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Threshold convergence results for a nonlocal time-delayed diffusion equation

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Abstract

In this paper we consider the asymptotic behavior for nonlocal dispersion Nicholson blowflies equation $u_t = D(J*u-u)-du+pu(t-\tau,x)e^{-au(t-\tau,x)}$ in the whole $\mathbb{R}^N$. By the method of Fourier transform, we first derive the decay estimates for the fundamental solutions with time-delay. Then, we obtain the threshold results with optimal convergence rates for the original solution to the constant equilibrium. Namely, when $0<\frac{p}{d}<1$, the solution $u(t,x)$ globally converges to the equilibrium 0 in the time-exponential form; when $\frac{p}{d}=1$, the solution $u(t,x)$ globally converges to 0 in the time-algebraical form; when $1<\frac{p}{d}\le e$, the solution $u(t,x)$ globally converges to u_+ in the time-exponential form; and when $e<\frac{p}{d}<e^2$, it locally converges to u_+ in the time-exponential form. This indicates that when the death rate is bigger than the birth rate, the blowflies will disappear in future. While, when the birth rate is bigger than the death rate in a certain range, then the blowflies population will reach an equilibrium after long time. The lower-higher frequency analysis plays a crucial role in the proof.

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1. Introduction

In this paper we mainly study the asymptotic behavior of solutions to the following Cauchy problem of nonlocal time-delayed diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - D(J * u - u) + du = b(u(t-\tau, x)), & t > 0,\ x \in \mathbb{R}^N, \\ u|_{t=s} = u_0(s, x), & s \in [-\tau, 0],\ x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

This model represents the spatial dynamics of a single-species population with age-structure and nonlocal diffusion such as the Australian blowflies population distribution [3,4,6,7,11,14,21,24]. Here, $u(t,x)$ denotes the total mature population of the species (after the maturation age $\tau > 0$) at time t and position x. $D > 0$ is the diffusion rate of the species, du is the death rate function for $d > 0$ representing the per capita daily adult death rate, $b(u) = pue^{-au}$ is the birth rate function, where $p > 0$ is the maximum per capita daily egg production rate, $\frac{1}{a} > 0$ is the size at which the blowfly population reproduces at its maximum rate. $J(x)$ is a nonnegative, unit and radial kernel

$$(J * u)(t,x) = \int_{\mathbb{R}^N} J(x-y)u(t,y)dy,$$

where $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, and $(J * u)(t,x) = \int_{\mathbb{R}^N} J(x-y)u(t,y)dy$ is the rate at which individuals are arriving to position x from all other places. While

$$-u(t,x) = -\int_{\mathbb{R}^N} J(x-y)u(t,x)dy$$

stands the rate at which they are leaving the location x to all other sites. We also assume that the kernel $J(x)$ satisfies the following two conditions:

(J_1) $J(x) = \prod_{i=1}^{N} J_i(x_i)$, where $J_i(x_i)$ is smooth, and $J_i(x_i) = J_i(|x_i|) \geq 0$ and $\int_{\mathbb{R}} J_i(x_i)dx_i = 1$ for $i = 1, 2, \cdots, N$.

(J_2) Fourier transform of $J(x)$ satisfies $\hat{J}(\xi) = 1 - \kappa|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ with $\alpha \in (0, 2]$ and some constant $\kappa > 0$.

The equation (1.1) is called nonlocal diffusion equation since the diffusion on the density $u(t,x)$ at a point x and time t does not only depend on $u(t,x)$, but also on all the values of u in a neighborhood of x through the convolution term $J * u$, see [1,2,8,9,12,13,15,16,32,33] and references therein. When J is nonnegative and compactly supported, this equation shares many properties with the classical heat equation $u_t = D\Delta u$. For example, the bounded stationary solutions are constant, the maximum principle holds for both of them and perturbations propagate with infinite speed [17]. However, there is no regularizing effect in general. In fact, if the equation (1.1) is simplified to the form of $\partial_t u = J * u - u$, and J is rapidly decaying (or compactly supported), then the singularity of the source solution, that is a solution with a delta measure initial condition $u_0 = \delta_0$, remains with an exponential decay. This fundamental solution can be decomposed as $w(t,x) = e^{-t}\delta_0 + v(t,x)$, where $v(t,x)$ is smooth [7]. In this way we see that there is no regularizing effect since the solution u can be written as $u = w * u_0 = e^{-t}u_0 + v * u_0$ with smooth v, which means that u is as regular as u_0, and no more better regularity.

The nonlocal dispersion equation (1.1) has been extensively studied recently. For the nonlocal dispersion equation

$$u_t = J * u - u + F(u). \tag{1.2}$$

García-Melián and Quirós [18] investigated the blow up phenomenon of the solutions to the equation (1.2) with $F(u) = u^p$, and gave the Fujita critical exponent. Regarding the structure of special solutions to the equation (1.2) like traveling wave solutions, Coville et al. [8–10] studied the existence and uniqueness (up to a shift) of traveling wave solution. One can see also the existence/nonexistence of traveling waves by Yagisita [36] and the existence of almost periodic traveling waves by Chen [5]. Furthermore, the global stability of traveling waves with exponential convergence rate for noncritical planar wavefronts, and algebraic convergence rate for critical wavefronts, were obtained by Huang et al. in [21]. When the equation is crossing-monostable, the equation and the traveling waves both lose their monotonicity, and the traveling waves are oscillating as the time-delay is big. Huang et al. [22] proved that all non-critical traveling waves, including those oscillatory waves, are time-exponentially stable, when the initial perturbations around the waves are small. Later, Xu et al. [34] proved the global stability of critical oscillatory traveling waves for a class of nonlocal dispersion equations with time-delay. For the long time behavior and stability of traveling waves for other type of diffusion, for example degenerate diffusion, we refer the readers to [19,20,23,35] and the references therein.

As for the asymptotic behavior as $t \to \infty$ for the nonlocal model (1.2) without external or internal sources, Chasseigne et al. [7] and Cortazar et al. [13] showed that the long time behavior of the solutions is determined by the behavior of the Fourier transform of J near the origin. If $\hat{J}(\xi) = 1 - \kappa|\xi|^\alpha + o(|\xi|^\alpha)(0 < \alpha \le 2)$, the asymptotic behavior is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the Laplacian. In particular, the asymptotic behavior is the same as the one for the heat equation for J is a compactly supported kernel. Ignat and Rossi [25] further obtained the asymptotic behavior of the solutions to the nonlocal equation that takes into account convective and diffusive effects. Ignat and Rossi [26] also proved that every solution to (1.1) with an initial condition $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ has an asymptotic behavior given by $\|u(t)\|_{L^q(\mathbb{R}^N)} \le Ct^{-\alpha}$, but it should be noted that this estimate is not optimal. Our main task here is to develop an new method to obtain more accurate decay estimate.

In the present paper, we are mainly interested in the asymptotic behavior for the nonlocal Cauchy problem (1.1) with time delay $\tau > 0$. We obtain the threshold results with optimal convergence rates for the original solution to the constant equilibrium. Namely, when $0 < \frac{p}{d} < 1$, the solution $u(t,x)$ globally converges to the equilibrium 0 in the time-exponential form; when $\frac{p}{d} = 1$, the solution $u(t,x)$ globally converges to 0 in the time-algebraical form; when $1 < \frac{p}{d} \le e$, the solution $u(t,x)$ globally converges to $u_+ > 0$ in the time-exponential form; and when $e < \frac{p}{d} < e^2$, the solution $u(t,x)$ locally converges to u_+ in the time-exponential form, once the initial perturbation around u_+ is small enough. The adopted approach is the optimal energy estimates by taking Fourier transform to the fundamental solutions. The lower-higher frequency analysis plays a crucial role in the proof. These results indicate that, from the ecological point of view, when the death rate is bigger than the birth rate, the blowflies will disappear in future. While, when the birth rate is bigger than the death rate in a certain range, then the blowflies population will reach an equilibrium after long time.

Roughly speaking, our first result states that the decay rate as t goes to infinity of solutions of this nonlocal problem is determined by the behavior of the Fourier transform of J near the origin. The asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the Fractional Laplacian. As mentioned in [7], a simple way to understand our results follows that, by the assumption (J_2), at low frequencies ($\xi \sim 0$), the operator is very much like the fractional Laplacian (usual Laplacian if $\alpha = 2$). Now, as time evolves, diffusion occurs and high frequencies of the initial data go to zero. This is reflected in the explicit frequency solution

$$\hat{u}(t,\xi) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

Indeed, if J is a L^1 function, then it happens that $\hat{J}(\xi) \to 0$ as $\xi \to \infty$, so that for $|\xi| \gg 1$, the high frequencies of u_0 are multiplied by something decreasing exponentially fast in time (this could be different in the case when J is a measure, but we do not consider such a case here). Thus, roughly speaking, only low frequencies of the solution will play an important role in the asymptotic behavior as $t \to \infty$, which explains why we obtain something similar to the heat equation. In fact, what we do in the proof of the linearized problem (Theorem 2.1) is precisely to separate the low frequencies, where we use the expansion (J_2), from the high frequencies that we control since they tend to zero fast enough in a suitable time scale. To this end, we first analyze the decay of solutions of the linearized problem. These solutions have a similar decay rate as the one that holds for the heat equation, see [7] and [26] where the Fourier transform plays a key role. This is a crucial step for getting the optimal convergence for the solution of (1.1) to the constant equilibrium 0 and $u_+ = \frac{1}{a}\ln\frac{p}{d}$.

