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Dirichlet problem for the Nicholson's blowflies equation with density-dependent diffusion

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ABSTRACT

This paper is concerned with the time delayed Nicholson's blowflies equation with degenerate diffusion. We prove the existence and uniqueness of the positive steady state solution under the Dirichlet boundary condition and we show the stability of the nontrivial steady state.

1. Introduction and preliminaries

We consider the following degenerate diffusion Nicholson's blowflies equation with time delay under homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = D\Delta u^m(t,x) - du(t,x) + pu(t-r,x)e^{-au(t-r,x)}, & x \in \Omega,\ t > 0, \\ u(t,x) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(s,x) = u_0(s,x), \quad x \in \Omega,\ s \in [-r,0], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ (for the spatial dimension $n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, $r \geq 0$, $m > 1$, $D > 0$, $p > d > 0$, $a > 0$, and the initial condition $u_0(s,x) \geq, \not\equiv 0$. For the sake of convenience, we denote the Nicholson's birth rate function $b(u) := pue^{-au}$. Here $u(t,x)$ is the mature population of a species at time t and location x, $b(u(t-r,x))$ is the birth function, $r \geq 0$ is the time delay, $D > 0$ represents the

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diffusivity of the population, and $d(u)$ is the death rate function. The diffusion term $D\Delta u^m$ (with $m > 1$) is considered to be in the form of porous medium type, which is dependent on the population density due to the population pressure [1,2].

In contrast to the linear diffusion in earlier theoretical studies, many species exhibit positive density-dependent dispersal [3,4]. Individuals tends to migrate from high-populated regions to the sparse regions due to the population pressure and fierce competition for resources. Aronson [3] described this diffusive mechanism with a density-dependent diffusion coefficient $D(u) = u^{m-1}, m > 1$. Hence, a natural question is to investigate the population dynamics model with density-dependent diffusion and time delay.

The steady states $\phi(x)$ of (1.1) satisfy the following degenerate elliptic Dirichlet problem:

$$\begin{cases} -D\Delta\phi^m(x) + d\phi(x) = p\phi(x)e^{-a\phi(x)}, \quad x \in \Omega, \\ \phi(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

For the linear diffusion case of (1.1) (i.e., $m = 1$), it is proved that the elliptic problem (1.2) admits a unique positive solution if and only if $p - d > D\lambda_1$, where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ in Ω with Dirichlet boundary condition, see Hess [5], So and Yang [6]. More importantly, it is shown in [6] that the asymptotic behavior of solutions $u(t,x)$ for (1.1) depends on the birth rate p and death rate d: (i) if $p - d < D\lambda_1$, then $u(t,x)$ converges to the zero solution; (ii) if $p - d > D\lambda_1$ and $p/d \in (1, e^2)$, then $u(t,x)$ converges to the positive steady states $\phi(x)$ of (1.2). Roughly speaking, the zero solution is globally attractive if the growth rate $p - d$ is small while the positive steady state is globally attractive if $p - d$ is large (with $p/d < e^2$).

Here for the degenerate diffusion case (i.e., $m > 1$), we show a different dynamical behavior of solutions to (1.1). The degenerate elliptic problem (1.2) always admits a unique positive steady state solution $\phi(x)$ for all the $p > d$, and this steady state $\phi(x)$ is time-globally stable when $p/d \in (1, e)$. In contrast to the linear diffusion case, the zero solution is not attractive no matter how small the growth rate $p - d$ is. This indicates that the nonlinear diffusion allows the species to survive even for small growth rate.

2. Main results

Since the problems (1.1) and (1.2) are both degenerate at where $u(t,x) = 0$ (including the boundary $\partial\Omega$ due to the homogeneous Dirichlet boundary condition), we employ the following definitions of (weak) solutions.

Definition 2.1. A function $u \in L^\infty((0,+\infty) \times \Omega)$ is called a weak solution of (1.1) if $u \geq 0$, $\nabla u^m \in L^\infty(0,+\infty; L^2(\Omega))$, $u(t,x) = 0$ in the sense of traces at $(t,x) \in (0,+\infty) \times \partial\Omega$, and for any $T > 0$ and $\psi \in C_0^\infty((-r,T) \times \Omega)$

$$\begin{aligned} &-\int_0^T\int_\Omega u(t,x)\frac{\partial\psi}{\partial t}dxdt + D\int_0^T\int_\Omega \nabla u^m \cdot \nabla\psi dxdt + \int_0^T\int_\Omega du(t,x)\psi(t,x)dxdt \\ &= \int_\Omega u_0(0,x)\psi(0,x)dx + \int_r^{\max\{T,r\}}\int_\Omega b(u(t-r,x))\psi(t,x)dxdt \\ &\quad + \int_0^{\min\{T,r\}}\int_\Omega b(u_0(t-r,x))\psi(t,x)dxdt. \end{aligned}$$

