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LONG-TIME BEHAVIOR OF SOLUTIONS TO THE BIPOLAR HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH BOUNDARY EFFECT*

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Abstract. For a bipolar hydrodynamic model of semiconductors in the form of Euler–Poisson equations with Dirichlet or Neumann boundary conditions, in this paper we first heuristically analyze the most probable asymptotic profile (the so-called diffusion waves) and then prove this long-time behavior rigorously. For this, we construct correction functions to show the convergence of the original solution to the diffusion wave with optimal convergence rates by the energy method. Moreover, in the case with Dirichlet boundary condition, when the initial perturbation is in some weighted L^1 space, a faster and optimal convergence rate is also given.

Key words. bipolar hydrodynamic model, semiconductor, nonlinear damping, nonlinear diffusion waves, asymptotic behavior, convergence rates

AMS subject classifications. 35L50, 35L60, 35L65, 76R50

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1. Introduction.

This is a continuation of our study on the stability of diffusion waves for the bipolar hydrodynamic system of semiconductors. Based on the work [15] on the initial value problem (IVP), in this paper we consider the initial-boundary value problem (IBVP) of the 5×5 Euler–Poisson system in the half-space

$$\begin{cases} n_{1t} + J_{1x} = 0, \\ J_{1t} + \left(\frac{J_1^2}{n_1} + p(n_1)\right)_x = n_1 E - J_1, \\ n_{2t} + J_{2x} = 0, \\ J_{2t} + \left(\frac{J_2^2}{n_2} + q(n_2)\right)_x = -n_2 E - J_2, \\ E_x = n_1 - n_2, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.1}$$

with the initial data

$$(n_1, n_2, J_1, J_2)|_{t=0} = (n_{10}, n_{20}, J_{10}, J_{20})(x), \tag{1.2}$$

and either the Dirichlet boundary condition

$$J_1|_{x=0} = J_2|_{x=0} = E|_{x=0} = 0 \tag{1.3}$$

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or the Neumann boundary condition

$$J_{1x}|_{x=0} = J_{2x}|_{x=0} = 0 \quad \text{and} \quad E(+\infty, t) = a(t). \tag{1.4}$$

Here, for the later case, two situations will be considered, that is, either

$$|a(t)| = O(1)e^{-\eta t} \text{ for } \eta > 0 \tag{1.5}$$

or

$$|a(t)| = O(1)(1+t)^{-\theta} \text{ for } \theta > \frac{3}{2}. \tag{1.6}$$

In the above system, $n_1(x,t)$, $n_2(x,t)$, $J_1(x,t)$, $J_2(x,t)$, and $E(x,t)$ represent the electron density, the hole density, the current density of electrons, the current density of holes, and the electric field, respectively. For detail of physical background, see the textbooks [20, 29, 38, 46]. The nonlinear functions $p(s)$ and $q(s)$ denote the pressures of the electrons and the holes which are assumed to be identical and smooth and satisfy

$$p(s) = q(s) \geq 0, \quad p'(s) = q'(s) > 0 \quad \text{for } s > 0. \tag{1.7}$$

Moreover, we assume the initial data satisfying

$$\lim_{x \to +\infty} (n_{10}(x), J_{10}(x), n_{20}(x), J_{20}(x), E(x,0)) = (n_+, J_+, n_+, J_+, E_+), \tag{1.8}$$

together with the compatibility condition with the Dirichlet boundary condition,

$$J_{10}(0) = J_{20}(0) = 0, \quad \text{and} \quad E_+ = \int_{\mathbb{R}_+} [n_{01}(x) - n_{02}(x)]dx, \tag{1.9}$$

or the compatibility condition with the Neumann boundary condition,

$$J'_{i0}(0) = 0, \; n_i|_{x=0} = n_{i0}(0) \quad \text{for } i = 1, 2, \quad \text{and } a(0) = E_+. \tag{1.10}$$

Here $(n_+, J_+, n_+, J_+, E_+)$ are some constant states. See (3.3) for deriving the expression of E_+ in (1.9).

According to Darcy's law, in long-time the solutions to the Euler–Poisson system behave similarly to those solutions to the porous media equations because of the frictional damping, like the Euler equation with damping that has been extensively studied; cf. [10]. In fact, this can be verified by variables scaling [14, 26, 33, 35]

$$x \to x/\varepsilon, \quad t \to t/\varepsilon^2, \quad n_i \to \bar{n}_i, \quad J_i \to \varepsilon \bar{J}_i, \quad E \to \bar{E},$$

for $i = 1, 2$ with an arbitrarily small number $\varepsilon > 0$; then the system (1.1) becomes

$$\begin{cases} \bar{n}_{1t} + \bar{J}_{1x} = 0, \\ \varepsilon^3 \bar{J}_{1t} + \varepsilon^3 \left(\frac{\bar{J}_1^2}{\bar{n}_1}\right)_x + \varepsilon(p(\bar{n}_1))_x = \bar{n}_1 \bar{E} - \varepsilon \bar{J}_1, \\ \bar{n}_{2t} + \bar{J}_{2x} = 0, \\ \varepsilon^3 \bar{J}_{2t} + \varepsilon^3 \left(\frac{\bar{J}_2^2}{\bar{n}_2}\right)_x + \varepsilon(p(\bar{n}_2))_x = -\bar{n}_2 \bar{E} - \varepsilon \bar{J}_2, \\ \varepsilon \bar{E}_x = \bar{n}_1 - \bar{n}_2. \end{cases} \tag{1.11}$$

Neglecting the small terms with ε, (1.11) gives

$$\begin{cases} \bar{n}_{it} + \bar{J}_{ix} = 0, \ i = 1, 2, \\ 0 = \bar{n}_1 \bar{E}, \\ 0 = \bar{n}_2 \bar{E}, \\ 0 = \bar{n}_1 - \bar{n}_2, \end{cases}$$

which implies

$$\bar{n}_1 = \bar{n}_2 =: \bar{n}, \quad \bar{J}_1 = \bar{J}_2 =: \bar{J}, \quad \bar{E} = 0. \tag{1.12}$$

Thus, substituting (1.12) into (1.11) yields

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ \varepsilon^2 \bar{J}_t + \varepsilon^2 \left(\frac{\bar{J}^2}{\bar{n}} \right)_x + p(\bar{n})_x = -\bar{J}. \end{cases}$$

Again, by omitting the small terms with ε, we obtain the following equations which can be used to describe the asymptotic profiles of (1.1):

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ p(\bar{n})_x = -\bar{J}, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar{n}_t - p(\bar{n})_{xx} = 0, & \text{porous media equation,} \\ p(\bar{n})_x = -\bar{J}, & \text{Darcy's law.} \end{cases}$$

Therefore, we expect that the solution $(n_1, J_1, n_2, J_2, E)(x, t)$ converges to $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x, t)$, where $(\bar{n}, \bar{J})(x, t)$ satisfies the above porous media equations.

In most of the previous works, the asymptotic profiles for the damped Euler–Poisson equations (1.1) are chosen to be the self-similar solutions $(\bar{n}, \bar{J})(x/\sqrt{1+t})$ of the above porous media equations, called diffusion waves. On the other hand, as indicated in [26, 28] for the IBVP to the damped p-system (see [33] for the IVP), a better asymptotic profile is the solution of the corresponding IBVP to the porous media equations itself, rather than the self-similar solution. Based on this thinking, for the problems considered in this paper, we expect the asymptotic profile of the Dirichlet IBVP (1.1)–(1.3) satisfying

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ p(\bar{n})_x = -\bar{J}, \\ \bar{n}|_{t=0} = \bar{n}_0(x), \\ \bar{J}|_{x=0} = 0, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar{n}_t - p(\bar{n})_{xx} = 0, \\ p(\bar{n})_x = -\bar{J}, \\ \bar{n}|_{t=0} = \bar{n}_0(x), \\ \bar{n}_x|_{x=0} = 0, \end{cases} \tag{1.13}$$

while the asymptotic profile of the Neumann IBVP (1.1), (1.2), and (1.4) satisfies

$$\begin{cases} \bar{n}_t + \bar{J}_x = 0, \\ p(\bar{n})_x = -\bar{J}, \\ \bar{n}|_{t=0} = \bar{n}_0(x), \\ \bar{J}_x|_{x=0} = 0, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \bar{n}_t - p(\bar{n})_{xx} = 0, \\ p(\bar{n})_x = -\bar{J}, \\ \bar{n}|_{t=0} = \bar{n}_0(x), \\ \bar{n}|_{x=0} = \bar{n}_0(0). \end{cases} \tag{1.14}$$

Here, the initial data $\bar{n}_0(x) \to n_+$ as $x \to \infty$ will be specified later in (3.41) for the Dirichlet boundary and (4.32) for the Neumann boundary, respectively. We also call such a particular solution $(\bar{n}, \bar{J})(x, t)$ for the IBVPs (1.13) or (1.14) as a diffusion wave without any confusion.