Throughout this paper, $C > 0$ denotes a generic constant, and denote specific positive constant by $C_i > 0 (i = 0, 1, 2, \cdots)$. $k = (k_1, k_2, \cdots, k_N)$ denotes a multi-index with nonnegative integers $k_i \geq 0 (i = 1, 2, \cdots, N)$, and $|k| = k_1 + k_2 + \cdots + k_N$. The derivatives for multi-dimensional function are denoted as

$$\partial_x^k f(x) := \partial_{x_1}^{k_1} \cdots \partial_{x_N}^{k_N} f(x).$$

For a $N - D$ function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx, \quad i := \sqrt{-1},$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}\hat{f}(\xi)d\xi.$$

Let $L^p(\mathbb{R}^N) (p \geq 1)$ be the Lebesgue space of the integrable functions defined on $\mathbb{R}^N$, $W^{m,p}(\mathbb{R}^N) (m \geq 0, p \geq 1)$ denoted by

$$W^{m,p}(\mathbb{R}^N) = \left\{ f(x) \in L^p(\mathbb{R}^N) \mid \partial_x^k f(x) \in L^p(\mathbb{R}^N),\ |k| \leq m \right\},$$

with the norm given by

$$\|f\|_{W^{m,p}} = \Big(\sum_{|k|=0}^{m} \int_{\mathbb{R}^N} \Big| \partial_x^k f(x) \Big|^p \Big)^{\frac{1}{p}},$$

and in particular, we denote $W^{m,2}(\mathbb{R}^N)$ as $H^m(\mathbb{R}^N)$. Further, $H^s(\mathbb{R}^N)$ denotes the Sobolev space of substantial number which defined by Fourier transform, for any nonnegative real number α,

$$H^s(\mathbb{R}^N) = \Big\{ f \in L^2(\mathbb{R}^N) \mid (1+|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^N) \Big\}.$$

The rest of the paper is organized as follows. In Section 2, we first give some preliminaries about the estimates for the linear delayed ODEs that will be used to give explicit formula of solutions as well as for the proof of the asymptotic behavior. Then we list the main results of this paper. In Section 3, we consider the existence, uniqueness and asymptotic behavior of the solutions to the linear problem, and prove Theorem 2.1. In Section 4, we consider the asymptotic behavior of the solutions to the original nonlinear problem, and prove Theorems 2.2 and 2.3. Finally, in Section 5, as a direct application of the results of this paper, we give the corresponding results for the Nicholson's blowflies type equation with local dispersion.

2. Preliminaries and main results

In this section, we first give some basic results for a class of linear delayed ODEs, which will be used to derive the asymptotic behavior of the solutions for the linearized problem with nonlocal diffusion and time delay. Then, we present the main results of this paper, that is, the asymptotic behavior of the original problem (1.1) for two cases of $0 < \frac{p}{d} \le 1$ and $1 < \frac{p}{d} < e^2$.

Now, as preliminaries, we recall the linear delayed ODEs and list some basic properties of the solutions as shown in [27] and [30].

Lemma 2.1 *([27]). Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $\tau > 0$*

$$\begin{cases} \dfrac{d}{dt} z(t) + \beta z(t) = \gamma z(t-\tau), \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Then

$$z(t) = e^{-\beta(t+\tau)} e_\tau^{\bar{\gamma} t} z_0(-\tau) + \int_{-\tau}^{0} e^{-\beta(t-s)} e_\tau^{\bar{\gamma}(t-\tau-s)} [z_0'(s) + \beta z_0(s)] ds,$$

where

$$\bar{\gamma} := \gamma e^{\beta\tau},$$

*and $e_\tau^{\bar{\gamma} t}$ is the so-called **delayed exponential function** in the form*

$$e_\tau^{\bar{\gamma}t} := \begin{cases} 0, & t \in (-\infty, -\tau), \\ 1, & t \in [-\tau, 0], \\ 1 + \frac{\bar{\gamma}t}{1!}, & t \in [0, \tau], \\ 1 + \frac{\bar{\gamma}t}{1!} + \frac{\bar{\gamma}^2}{2!}(t-\tau)^2, & t \in [\tau, 2\tau], \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 + \frac{\bar{\gamma}t}{1!} + \frac{\bar{\gamma}^2}{2!}(t-\tau)^2 + \frac{\bar{\gamma}^3}{3!}(t-2\tau)^3 + \cdots + \frac{\bar{\gamma}^m}{m!}[t-(m-1)\tau]^m, & t \in [(m-1)\tau, m\tau], \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{cases}$$

and $e_\tau^{\bar{\gamma}t}$ is the fundamental solution to

$$\begin{cases} \dfrac{d}{dt}z(t) = \bar{\gamma}z(t-\tau), \\ z(s) \equiv 1, s \in [-\tau, 0]. \end{cases} \tag{2.2}$$

Note that, different from the exponential function $e^{(k_1+k_2)t} = e^{k_1 t}e^{k_2 t}$, here we have

$$e_\tau^{(k_1+k_2)t} \neq e_\tau^{k_1 t}e_\tau^{k_2 t}.$$

Furthermore, such a solution captures the following asymptotic behavior.

Lemma 2.2 *([30]). Let $\beta > 0$ and $\gamma > 0$. Then the solution $z(t)$ to (2.1) satisfies*

$$|z(t)| \le C_0 e^{-\beta t} e_\tau^{\bar{\gamma}t},$$

where

$$C_0 := e^{-\beta\tau}|z_0(-\tau)| + \int_{-\tau}^{0} e^{\beta s}|z_0'(s) + \beta z_0(s)|ds,$$

and the fundamental solution $e_\tau^{\bar{\gamma}t}$ with $\bar{\gamma} > 0$ to (2.2) satisfies

$$e_\tau^{\bar{\gamma}t} \le C(1+t)^{-\delta}e^{\bar{\gamma}t}, \quad t > 0$$

for arbitrary number $\delta > 0$.

Furthermore, when $\beta \ge \gamma \ge 0$, there exists a constant $\epsilon_1 = \epsilon_1(\beta, \gamma, \tau)$ with $0 < \epsilon_1 < 1$ for $\tau > 0$, and $\epsilon_1 = 1$ for $\tau = 0$, and $\epsilon_1 = \epsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$, such that

$$e_\tau^{\bar{\gamma}t}e^{-\beta t} \le Ce^{-\epsilon_1(\beta-\gamma)t}, \quad t > 0,$$

and the solution $z(t)$ to (2.1) satisfies

$$|z(t)| \le Ce^{-\epsilon_1(\beta-\gamma)t}, \quad t > 0.$$

Remark 2.1. We can also refer to the textbooks [31] that, when $\gamma < 0 < \beta$ and $|\gamma| \le \beta$, for any time-delay $\tau > 0$, it holds that

$$|z(t)| \le Ce^{-\nu t}, \quad t > 0,$$

for some $\nu > 0$.

Next we consider the following nonlocal linear time-delayed dispersion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - D(J * u - u) + \beta u = \gamma u(t-\tau, x), & t > 0,\ x \in \mathbb{R}^N, \\ u|_{t=s} = u_0(s, x), & s \in [-\tau, 0],\ x \in \mathbb{R}^N. \end{cases} \tag{2.3}$$

By Lemmas 2.1 and 2.2, and using the Fourier transform method, we derive the asymptotic behavior of solution of (2.3) with the optimal decay rates as follows.

Theorem 2.1. *Let $\beta \ge |\gamma|$ and $u(t,x)$ be the solution of the problem (2.3), assume that $u_0(s,\cdot) \in C([-\tau,0]; H^{m+\alpha}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ and $\partial_s u_0(s,\cdot) \in L^1([-\tau,0]; H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ with $m \ge 0$, $\alpha \in (0,2]$, then we have*

$$\left\| \partial_x^k u(t) \right\|_{L^2(\mathbb{R}^N)} \le C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_2(\beta-|\gamma|)t},\ t > 0, \tag{2.4}$$

for $|k| = 0, 1, 2, \cdots, [m]$, and $\epsilon_2 = \epsilon_2(\beta,\gamma) > 0$,

$$\begin{aligned} \mathcal{M}_{u_0}^k :=& \|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0(-\tau)\|_{H^{|k|}(\mathbb{R}^N)} \\ &+ \int_{-\tau}^{0} \left[\|u_0(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|+\alpha}(\mathbb{R}^N)} + \|u_0'(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|}(\mathbb{R}^N)} \right] ds. \end{aligned}$$

Furthermore, if $m > \frac{N}{2} + |k|$,

$$\left\| \partial_x^k u(t) \right\|_{L^\infty(\mathbb{R}^N)} \le C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{\alpha}} e^{-\epsilon_2(\beta-|\gamma|)t},\ t > 0. \tag{2.5}$$

As a direct application of Theorem 2.1, by a detailed analysis as sketched before in the introduction, we can further obtain the following asymptotic behavior of the solution of the problem (1.1).

Theorem 2.2. *When $0 < \frac{p}{d} \le 1$, suppose that initial data u_0 satisfies $0 \le u_0(s,x) \le \frac{1}{a}$, $u_0 \in C([-\tau,0]; H^{m+\alpha}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ and $\partial_s u_0(s,\cdot) \in L^1([-\tau,0]; H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ with $m \ge 0$, $\alpha \in (0,2]$. Then the problem (1.1) admits a unique solution $u(t,x)$, such that:*

- *when $0 < \frac{p}{d} < 1$, then $u(t,x)$ converges globally to $u = 0$ time-exponentially*

$$\left\| \partial_x^k u(t) \right\|_{L^2(\mathbb{R}^N)} \le C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_3(d-p)t},\ t > 0,$$

where constant $\epsilon_3 = \epsilon_3(d, p) > 0$*;*

- *when* $\frac{p}{d} = 1$*, then* $u(t, x)$ *converges globally to* $u = 0$ *time-algebraically*

$$\left\| \partial_x^k u(t) \right\|_{L^2(\mathbb{R}^N)} \leq C \mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}}, \ t > 0,$$

with $|k| < \frac{N}{2}$ *and* $N > \alpha$*.*

Theorem 2.3. *When* $1 < \frac{p}{d} < e^2$*, suppose that* $0 \leq u_0 \leq u_+$*,* $v_0 := u_0 - u_+ \in C([-\tau, 0]; H^{m+\alpha}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ *and* $\partial_s(u_0(s, \cdot) - u_+) \in L^1([-\tau, 0]; H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ *with* $m \geq 0$*,* $\alpha \in (0, 2]$*. Then the problem (1.1) admits a unique solution* $u(t, x)$*, such that:*

- *when* $1 < \frac{p}{d} \leq e$*, then* $u(t, x)$ *converges globally to* $u_+ = \frac{1}{a} \ln \frac{p}{d}$ *in the exponential form*

$$\left\| \partial_x^k (u - u_+)(t) \right\|_{L^2(\mathbb{R}^N)} \leq C \mathcal{M}_{v_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_4 d \ln \frac{p}{d} t}, \ t > 0,$$

for some $\epsilon_4 = \epsilon_4(d, p) > 0$*;*

- *when* $e < \frac{p}{d} < e^2$*, then* $u(t, x)$ *converges locally to* $u_+ = \frac{1}{a} \ln \frac{p}{d}$ *in the exponential form*

$$\left\| \partial_x^k (u - u_+)(t) \right\|_{L^2(\mathbb{R}^N)} \leq C \mathcal{M}_{v_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d (2 - \ln \frac{p}{d}) t}, \ t > 0$$

for some positive number ϵ_5*.*

Remark 2.2. In Theorem 2.3, when $e < \frac{p}{d} < e^2$, we said that the solution $u(t, x)$ converges locally to u_+ means that, different from Theorem 2.2 which has no restraint on the initial perturbation, here we need to suppose the initial perturbation of u_0 with respect to u_+, that is, $M_{v_0}^k$ small enough.