Definition 2.2. A function $\phi \in L^\infty(\Omega)$ is called a bounded positive weak solution of (1.2) if $\phi(x) \geq 0$, $\nabla\phi^m \in L^2(\Omega)$, $\phi(x) = 0$ in the sense of traces at $x \in \partial\Omega$, and for any $\psi \in C_0^\infty(\Omega)$

$$D\int_\Omega \nabla\phi^m \cdot \nabla\psi dx + \int_\Omega d\phi(x)\psi(x)dx = \int_\Omega p\phi(x)e^{-a\phi(x)}\psi(x)dx.$$

Our main results are stated as follows.

Theorem 2.1. *For any $p > d$, the degenerate elliptic problem (1.2) admits a unique positive solution $\phi(x)$ as defined in Definition 2.2.*

Theorem 2.2. *If $p/d \in (1, e)$ and the initial condition $0 \le u_0 \in L^\infty$, $u_0^m(s,x) \ge \delta\phi_1(x)$ for $s \in [-r, 0]$ and $x \in \Omega$, where $\delta > 0$ and $\phi_1(x)$ is the principal eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition. Then the solutions of (1.1) converge to the positive steady state $\phi(x)$ of (1.2) in $C(\Omega)$.*

We formulate the comparison principle of the degenerate diffusion equation (1.1) and the degenerate elliptic problem (1.2).

Lemma 2.1. *Let $\phi_1(x)$ and $\phi_2(x)$ be non-negative functions such that $\phi_i^m(x) \in H^1(\Omega)$ for $i = 1, 2$, $\phi_1(x) \ge \phi_2(x)$ in the sense of traces at $x \in \partial\Omega$, and*

$$-D\Delta\phi_1^m(x) + d\phi_1(x) + K\phi_1(x) \ge -D\Delta\phi_2^m(x) + d\phi_2(x) + K\phi_2(x),$$

in the sense of distributions, where $K \ge 0$ is a constant. Then $\phi_1(x) \ge \phi_2(x)$ a.e. on Ω.

Proof. This is proved by testing the differential inequality with $\psi(x) := (\phi_2^m(x) - \phi_1^m(x))_+$, since $(\phi_2^m(x) - \phi_1^m(x))_+ \in H_0^1(\Omega)$. $\square$

For the degenerate diffusion equation, we apply the approximate Hohmgren's approach (see Theorem 6.5 in [7], Chapter 1.3 and Chapter 3.2 in [8]) to derive the comparison principle. See also the comparison principle in [9–11] for degenerate diffusion equations.

Lemma 2.2. *Let $T > 0$, $Q_T := (0,T) \times \Omega$, and the function space $E = \{u \in L^\infty(Q_T); u \ge 0, \nabla u^m \in L^2((0,T); L^2(\Omega))\}$, $u_1, u_2 \in E$, $u_1(t,x) \ge u_2(t,x)$ on $(0,T) \times \partial\Omega$ and $u_1(0,x) \ge u_2(0,x)$ on Ω in the sense of traces, and u_1, u_2 satisfy the following differential inequality*

$$\frac{\partial u_1}{\partial t} - D\Delta u_1^m + du_1 \ge \frac{\partial u_2}{\partial t} - D\Delta u_2^m + du_2, \quad x \in \Omega,\ t > 0,$$

in the sense of distributions. Then $u_1(t,x) \ge u_2(t,x)$ almost everywhere in Q_T.

Proof. This lemma can be regarded as a simple case of a variation of Lemma 4.1 in [10]. Here we omit the details for simplicity. $\square$

For any $p > d$, we apply the comparison principle Lemma 2.1 and the monotone iteration method to show the existence of positive steady state solution of the degenerate elliptic problem (1.2). The following upper and lower solutions are defined

$$\overline{\phi}(x) := \frac{1}{a}\ln\frac{p}{d}, \quad \underline{\phi}(x) := \varepsilon\phi_1^{\frac{1}{m}}(x), \quad x \in \Omega, \tag{2.1}$$

where $\varepsilon > 0$ and $\phi_1(x)$ is the principal eigenfunction (corresponding to the principal eigenvalue $\lambda_1 > 0$) of $-\Delta$ in Ω with Dirichlet boundary condition.