Let us now review previous work on the Euler–Poisson equations. For the unipolar hydrodynamic model, the existence of steady-state solutions was studied in [4, 5, 6], and their stability was given in [7, 11, 16, 17, 21, 27, 39] in different settings. Moreover, the global existence of classical and/or the entropy weak solutions as well as the zero relaxation limits were investigated in [1, 2, 9, 12, 11, 21, 22, 23, 31, 43, 48]. On the other hand, for the bipolar hydrodynamic model, there is much less related research. For this, Natalini [37] and Hsiao and Zhang [12, 13] established the global entropy weak solutions in the framework of compensated compactness on the whole real line and spatial bounded domain, respectively. Zhu and Hattori [50] proved the stability of steady-state solutions for a recombined bipolar hydrodynamic model. Y.-P. Li [24] studied the relaxation limit of a bipolar isentropic hydrodynamic model with small momentum relaxation time. For the stability of diffusion waves, Gasser, Hsiao, and H.-L. Li [8] and Huang and Y.-P. Li [14] studied it in the strong and the weak sense, respectively, under a restrictive condition

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]dx = 0. \tag{1.15}$$

Note that this together with $(1.1)_5$ implies

$$E(+\infty, 0) - E(-\infty, 0) = 0.$$

This is called the switch-off case for the device because there is no voltage. For the switch-on case

$$E(+\infty, 0) - E(-\infty, 0) \neq 0, \quad \text{or equivalently} \quad \int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]dx \neq 0,$$

there is some difference between the original solutions and the diffusion waves at the far fields so that the perturbation of a diffusion wave cannot be in L^2 space. Note that the L^∞-stability of diffusion waves remained open. To overcome this difficulty, by constructing some correction functions to take care of the difference, we proved in [15] the stability of diffusion waves for the IVP of (1.1).

Note that for the IVP studied in [15] the far field layer due to the fact that $J_i|_{x=\infty} - J_i|_{x=-\infty} \neq 0$ and $E|_{x=\infty} - E|_{x=-\infty} \neq 0$ was analyzed. With the boundary effect (Dirichlet or Neumann) in the half-space considered in this paper, we also need to take care of the boundary layer. For the one-side layer at the far field $x = \infty$, the correction function for the IBVPs in some sense can be constructed in a more straightforward way than the IVP in the full space. In fact, we can linearize $n_i(x,t)$ around the constant state n_+ and derive the explicit integral equations with Green functions to the corresponding IBVPs (see (3.66) and (3.71), and (4.39) and (4.40), respectively). With this advantage, by using the energy estimates for the elementary solutions to the linear damped wave equations introduced by Matsumura [32] and Ikehata [18] (see also the improvements in [26, 28, 30, 44] for nonlinear damped p-system), together with the new techniques in [15, 16, 17, 36] on the construction of correction functions, we succeed in proving the following optimal convergence results:

1. In the Dirichlet boundary case (1.3), the solution $(n_1, J_1, n_2, J_2, E)(x,t)$ of the Dirichlet IBVP (1.1)–(1.3) converges to its corresponding diffusion wave $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x,t)$ of IBVP (1.13) with the optimal convergence rates

$$\begin{aligned} &\|(n_1 - \bar{n}, J_1 - \bar{J}, n_2 - \bar{n}, J_2 - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)} \\ &\quad = O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, e^{-\nu t}) \end{aligned} \tag{1.16}$$

for the initial perturbations in $H^3(\mathbb{R}_+)\times H^2(\mathbb{R}_+)$, where $0<\nu<\frac{1}{2}$, and

$$\begin{aligned}&\|(n_1-\bar n, J_1-\bar J, n_2-\bar n, J_2-\bar J, E)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &\quad = O(1)(t^{-1}, t^{-\frac{3}{2}}, t^{-1}, t^{-\frac{3}{2}}, e^{-\nu t})\end{aligned}\tag{1.17}$$

for the initial perturbation in $(H^2(\mathbb{R}_+)\times H^1(\mathbb{R}_+))\cap L^1(\mathbb{R}_+)$. Furthermore,

$$\begin{aligned}&\|(n_1-\bar n, J_1-\bar J, n_2-\bar n, J_2-\bar J, E)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &\quad = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}}, t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}}, e^{-\nu t})\end{aligned}\tag{1.18}$$

for the initial perturbation in the weighted space $(H^2(\mathbb{R}_+)\times H^1(\mathbb{R}_+))\cap L^{1,\gamma}(\mathbb{R}_+)$ with $0<\gamma\le\frac{1}{4}$.

2. In the Neumann boundary case (1.4), the solution $(n_1, J_1, n_2, J_2, E)(x,t)$ of the Neumann IBVP (1.1), (1.2), and (1.4) converges to its corresponding diffusion wave $(\bar n, \bar J, \bar n, \bar J, 0)(x,t)$ of IBVP (1.14) with the optimal convergence rates

$$\begin{aligned}&\|(n_1-\bar n, J_1-\bar J, n_2-\bar n, J_2-\bar J, E)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &= \begin{cases} O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, e^{-\nu t}) & \text{as } |a(t)| = O(e^{-\eta t}), \eta>0,\\ O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\theta}) & \text{as } |a(t)| = O(t^{-\theta}), \theta>\frac{3}{2},\end{cases}\end{aligned}\tag{1.19}$$

for the initial perturbations in $H^3(\mathbb{R}_+)\times H^2(\mathbb{R}_+)$, where $0<\nu<\min\{\gamma,\frac{1}{2}\}$, and

$$\begin{aligned}&\|(n_1-\bar n, J_1-\bar J, n_2-\bar n, J_2-\bar J, E)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &= \begin{cases} O(1)(t^{-1}, t^{-\frac{3}{2}}, t^{-1}, t^{-\frac{3}{2}}, e^{-\nu t}) & \text{as } |a(t)| = O(e^{-\eta t}), \eta>0,\\ O(1)(t^{-1}, t^{-\frac{3}{2}}, t^{-1}, t^{-\frac{3}{2}}, t^{-\theta}) & \text{as } |a(t)| = O(t^{-\theta}), \theta>\frac{3}{2},\end{cases}\end{aligned}\tag{1.20}$$

for the initial perturbation in $(H^2(\mathbb{R}_+)\times H^1(\mathbb{R}_+))\cap L^1(\mathbb{R}_+)$. Note that unlike the Dirichlet IVBP, there is no weighted $L^{1,\gamma}$-decay obtained for this case.

Finally, for the study on diffusion phenomena to damped p-system or relaxation models, we refer to [3, 10, 25, 26, 33, 34, 40, 41, 42, 47, 49] and the references therein.

The rest of the paper is organized as follows. First, at the end of this section, we introduce some notation for readers' convenience. In section 2, we will introduce some known results about the diffusion waves and the properties of solutions to the Dirichlet and Neumann IBVPs of linear damped wave equations. In section 3, inspired by [15], we first heuristically study the behavior of the solution for the Dirichlet IBVP (1.1)–(1.3) at the far field $x=\infty$ and the difference between the original solution and the possible asymptotic profiles. We then construct the correction function to take care of the difference. Here, the correction function depends on the original initial data (1.2). Later on, after we show how to find the most suitable asymptotic profiles by carefully choosing the initial data for the IBVP of porous media equations (1.13), we prove the convergence of the original IBVP solution (1.1)–(1.3) to the selected asymptotic profile (1.13). In section 4, instead of the Dirichlet boundary condition (1.3), we consider the Neumann boundary case $J_{ix}|_{x=0}$ with $E|_{x=\infty}=0$ for the system (1.1). After constructing some correction function and determining the asymptotic profile of (1.14) with some specified initial data $\bar n_0(x)$, we prove the convergence of the solutions for the Neumann IBVP (1.1), (1.2), and (1.4) to the chosen diffusion wave satisfying (1.14).