Remark 2.3. We need to indicate that in our main results, the decay rates are optimal in the sense by comparing to the linearized problem. Since for the original nonlinear problem, when $0 < \frac{p}{d} \leq e$, the birth function is convex, $b''(u) \leq 0$ for $u \in [0, u_+]$, and we can neglect the higher order terms in the decay estimations of the solution. While for the case of $e < \frac{p}{d} < e^2$, no higher order terms are neglected in the decay estimations, but as stated in Remark 2.2, the convergence is locally in this case.

Remark 2.4. The analysis for nonlinear problem is more involved than the linear case, and here we cannot use the Fourier transform directly (by the presence of the nonlinear term). Our strategy is to decompose the nonlinear equation into a linear part and a higher-order part. Then the desired estimates can be proved by using the fact that the higher-order part decays faster than the linear part.

3. Linear nonlocal reaction-diffusion equations with time-delay

The aim of this section is to prove Theorem 2.1, we will derive the solution formula for the nonlocal delayed dispersion equation (2.3), as well as its asymptotic behavior, which will play a key role to prove the main results for our original nonlinear problem in the next section. The main ingredient for our proofs is the Fourier splitting method introduced by Schonoced, see [33].

The following lemmas will be used to estimate the optimal decay rates for the solutions of the problem (2.3).

Lemma 3.1. *Let $\beta \geq \gamma \geq 0$, and $u_\pm$ be the solutions to the following problems*

$$\begin{cases} u'_\pm(t) + \beta u_\pm(t) = \pm\gamma u_\pm(t-\tau), \\ u_\pm|_{t=s} = u_{\pm,0}(s), \quad s \in [-\tau, 0], \end{cases}$$

with

$$|u_{-,0}(s)| \leq u_{+,0}(s), \quad s \in [-\tau, 0],$$

then

$$|u_-(t)| \leq u_+(t).$$

Proof. We first prove that $u_+(t) \geq 0$ for all $t > 0$. For $t \in [0, \tau]$, i.e., $t - \tau \in [-\tau, 0]$, then $u_+(t)$ satisfies

$$\begin{cases} u'_+(t) + \beta u_+(t) = \gamma u_+(t-\tau) \geq 0, \\ u_+|_{t=s} = u_{+,0}(s) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

According to maximal principle, we have

$$u_+(t) \geq 0, \ \text{for } t \in [0, \tau].$$

Repeating the same procedure on $[\tau, 2\tau], [2\tau, 3\tau], \cdots, [m\tau, (m+1)\tau]$, we can prove

$$u_+(t) \geq 0, \ \text{for } t \in [m\tau, (m+1)\tau],$$

and finally

$$u_+(t) \geq 0, \ \text{for } t \in \mathbb{R}_+.$$

Next, we prove $|u_-(t)| \leq u_+(t)$ for $t > 0$. Let

$$\overline{u}(t) := u_+(t) + u_-(t) \quad \text{and} \quad \underline{u}(t) := u_+(t) - u_-(t).$$

We will prove $\overline{u}(t) \geq 0$ and $\underline{u}(t) \geq 0$. When $t \in [0, \tau]$, then $\overline{u}(t)$ satisfies the following equation

$$\begin{cases} \overline{u}'(t) + \beta\overline{u}(t) = \gamma(u_+(t-\tau) - u_-(t-\tau)) \geq 0, \\ \overline{u}|_{t=s} = u_{+,0}(s) + u_{-,0}(s) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

By maximal principle, we have

$$\overline{u}(t) \geq 0, \ \text{for } t \in [0, \tau].$$

Repeat the same procedure on $[\tau, 2\tau], [2\tau, 3\tau], \cdots$, we have

$$\overline{u}(t) \geq 0, \ \text{for } t \in [m\tau, (m+1)\tau],$$

and then

$$\overline{u}(t) \geq 0, \ \text{for } t \in \mathbb{R}_+.$$

Similarly, we can also prove that $\underline{u}(t) \geq 0$, for $t \in \mathbb{R}_+$. Based on the above analysis, we immediately know that

$$|u_-(t)| \leq u_+(t).$$

The proof of this lemma is complete. □

In what follows, we give a comparison principle.

Lemma 3.2. *Let $\beta_1 > \beta_2 \geq \gamma \geq 0$, and let $u_i(t)$ $(i = 1, 2)$ be the solutions to the following problems*

$$\begin{cases} u_i'(t) + \beta_i u_i(t) = \gamma u_i(t-\tau), \\ u_i|_{t=s} = u_0(s) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

Then

$$0 \leq u_1(t) \leq u_2(t).$$

Proof. We first prove that $u_1(t) \geq 0$. When $t \in [0, \tau]$, we have $t - \tau \in [-\tau, 0]$, then $u_+(t)$ satisfies

$$\begin{cases} u_1'(t) + \beta u_1(t) = \gamma u_1(t-\tau) \geq 0, \\ u_1|_{t=s} = u_1(s) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

By the maximal principle, we have

$$u_1(t) \geq 0, \ \text{for } t \in [0, \tau].$$

Similarly, we can prove

$$u_1(t) \geq 0, \ \text{for } t \in [m\tau, (m+1)\tau],$$

and finally

$$u_1(t) \geq 0, \ \text{for } t \in \mathbb{R}_+.$$

Next we prove that $u_2(t) \geq u_1(t)$. Let $\tilde{u}(t) := u_2(t) - u_1(t)$, then $\tilde{u}$ satisfies the following equation

$$\begin{cases} \tilde{u}'(t) + \beta_2 u_2(t) - \beta_1 u_1(t) = \gamma \tilde{u}(t-\tau), \\ \tilde{u}|_{t=s} = 0, \quad s \in [-\tau, 0]. \end{cases}$$

When $t \in [0, \tau]$, we have

$$\tilde{u}'(t) + \beta_2 u_2(t) - \beta_1 u_1(t) = 0,$$

that is,

$$\tilde{u}'(t) + \beta_2 \tilde{u}(t) = (\beta_1 - \beta_2) u_1(t) \geq 0.$$

Then we can obtain

$$\tilde{u}(t) \geq e^{-\beta_2 t}, \quad \text{for } t \in [0, \tau].$$

Repeating this procedure step by step, we can prove

$$\tilde{u}(t) \geq e^{-\beta_2 t}, \quad \text{for } t \in [m\tau, (m+1)\tau],$$

and further

$$\tilde{u}(t) \geq e^{-\beta_2 t} \geq 0, \quad \text{for } t \in \mathbb{R}_+.$$

Namely,

$$u_2(t) \geq u_1(t).$$

The proof of this lemma is complete. □

Lemma 3.3. *Let $\beta_1 \geq \beta_2 \geq |\gamma|$, and u_1 solves the problem*

$$\begin{cases} u_1'(t) + \beta_1 u_1(t) = \gamma u_1(t-\tau), \\ u_1|_{t=s} = u_0(s), \ s \in [-\tau, 0], \end{cases}$$

and u_2 solves the following problem

$$\begin{cases} u_2'(t) + \beta_2 u_2(t) = |\gamma| u_2(t-\tau), \\ u_2|_{t=s} = |u_0(s)|, \ s \in [-\tau, 0]. \end{cases}$$

Then

$$|u_1(t)| \leq u_2(t).$$

Proof. As a consequence of Lemma 3.1 and Lemma 3.2, we can immediately deduce Lemma 3.3, and we omit the details here. □

Note that $e_\tau^{\bar{\gamma}t} e^{-\beta t}$ is the solution of the following problem

$$\begin{cases} u'(t) + \beta u(t) = \gamma u(t-\tau), \\ u|_{t=s} = e^{-\beta s}, \ s \in [-\tau, 0], \end{cases}$$

where $\bar{\gamma} := \gamma e^{\beta\tau}$ and $e_\tau^{\bar{\gamma}t}$ be defined in Lemma 2.1. We have the following decay estimates which plays an key role in the asymptotic behavior of the solution of the problem (2.3).

Lemma 3.4. *Let $\beta_1 \geq \beta_2 \geq |\gamma|$. It holds*

$$\left| e_\tau^{\bar{\gamma}_1 t} e^{-\beta_1 t} \right| \leq C \left| e_\tau^{\bar{\gamma}_2 t} e^{-\beta_2 t} \right| \leq C e^{-\epsilon_1(\beta_2 - |\gamma|)t}, \tag{3.1}$$

where $\bar{\gamma}_i := \gamma e^{\beta_i \tau}, i = 1, 2$. Let $\beta \geq |\gamma|$, we have

$$\left| e_\tau^{\bar{\gamma}(\xi)t} e^{-\beta t} \right| \leq C \left| e_\tau^{\bar{\gamma}t} e^{-\beta t} \right| \leq C e^{-\epsilon_1(\beta - |\gamma|)t}, \quad \textit{for } |\bar{\gamma}(\xi)| \geq \bar{\gamma}, \tag{3.2}$$

where $\epsilon_1 = \epsilon_1(\beta, \gamma) > 0$ and

$$|\bar{\gamma}(\xi)| = \left| \gamma e^{[A(\xi)+\beta]\tau} \right| = \left| \gamma e^{\left[D\left(1-\hat{J}(\xi)\right)+\beta\right]\tau} \right|.$$

Proof. First, by Lemma 2.2, for $\beta \geq \gamma \geq 0$, we have

$$\left| e_\tau^{\bar{\gamma}t} e^{-\beta t} \right| \leq C e^{-\epsilon_1(\beta-\gamma)t}, \tag{3.3}$$

where $0 < \epsilon_1 := \epsilon_1(\beta, \gamma) < 1$. By Lemma 3.1, we obtain that (3.3) is also established for $\beta \geq |\gamma|$, here γ may be negative. Therefore, (3.1) is derived immediately from Lemma 3.2.