Definition 2.3. A function $\phi \in L^\infty(\Omega)$ is called a bounded positive weak lower (or upper, respectively) solution of (1.2) if $\phi(x) \ge 0$, $\nabla\phi^m \in L^2(\Omega)$, $\phi(x) \ge 0$ ($\phi(x) = 0$) in the sense of traces at $x \in \partial\Omega$, and for any $\psi \in C_0^\infty(\Omega)$

$$D\int_\Omega \nabla\phi^m \cdot \nabla\psi dx + \int_\Omega d\phi(x)\psi(x)dx \ge (\le) \int_\Omega p\phi(x)e^{-a\phi(x)}\psi(x)dx.$$

Lemma 2.3. *The functions $\overline{\phi}(x)$ and $\underline{\phi}(x)$ defined in (2.1) are upper and lower solutions for the degenerate elliptic problem (1.2) and $\underline{\phi}(x) \le \overline{\phi}(x)$ for $x \in \Omega$ provided that ε is sufficiently small.*

Proof. The constant function $\overline{\phi}(x)$ satisfies that $d\overline{\phi}(x) \equiv p\overline{\phi}(x)e^{-a\overline{\phi}(x)}$. We only need to check the lower solution $\underline{\phi}(x)$ such that

$$\begin{aligned} -D\Delta\underline{\phi}^m(x) + d\underline{\phi}(x) &= -D\varepsilon^m \Delta\phi_1(x) + d\varepsilon\phi_1^{\frac{1}{m}}(x) \\ &= D\varepsilon^m\lambda_1\phi_1(x) + d\varepsilon\phi_1^{\frac{1}{m}}(x) \le p\varepsilon\phi_1^{\frac{1}{m}}(x)e^{-a\varepsilon\phi_1^{\frac{1}{m}}(x)}. \end{aligned}$$

A sufficient condition is that

$$0 < \varepsilon \le \min\left\{ \left(\frac{p-d}{2D\lambda_1 M_0^{1-1/m}} \right)^{1/(m-1)}, \frac{1}{aM_0^{1/m}} \ln\frac{2p}{p+d} \right\},$$

where $M_0 := \sup_{x\in\Omega}\phi_1(x)$. In fact, $pe^{-a\varepsilon\phi_1^{\frac{1}{m}}(x)} \ge \frac{p+d}{2}$, and then

$$D\varepsilon^m\lambda_1\phi_1(x) \le \frac{p-d}{2}\varepsilon\phi_1^{\frac{1}{m}}(x) \le p\varepsilon\phi_1^{\frac{1}{m}}(x)e^{-a\varepsilon\phi_1^{\frac{1}{m}}(x)} - d\varepsilon\phi_1^{\frac{1}{m}}(x).$$

The proof is completed if we further assume that $0 < \varepsilon \le \ln(p/d)/(aM_0^{1/m})$. □

Let $K := \|b(\cdot)\|_{\mathrm{Lip}} > 0$ be the Lipschitz constant of $b(u) = pue^{-au}$ on $[0, +\infty)$. We show that the following degenerate elliptic problem is solvable.

Lemma 2.4. *For any $0 \le \psi(x) \in L^\infty(\Omega)$, the following auxiliary problem*

$$\begin{cases} -D\Delta\phi^m(x) + d\phi(x) + K\phi(x) = K\psi(x) + p\psi(x)e^{-a\psi(x)}, & x \in \Omega, \\ \phi(x) = 0, \quad x \in \partial\Omega \end{cases} \tag{2.2}$$

admits a unique non-negative weak solution $0 \le \phi \in L^\infty(\Omega)$ such that $\nabla\phi^m(x) \in L^2(\Omega)$. We define a nonlinear operator $T[\psi] = \phi$ such that $\phi(x)$ is the unique solution of (2.2) corresponding to $\psi(x)$.