Notation. Throughout the paper, $(\bar{n}, \bar{J}, \bar{E})(x,t)$ denotes the diffusion wave, that is, the solution of the IBVP (1.13) or (1.14), and $(\hat{n}_1, \hat{J}_1, \hat{n}_2, \hat{J}_2, \hat{E})(x,t)$ denotes the correction function. $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \dots)$ represents some specific constant. The derivatives of f are denoted by f_x, f_{xx}, or $\partial_x^k f$, $k = 0, 1, 2, \dots$. $L^p(\mathbb{R}_+)$ $(1 \le p \le \infty)$ is the usual Lebesque space with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}_+} |f(x)|^p dx \right)^{1/p} \quad \text{for} \ \ 1 \le p < \infty \ \ \text{and} \ \ \|f\|_{L^\infty} = \sup_{x \in \mathbb{R}_+} |f(x)|.$$

Sometimes, the integral region $\mathbb{R}_+$ will be omitted for brevity. $L^{p,\gamma}(\mathbb{R}_+)$ with $\gamma > 0$ and $1 \le p \le \infty$ is a weighted $L^p(\mathbb{R}_+)$ space with the weight $(1+x)^\gamma$. Its norm is

$$\|f\|_{L^{p,\gamma}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} (1+x)^\gamma |f(x)|^p dx \right)^{1/p}, \qquad 1 \le p \le \infty.$$

$H^k(\mathbb{R}_+)$ $(k \ge 0)$ is the usual Sobolev space with the norm

$$\|f\|_{H^k} = \left(\sum_{i=0}^{k} \int_{\mathbb{R}_+} |\partial_x^i f|^2 dx \right)^{1/2}.$$

For simplicity, we also denote $\|(f,g,h)\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + \|h\|_{L^2}^2$ and $\|(f,g,h)\|_{H^k}^2 = \|f\|_{H^k}^2 + \|g\|_{H^k}^2 + \|h\|_{H^k}^2$. Let $T > 0$ and $\mathcal{B}$ be a Banach space. $C^0([0,T];\mathcal{B})$ is the space of $\mathcal{B}$-valued continuous functions on $[0,T]$, and $L^2([0,T];\mathcal{B})$ is the space of $\mathcal{B}$-valued L^2-functions on $[0,T]$. The other spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ can be defined similarly.

2. Preliminaries.

In this section, we state some known results which will be used for the proof of the convergence in sections 3 and 4.

LEMMA 2.1 (see [19, 30]). *Let $\bar{n}_0 - n_+ \in L^1(\mathbb{R}_+) \cap H^m(\mathbb{R}_+)$ for some positive integer m. Then the solution $(\bar{n}, \bar{J})(x,t)$ of the IBVP (1.13) globally and uniquely exists and satisfies*

$$\|\partial_x^k \partial_t^j (\bar{n} - n_+)(t)\|_{L^2(\mathbb{R}_+)} = O(1)\delta_2 (1+t)^{-\frac{4j+2k+1}{4}}, \qquad 0 \le 2k + j \le m, \tag{2.1}$$

$$\|\partial_x^k \partial_t^j (\bar{n} - n_+)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)\delta_2 (1+t)^{-\frac{2j+k+1}{2}}, \qquad 0 \le 2k + j \le m, \tag{2.2}$$

$$\|(\bar{n} - n_+)_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(1)\delta_2 (1+t)^{-\frac{3}{2}}, \tag{2.3}$$

where $\delta_2 := \max_{x \in \mathbb{R}_+} |\bar{n}_0(x) - n_+|$.

Furthermore, for the linear damped wave equation on the first quadrant with null Dirichlet boundary condition

$$\begin{cases} \phi_{tt} + \alpha \phi_t - \mu \phi_{xx} = g(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\phi, \phi_t)|_{t=0} = (\phi_0, \phi_1)(x), & x \in \mathbb{R}_+, \\ \phi|_{x=0} = 0, & t \in \mathbb{R}_+, \end{cases} \tag{2.4}$$

where $\alpha > 0$ and $\mu > 0$, as shown in [30] (originally, we refer to [32] for the IVP case), its solution can be expressed as

$$
\begin{aligned}
\phi(x,t) = & \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_0(y)dy \\
& + \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\phi_1(y)dy \\
& + \int_0^t \int_0^\infty [K_1(x-y,t-s) - K_1(x+y,t-s)]g(y,s)dyds,
\end{aligned} \tag{2.5}
$$

where $K_j(x,t)$ $(j=0,1)$ are the fundamental solutions of the homogenous equation

$$
\partial_{tt} K_j + \alpha \partial_t K_j - \mu \partial_{xx} K_j = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}_+,
$$

with

$$
\begin{cases} K_0(x,0) = \delta(x), \\ \frac{\partial}{\partial t} K_0(x,0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} K_1(x,0) = 0, \\ \frac{\partial}{\partial t} K_1(x,0) = \delta(x), \end{cases}
$$

where $\delta(x)$ is the Dirac-delta function. The Fourier transforms of $K_i(x,t)$ $(i=0,1)$, denoted by $\hat{K}_i(\xi,t)$ $(i=0,1)$, are given explicitly by

$$
\hat{K}_1(\xi,t) = \begin{cases} \frac{2e^{-\alpha t/2}}{\sqrt{\alpha^2-4\mu\xi^2}} \sinh\left(\frac{\sqrt{\alpha^2-4\mu\xi^2}}{2} t\right), & |\xi| < \frac{\alpha}{2\sqrt{\mu}}, \\ te^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\mu}}, \\ \frac{2e^{-\alpha t/2}}{\sqrt{4\mu\xi^2-\alpha^2}} \sin\left(\frac{\sqrt{4\mu\xi^2-\alpha^2}}{2} t\right), & |\xi| > \frac{\alpha}{2\sqrt{\mu}}, \end{cases}
$$

and

$$
\hat{K}_0(\xi,t) = \frac{\alpha}{2}\hat{K}_1(\xi,t) + R_2(\xi,t),
$$

where

$$
R_2(\xi,t) = \begin{cases} e^{-\alpha t/2} \cosh\left(\frac{\sqrt{\alpha^2-4\mu\xi^2}}{2} t\right), & |\xi| < \frac{\alpha}{2\sqrt{\mu}}, \\ e^{-\alpha t/2}, & |\xi| = \frac{\alpha}{2\sqrt{\mu}}, \\ e^{-\alpha t/2} \cos\left(\frac{\sqrt{4\mu\xi^2-\alpha^2}}{2} t\right), & |\xi| > \frac{\alpha}{2\sqrt{\mu}}. \end{cases}
$$

For the linear damped wave equation with the null Neumann boundary condition

$$
\begin{cases} \phi_{tt} + \alpha\phi_t - \mu\phi_{xx} = g(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\phi,\phi_t)|_{t=0} = (\phi_0,\phi_1)(x), & x \in \mathbb{R}_+, \\ \phi_x|_{x=0} = 0, & t \in \mathbb{R}_+, \end{cases} \tag{2.6}
$$

its solution can be expressed as

$$\begin{aligned}\phi(x,t) = &\int_0^\infty [K_0(x-y,t)+K_0(x+y,t)]\phi_0(y)dy\\ &+\int_0^\infty [K_1(x-y,t)+K_1(x+y,t)]\phi_1(y)dy\\ &+\int_0^t\int_0^\infty [K_1(x-y,t-s)+K_1(x+y,t-s)]g(y,s)dyds. \end{aligned}\tag{2.7}$$

The decay rates for the solutions are also obtained in [32] for the Cauchy problem and extended in [30] for the Dirichlet IBVP (2.4). Without any difficulty, the same decay rates for the solution to the Neumann IBVP (2.6) can be derived similarly. Now we summarize these decay estimates as follows.