As for (3.2), by the assumption (J_2), we have $\hat{J}(\xi) = 1 - \kappa|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ with $\alpha \in (0, 2]$ and $\kappa > 0$. Then there exist $0 < m_1 < m_2$, $0 < \eta < 1$ and $\tilde{a} > 0$, such that

$$\begin{cases} m_1|\xi|^\alpha \leq 1 - \hat{J}(\xi) \leq m_2|\xi|^\alpha, & \text{as } |\xi| \leq \tilde{a}, \\ \eta := m_1\tilde{a}^\alpha \leq 1 - \hat{J}(\xi) \leq m_2|\xi|^\alpha, & \text{as } |\xi| \geq \tilde{a}. \end{cases} \tag{3.4}$$

Then, a simple calculation gives that

$$|A(\xi) + \beta| \geq \left| D\left(1 - \hat{J}(\xi)\right) + \beta \right| \geq \left| D|0|^\alpha + \beta \right| = \beta, \quad \xi \in \mathbb{R}^N,$$

and (3.2) followers by Lemma 3.2 immediately. The proof of this lemma is complete. □

Now, we are in the position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. By taking Fourier transform to (2.3), and denoting the Fourier transform of $u(t,x)$ by $\hat{u}(t,\xi)$, we have

$$\begin{cases} \dfrac{d\hat{u}}{dt} + (A(\xi)+\beta)\hat{u} = \gamma\hat{u}(t-\tau,\xi), \\ \hat{u}|_{t=s} = \hat{u}(s,\xi), \ s\in[-\tau,0], \end{cases} \tag{3.5}$$

where

$$A(\xi) = -D\left(\hat{J}(\xi)-1\right).$$

From (2.2), the time-delayed equation (3.5) can be solved by

$$\begin{aligned} \hat{u}(t,\xi) =& e_{\tau}^{\bar{\gamma}(\xi)t} e^{-[A(\xi)+\beta](t+\tau)}\hat{u}_0(-\tau,\xi) \\ &+ \int_{-\tau}^{0} e_{\tau}^{\bar{\gamma}(\xi)(t-\tau-s)} e^{-[A(\xi)+\beta](t-s)}\left[\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right]ds \\ =:& \hat{G}(t,\xi)\hat{u}_0(-\tau,\xi) \\ &+ \int_{-\tau}^{0} \hat{G}(t-\tau-s,\xi)\left[\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right]ds, \end{aligned} \tag{3.6}$$

where

$$\bar{\gamma}(\xi) := \gamma e^{(A(\xi)+\beta)\tau},$$

and

$$\hat{G}(t,\xi) := e_{\tau}^{\bar{\gamma}(\xi)t} e^{-[A(\xi)+\beta](t+\tau)}.$$

Taking the inverse Fourier transform to (3.6), then

$$\begin{aligned} u(t,x) =& G(t,\cdot) * u(-\tau,\cdot) \\ &+ \int_{-\tau}^{0} G(t-\tau-s,\cdot) * \left[\frac{d}{ds}u_0(s) - D(J*u_0-u_0) + \beta u_0(s)\right]ds \\ =& \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} e_{\tau}^{\bar{\gamma}(\xi)t} e^{-[A(\xi)+\beta](t+\tau)}\hat{u}_0(-\tau,\xi)d\xi \\ &+ \frac{1}{(2\pi)^N}\int_{-\tau}^{0}\int_{\mathbb{R}^N} e^{ix\cdot\xi} e_{\tau}^{\bar{\gamma}(\xi)(t-\tau-s)} e^{-[A(\xi)+\beta](t-s)} \end{aligned}$$

$$\times \left[\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right] d\xi ds,$$

and its derivatives

$$\begin{aligned}
\partial_x^k u(t,x) =& \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \prod_{j=1}^N (i\xi_j)^{k_j} e^{ix\cdot\xi} e_\tau^{\bar{\gamma}(\xi)t} e^{-[A(\xi)+\beta](t+\tau)} \hat{u}_0(-\tau,\xi) d\xi \\
&+ \frac{1}{(2\pi)^N} \int_{-\tau}^0 \int_{\mathbb{R}^N} \prod_{j=1}^N (i\xi_j)^{k_j} e^{ix\cdot\xi} e_\tau^{\bar{\gamma}(\xi)(t-\tau-s)} e^{-[A(\xi)+\beta](t-s)} \\
&\qquad \times \left[\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right] d\xi ds \\
=:& \mathcal{F}^{-1}[I_1](t,x) + \int_{-\tau}^0 \mathcal{F}^{-1}[I_2](t-\tau-s,x) ds
\end{aligned}$$

for $k_j = 0, 1, \cdots$, and $j = 1, 2, \cdots, N$. Using Parseval's equality, we have

$$\begin{aligned}
\left\|\partial_x^k u(t)\right\|_{L^2(\mathbb{R}^N)} \le& \left\|\mathcal{F}^{-1}[I_1](t)\right\|_{L^2(\mathbb{R}^N)} + \int_{-\tau}^0 \left\|\mathcal{F}^{-1}[I_2](t-\tau-s)\right\|_{L^2(\mathbb{R}^N)} ds \\
=& \|I_1(t)\|_{L^2(\mathbb{R}^N)} + \int_{-\tau}^0 \|I_2(t-\tau-s)\|_{L^2(\mathbb{R}^N)} ds. \qquad (3.7)
\end{aligned}$$

Next we are going to obtain the decay estimates of $\|I_1(t)\|_{L^2(\mathbb{R}^N)}$ and $\|I_2(t-\tau-s)\|_{L^2(\mathbb{R}^N)}$. By (3.7), we have

$$\begin{aligned}
\|I_1(t)\|^2_{L^2(\mathbb{R}^N)} =& \int_{\mathbb{R}^N} \left| \prod_{j=1}^N (i\xi_j)^{k_j} e_\tau^{\bar{\gamma}(\xi)t} e^{-[A(\xi)+\beta](t+\tau)} \hat{u}_0(-\tau,\xi) \right|^2 d\xi \\
=& \int_{\mathbb{R}^N} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} \left| e_\tau^{\bar{\gamma}(\xi)t} e^{-\beta(t+\tau)} \right|^2 \left| e^{-D\left(1-\hat{J}(\xi)\right)(t+\tau)} \hat{u}_0(-\tau,\xi) \right|^2 d\xi \\
\le& \int_{\mathbb{R}^N} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} \left| e_\tau^{\bar{\gamma}t} e^{-\beta(t+\tau)} \right|^2 \left| e^{-D\left(1-\hat{J}(\xi)\right)(t+\tau)} \hat{u}_0(-\tau,\xi) \right|^2 d\xi \\
\le& Ce^{-2\epsilon_2(\beta-|\gamma|)t} \int_{\mathbb{R}^N} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} \left| e^{-D\left(1-\hat{J}(\xi)\right)(t+\tau)} \hat{u}_0(-\tau,\xi) \right|^2 d\xi, \qquad (3.8)
\end{aligned}$$

where $\epsilon_2 = \epsilon_2(\beta, \gamma) > 0$. Furthermore, according to (3.4), we split (3.8) into two parts: low frequency and high frequency

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2D\left(1-\hat{J}(\xi)\right)(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&= \int_{|\xi|\le\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2D\left(1-\hat{J}(\xi)\right)(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&\quad + \int_{|\xi|\ge\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2D\left(1-\hat{J}(\xi)\right)(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&\le \int_{|\xi|\le\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2Dm_1|\xi|^\alpha(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi + \int_{|\xi|\ge\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2D\eta(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&\le C \sup_{\xi\in\mathbb{R}^N} |\hat{u}_0(-\tau,\xi)|^2 \int_{|\xi|\le\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2Dm_1|\xi|^\alpha(1+t)} d\xi \\
&\quad + \int_{|\xi|\ge\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} e^{-2D\eta(t+\tau)} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&\le C \sup_{\xi\in\mathbb{R}^N} |\hat{u}_0(-\tau,\xi)|^2 (1+t)^{-\frac{N+2|k|}{\alpha}} \int_{|\xi|\le\tilde{a}} \prod_{j=1}^{N} \left|\xi_j(1+t)^{\frac{1}{\alpha}}\right|^{2k_j} e^{-2Dm_1\left|\xi(1+t)^{\frac{1}{\alpha}}\right|^\alpha} d\xi (1+t)^{\frac{1}{\alpha}} \\
&\quad + e^{-2D\eta(t+\tau)} \int_{|\xi|\ge\tilde{a}} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} |\hat{u}_0(-\tau,\xi)|^2 d\xi \\
&\le C \left(\|u_0(-\tau)\|^2_{L^1(\mathbb{R}^N)} + \|u_0(-\tau)\|^2_{H^{|k|}(\mathbb{R}^N)} \right) (1+t)^{-\frac{N+2|k|}{\alpha}}. \qquad (3.9)
\end{aligned}
$$

Here we have used the property of Fourier transform that

$$
\sup_{\xi\in\mathbb{R}^N} |\hat{u}_0(-\tau,\xi)|^2 \le \left(\int_{\mathbb{R}^N} |u_0(-\tau,\xi)| d\xi \right)^2 = \|u_0(-\tau)\|^2_{L^1(\mathbb{R}^N)},
$$

and

$$
\int_{\mathbb{R}^N} \left| \prod_{j=1}^{N} (\xi_j)^{k_j} \hat{u}_0(-\tau,\xi) \right|^2 d\xi = \int_{\mathbb{R}^N} \left| \partial_x^k \widehat{u_0(-\tau}, \xi) \right|^2 d\xi
$$

$$= \int_{\mathbb{R}^N} \left|\partial_x^k u_0(-\tau, x)\right|^2 dx \le \|u_0(-\tau)\|^2_{H^{|k|}(\mathbb{R}^N)}.$$

Substituting (3.9) into (3.8), we immediately obtain

$$\|I_1(t)\|_{L^2(\mathbb{R}^N)} \le C\left(\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0(-\tau)\|_{H^{|k|}(\mathbb{R}^N)}\right)(1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_2(\beta-|\gamma|)t}. \quad (3.10)$$