Proof. The unique solvability of the problem (2.2) follows from a standard regularization process. Here we omit the approximation process. We note that the comparison principle Lemma 2.1 is also applicable. □

Lemma 2.5. *The functions $\overline{\phi}(x)$ and $\underline{\phi}(x)$ defined in (2.1) are the upper and lower solutions for the degenerate elliptic problem (1.2) and $\underline{\phi}(x) \le \overline{\phi}(x)$ for $x \in \Omega$ according to Lemma 2.3. Then $\overline{\phi}_1(x) := T[\overline{\phi}]$, $\underline{\phi}_1(x) := T[\underline{\phi}]$, $\overline{\phi}_i(x) := T[\overline{\phi}_{i-1}]$, $\underline{\phi}_i(x) := T[\underline{\phi}_{i-1}]$ for $i = 2, 3, 4, \ldots$, satisfy*

$$0 \le \underline{\phi}(x) \le \underline{\phi}_1(x) \le \cdots \le \underline{\phi}_i(x) \le \cdots \le \overline{\phi}_i(x) \le \cdots \le \overline{\phi}_1(x) \le \overline{\phi}(x), \quad x \in \Omega$$

for all $i \in \mathbb{Z}^+$, where T is the nonlinear operator defined in Lemma 2.4. Furthermore, there exist functions $\overline{\phi}_0(x) = \lim_{i\to\infty}\overline{\phi}_i(x)$ and $\underline{\phi}_0(x) = \lim_{i\to\infty}\underline{\phi}_i(x)$ (may be the same one) such that $\overline{\phi}_0(x)$ and $\underline{\phi}_0(x)$ are positive solutions to the problem (1.2).

Proof. The order of $\overline{\phi}_i(x)$ and $\underline{\phi}_i(x)$ follows from the standard monotone iteration method and the comparison principle Lemma 2.1 is applicable. □

We need to show that the positive solution to the problem (1.2) is actually unique.

Lemma 2.6. *The positive solution to the problem (1.2) is unique.*

Proof. We first prove that the positive solution $\phi(x)$ is bounded upward by $\overline{\phi}(x)$. Let $\Omega_0 := \{x \in \Omega; \phi(x) > \overline{\phi}(x)\} \subset \Omega$. If Ω_0 is non-empty, then $\phi(x)$ satisfies $-D\Delta\phi^m(x) + d\phi(x) = b(\phi(x))$ on Ω_0, which is uniformly elliptic as $\overline{\phi}(x) < \phi(x) \leq \|\phi\|_{L^\infty(\Omega)}$. Thus, $\phi(x)$ is continuous in Ω_0 according to the classical elliptic regularity theory. Let $x_0 \in \Omega_0$ be the maximum point of $\phi(x)$ over $\overline{\Omega}_0$. Then $\nabla\phi(x_0) = 0$ and $\Delta\phi^m(x_0) \leq 0$ as $\phi^m(x)$ also attains its maximum at x_0. Then at the point x_0, (1.2) tells us that

$$p\phi(x_0)e^{-a\phi(x_0)} = -D\Delta\phi^m(x_0) + d\phi(x_0) \geq d\phi(x_0),$$

that is, $\max_{x\in\overline{\Omega}} \phi(x) = \phi(x_0) \leq \frac{1}{a}\ln\frac{p}{d} = \overline{\phi}(x)$.

Next, we show that $\phi(x) \geq \underline{\phi}(x)$ as we may change the choice of ε in (2.1) smaller if necessary. We can regard $\phi^m(x)$ as a super-harmonic function since $-D\Delta\phi^m(x) = p\phi(x)e^{-a\phi(x)} - d\phi(x) \geq 0$ as $\phi(x) \leq \overline{\phi}(x)$. Then the classical strong maximum principle implies that $\frac{\partial\phi^m(x)}{\partial\nu} < 0$ at $x \in \partial\Omega$, where ν is the unit outward normal vector. It follows that $\phi^m(x) \geq \varepsilon^m\phi_1(x) = \underline{\phi}^m(x)$ provided that ε is small enough.

Now we argue by contradiction and suppose that the positive solutions to the problem (1.2) are not unique. We have already proved that they lie between $\overline{\phi}(x)$ and $\underline{\phi}(x)$. Then the limiting functions $\overline{\phi}_0(x)$ and $\underline{\phi}_0(x)$ in Lemma 2.5 are not identically equal and they are ordered as $\overline{\phi}_0(x) \geq, \not\equiv \underline{\phi}_0(x)$. We multiple Eqs. (1.2) of $\overline{\phi}_0(x)$ and $\underline{\phi}_0(x)$ with the mth power of each other, and integrate over Ω to get

$$\int_\Omega (p\overline{\phi}_0 e^{-a\overline{\phi}_0} - d\overline{\phi}_0)\underline{\phi}_0^m dx = \int_\Omega D\nabla\overline{\phi}_0^m \cdot \nabla\underline{\phi}_0^m dx = \int_\Omega (p\underline{\phi}_0 e^{-a\underline{\phi}_0} - d\underline{\phi}_0)\overline{\phi}_0^m dx.$$

