hat{E}(x,0) = E_+ \displaystyle\int_{-\infty}^{x} m_0(y)dy, \\ \hat{J}_i(x,0) = J_{i-} + (J_{i+} - J_{i-}) \displaystyle\int_{-\infty}^{x} m_0(y)dy, \quad i = 1,2, \end{cases} \tag{4.21}$$

where $m_0(x)$ is chosen as $m_0(x) \geq 0$, $m_0(x) \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} m_0(y)dy = 1$. Clearly, $\check{n}$ lies between n_- and n_+. Differentiating $(4.20)_5$ with respect to t, by $(4.20)_1$ and $(4.20)_3$, we get

$$\hat{E}_{tx} = -(\hat{J}_1 - \hat{J}_2)_x.$$

Integrating this equation over $(-\infty, x)$, using (4.7), we have

$$\hat{E}_t = -(\hat{J}_1 - \hat{J}_2) + (J_{1-} - J_{2-})e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

Adding $(4.20)_2$ to $(4.20)_4$, by $(4.21)_3$, we get

$$\begin{cases} \dfrac{d}{dt}\big[\hat{J}_1 + \hat{J}_2\big] = -\dfrac{1}{(1+t)^\lambda}\big[\hat{J}_1 + \hat{J}_2\big], \\ \big[\hat{J}_1 + \hat{J}_2\big]\big|_{t=0} = (J_{1-} + J_{2-}) + \big[(J_{1+} - J_{1-}) + (J_{2+} - J_{2-})\big] \displaystyle\int_{-\infty}^{x} m_0(y)dy, \end{cases}$$

which can be solved as

$$\begin{aligned} &\hat{J}_1(x,t) + \hat{J}_2(x,t) \\ &= e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]} \left[(J_{1-} + J_{2-}) + \big(J_{1+} - J_{1-} + J_{2+} - J_{2-}\big) \int_{-\infty}^{x} m_0(y)dy \right]. \end{aligned} \tag{4.22}$$

On the other hand, subtracting $(4.20)_4$ from $(4.20)_2$ gives

$$\begin{cases} \dfrac{d}{dt}\big[\hat{J}_1(t) - \hat{J}_2(t)\big] = 2\check{n}\hat{E}(t) - \dfrac{1}{(1+t)^\lambda}\big[\hat{J}_1(t) - \hat{J}_2(t)\big], \\ \big[\hat{J}_1(t) - \hat{J}_2(t)\big]\big|_{t=0} = (J_{1-} - J_{2-}) + \big[(J_{1+} - J_{1-}) - (J_{2+} - J_{2-})\big] \displaystyle\int_{-\infty}^{x} m_0(y)dy, \end{cases}$$

which, in combination with (4.21), as in (4.13), yields

$$\begin{cases} \dfrac{d^2\hat{E}}{dt^2} + \dfrac{1}{(1+t)^\lambda}\dfrac{d\hat{E}}{dt} + 2\check{n}\hat{E} = 0, \\ \hat{E}(x,0) = E_+ \displaystyle\int_{-\infty}^{x} m_0(y)dy, \\ \dfrac{d\hat{E}}{dt}\Big|_{t=0} = [(J_{2+} - J_{2-}) - (J_{1+} - J_{1-})] \displaystyle\int_{-\infty}^{x} m_0(y)dy. \end{cases} \tag{4.23}$$

We next establish the decay estimates for $\hat{E}$.

Case 1: $0 \le \lambda < 1$. Multiplying (4.23) by $2\hat{E}_t + (1+t)^{-\lambda}\hat{E}$, and applying the same argument as that for $E^+(t)$, by Gronwall's inequality, we get

$$|\hat{E}| + |\hat{E}_t| \le C\delta \int_{-\infty}^{x} m_0(y)dy e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]} \text{ for some constant } \sigma > 0.$$

Similarly, differentiating (4.23) in x and multiplying it by $2\hat{E}_{xt} + (1+t)^{-\lambda}\hat{E}_x$, we get

$$|\hat{E}_x| \le C\delta m_0(x) e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

By (4.22) and (4.21), we have

$$\begin{aligned} \hat{J}_1(t) &= -\frac{1}{2}\hat{E}_t + \left[J_{1-} + \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-}) \int_{-\infty}^{x} m_0(y)dy \right] e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]} \\ &\le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]}, \\ \hat{J}_2(t) &= \frac{1}{2}\hat{E}_t + \left[J_{2-} + \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-}) \int_{-\infty}^{x} m_0(y)dy \right] e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]} \\ &\le C\delta e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]}. \end{aligned}$$

And then

$$\hat{J}_{1x}(t) = -\frac{1}{2}\hat{E}_{xt} + \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-}) m_0(x) e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

Owing to $(4.20)_1$,

$$\hat{n}_{1t}(t) = \frac{1}{2}\hat{E}_{xt} - \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-}) m_0(x) e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]}.$$

Thus

$$\begin{aligned} \hat{n}_1(t) &= \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-}) m_0(x) \int_{t}^{+\infty} e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]} ds + \frac{1}{2}\hat{E}_x \\ &\le C\delta m_0(x) e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]}. \end{aligned} \tag{4.24}$$

Similarly,

$$\hat{J}_{2x}(t) = \frac{1}{2}\hat{E}_{xt} + \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-})m_0(x)e^{\frac{1}{1-\lambda}[1-(1+t)^{1-\lambda}]},$$

and

$$\begin{aligned}\hat{n}_2(t) &= \frac{1}{2}(J_{1+} - J_{1-} + J_{2+} - J_{2-})m_0(x)\int_t^{+\infty} e^{\frac{1}{1-\lambda}[1-(1+s)^{1-\lambda}]}ds - \frac{1}{2}\hat{E}_x \\ &\leq C\delta m_0(x)e^{\frac{\sigma}{1-\lambda}[1-(1+t)^{1-\lambda}]}.\end{aligned} \tag{4.25}$$

Case 2: $-1 < \lambda < 0$. Multiplying (4.23) by $2\hat{E}_t + (1+t)^\lambda\hat{E}$, as in the case 1, we have

$$|\hat{E}| + |\hat{E}_t| \leq C\delta\int_{-\infty}^{x} m_0(y)dye^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]} \quad \text{for some constant } \nu > 0,$$

$$|\hat{E}_x| + |\hat{J}_1(t)| + |\hat{J}_2(t)| \leq C\delta e^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]},$$

$$\hat{n}_1(t) \leq C\delta m_0(x)e^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]}, \quad \hat{n}_2(t) \leq C\delta m_0(x)e^{\frac{\nu}{1+\lambda}[1-(1+t)^{1+\lambda}]}.$$

From the construction of the correction functions $(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})$ as shown above, we immediately obtain Lemma 2.2.

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