As for $\|I_2(t-\tau-s)\|_{L^2(\mathbb{R}^N)}$, we also have

$$\begin{aligned}
&\|I_2(t-\tau-s)\|^2_{L^2(\mathbb{R}^N)} \\
=&\int_{\mathbb{R}^N} \prod_{j=1}^N \left|i\xi_j\right|^{2k_j} \left|e_\tau^{\bar{\gamma}(\xi)(t-\tau-s)} e^{-[A(\xi)+\beta](t-s)}\right|^2 \left|\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right|^2 d\xi \\
\le& Ce^{-2\epsilon_2(\beta-|\gamma|)(t-s)} \int_{\mathbb{R}^N} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} e^{-2D\left(1-\hat{J}(\xi)\right)(t-s)} \left|\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right|^2 d\xi.
\end{aligned} \quad (3.11)$$

Similar to the approach as (3.9), we divide the last formula above into low frequency and high frequency two parts

$$\begin{aligned}
&\int_{\mathbb{R}^N} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} e^{-2D\left(1-\hat{J}(\xi)\right)(t-s)} \left|\frac{d}{ds}\hat{u}_0(s,\xi) + (A(\xi)+\beta)\hat{u}_0(s,\xi)\right|^2 d\xi \\
\le& C\int_{|\xi|\le\tilde{a}} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} e^{-2Dm_1|\xi|^\alpha(t-s)} \left(\left|\frac{d}{ds}\hat{u}_0(s,\xi)\right|^2 + \left|(Dm_2|\tilde{a}|^\alpha+\beta)\hat{u}_0(s,\xi)\right|^2\right) d\xi \\
&+ C\int_{|\xi|\ge\tilde{a}} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} e^{-2D\eta(t-s)} \left(\left|\frac{d}{ds}\hat{u}_0(s,\xi)\right|^2 + \left|(Dm_2|\xi|^\alpha+\beta)\hat{u}_0(s,\xi)\right|^2\right) d\xi \\
\le& C\sup_{\xi\in\mathbb{R}^N} \left(\left|\frac{d}{ds}\hat{u}_0(s,\xi)\right|^2 + |\hat{u}_0(s,\xi)|^2\right)(t-s)^{-\frac{N+2|k|}{\alpha}} \\
&\times \int_{|\xi|\le\tilde{a}} \prod_{j=1}^N \left|\xi_j(t-s)^{\frac{1}{\alpha}}\right|^{2k_j} e^{-2Dm_1\left|\xi(t-s)^{\frac{1}{\alpha}}\right|^\alpha} d\xi(t-s)^{\frac{1}{\alpha}} \\
&+ Ce^{-2D\eta(t-s)} \int_{|\xi|\ge\tilde{a}} \prod_{j=1}^N \left|\xi_j\right|^{2k_j} \left(\left|\frac{d}{ds}\hat{u}_0(s,\xi)\right|^2 + \left|(Dm_2|\xi|^\alpha+\beta)\hat{u}_0(s,\xi)\right|^2\right) d\xi \\
\le& C\left(\|u_0'(s)\|^2_{L^1(\mathbb{R}^N)} + \|u_0(s)\|^2_{L^1(\mathbb{R}^N)} + \|u_0'(s)\|^2_{H^{|k|}(\mathbb{R}^N)} + \|u_0(s)\|^2_{H^{|k|+\alpha}(\mathbb{R}^N)}\right)(t-s)^{-\frac{N+2|k|}{\alpha}},
\end{aligned} \quad (3.12)$$

where we have used

$$\int_{\mathbb{R}^N} \prod_{j=1}^{N} \left|\xi_j\right|^{2k_j} |\xi|^{2\alpha} |\hat{u}_0(s,\xi)|^2 d\xi = \int_{\mathbb{R}^N} |\xi|^{2\alpha} \left|\widehat{\partial_x^k u_0(s,\xi)}\right|^2 d\xi \le \left\|\partial_x^k u_0(s)\right\|_{H^\alpha(\mathbb{R}^N)}.$$

Combining (3.11) and (3.12), then we obtain

$$\begin{aligned}
&\int_{-\tau}^{0} \|I_2(t-\tau-s)\|_{L^2(\mathbb{R}^N)} ds \\
\le& C \int_{-\tau}^{0} \left(\|u_0(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|+\alpha}(\mathbb{R}^N)} + \|u_0'(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|}(\mathbb{R}^N)}\right) (t-s)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_2(\beta-|\gamma|)(t-s)} ds \\
\le& C(1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_2(\beta-|\gamma|)t} \int_{-\tau}^{0} \left(\|u_0(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|+\alpha}(\mathbb{R}^N)} + \|u_0'(s)\|_{L^1(\mathbb{R}^N)\cap H^{|k|}(\mathbb{R}^N)}\right) \\
&\qquad \times \left(\frac{t-s}{1+t}\right)^{-\frac{N+2|k|}{2\alpha}} e^{\epsilon_2(\beta-|\gamma|)s} ds \\
\le& C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_2(\beta-|\gamma|)t}.
\end{aligned} \tag{3.13}$$

Substituting (3.10) and (3.13) into (3.7), we immediately obtain (2.4). As for (2.5), taking a similar approach, we can first obtain the estimates for higher order term. Then, (2.5) can be concluded from the Sobolev's embedding theorem. The proof of this theorem is complete. □

4. Case of $0 < \frac{p}{d} \le 1$: convergence to 0

In this section, basing on the results for linear problem in Theorem 2.1, we further consider the asymptotic behavior of the solutions for the nonlinear problem (1.1). Note that the constant equilibria for the equation (1.1) can be found by solving $du = pue^{-au}$. This equation admits only two roots:

$$u_- = 0, \quad u_+ = \frac{1}{a} \ln \frac{p}{d}.$$

Next, we first study the decay estimate for case of $0 < \frac{p}{d} \le 1$, and derive the decay rate of the solution $u(t,x)$ to constant equilibrium state 0.

Lemma 4.1 *(Boundedness). Let $u(t,x)$ be a solution of the problem (1.1), then it holds*

$$0 \le u(t,x) \le \frac{1}{a}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

Proof. For $t \in [0, \tau]$, i.e., $t - \tau \in [-\tau, 0]$, then we have

$$\begin{cases} u_t - D(J * u - u) + du = b(u(t-\tau, x)) = b(u_0(t-\tau, x)) \geq 0, \\ u|_{t=s} = u_0(s, x) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

By maximal principle, we have

$$u(t, x) \geq 0, \quad \text{for } (t, x) \in [0, \tau] \times \mathbb{R}^N.$$

Repeating the same procedure on $[\tau, 2\tau], [2\tau, 3\tau], \cdots$, we can prove

$$u(t, x) \geq 0, \quad \text{for } (t, x) \in [m\tau, (m+1)\tau] \times \mathbb{R}^N,$$

and finally

$$u(t, x) \geq 0, \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

Next, we prove $u(t, x) \leq \frac{1}{a}$. Let

$$v(t, x) = \frac{1}{a} - u(t, x),$$

then

$$\begin{cases} v_t - D(J * v - v) + dv = \dfrac{d}{a} - b(u(t-\tau, x)), \\ v|_{t=s} = \dfrac{1}{a} - u_0(s, x) \geq 0, \ s \in [-\tau, 0]. \end{cases}$$

Since

$$\frac{d}{a} - b(u(t-\tau, x)) \geq \frac{d}{a} - \frac{p}{ae} = \frac{d}{a}\left[1 - \frac{p}{de}\right] \geq 0,$$

here we have used the condition $0 < \frac{p}{d} \leq 1$. Therefore, similar to the above procedure, we obtain $v(t, x) \geq 0$. i.e., $u(t, x) \leq \frac{1}{a}$. The proof of this lemma is complete. □

Motivated by [21], we will use the framework of Banach's fixed point theorem to prove the existence and uniqueness of the solution. Clearly, for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$, the solution u of the equation (1.1) satisfies

$$\begin{aligned} v(t, x) =& e^{-\mu t} u(0, x) \\ &+ \int_0^t e^{-\mu(t-s)} \Big[D \int_{\mathbb{R}^{\mathbb{N}}} J(x-y) u(s, y) dy + (\mu - D - d) u(s, x) + b(u(s-\tau, x)) \Big] ds, \end{aligned} \tag{4.1}$$

where $\mu := 1 + d + p$. Fix $T > 0$ and define a space

$$X_T := \Big\{ u | u(t,x) \in C([-\tau,T]\times\mathbb{R}^N), 0 \le u(t,x) \le \frac{1}{a},$$
$$u(s,x) = u_0(s,x), (s,x) \in [-\tau,0]\times\mathbb{R}^N \Big\}$$

equipped with the norm

$$\|u\|_{X_T} = \sup_{t\in[0,T]} e^{-\mu t}\|u(t)\|_{L^\infty(\mathbb{R}^N)}.$$

It is clear that X_T is a Banach space. Then, we define an operator $S_T : X_T \to X_T$ as follows:

$$S_T u = \begin{cases} e^{-\mu t}u(0,x) + \int_0^t e^{-\mu(t-s)} \\ \quad \times\Big[D\int_{\mathbb{R}^N} J(x-y)u(s,y)dy + (\mu - D - d)u(s,x) + b(u(s-\tau,x))\Big]ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[0,T]\times\mathbb{R}^N, \\ u_0(s,x), \qquad\qquad\qquad\qquad (s,x)\in[-\tau,0]\times\mathbb{R}^N. \end{cases}$$

Next, we will obtain the existence and uniqueness of the solution as a fixed point of the operator S_T.

Lemma 4.2 *(Existence and uniqueness). Let $u_0(t,x) \in C([-\tau,0]; C(\mathbb{R}^N))$ with $0 \le u_0(t,x) \le \frac{1}{a}$ for $(t,x)\in[-\tau,0]\times\mathbb{R}^N$, then the solution to (1.1) uniquely and globally exists, and satisfies that $u \in C^1([0,\infty]; C(\mathbb{R}^N))$, and $0 \le u(t,x) \le \frac{1}{a}$ for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^N$.*

Proof. Now we are going to prove that S_T is a contractive operator from X_T to X_T. We first prove that S_T maps X_T to X_T. In fact, if $u \in X_T$, using the fate that $\int_{\mathbb{R}^N} J(x)dx = 1$, and $\mu - 1 - d > 0$, then we have

$$\begin{aligned} 0 \le & S_T u \\ \le & e^{-\mu t}\frac{1}{a} + \int_0^t e^{-\mu(t-s)}\left[D\int_{\mathbb{R}^N} J(x-y)\frac{1}{a}dy + (\mu - D - d)\frac{1}{a} + b(\frac{1}{a})\right]ds \\ \le & e^{-\mu t}\frac{1}{a} + \int_0^t e^{-\mu(t-s)}\left[\frac{\mu - d}{a} + \frac{p}{ae}\right]ds \\ \le & \frac{1}{a}, \end{aligned}$$

which combined with the continuity of S_T proves $S_T u \in X_T$, namely, $S_T(X_T) \subseteq X_T$.