That is,

$$\int_\Omega \overline{\phi}_0^m \underline{\phi}_0^m \left(\frac{p\overline{\phi}_0 e^{-a\overline{\phi}_0} - d\overline{\phi}_0}{\overline{\phi}_0^m} - \frac{p\underline{\phi}_0 e^{-a\underline{\phi}_0} - d\underline{\phi}_0}{\underline{\phi}_0^m} \right) dx = 0. \tag{2.3}$$

We note that the function $\frac{pue^{-au} - du}{u^m} = \frac{pe^{-au} - d}{u^{m-1}}$ is strictly monotonically decreasing in $(0, \ln(p/d)/a]$, which means that the integrand in (2.3) is non-positive and not identical to zero. This contradiction completes the proof. $\square$

Proof of Theorem 2.1. The existence and uniqueness of the positive steady state solution $\phi(x)$ follow from Lemmas 2.5 and 2.6. $\square$

Using the monotone method, we present the following convergence result. The proof is based on the appropriate modification in So and Yang [6] suitable for the case with time delay.

Lemma 2.7. *Assume the assumptions in Theorem 2.2 hold. Then $u(t,x) \geq 0$ for all $x \in \overline{\Omega}$ and $t > 0$, $u(t,x) > 0$ for all $x \in \Omega$ and $t > r$, and $\limsup_{t\to\infty} u(t,x) \leq \frac{p}{ade}$.*

Proof. This proof is similar to that of Lemma 5.1 in [6] except that the comparison principle is replaced by Lemma 2.2. Here we omit the details. $\square$

Proof of Theorem 2.2. The problem (1.1) is solved by a standard regularization method and the uniform estimate $\nabla u^m \in L^\infty(0, +\infty; L^2(\Omega))$ holds. Lemma 2.7 shows that for large time t, the solution $0 < u(t,x) \leq 1/a$ for all $x \in \Omega$ since $p/d < e$. We note that the function $b(u) = pue^{-au}$ is monotonically increasing on $[0, 1/a]$. In Lemma 2.3, we constructed upper and lower solutions $\overline{\phi}(x) = \frac{1}{a}\ln\frac{p}{d}$ and $\underline{\phi}(x) = \varepsilon\phi_1^{\frac{1}{m}}(x)$ for the steady state problem (1.2). We can take ε even smaller such that

$$-D\Delta\underline{\phi}^m(x) + d\underline{\phi}(x) < p\underline{\phi}(x)e^{-a\underline{\phi}(x)}, \quad x \in \Omega, \tag{2.4}$$

$\underline{\phi}(x) \leq u_0(s,x)$ for $s \in [-r, 0]$ and $x \in \Omega$. Let $\underline{u}(t,x)$ be the solution of (1.1) with initial data $\underline{\phi}(x)$.

We assert that $\partial\underline{u}/\partial t \geq 0$ for all $t \geq 0$ and $x \in \Omega$. Define $S = \{t \geq 0; \partial\underline{u}/\partial t \geq 0, \forall x \in \Omega\}$. Then, S is not empty since $0 \in S$. The first step of the modification of monotone method suitable for the case with time delay is to prove that $(0, r) \subset S$. For $t \in (0, r)$, let $w_h(t, x) = \underline{u}(t+h, x) - \underline{u}(t, x)$, where $h > 0$ is sufficiently small such that $t + h \in (0, r]$ and $\underline{u}(h, x) - \underline{u}(0, x) \geq 0$. If such kind of h exists, then we have

$$
\begin{aligned}
\frac{\partial\underline{u}(t+h,x)}{\partial t} &- D\Delta\underline{u}^m(t+h,x) + d\underline{u}(t+h,x) = p\underline{\phi}(x)e^{-a\underline{\phi}(x)} \\
&= \frac{\partial\underline{u}(t,x)}{\partial t} - D\Delta\underline{u}^m(t,x) + d\underline{u}(t,x), \quad x \in \Omega,\ t \in (0, r-h),
\end{aligned}
$$

with the initial condition $\underline{u}(h, x) \geq \underline{u}(0, x)$ and the same Dirichlet boundary condition. Applying the comparison principle Lemma 2.2, we get $\underline{u}(t + h, x) \geq \underline{u}(t, x)$ for all $t \in (0, r - h)$ and $x \in \Omega$. It follows that $w_h(t, x) \geq 0$ and hence $\partial\underline{u}/\partial t \geq 0$. Since the choice of h can be as small as we want, we see that $(0, r) \subset S$. Noticing that S is a closed set, we have $[0, r] \subset S$ as well. We obtain by induction $[0, nr] \subset S$ for any integer $n \geq 0$. Hence $[0, +\infty) \subset S$ and $\partial\underline{u}/\partial t \geq 0$ for all $t \geq 0$ and $x \in \Omega$. Therefore, $\underline{u}(t, x)$ is monotonically increasing and converges to the unique positive steady state solution $\phi(x)$ as $t \to \infty$.