LEMMA 2.2 (see [32, 30]). *If $f \in L^1(\mathbb{R}_+)\cap H^{j+k}(\mathbb{R}_+)$, then*

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_1(x-y,t)\pm K_1(x+y,t)]f(y)dy\right\|_{L^2(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|f\|_{L^1(\mathbb{R}_+)}+\|f\|_{H^{j+k-1}(\mathbb{R}_+)}\Big],\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_1(x-y,t)\pm K_1(x+y,t)]f(y)dy\right\|_{L^\infty(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|f\|_{L^1}+\|f\|_{H^{j+k}(\mathbb{R}_+)}\Big].\end{aligned}\tag{2.9}$$

If $f \in L^1(\mathbb{R}_+)\cap H^{j+k+1}(\mathbb{R}_+)$, then

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_0(x-y,t)\pm K_0(x+y,t)]f(y)dy\right\|_{L^2(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|f\|_{L^1}+\|f\|_{H^{j+k}(\mathbb{R}_+)}\Big],\end{aligned}\tag{2.10}$$

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_0(x-y,t)\pm K_0(x+y,t)]f(y)dy\right\|_{L^\infty(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|f\|_{L^1(\mathbb{R}_+)}+\|f\|_{H^{j+k+1}(\mathbb{R}_+)}\Big].\end{aligned}\tag{2.11}$$

Furthermore, as shown in [18] (see also [28, 44]), for the Dirichlet IBVP (2.4), we have the following faster decay estimates if the initial data belongs to the weighted space $L^{1,\gamma}(\mathbb{R}_+)$.

LEMMA 2.3 (see [18, 28, 44]). *Let $\gamma \in [0,1]$. If $f \in L^{1,\gamma}(\mathbb{R}_+)\cap H^{j+k}(\mathbb{R}_+)$, then*

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_1(x-y,t)- K_1(x+y,t)]f(y)dy\right\|_{L^2(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}}\Big[\|f\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|f\|_{H^{j+k-1}(\mathbb{R}_+)}\Big],\end{aligned}\tag{2.12}$$

$$\begin{aligned}&\left\|\partial_t^j\partial_x^k\int_0^\infty [K_1(x-y,t)- K_1(x+y,t)]f(y)dy\right\|_{L^\infty(\mathbb{R}_+)}\\ &\qquad\le C(1+t)^{-j-\frac{k+1}{2}-\frac{\gamma}{2}}\Big[\|f\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|f\|_{H^{j+k}(\mathbb{R}_+)}\Big].\end{aligned}\tag{2.13}$$

If $f \in L^{1,\gamma}(\mathbb{R}_+) \cap H^{j+k+1}(\mathbb{R}_+)$, *then*

$$\begin{aligned} &\left\| \partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)] f(y) dy \right\|_{L^2(\mathbb{R}_+)} \\ &\qquad \le C(1+t)^{-j-\frac{2k+1}{4}-\frac{\gamma}{2}} \left[\|f\|_{L^{1,\gamma}(\mathbb{R}_+)} + \|f\|_{H^{j+k}(\mathbb{R}_+)} \right], \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\left\| \partial_t^j \partial_x^k \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)] f(y) dy \right\|_{L^\infty(\mathbb{R}_+)} \\ &\qquad \le C(1+t)^{-j-\frac{k+1}{2}-\frac{\gamma}{2}} \left[\|f\|_{L^{1,\gamma}(\mathbb{R}_+)} + \|f\|_{H^{j+k+1}(\mathbb{R}_+)} \right]. \end{aligned} \tag{2.15}$$

Remark 1. As shown in [18], for the Neumann IBVP (2.6), the solution does not possess such faster decay rates (2.12)–(2.15), no matter the initial data is $(\phi_0, \phi_1) \in L^{1,\gamma}(\mathbb{R}_+)$ or not.

Finally, we state a known and useful auxiliary lemma as follows.

LEMMA 2.4 (see [45]). *Let* $a > 0$, $b > 0$. *Then*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds \le \begin{cases} C(1+t)^{-\min(a,b)}, & \max(a,b) > 1, \\ C(1+t)^{-\min(a,b)} \ln(2+t), & \max(a,b) = 1, \\ C(1+t)^{1-a-b}, & \max(a,b) < 1. \end{cases} \tag{2.16}$$

3. Dirichlet initial-boundary value problem. In this section, we will first investigate the exact difference between the original solution of (1.1)–(1.3) and the diffusion wave of (1.13) at the far field. Then, as shown in [15], we will construct the correction functions to take care of this difference so that the estimation can be done in an L^2-framework. Finally, for a given initial data (1.2) to the IBVP (1.1)–(1.3), we will specify the solution to the IBVP for the porous media equations (1.13) as the asymptotic profile of the original IBVP (1.1)–(1.3) and then prove the convergence of the solution to the IBVP (1.1)–(1.3) to the selected diffusion wave to (1.13).

3.1. Correction functions. In order to show the convergence of the solution $(n_1, J_1, n_2, J_2, E)(x,t)$ of the IBVP (1.1)–(1.3) to the solution $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x,t)$ of the IBVP for the corresponding porous media equations, we now heuristically study the difference between these two solutions at the far field.

Let us first investigate the solution $(n_1, J_1, n_2, J_2, E)(x,t)$ of the IBVP (1.1)–(1.3) at the far field $x = +\infty$. Define

$$(n_1^+, J_1^+, n_2^+, J_2^+, E^+)(t) := (n_1, J_1, n_2, J_2, E)(+\infty, t). \tag{3.1}$$

Integrating $(1.1)_5$ over $[0,\infty)$ with respect to x, we have

$$E^+(t) - E(0,t) = \int_{\mathbb{R}_+} [n_1(x,t) - n_2(x,t)] dx, \tag{3.2}$$

which implies, with the boundary condition $E|_{x=0} = 0$, that

$$E_+ = E(+\infty, 0) = E^+|_{t=0} = \int_{\mathbb{R}_+} [n_{10}(x) - n_{20}(x)] dx. \tag{3.3}$$

Noting the boundary conditions (1.3), and differentiating $(1.1)_5$ with respect to t and using $(1.1)_1$ and $(1.1)_3$, we get

$$\frac{d}{dt} E^+(t) = \int_{\mathbb{R}_+} \partial_t [n_1(x,t) - n_2(x,t)] dx = -[J_1^+(t) - J_2^+(t)]. \tag{3.4}$$

So, taking limit in the equations $(1.1)_1$-$(1.1)_4$ as $x \to +\infty$, and combining (3.4), we have

$$
\begin{cases}
n_1^+(t) = n_+, \\
\frac{d}{dt} J_1^+(t) = n_+ E^+(t) - J_1^+(t), \\
n_2^+(t) = n_+, \\
\frac{d}{dt} J_2^+(t) = -n_+ E^+(t) - J_2^+(t), \\
\frac{d}{dt} E^+(t) = -J_1^+(t) + J_2^+(t), \\
(J_1^+, J_2^+, E^+)|_{t=0} = (J_{1+}, J_{2+}, E_+).
\end{cases}
\tag{3.5}
$$

Differentiating $(3.5)_5$ with respect to t and using $(3.5)_2$ and $(3.5)_4$, we obtain

$$
\begin{cases}
\frac{d^2}{dt^2} E^+(t) + \frac{d}{dt} E^+(t) + 2n_+ E^+(t) = 0, \\
E^+|_{t=0} = E_+, \\
\frac{dE^+}{dt}|_{t=0} = -(J_{1+} - J_{2+}),
\end{cases}
\tag{3.6}
$$

which possesses the eigenvalues

$$
\lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 8n_+}}{2}. \tag{3.7}
$$

By a similar computation but simpler than in [15] for the IVP case, we can solve the above ODEs in the following three cases.