Now, we prove that S_T is a contractive operator. In fact, let $u_1, u_2 \in X_T$, and $v = u_1 - u_2$, then we have

$$S_T(u_1) - S_T(u_2) = \int_0^t e^{-\mu(t-s)}\Big[D\int_{\mathbb{R}^N} J(x-y)v(s,y)dy + (\mu - D - d)v(s,x)$$
$$+ (b(u_1(s-\tau,x)) - b(u_2(s-\tau,x)))\Big]ds.$$

So, we have

$$\begin{aligned}
&|S_T(u_1) - S_T(u_2)|\, e^{-\mu t} \\
\le& \int_0^t e^{-2\mu(t-s)}\,(\mu+d)\,\|v\|_{X_T} ds \\
&+ \max_{u\in[0,u_+]} |b'(u)| \begin{cases} e^{2\mu\tau}\int_0^{t-\tau} e^{-2\mu(t-s)}\|v\|_{X_T}ds, & \text{for } t\ge\tau \\ 0, & \text{for } 0\le t\le\tau \end{cases} \\
\le& \frac{1}{2\mu}\Big((\mu+d)(1-e^{-2\mu t}) + pe^{2\mu\tau}(e^{-2\mu\tau} - e^{-2\mu t})\Big)\|v\|_{X_T} \\
\le& \frac{\mu+d+p}{2\mu}\|v\|_{X_T} \\
=& \frac{2\mu-1}{2\mu}\|v\|_{X_T} \\
=:& \rho\|v\|_{X_T}
\end{aligned}$$

for $0 < \rho := \frac{2\mu-1}{2\mu} < 1$, namely, we prove that the map S_T is contractive

$$\|S_T(u_1) - S_T(u_2)\|_{X_T} \le \rho\|u_1 - u_2\|_{X_T}.$$

Hence, by the Banach fixed-point theorem, S_T has a unique fixed point u in X_T, i.e., the integral equation (4.1) has a unique solution on $[0,T]$ for any given $T > 0$. Differentiating (4.1) with respect to t, we get the unique solvability of the original problem (1.1). By the equation itself, we can easily confirm that $u \in C^1([0,\infty]; C(\mathbb{R}^N))$. The proof of this lemma is complete. □

Now, we are in the position to give the proof of Theorem 2.2.

Proof of Theorem 2.2. For $0 \le u(t,x) \le \frac{1}{a}$, linearize $f(u)$ by Taylor's formula, then

$$\begin{aligned}
b(u(t-\tau,x)) =& b(0) + b'(0)u(t-\tau,x) + \frac{b''(\tilde{u})}{2}u^2(t-\tau,x) \\
=& pu(t-\tau,x) - o(1)u^2(t-\tau,x) \\
\le& pu(t-\tau,x),
\end{aligned}$$

provided that $\tilde{u} \in (0, u(t-\tau,x))$. So the problem (1.1) can be rewritten as

$$\begin{cases} u_t - D(J*u-u) + du - pu(t-\tau,x) = \dfrac{b''(\tilde{u})}{2}u^2(t-\tau,x), \\ u|_{t=s} = u_0(s,x). \end{cases} \tag{4.2}$$

Noting that $b(u) = pue^{-au}$, then $b(u)$ concave downward in $[0, \frac{2}{a}]$, and $b''(u) < 0$ for $u \in [0, \frac{1}{a}]$. Applying the fundamental solution formula to (4.2), we have

$$\begin{aligned} u(t,x) = & \int_{\mathbb{R}^N} G(t,0;x,y)u_0(-\tau,y)dy \\ & + \int_{-\tau}^{0}\int_{\mathbb{R}^N} G(t,s;x,y)\left[\frac{d}{ds}u_0(s,y) - D(J*u_0-u_0) + du_0(s,y)\right]dyds \\ & + \int_{0}^{t}\int_{\mathbb{R}^N} G(t,s;x,y)\frac{b''(\tilde{u})}{2}u^2(s-\tau,y)dyds \\ \le & \int_{\mathbb{R}^N} G(t,0;x,y)u_0(-\tau,y)dy \\ & + \int_{-\tau}^{0}\int_{\mathbb{R}^N} G(t,s;x,y)\left[\frac{d}{ds}u_0(s,y) - D(J*u_0-u_0) + du_0(s,y)\right]dyds, \end{aligned}$$

where $G(t;x)$ is the fundamental solution of the time-delayed dispersal equation

$$u_t - D(J*u-u) + du = pu(t-\tau,x).$$

On the concrete expression and properties of $G(t,x)$, we can refer to Lemma 2.1. Furthermore, by Theorem 2.1 and the properties of convolution, we have the following L^2-decay estimate,

$$\begin{aligned} & \|u(t)\|_{L^2(\mathbb{R}^N)} \\ \le & \|G(t)*u_0(-\tau)\|_{L^2(\mathbb{R}^N)} + \int_{-\tau}^{0}\left\|G(t,s)*\left[\frac{d}{ds}u_0(s) - D(J*u_0-u_0) + du_0(s)\right]\right\|_{L^2(\mathbb{R}^N)} ds \\ \le & C\|G(t)\|_{L^2(\mathbb{R}^N)}\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} \\ & + \int_{-\tau}^{0}\|G(t,s)\|_{L^2(\mathbb{R}^N)}\left\|\frac{d}{ds}u_0(s) - D(J*u_0-u_0) + du_0(s)\right\|_{L^1(\mathbb{R}^N)} ds \\ \le & C\mathcal{M}_{u_0}^0(1+t)^{-\frac{N}{2\alpha}}e^{-\epsilon_3(d-p)t}, \end{aligned} \tag{4.3}$$

where $\epsilon_3 = \epsilon_3(d,p) > 0$. Next, we prove the decay estimate of the higher order derivation. Note that

$$\partial_x^k u(t,x) = \int_{\mathbb{R}^N} \partial_x^k G(t,0;x,y)u_0(-\tau,y)dy$$
$$+ \int_{-\tau}^{0}\int_{\mathbb{R}^N} \partial_x^k G(t,s;x,y)\left[\frac{d}{ds}u_0(s,y) - D(J*u_0-u_0) + du_0(s,y)\right]dyds$$
$$+ \int_{0}^{t}\int_{\mathbb{R}^N} \partial_x^k G(t,s;x,y)\left[\frac{b''(\tilde{u})}{2}u^2(s-\tau,y)\right]dyds.$$

Then, combining with (4.3), for $N > \alpha$ and $|k| < \frac{N}{2}$, we have

$$\begin{aligned}
&\left\|\partial_x^k u(t)\right\|_{L^2(\mathbb{R}^N)} \\
\le &\left\|\partial_x^k G(t) * u_0(-\tau)\right\|_{L^2(\mathbb{R}^N)} \\
&+ \int_{-\tau}^{0} \left\|\partial_x^k G(t,s) * \left[\frac{d}{ds}u_0(s) - D(J*u_0-u_0) + du_0(s)\right]\right\|_{L^2(\mathbb{R}^N)} ds \\
&+ C\int_0^t \left\|\partial_x^k G(t,s) * u^2(s-\tau)\right\|_{L^2(\mathbb{R}^N)} ds \\
\le &\left\|\partial_x^k G(t)\right\|_{L^2(\mathbb{R}^N)} \|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} \\
&+ \int_{-\tau}^{0} \left\|\partial_x^k G(t,s)\right\|_{L^2(\mathbb{R}^N)} \left\|\frac{d}{ds}u_0(s) - D(J*u_0-u_0) + du_0(s)\right\|_{L^1(\mathbb{R}^N)} ds \\
&+ C\int_0^t \left\|\partial_x^k G(t,s)\right\|_{L^2(\mathbb{R}^N)} \left\|u^2(s-\tau)\right\|_{L^1(\mathbb{R}^N)} ds \\
\le &C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_3(d-p)t} \\
&+ C\mathcal{M}_{u_0}^k \int_0^t e^{-\epsilon_3(d-p)(t-s)}(1+t-s)^{-\frac{N+2|k|}{2\alpha}} \|u(s-\tau)\|_{L^2(\mathbb{R}^N)}^2 ds \\
\le &C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_3(d-p)t} \\
&+ C\mathcal{M}_{u_0}^k \int_0^t (1+t-s)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_3(d-p)(t-s)}(1+s-\tau)^{-\frac{N}{\alpha}} e^{-2\epsilon_3(d-p)(s-\tau)} ds \\
\le &C\mathcal{M}_{u_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_3(d-p)t}
\end{aligned}$$

$$
\begin{aligned}
&+C\mathcal{M}_{u_0}^{k}e^{-\epsilon_3(d-p)t}\int_0^t e^{-\epsilon_3(d-p)(s-2\tau)}\left[(1+t-s)^{-\frac{N+2|k|}{2\alpha}}(1+s-\tau)^{-\frac{N}{\alpha}}\right]ds\\
&\le\begin{cases} C\mathcal{M}_{u_0}^{k}(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_3(d-p)t}, & \text{if } d>p,\\ C\mathcal{M}_{u_0}^{k}(1+t)^{-\frac{N+2|k|}{2\alpha}}, & \text{if } d=p.\end{cases}
\end{aligned}
$$

Here, we have used the following inequality

$$
\int_0^t (1+t-s)^{-a}(1+s)^{-b}ds \le \begin{cases} C(1+t)^{-\min\{a,b\}}, & \text{if } \max\{a,b\}>1,\\ C(1+t)^{-\min\{a,b\}}\ln(2+t), & \text{if } \max\{a,b\}=1,\\ C(1+t)^{1-a-b}, & \text{if } \max\{a,b\}<1.\end{cases}
$$

The proof of this theorem is complete. □

5. Case of $1<\frac{p}{d}<e^2$: convergence to u_+

In what follows, we consider the asymptotic behavior of the solutions for the case of $1<\frac{p}{d}<e^2$, and give the proof of Theorem 2.3. Let $u(t,x)$ be the solution of the problem (1.1), and denote

$$
\begin{cases} v(t,x):=u(t,x)-u_+, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,\\ v_0(s,x):=u_0(s,x)-u_+, & s\in[-\tau,0],x\in\mathbb{R}^N.\end{cases}
$$

Then $v(t,x)$ satisfies

$$
\begin{cases} \dfrac{dv}{dt}-D(J*v-v)+dv-b'(u_+)v(t-\tau,x)=Q(v(t-\tau,x)),\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,\\ v|_{t=s}=v_0(s,x),\ s\in[-\tau,0],x\in\mathbb{R}^N,\end{cases} \tag{5.1}
$$

where

$$
Q(v(t-\tau,x)):=b(v(t-\tau,x)+u_+)-b(u_+)-b'(u_+)v(t-\tau,x).
$$

Obviously, $|b'(u_+)|=\left|d(1-\ln\frac{p}{d})\right|<d$, for $1<\frac{p}{d}<e^2$, and

$$
d-|b'(u_+)|=d-d\left|1-\ln\frac{p}{d}\right|=\begin{cases} d\ln\frac{p}{d}, & \text{for } 1<\frac{p}{d}\le e,\\ d\left(2-\ln\frac{p}{d}\right) & \text{for } e<\frac{p}{d}<e^2.\end{cases}
$$

Next, we give the proof of Theorem 2.3 in the two cases $1<\frac{p}{d}\le e$ and $e<\frac{p}{d}<e^2$ respectively.