If the assumption $\underline{u}(h, x) - \underline{u}(0, x) \geq 0$ for some $h > 0$ is not true, we modify the above procedure by an approximation process as follows. Consider the regularized problem

$$
\begin{cases}
\dfrac{\partial u_\eta(t,x)}{\partial t} = D\nabla\cdot(m(\eta + u_\eta^2(t,x))^{\frac{m-1}{2}}\nabla u_\eta(t,x)) - du_\eta(t,x) \\
\qquad\qquad + pu_\eta(t-r,x)e^{-au_\eta(t-r,x)} + \eta e^{-t}, \quad x \in \Omega,\ t > 0, \\
u_\eta(t,x) = 0, \quad x \in \partial\Omega,\ t > 0, \\
u_\eta(s,x) = \phi_\eta(x), \quad x \in \Omega,\ s \in [-r, 0],
\end{cases}
\tag{2.5}
$$

where $\eta > 0$ and $\phi_\eta(x)$ is a smooth approximation of $\underline{\phi}(x)$ such that

$$
-D\nabla\cdot(m(\eta + \phi_\eta^2(x))^{\frac{m-1}{2}}\nabla\phi_\eta(x)) + d\phi_\eta(x) \leq p\phi_\eta(x)e^{-a\phi_\eta(x)}
$$

since $\underline{\phi}(x)$ satisfies (2.4). The unique existence and regularity of the solution $u_\eta(t, x)$ of the problem (2.5) are ensured by the theory of uniformly parabolic equations. And $u_\eta(t, x)$ uniformly converges to $\underline{u}(t, x)$ on any compact set $Q_T = [0, T] \times \Omega$. Then for the function $u_\eta(t, x)$ we have $\partial u_\eta(0, x)/\partial t \geq \eta e^{-\eta} > 0$ and the Hölder regularity of $u_\eta(t, x)$ implies the existence of a constant $h > 0$ such that $\partial u_\eta(t, x)/\partial t \geq 0$ for $t \in [0, h]$ and $x \in \Omega$. Now for $u_\eta(t, x)$, we apply the above argument above to find that $\partial u_\eta(t, x)/\partial t \geq 0$ for all $t \geq 0$ and $x \in \Omega$. The locally uniformly convergence of $u_\eta(t, x)$ to $\underline{u}(t, x)$ shows that $\underline{u}(t, x)$ is monotonically increasing as well.

Similarly, let $\overline{u}(t, x)$ be the solution of (1.1) with initial data $\overline{\phi}(x)$. Then we use the same argument as above to obtain that $\underline{u}(t, x)$ is monotonically decreasing and converges to the unique positive steady state solution $\phi(x)$.

Therefore, we have proved the pointwise convergence of $u(t, x)$ to $\phi(x)$ and the lower bound $u(t, x) \geq \varepsilon\phi_1^{\frac{1}{m}}(x)$. For any $\mu > 0$ sufficiently small, let $\Omega_\mu := \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > \mu\}$. Then there exists a positive constant $\beta(\mu) > 0$ such that $u(t, x) \geq \varepsilon\phi_1^{\frac{1}{m}}(x) > \beta(\mu)$ for all $t > 0$ and $x \in \Omega_{\frac{\mu}{2}}$. We see that on $(0, +\infty) \times \Omega_{\frac{\mu}{2}}$, $u(t, x)$ satisfies a uniformly parabolic problem and the norm $\|u(t, x)\|_{C^\alpha(\overline{\Omega}_\mu)}$ is bounded by a constant $C(\mu)$ for some fixed $\alpha \in (0, 1)$ according to the inner regularity estimates. The uniform Hölder continuity and the pointwise convergence imply that $u(t, x)$ converges to $\phi(x)$ in $C(\overline{\Omega}_\mu)$. The proof is completed. □

CRediT authorship contribution statement

Shanming Ji: Formal analysis, Writing - original draft. **Ming Mei:** Conceptualization, Methodology. **Zejia Wang:** Writing - review & editing.

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