Case 1. When $1 - 8n_+ = 0$, then

$$
J_1^+(t) = \frac{J_{1+} + J_{2+}}{2} e^{-t} + \frac{1}{2} e^{-\frac{t}{2}} \left\{ (J_{1+} - J_{2+}) + \frac{1}{2}[(J_{2+} - J_{1+}) + \frac{1}{2}E_+]t \right\}, \tag{3.8}
$$

$$
J_2^+(t) = \frac{J_{1+} + J_{2+}}{2} e^{-t} - \frac{1}{2} e^{-\frac{t}{2}} \left\{ (J_{1+} - J_{2+}) + \frac{1}{2}[(J_{2+} - J_{1+}) + \frac{1}{2}E_+]t \right\}, \tag{3.9}
$$

$$
E^+(t) = e^{-\frac{t}{2}} \left\{ E_+ + [J_{2+} - J_{1+} + \frac{1}{2}E_+]t \right\}. \tag{3.10}
$$

Case 2. When $1 - 8n_+ < 0$, then

$$
\begin{aligned}
J_1^+(t) = \frac{1}{2}(J_{1+} + J_{2+}) e^{-t} + \frac{1}{4} e^{-\frac{t}{2}} \Bigg\{ & 2(J_{1+} - J_{1+}) \cos \frac{\sqrt{8n_+ - 1}}{2} t \\
& + \Bigg(\frac{2(J_{2+} - J_{1+}) + E_+}{\sqrt{8n_+ - 1}} \\
& \quad + \sqrt{8n_+ - 1} E_+ \Bigg) \sin \frac{\sqrt{8n_+ - 1}}{2} t \Bigg\},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
J_2^+(t) = \frac{1}{2}(J_{1+} + J_{2+}) e^{-t} - \frac{1}{4} e^{-\frac{t}{2}} \Bigg\{ & 2(J_{1+} - J_{2+}) \cos \frac{\sqrt{8n_+ - 1}}{2} t \\
& + \Bigg(\frac{2(J_{2+} - J_{1+}) + E_+}{\sqrt{8n_+ - 1}} \\
& \quad + \sqrt{8n_+ - 1} E_+ \Bigg) \sin \frac{\sqrt{8n_+ - 1}}{2} t \Bigg\},
\end{aligned}
\tag{3.12}
$$

$$E^+(t) = e^{-\frac{t}{2}}\left\{E_+\cos\frac{\sqrt{8n_+-1}}{2}t + \frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}}\sin\frac{\sqrt{8n_+-1}}{2}t\right\}. \tag{3.13}$$

Case 3. When $1-8n_+ > 0$, then

$$J_1^+(t) = \frac{1}{2}(J_{1+}+J_{2+})e^{-t} - \frac{1}{2}\left\{\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\right\}, \tag{3.14}$$

$$J_2^+(t) = \frac{1}{2}(J_{1+}+J_{2+})e^{-t} + \frac{1}{2}\left\{\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\right\}, \tag{3.15}$$

$$E^+(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \tag{3.16}$$

where A and B are defined as

$$A = \frac{(J_{2+}-J_{1+})-\lambda_2 E_+}{\sqrt{1-8n_+}}, \quad B = -\frac{(J_{2+}-J_{1+})-\lambda_1 E_+}{\sqrt{1-8n_+}}.$$

However, according to the properties of the solutions to the IBVP of porous media equations (1.13), the expected asymptotic profiles $(\bar n, \bar J, \bar n, \bar J, \bar E)(x,t) = (\bar n, \bar J, \bar n, \bar J, 0)(x,t)$ at far field $x = +\infty$ are

$$(\bar n, \bar J, \bar n, \bar J, \bar E)(+\infty, t) = (n_+, 0, n_+, 0, 0). \tag{3.17}$$

From (3.8)–(3.16) and (3.17), it holds that

$$\begin{cases} |n_i(\infty,t) - \bar n(\infty,t)| = 0, & i=1,2,\\ |J_i(+\infty,t) - \bar J(+\infty,t)| = O(1)e^{-\nu_0 t} \neq 0, & i=1,2,\\ |E(+\infty,t) - 0| = O(1)e^{-\nu_0 t} \neq 0 \end{cases}$$

for some constant $0 < \nu_0 < \frac{1}{2}$. So, there are some differences between the original solutions and their asymptotic profiles such that

$$J_i(x,t) - \bar J(x,t) \quad\text{and}\quad E(x,t) \notin L^2(\mathbb{R}_+).$$

Hence, we need to introduce some correction functions to take care of this difference. As shown in [15], we may similarly construct the correction functions $(\hat n_1, \hat J_1, \hat n_2, \hat J_2, \hat E)(x,t)$ such that

$$\begin{cases} \hat n_{1t} + \hat J_{1x} = 0,\\ \hat J_{1t} = n_+\hat E - \hat J_1,\\ \hat n_{2t} + \hat J_{2x} = 0,\\ \hat J_{2t} = -n_+\hat E - \hat J_2,\\ \hat E_x = \hat n_1 - \hat n_2, \end{cases} \quad\text{with}\quad \begin{cases} \hat n_i(x,t) \to 0,\\ \hat J_i(x,t) \to J_i^+(t),\\ \hat E(x,t) \to E^+(t), \end{cases} \quad \text{as } x \to \infty, \tag{3.18}$$

for $i = 1,2$. Let us select the initial data for the system (3.18) as follows:

$$\begin{cases} \hat J_i(x,0) = J_{i+}\int_0^x m_0(y)dy, \quad i=1,2,\\ \hat E(x,0) = E_+\int_0^x m_0(y)dy, \end{cases} \tag{3.19}$$

where $m_0(x)$ is a smooth function satisfying

$$m_0(x) \geq 0, \quad m_0 \in C_0^\infty(\mathbb{R}_+), \quad \int_{\mathbb{R}_+} m_0(y)dy = 1.$$

Then the correction functions can be constructed as follows.

Case 1. When $1-8n_+=0$, then

$$\hat{J}_1(x,t) = J_{1-}e^{-t} + \frac{1}{2}\Big\{(J_{2+}+J_{1+})e^{-t} + e^{-\frac{t}{2}}\Big[(J_{1+}-J_{2+}) + \frac{1}{2}(J_{2+}-J_{1+}+\frac{1}{2}E_+)t\Big]\Big\}\int_0^x m_0(y)dy, \tag{3.20}$$

$$\hat{J}_2(x,t) = J_{2-}e^{-t} + \frac{1}{2}\Big\{(J_{2+}+J_{1+})e^{-t} - e^{-\frac{t}{2}}\Big[J_{1+}-J_{2+} + \frac{1}{2}[(J_{2+}-J_{1+})+\frac{1}{2}E_+]t\Big]\Big\}\int_0^x m_0(y)dy, \tag{3.21}$$

$$\hat{E}(x,t) = e^{-\frac{t}{2}}\Big\{E_+ + \Big[J_{2+}-J_{1+}+\frac{1}{2}E_+\Big]t\Big\}\int_0^x m_0(y)dy, \tag{3.22}$$

$$\hat{n}_1(x,t) = \frac{m_0(x)}{2}\Big\{(J_{2+}+J_{1+})e^{-t} + e^{-\frac{t}{2}}\Big(E_+ + \Big(J_{2+}-J_{1+}+\frac{E_+}{2}\Big)t\Big)\Big\}, \tag{3.23}$$

$$\hat{n}_2(x,t) = \frac{m_0(x)}{2}\Big\{(J_{2+}+J_{1+})e^{-t} - e^{-\frac{t}{2}}\Big(E_+ + \Big(J_{2+}-J_{1+}+\frac{E_+}{2}\Big)t\Big)\Big\}. \tag{3.24}$$

Case 2. When $1-8n_+<0$, then

$$\begin{aligned}\hat{J}_1(x,t) = {} & \frac{J_{2+}+J_{1+}}{2}e^{-t}\int_0^x m_0(y)dy \\ & + \frac{1}{4}e^{-\frac{t}{2}}\Big\{2(J_{1+}-J_{2+})\cos\frac{\sqrt{8n_+-1}}{2}t \\ & \qquad + \Big(\frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}} + \sqrt{8n_+-1}E_+\Big) \\ & \qquad \times \sin\frac{\sqrt{8n_+-1}}{2}t\Big\}\int_0^x m_0(y)dy,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\hat{J}_2(x,t) = {} & \frac{J_{2+}+J_{1+}}{2}e^{-t}\int_0^x m_0(y)dy \\ & - \frac{1}{4}e^{-\frac{t}{2}}\Big\{2(J_{1+}-J_{2+})\cos\frac{\sqrt{8n_+-1}}{2}t \\ & \qquad + \Big(\frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}} + \sqrt{8n_+-1}E_+\Big) \\ & \qquad \times \sin\frac{\sqrt{8n_+-1}}{2}t\Big\}\int_0^x m_0(y)dy,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\hat{E}^+(x,t) = {} & e^{-\frac{t}{2}}\Big\{E_+\cos\frac{\sqrt{8n_+-1}}{2}t \\ & + \frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}}\sin\frac{\sqrt{8n_+-1}}{2}t\Big\}\int_0^x m_0(y)dy,\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
\hat{n}_1(x,t) = {} & \frac{J_{2+}+J_{1+}}{2} m_0(x) e^{-t} \\
& + \frac{1}{16n_+} m_0(x) e^{-\frac{t}{2}} \Big\{ 8n_+ E_+ \cos \frac{\sqrt{8n_+-1}}{2} t \\
& \qquad + \Big[\sqrt{8n_+-1}(E_+ + 2(J_{2+}-J_{1+})) \\
& \qquad\quad + \frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}} \Big] \times \sin \frac{\sqrt{8n_+-1}}{2} t \Big\},
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
\hat{n}_2(x,t) = {} & \frac{J_{2+}+J_{1+}}{2} m_0(x) e^{-t} \\
& - \frac{1}{16n_+} m_0(x) e^{-\frac{t}{2}} \Big\{ 8n_+ E_+ \cos \frac{\sqrt{8n_+-1}}{2} t \\
& \qquad + \Big[\sqrt{8n_+-1}(E_+ + 2(J_{2+}-J_{1+})) \\
& \qquad\quad + \frac{2(J_{2+}-J_{1+})+E_+}{\sqrt{8n_+-1}} \Big] \times \sin \frac{\sqrt{8n_+-1}}{2} t \Big\}.
\end{aligned} \tag{3.29}
$$