5.1. *Case of $1<\frac{p}{d}\le e$: global convergence to u_+*

When $1<\frac{p}{d}\le e$, the birth rate function $b(u)=pue^{-au}$ is monotone. Let $w(t,x):=-v(t,x)$, i.e., $w(t,x):=u_+-u(t,x)$, we immediately obtain that $w\in[0,u_+]$. According to (5.1), we have

$$\begin{cases} \dfrac{dw}{dt} - D(J*w-w) + dw - b'(u_+)w(t-\tau,x) = \dfrac{b''(\hat{u})}{2}w^2(t-\tau,x), \ (t,x)\in\mathbb{R}^+\times\mathbb{R}^N, \\ w|_{t=s} = w_0(s,x), \ s\in[-\tau,0], x\in\mathbb{R}^N, \end{cases} \tag{5.2}$$

where $\hat{u} \in (w(t-\tau,x),u_+)$. Applying the fundamental solution formula to (5.2), we have

$$\begin{aligned} w(t,x) =& \int_{\mathbb{R}^N} S(t,0;x,y)w_0(-\tau,y)dy \\ &+ \int_{-\tau}^{0}\int_{\mathbb{R}^N} S(t,s;x,y)\left[\frac{d}{ds}w_0(s,y) - D(J*w_0-w_0) + dw_0(s,y)\right]dyds \\ &+ \int_{0}^{t}\int_{\mathbb{R}^N} S(t,s;x,y)\frac{b''(\hat{u})}{2}w^2(s-\tau,y)dyds \\ \le& \int_{\mathbb{R}^N} S(t,0;x,y)w_0(-\tau,y)dy \\ &+ \int_{-\tau}^{0}\int_{\mathbb{R}^N} S(t,s;x,y)\left[\frac{d}{ds}w_0(s,y) - D(J*w_0-w_0) + dw_0(s,y)\right]dyds, \end{aligned}$$

provide that $b''(u) < 0$ for $u \in [0,u_+]$, since $1 < \frac{p}{d} \le e$. $S(t;x)$ is the fundamental solution of the time-delayed dispersal equation

$$w_t - D(J*w-w) + dw = b'(u_+)w(t-\tau,x).$$

Similar to the proof of Theorem 2.2, we have the following L^2 decay estimate

$$\begin{aligned} &\|w(t)\|_{L^2(\mathbb{R}^N)} \\ \le& \|S(t)*w_0(-\tau)\|_{L^2(\mathbb{R}^N)} + \int_{-\tau}^{0}\left\|S(t,s)*\left[\frac{d}{ds}w_0(s) - D(J*w_0-w_0) + dw_0(s)\right]\right\|_{L^2(\mathbb{R}^N)} ds \\ \le& C\|S(t)\|_{L^2(\mathbb{R}^N)}\|w_0(-\tau)\|_{L^1(\mathbb{R}^N)} \\ &+ \int_{-\tau}^{0}\|S(t,s)\|_{L^2(\mathbb{R}^N)}\left\|\frac{d}{ds}w_0(s) - D(J*w_0-w_0) + dw_0(s)\right\|_{L^1(\mathbb{R}^N)} ds \\ \le& C\mathcal{M}^0_{w_0}(1+t)^{-\frac{N}{2\alpha}}e^{-\epsilon_4 d\ln\frac{p}{d}t} \end{aligned}$$

and

$$
\begin{aligned}
&\left\|\partial_x^k w(t)\right\|_{L^2(\mathbb{R}^N)}\\
\leq &\left\|\partial_x^k S(t) * w_0(-\tau)\right\|_{L^2(\mathbb{R}^N)}\\
&+\int_{-\tau}^{0}\left\|\partial_x^k S(t,s) * \left[\frac{d}{ds}w_0(s)-D(J*w_0-w_0)+dw_0(s)\right]\right\|_{L^2(\mathbb{R}^N)} ds\\
&+C\int_0^t\left\|\partial_x^k S(t,s) * w^2(s-\tau)\right\|_{L^2(\mathbb{R}^N)} ds\\
\leq &\left\|\partial_x^k S(t)\right\|_{L^2(\mathbb{R}^N)}\|w_0(-\tau)\|_{L^1(\mathbb{R}^N)}\\
&+\int_{-\tau}^{0}\left\|\partial_x^k S(t,s)\right\|_{L^2(\mathbb{R}^N)}\left\|\frac{d}{ds}w_0(s)-D(J*w_0-w_0)+dw_0(s)\right\|_{L^1(\mathbb{R}^N)} ds\\
&+C\int_0^t\left\|\partial_x^k S(t,s)\right\|_{L^2(\mathbb{R}^N)}\left\|w^2(s-\tau)\right\|_{L^1(\mathbb{R}^N)} ds\\
\leq &C\mathcal{M}_{w_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_4 d\ln\frac{p}{d}t}\\
&+C\mathcal{M}_{w_0}^k\int_0^t(1+t-s)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_4 d\ln\frac{p}{d}(t-s)}(1+s-\tau)^{-\frac{N}{\alpha}}e^{-2\epsilon_4 d\ln\frac{p}{d}(s-\tau)}ds\\
\leq &C\mathcal{M}_{w_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_4 d\ln\frac{p}{d}t}\\
&+C\mathcal{M}_{w_0}^k e^{-\epsilon_4 d\ln\frac{p}{d}t}\int_0^t e^{-\epsilon_4 d\ln\frac{p}{d}(s-2\tau)}\left[(1+t-s)^{-\frac{N+2|k|}{2\alpha}}(1+s-\tau)^{-\frac{N}{\alpha}}\right]ds\\
\leq &\begin{cases} C\mathcal{M}_{w_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_4 d\ln\frac{p}{d}t}, & \text{if } 1<\frac{p}{d}<e,\\ C\mathcal{M}_{w_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_4 dt}, & \text{if } \frac{p}{d}=e,\end{cases}
\end{aligned}
$$

with $N>\alpha$ and $|k|<\frac{N}{2}$. Then, we complete the proof of Theorem 2.3 in the case of $1<\frac{p}{d}\leq e$.

5.2. *Case of $e<\frac{p}{d}<e^2$: local convergence to u_+*

When $e<\frac{p}{d}<e^2$, the birth rate function loses its monotonicity, and the above method for monotone birth rate function is no longer applicable. Here we adopt a continuous extension method. For $T>0$, we define the solution space for (5.1) as follows

$$
\begin{aligned}
\mathbb{X}(r-\tau,T+r)=\Big\{v|v(t,x)\in C([r-\tau,T+r];H^{m+2}(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)),\\
\partial_s u_0(s,\cdot)\in L^1([r-\tau,T+r];H^m(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)),
\end{aligned}
$$

$$\sup_{t\in[r-\tau,T+r]}\sum_{|k|=0}^{m}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}<\infty,$$

$$0\le |k|\le m,\ \epsilon_5=\epsilon_5(d,p)>0\Big\},$$

equipped with the norm

$$\mathcal{N}_r(T)=\sup_{t\in[r-\tau,T+r]}\sum_{|k|=0}^{m}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}.$$

Particularly, $\mathcal{N}(T):=\mathcal{N}_0(T)$ for $r=0$. It is easy to see that Theorem 2.3 for the original problem (1.1) in this case is equivalent to the following theorem for the problem (5.1).

Theorem 5.1. *Let $e<\frac{p}{d}<e^2$, $v_0(s,x)\in\mathbb{X}(-\tau,0)$, then there exists a constant $\delta_0=\delta_0(d,p)\ll 1$ such that, when $\mathcal{M}_{v_0}^k\le\delta_0$, the solution $v(t,x)$ of (5.1) uniquely and globally exists in $\mathbb{X}(-\tau,\infty)$ and satisfies*

$$(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\le C\mathcal{M}_{v_0}^k,$$

for $t\in[0,\infty)$.

The proof of Theorem 5.1 is based on the following local existence and the a priori energy estimates. By a standard approach, we first have the local existence as follows.

Proposition 5.1 *(Local existence). Suppose that $v_0\in\mathbb{X}(-\tau,0)$, and $\mathcal{M}_{v_0}^k\le\delta_1$ for a given positive constant $\delta_1>0$. Then there exists a small $t_0=t_0(\delta_1)>0$ such that the local solution $v(t,x)$ of (5.1) uniquely exists for $t\in[-\tau,t_0]$ and satisfies $v\in\mathbb{X}(-\tau,t_0)$ and $\mathcal{N}_r(t_0)\le C_1\mathcal{N}_r(0)$ for some constant C_1.*

Next, we will pay more attention on the following a priori energy estimates.

Proposition 5.2 *(A priori estimates). Assume that $e<\frac{p}{d}<e^2$. Let $v\in\mathbb{X}(-\tau,T)$ be a local solution of the problem (5.1) for a given constant $T>0$, then there exist positive constants $0<\delta_2\ll 1$ and C_2 independent of T such that $\mathcal{N}(T)\le\delta_2$, which implies*

$$(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\le C_2\mathcal{M}_{v_0}^k,\quad 0\le t\le T.$$

Proof. The solution of the problem (5.1) can be expressed by

$$v(t,x)=\int_{\mathbb{R}^N}K(t,0;x,y)v_0(-\tau,y)dy$$
$$+\int_{-\tau}^{0}\int_{\mathbb{R}^N}K(t,s;x,y)\left[\frac{d}{ds}v_0(s,y)-D(J*v_0-v_0)+dv_0(s,y)\right]dyds$$

$$+\int_0^t \int_{\mathbb{R}^N} K(t,s;x,y)Q(v(s-\tau,y))dyds,$$

and its derivatives

$$\partial_x^k v(t,x) = \int_{\mathbb{R}^N} \partial_x^k K(t,0;x,y)v_0(-\tau,y)dy$$
$$+\int_{-\tau}^0 \int_{\mathbb{R}^N} \partial_x^k K(t,s;x,y)\left[\frac{d}{ds}v_0(s,y) - D(J*v_0 - v_0) + dv_0(s,y)\right]dyds$$
$$+\int_0^t \int_{\mathbb{R}^N} \partial_x^k K(t,s;x,y)Q(v(s-\tau,y))dyds,$$

where $K(t;x)$ is the fundamental solution of the equation

$$v_t - D(J*v - v) + dv = b'(u_+)v(t-\tau,x).$$

By Theorem 3.1, we obtain the L^2-decay estimate of fundamental solution $K(t;x)$. Furthermore, we have

$$\begin{aligned}
&\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)} \\
\le &\left\|\partial_x^k K(t)\right\|_{L^2(\mathbb{R}^N)} \|v_0(-\tau)\|_{L^1(\mathbb{R}^N)} \\
&+\int_{-\tau}^0 \left\|\partial_x^k K(t,s)\right\|_{L^2(\mathbb{R}^N)} \left\|\frac{d}{ds}v_0(s) - D(J*v_0 - v_0) + dv_0(s)\right\|_{L^1(\mathbb{R}^N)} ds \\
&+\int_0^t \left\|\partial_x^k K(t,s)\right\|_{L^2(\mathbb{R}^N)} \|Q(v(s-\tau))\|_{L^1(\mathbb{R}^N)} ds \\
\le & C\mathcal{M}_{v_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t} \\
&+ C\mathcal{M}_{v_0}^k \int_0^t (1+t-s)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(t-s)} \|v(s-\tau)\|_{L^2(\mathbb{R}^N)}^2 ds \\
\le & C\mathcal{M}_{v_0}^k (1+t)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t} \\
&+ C\mathcal{M}_{v_0}^k \int_0^t (1+t-s)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(t-s)} \left[(1+s-\tau)^{-\frac{N+2|k|}{2\alpha}} e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(s-\tau)}\right]^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad\times\Big[\sup_{0\le s\le T}(1+s-\tau)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(s-\tau)}\|v(s-\tau)\|_{L^2(\mathbb{R}^N)}\Big]^2 ds\\
&\le C\mathcal{M}_{v_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\\
&\quad + C\mathcal{M}_{v_0}^k e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\int_0^t e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(s-2\tau)}(1+t-s)^{-\frac{N+2|k|}{2\alpha}}(1+s-\tau)^{-\frac{N+2|k|}{\alpha}}\\
&\qquad\qquad\times\Big[\sup_{0\le s\le T}(1+s-\tau)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)(s-\tau)}\|v(s-\tau)\|_{L^2(\mathbb{R}^N)}\Big]^2 ds\\
&\le C\mathcal{M}_{v_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\\
&\quad + C\mathcal{M}_{v_0}^k(1+t)^{-\frac{N+2|k|}{2\alpha}}e^{-\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\Big[\sup_{0\le t\le T}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\Big]^2.
\end{aligned}
$$

When $N(T)\le\delta_2$, we have

$$
\begin{aligned}
&\sup_{0\le t\le T}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\\
&\le C\mathcal{M}_{v_0}^k + C\mathcal{M}_{v_0}^k\delta_2\sup_{0\le t\le T}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)},
\end{aligned}
$$

which implies that

$$
\left(1-C\mathcal{M}_{v_0}^k\delta_2\right)\sup_{0\le t\le T}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\le C\mathcal{M}_{v_0}^k.
$$

Taking $0<\delta_2\ll 1$ small enough, we obtain

$$
\sup_{0\le t\le T}(1+t)^{\frac{N+2|k|}{2\alpha}}e^{\epsilon_5 d\left(2-\ln\frac{p}{d}\right)t}\left\|\partial_x^k v(t)\right\|_{L^2(\mathbb{R}^N)}\le C_2\mathcal{M}_{v_0}^k.
$$

The proof of this proposition is complete. □

Based on the local existence and a priori estimates obtained in Propositions 5.1 and 5.2, similar to [28,29], we can employ the usual continuous extension method to give the proof of Theorem 5.1, which immediately implies Theorem 2.3 in the case of $e<\frac{p}{d}<e^2$.

6. Remarks

In this section, as a direct application of the results of Theorems 2.1, 2.2 and 2.3, we give the corresponding results for the following Nicholson's blowflies type equation with local dispersion

$$
\begin{cases}
\dfrac{\partial u}{\partial t}-D\Delta u+du=b(u(t-\tau,x)), & t>0,\ x\in\mathbb{R}^N,\\
u|_{t=s}=u_0(s,x), & s\in[-\tau,0],\ x\in\mathbb{R}^N,
\end{cases}
\tag{6.1}
$$

where $b(u(t-\tau,x)) = pu(t-\tau,x)e^{-au(t-\tau,x)}$. Clearly, there exist two constant equilibria $u_- = 0$ and $u_+ = \frac{1}{a}\ln\frac{p}{d}$. Similar to the proof of Theorems 2.1, 2.2 and 2.3, we first consider the following linear local diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} - D\Delta u + \beta u = \gamma u(t-\tau,x), & t>0,\ x\in\mathbb{R}^N, \\ u|_{t=s} = u_0(s,x), & s\in[-\tau,0],\ x\in\mathbb{R}^N. \end{cases} \tag{6.2}$$

Repeating the same procedure as in the proof of Theorem 2.1, we can derive a similar result for the linear problem (6.2) as follows.

Theorem 6.1. *Let $\beta \ge |\gamma|$, assume that $u_0(s,\cdot) \in C([-\tau,0];W^{2,1}(\mathbb{R}^N))$ and $\partial_s u_0(s,\cdot) \in L^1(\mathbb{R}^N)$ for all $s\in[-\tau,0]$, then there exists a constant $0<\epsilon_6=\epsilon_6(\beta,\gamma)<1$ such that the solution of the problem (6.2) satisfies*

$$\left\|\partial_x^k u(t)\right\|_{L^2(\mathbb{R}^N)} \le C\left(\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))}\right)(1+t)^{-\frac{N}{4}-\frac{|k|}{2}}e^{-\epsilon_6(\beta-|\gamma|)t},$$
$$t>0,$$
$$\left\|\partial_x^k u(t)\right\|_{L^\infty(\mathbb{R}^N)} \le C\left(\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))}\right)(1+t)^{-\frac{N}{2}-\frac{|k|}{2}}e^{-\epsilon_6(\beta-|\gamma|)t},$$
$$t>0.$$

For the nonlinear local diffusion problem (6.1), we have the following results.

Theorem 6.2. *When $0<\frac{p}{d}\le 1$, suppose that the initial data u_0 satisfies $0\le u_0(s,x)\le\frac{1}{a}$, $u_0\in L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))$ and $u_0(-\tau,\cdot)\in L^1(\mathbb{R}^N)$. Then the problem (6.1) admits a unique solution $u(t,x)$, such that:*

- *when $0<\frac{p}{d}<1$, then $u(t,x)$ converges globally to $u=0$ time-exponentially*

$$\left\|\partial_x^k u(t)\right\|_{L^2(\mathbb{R}^N)} \le C\left(\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^1([-\tau,0],W^{2,1}(\mathbb{R}^N))}\right)(1+t)^{-\frac{N}{4}-\frac{|k|}{2}}e^{-\epsilon_7(d-p)t},$$

  where constant $\epsilon_7=\epsilon_7(d,p)>0$;
- *when $\frac{p}{d}=1$, then $u(t,x)$ converges globally to $u=0$ time-algebraically*

$$\left\|\partial_x^k u(t)\right\|_{L^2(\mathbb{R}^N)} \le C\left(\|u_0(-\tau)\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^1([-\tau,0],W^{2,1}(\mathbb{R}^N))}\right)(1+t)^{-\frac{N}{4}-\frac{|k|}{2}},$$

  with $|k|<\frac{N}{2}$ and $N\ge 3$.

Theorem 6.3. *When $1<\frac{p}{d}<e^2$, suppose that $0\le u_0\le u_+$, $u_0-u_+\in L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))$ and $u_0(-\tau,\cdot)-u_+\in L^1(\mathbb{R}^N)$. Then the problem (6.1) exists a unique solution $u(t,x)$, such that:*

- *when* $1 < \frac{p}{d} \le e$, *then* $u(t,x)$ *converges globally to* $u_+ = \frac{1}{a}\ln\frac{p}{d}$ *in the exponential form*

$$\left\|\partial_x^k (u-u_+)(t)\right\|_{L^2(\mathbb{R}^N)} \le C\left(\|u(-\tau)-u_+\|_{L^1(\mathbb{R}^N)} + \|u_0-u_+\|_{L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))}\right) \times (1+t)^{-\frac{N}{4}-\frac{|k|}{2}} e^{-\epsilon_8 d \ln\frac{p}{d} t},\ t>0,$$

for some $\epsilon_8 = \epsilon_8(d,p) > 0$;
- *when* $e < \frac{p}{d} < e^2$, *then* $u(t,x)$ *converges locally to* $u_+ = \frac{1}{a}\ln\frac{p}{d}$ *in the exponential form*

$$\left\|\partial_x^k (u-u_+)(t)\right\|_{L^2(\mathbb{R}^N)} \le C\left(\|u(-\tau)-u_+\|_{L^1(\mathbb{R}^N)} + \|u_0-u_+\|_{L^1([-\tau,0];W^{2,1}(\mathbb{R}^N))}\right) \times (1+t)^{-\frac{N}{4}-\frac{|k|}{2}} e^{-\epsilon_9 d(2-\ln\frac{p}{d})t},\ t>0,$$

for some positive number ϵ_9.

Data availability

No data was used for the research described in the article.

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