Case 3. When $1-8n_+>0$, then

$$
\begin{aligned}
\hat{J}_1(x,t) = {} & \frac{J_{2+}+J_{1+}}{2} e^{-t} \int_{-\infty}^{x} m_0(y)dy \\
& - \frac{1}{2}\Big(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\Big) \int_0^x m_0(y)dy,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
\hat{J}_2(x,t) = {} & \frac{J_{2+}+J_{1+}}{2} e^{-t} \int_{0}^{x} m_0(y)dy \\
& + \frac{1}{2}\Big(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\Big) \int_0^x m_0(y)dy,
\end{aligned} \tag{3.31}
$$

$$
\hat{E}(x,t) = \Big(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\Big) \int_0^x m_0(y)dy, \tag{3.32}
$$

$$
\hat{n}_1(x,t) = \frac{1}{2}(J_{2+}+J_{1+}) m_0(x) e^{-t} + \frac{1}{2} m_0(x) \Big(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\Big), \tag{3.33}
$$

$$
\hat{n}_2(x,t) = \frac{1}{2}(J_{2+}+J_{1+}) m_0(x) e^{-t} - \frac{1}{2} m_0(x) \Big(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\Big). \tag{3.34}
$$

Summarizing the properties of these correction functions, we have the next lemma.

LEMMA 3.1. *It holds that*

$$
\|(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(t)\|_{L^\infty(\mathbb{R}_+)} \le C\delta_1 e^{-\nu_0 t}, \tag{3.35}
$$

$$
(\hat{n}_1,\hat{J}_1,\hat{n}_2,\hat{J}_2,\hat{E})|_{x=0} = (0,0,0,0,0), \tag{3.36}
$$

$$
(\hat{n}_1,\hat{J}_1,\hat{n}_2,\hat{J}_2,\hat{E})|_{x=\infty} = (0,J_1^+(t),0,J_2^+(t),E^+(t)), \tag{3.37}
$$

$$
\int_{\mathbb{R}_+} [\hat{n}_1(x,0)-\hat{n}_2(x,0)]dx = \int_{\mathbb{R}_+} [n_{10}(x)-n_{20}(x)]dx = E_+, \tag{3.38}
$$

where $\delta_1 := |J_{1+}|+|J_{2+}|+|E_+|$.

3.2. Asymptotic profiles. Now we turn to showing how to find the particular asymptotic profiles for the original solutions. Let us make a perturbation of the original system (1.1) around the specified diffusion waves (1.13) by adding the correction functions (3.18). Then we obtain

$$
\begin{cases}
(n_1-\hat{n}_1-\bar{n})_t+(J_1-\hat{J}_1-\bar{J})_x=0,\\
(J_1-\hat{J}_1-\bar{J})_t+\left(\frac{J_1^2}{n_1}+p(n_1)-p(\bar{n})\right)_x\\
\qquad\qquad = n_1E-n_+\hat{E}-(J_1-\hat{J}_1-\bar{J})+p(\bar{n})_{xt},\\
(n_2-\hat{n}_2-\bar{n})_t+(J_2-\hat{J}_2-\bar{J})_x=0,\\
(J_2-\hat{J}_2-\bar{J})_t+\left(\frac{J_2^2}{n_2}+p(n_2)-p(\bar{n})\right)_x\\
\qquad\qquad = -n_2E+n_+\hat{E}-(J_2-\hat{J}_2-\bar{J})+p(\bar{n})_{xt},\\
(E-\hat{E})_x=(n_1-\hat{n}_1-\bar{n})-(n_2-\hat{n}_2-\bar{n}).
\end{cases}
\tag{3.39}
$$

Integrating $(3.39)_1$ and $(3.39)_3$ over $\mathbb{R}_+$ with respect to x, and noting $J_i(\infty,t)=\hat{J}_i(\infty,t)=\bar{J}(\infty,t)=0$, $J_i(0,t)=\hat{J}_i(0,t)=\bar{J}(0,t)=0$ for $i=1,2$, we have

$$
\frac{d}{dt}\int_{\mathbb{R}_+}[n_i(x,t)-\hat{n}_i(x,t)-\bar{n}(x,t)]dx=0,\quad i=1,2.
$$

Integrating the above equation with respect to t, we further have

$$
\int_{\mathbb{R}_+}[n_i(x,t)-\hat{n}_i(x,t)-\bar{n}(x,t)]dx=\int_{\mathbb{R}_+}[n_{i0}(x)-\hat{n}_i(x,0)-\bar{n}_0(x)]dx=0 \tag{3.40}
$$

by selecting the initial data $\bar{n}_0(x)$ for the IBVP porous media equations (1.15) such that

$$
\int_{\mathbb{R}_+}[n_{i0}(x)-\hat{n}_i(x,0)-\bar{n}_0(x)]dx=0,\quad i=1,2, \tag{3.41}
$$

namely,

$$
\int_{\mathbb{R}_+}[n_{i0}(x)-\bar{n}_0(x)]dx=\int_{\mathbb{R}_+}\hat{n}_i(x,0)dx,\quad i=1,2. \tag{3.42}
$$

Here, the determined initial data $\bar{n}_0(x)$ in (3.42) is consistent for the pairs of $(n_{10}(x), \hat{n}_1(x,0))$ and $(n_{20}(x),\hat{n}_2(x,0))$ because from (3.38), with the same selected initial data $\bar{n}_0(x)$,

$$
\int_{\mathbb{R}_+}[n_{10}(x)-\bar{n}_0(x)]dx=\int_{\mathbb{R}_+}\hat{n}_2(x,0)dx
$$

is equivalent to

$$
\int_{\mathbb{R}_+}[n_{20}(x)-\bar{n}_0(x)]dx=\int_{\mathbb{R}_+}\hat{n}_1(x,0)dx.
$$

With this preparation, we can study the perturbation around the diffusion wave in the L^2-framework:

$$
n_i-\bar{n}-\hat{n}_i,\ J_i-\bar{J}-\hat{J}_i,\ E-\hat{E}\in L^2(\mathbb{R}_+),\quad i=1,2.
$$

CLAIM 1. *The asymptotic profile for the original solution* $(n_1,J_1,n_2,J_2,E)(x,t)$ *of the Dirichlet IBVP* (1.1)–(1.3) *is* $(\bar{n},\bar{J},\bar{n},\bar{J},0)(x,t)$, *where* $(\bar{n},\bar{J})(x,t)$ *is the solution of the IBVP* (1.13) *with a specified initial data* $\bar{n}_0(x)$ *satisfying* (3.41).

3.3. Convergence theorem. Let us define

$$\begin{cases} \phi_i(x,t) := \int_0^x [n_i(\xi,t) - \hat{n}_i(\xi,t) - \bar{n}(\xi,t)]d\xi, \\ \psi_i(x,t) := J_i(x,t) - \hat{J}_i(x,t) - \bar{J}(x,t), \quad i = 1,2, \\ \mathcal{H}(x,t) := E(x,t) - \hat{E}(x,t); \end{cases} \tag{3.43}$$

then we get the equations for perturbation

$$\begin{cases} \phi_{1t} + \psi_1 = 0, \\ \psi_{1t} + \left(\dfrac{(-\psi_{1t} + \hat{J}_1 + \bar{J})^2}{\phi_{1x} + \hat{n}_1 + \bar{n}} + p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\bar{n}) \right)_x \\ \qquad = (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} + (\phi_{1x} + \hat{n}_1 + \bar{n} - n_+)\hat{E} - \psi_1 + p(\bar{n})_{xt}, \\ \phi_{2t} + \psi_2 = 0, \\ \psi_{2t} + \left(\dfrac{(-\psi_{2t} + \hat{J}_2 + \bar{J})^2}{\phi_{2x} + \hat{n}_2 + \bar{n}} + p(\phi_{2x} + \hat{n}_2 + \bar{n}) - p(\bar{n}) \right)_x \\ \qquad = -(\phi_{2x} + \hat{n}_2 + \bar{n})\mathcal{H} - (\phi_{2x} + \hat{n}_2 + \bar{n} - n_+)\hat{E} - \psi_2 + p(\bar{n})_{xt}, \\ \mathcal{H} = \phi_1 - \phi_2, \end{cases} \tag{3.44}$$

with the initial data

$$\begin{cases} \phi_{i0}(x) := \phi_i(x,0) = \int_0^x [n_{i0}(\xi) - \hat{n}_i(\xi,0) - \bar{n}_0(\xi)]d\xi, \\ \psi_{i0}(x) := \psi_i(x,0) = J_{i0}(x) - \hat{J}_i(x,0) - \bar{J}(x,t), \quad i = 1,2, \\ \mathcal{H}_0(x) := \phi_{10}(x) - \phi_{20}(x) \end{cases} \tag{3.45}$$

and the boundary conditions

$$\begin{cases} (\phi_i, \psi_i, \mathcal{H})|_{x=0} = 0, \\ (\phi_i, \psi_i, \mathcal{H})|_{x=\infty} = 0, \end{cases} \quad i = 1,2. \tag{3.46}$$

Substituting $(3.44)_1$ into $(3.44)_2$ and $(3.44)_3$ into $(3.44)_4$, respectively, we have

$$\begin{aligned} \phi_{1tt} + \phi_{1t} - \Big(p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\bar{n}) \Big)_x + (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} \\ = -f_1 + g_{1x} - p(\bar{n})_{xt}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} \phi_{2tt} + \phi_{2t} - \Big(p(\phi_{2x} + \hat{n}_2 + \bar{n}) - p(\bar{n}) \Big)_x - (\phi_{2x} + \hat{n}_2 + \bar{n})\mathcal{H} \\ = f_2 + g_{2x} - p(\bar{n})_{xt}, \end{aligned} \tag{3.48}$$

where

$$f_i = [\phi_{ix} + (\hat{n}_i - n_+) + \bar{n}]\hat{E}, \quad g_i = \frac{(-\phi_{it} + \hat{J}_i + \bar{J})}{\phi_{ix} + \hat{n}_i + \bar{n}}, \quad i = 1,2, \tag{3.49}$$

with the IBVs

$$\phi_i|_{t=0} = \phi_{i0}(x), \ \phi_{it}|_{t=0} = -\psi_{i0}(x), \ \phi_i|_{x=0} = 0, \ \phi_i|_{x=\infty} = 0, \ i = 1,2. \tag{3.50}$$

Furthermore, subtracting (3.48) from (3.47), we get

$$\begin{cases} \mathcal{H}_{tt} + \mathcal{H}_t + 2\bar{n}\mathcal{H} = h := h_{1x} - h_2 - h_3 + h_{4x}, \\ \mathcal{H}|_{t=0} =: \mathcal{H}_0(x), \\ \mathcal{H}_t|_{t=0} = -[\psi_{10}(x) - \psi_{20}(0)] =: \mathcal{H}_1(x), \\ \mathcal{H}|_{x=0} = 0, \end{cases} \tag{3.51}$$

where

$$
\begin{cases}
h_1 = p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\phi_{2x} + \hat{n}_2 + \bar{n}), \\
h_2 = (\phi_{1x} + \phi_{2x} + \hat{n}_1 + \hat{n}_2)\mathcal{H}, \\
h_3 = [\phi_{1x} + \phi_{2x} + (\hat{n}_1 - n_+) + (\hat{n}_2 - n_+) + 2\bar{n}]\hat{E}, \\
h_4 = \dfrac{(-\phi_{1t} + \hat{J}_1 + \bar{J})^2}{\phi_{1x} + \hat{n}_1 + \bar{n}} - \dfrac{(-\phi_{2t} + \hat{J}_2 + \bar{J})^2}{\phi_{2x} + \hat{n}_2 + \bar{n}}.
\end{cases}
\tag{3.52}
$$

We now state the convergence results as follows.

THEOREM 3.2. *Let $(\phi_{i0}, \psi_{i0}) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ for $i = 1, 2$, $\delta := |J_{1+}| + |J_{1-}| + |J_{2+}| + |J_{2-}| + |E_+| + \max_{x\in\mathbb{R}_+}[|n_{10}(x) - n_+| + |n_{20}(x) - n_+|]$, and $\Phi_0 := \|(\phi_{10}, \phi_{20})\|_{H^3(\mathbb{R}_+)} + \|(\psi_{10}, \psi_{20})\|_{H^2(\mathbb{R}_+)}$. Then there exists $\delta_0 > 0$ such that if $\delta + \Phi_0 \le \delta_0$, the solution (n_1, n_2, J_1, J_2, E) of the Dirichlet IBVP (1.1)–(1.3) is unique and globally exists. And the following optimal decay rates (in L^2-sense of initial data), for $i = 1, 2$, hold:*

$$
\begin{cases}
\|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, & k = 0, 1, 2, 3, \\
\|\partial_x^k \psi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, & k = 0, 1, \\
\|\partial_x^k \partial_t^l \mathcal{H}(t)\|_{L^2(\mathbb{R})} = O(1)e^{-\nu t}, & 0 \le k + l \le 2, \\
\|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, & k = 0, 1, 2, \\
\|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}, & \\
\|\partial_x^k \partial_t^l \mathcal{H}(t)\|_{L^\infty(\mathbb{R})} = O(1)e^{-\nu t}, & 0 \le k + l \le 1.
\end{cases}
\tag{3.53}
$$

Moreover, if $(\phi_{i0}, \psi_{i0}) \in L^1(\mathbb{R}_+)$ for $i = 1, 2$, then the optimal decay rates (in the L^1-sense of initial data) hold:

$$
\begin{cases}
\|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, & k = 0, 1, 2, \\
\|\psi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}, & \\
\|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(k+1)/2}, & k = 0, 1, \\
\|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-3/2}. &
\end{cases}
\tag{3.54}
$$

Furthermore, if $(\phi_{i0}, \psi_{i0}) \in \cap L^{1,\gamma}(\mathbb{R}_+)$ for $i = 1, 2$, where $0 \le \gamma \le \frac{1}{4}$, then faster decay rates hold:

$$
\begin{cases}
\|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{2k+1}{4} - \frac{\gamma}{2}}, & k = 0, 1, 2, \\
\|\psi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{5}{4} - \frac{\gamma}{2}}, & \\
\|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{k+1}{2} - \frac{\gamma}{2}}, & k = 0, 1, \\
\|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{3}{2}}. &
\end{cases}
\tag{3.55}
$$

Notice that $\phi_{ix} = n_i - \bar{n} - \hat{n}_i$ and $\psi_i = J_i - \bar{J} - \hat{J}_i$ for $i = 1, 2$, and the correction functions $(\hat{n}_i, \hat{J}_i, \hat{E})(x, t)$ decay exponentially (see Lemma 3.1). We then immediately obtain the following optimal convergence to the diffusion waves in L^∞-norm.

COROLLARY 3.3 (convergence to diffusion waves).

1. *When $(\phi_{i0}, \psi_{i0}) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ for $i = 1, 2$, then*

$$
\|(n_i - \bar{n}, J_i - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, e^{-\nu t}), \tag{3.56}
$$

$$
\|(n_1 - n_2, J_1 - J_2, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(e^{-\nu t}, e^{-\nu t}, e^{-\nu t}). \tag{3.57}
$$

2. *When* $(\phi_{i0}, \psi_{i0}) \in L^1(\mathbb{R}_+) \cap (H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+))$ *for* $i = 1, 2$, *then*

$$\|(n_i - \bar{n}, J_i - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(t^{-1}, t^{-\frac{3}{2}}, e^{-\nu t}), \tag{3.58}$$

$$\|(n_1 - n_2, J_1 - J_2, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(e^{-\nu t}, e^{-\nu t}, e^{-\nu t}). \tag{3.59}$$

3. *When* $(\phi_{i0}, \psi_{i0}) \in L^{1,\gamma}(\mathbb{R}_+) \cap (H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+))$ *for* $i = 1, 2$, *where* $0 \leq \gamma \leq \frac{1}{4}$, *then*

$$\|(n_i - \bar{n}, J_i - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}}, e^{-\nu t}), \tag{3.60}$$

$$\|(n_1 - n_2, J_1 - J_2, E)(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(e^{-\nu t}, e^{-\nu t}, e^{-\nu t}). \tag{3.61}$$

Remark 2.

1. By comparison with linear heat equations, the convergence rates shown of (3.53) and (3.54) are optimal, in the cases of the initial perturbations in L^2 and L^1, respectively. Moreover, the convergence rates of (3.55) are also optimal in the case of the initial perturbation in the weighted space $L^{1,\gamma}$. In particular, $\|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = \|\phi_{it}(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)t^{-3/2}$ is optimal, due to the slower decay of the nonlinear source term caused by the diffusion wave $\|p(\bar{n})_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(1)t^{-3/2}$. In fact, it can be easily seen from the following example. Consider the linear heat equation

$$\begin{cases} \phi_t - \phi_{xx} = (1+t)^{-3/2} m_0(x), \\ \phi|_{t=0} = 0, \\ \phi|_{x=0} = 0 \end{cases} \tag{3.62}$$

with sufficiently smooth initial data $\phi(x, 0) = 0$, where $m_0(x)$ is in $C_0^\infty(\mathbb{R}_+)$. Its solution can be expressed by

$$\phi(x,t) = \int_0^t \int_{\mathbb{R}_+} [G(x-y, t-s) - G(x-y, t+s)](1+s)^{-3/2} m_0(y) dy ds,$$

where $G(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{t}}$ is the heat kernel. A simple computation gives

$$|\phi_t(x,t)| \sim O(1)t^{-3/2}, \quad |\partial_x^k \partial_t \phi(x,t)| \sim O(1)t^{-3/2}.$$

This confirms that even for the linear heat equation (3.62), the optimal decay rates of the solution in the higher order derivatives ϕ_{xt}, ϕ_{xxx}, and ϕ_{xxt}, etc., are of the order $t^{-3/2}$ only.

2. Although the diffusion wave $(\bar{n}, \bar{J}, \bar{E})(x,t)$ is the asymptotic profile to the original solution $(n_1, J_1, n_2, J_2, E)(x,t)$ to the IBVP (1.1)–(1.3), comparing (3.56), (3.58), and (3.60) to (3.57), (3.59), and (3.61), we can see that the solutions (n_1, J_1) and (n_2, J_2) are closer to each other. This is due to the dispersion effect of the Klein–Gordon equation (3.51) with the null Dirichlet boundary.

3.4. Proof of Theorem 3.2. This subsection is devoted to the proof of Theorem 3.2. The global existence and uniqueness of solution for evolution equations as well as the corresponding energy decay estimates usually can be obtained by the elementary energy method and a continuity extension argument, which is based on the local existence and the a priori estimates. Establishing the a priori estimates plays a key role in the whole proof. Here we focus on establishing the a priori energy estimates in different solution spaces.

Step 1: *Proof of* (3.53). Let the initial perturbation be $(\phi_{i0}, \psi_{i0}) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ for $i = 1, 2$. Let $T \in (0, +\infty]$; we define the solution space for (3.47), (3.48), (3.50), and (3.51) as follows:

$$X_1(0,T) = \Big\{(\phi_1, \phi_{1t}, \phi_2, \phi_{2t}, \mathcal{H})(x,t) \Big| \partial_t^j \phi_i \in C(0,T; H^{3-j}(\mathbb{R}_+)), i = 1,2,\ j = 0,1,$$
$$\partial_t^j \mathcal{H} \in C(0,T; H^{2-j}(\mathbb{R}_+)),\ j = 0,1,\ 0 \le t \le T\Big\}$$

with the norm

$$N_1(T)$$
$$= \sup_{0\le t\le T} \left\{ \sum_{k=0}^{3} (1+t)^{\frac{k}{2}} \|\partial_x^k(\phi_1, \phi_2)(t)\|_{L^2(\mathbb{R}_+)} \right.$$
$$\left. + \sum_{k=0}^{2} (1+t)^{\frac{k+2}{2}} \|\partial_x^k(\phi_{1t}, \phi_{2t})(t)\|_{L^2(\mathbb{R}_+)} + \sum_{k+j=0}^{2} e^{\nu t} \|\partial_t^j \partial_x^k \mathcal{H}(t)\|_{L^2(\mathbb{R}_+)} \right\}.$$

As shown in [15], by the basic energy method but a tedious computation, we can prove the following lemma. The details of the proof are omitted for brevity. Interested readers can refer to [15].

LEMMA 3.4. *When $(\phi_{10}, \phi_{20}, \psi_{10}, \psi_{20}) \in H^3(\mathbb{R}_+) \times H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$, there exists a unique and global solution $(\phi_1, \psi_1, \phi_2, \psi_2, \mathcal{H})(x,t) \in X_1(0,T)$ for the IBVP* (3.47)–(3.51) *satisfying*

$$\|(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_t, \mathcal{H}_{xx}, \mathcal{H}_{xt}, H_{tt})(t)\|_{L^2(\mathbb{R}_+)} \le C(\delta + \Phi_0) e^{-\nu t}, \tag{3.63}$$

$$\|\partial_x^k(\phi_1, \phi_2)(t)\|_{L^2(\mathbb{R}_+)} \le C(\delta + \Phi_0)(1+t)^{-\frac{k}{2}}, \quad 0 \le k \le 3, \tag{3.64}$$

$$\|\partial_x^k(\psi_1, \psi_2)(t)\|_{L^2(\mathbb{R}_+)} \le C(\delta + \Phi_0)(1+t)^{-\frac{k+1}{2}}, \quad 0 \le k \le 2, \tag{3.65}$$

provided $\delta + N_1(T) \ll 1$.

Step 2: *Proof of* (3.54). Let the initial perturbation be $(\phi_{i0}, \psi_{i0}) \in L^1(\mathbb{R}_+) \cap (H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+))$ for $i = 1, 2$. First, we already have all the decay estimates (3.63)–(3.65). Now what we need is to improve the decay rates for $(\phi_i, \psi_i)(x,t)$ as in (3.54) when the initial perturbation is in L^1. Let $T \in (0, +\infty]$; we define the norm $N_2(T)$ for the solution space $X_2(0,T)$ as follows:

$$X_2(0,T) = \{(\phi_1, \phi_2)(x,t) | \phi_i \in C(0,T; H^2(\mathbb{R}_+)), \phi_{it} \in C(0,T; L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+))\}$$

and

$$N_2(T) = \sup_{0\le t\le T} \left\{ \sum_{k=0}^{2} (1+t)^{\frac{2k+1}{4}} \|\partial_x^k(\phi_1, \phi_2)(t)\|_{L^2(\mathbb{R}_+)} \right.$$
$$+ \sum_{k=0}^{1} (1+t)^{\frac{k+1}{2}} \|\partial_x^k(\phi_1, \phi_2)(t)\|_{L^\infty(\mathbb{R}_+)}$$
$$\left. + (1+t)^{\frac{5}{4}} \|(\phi_{1t}, \phi_{2t})(t)\|_{L^2(\mathbb{R}_+)} + (1+t)^{\frac{3}{2}} \|(\phi_{1t}, \phi_{2t})(t)\|_{L^\infty(\mathbb{R}_+)} \right\}.$$

Let us rewrite (3.47) and (3.48) as follows: for $i = 1, 2$,

$$\begin{cases} \phi_{itt} + \phi_{it} - p'(n_+)\phi_{ixx} = F_i, \\ (\phi_i, \phi_{it})|_{t=0} = (\phi_{i0}, -\psi_{i0})(x), \\ \phi_i|_{x=0} = 0, \end{cases} \tag{3.66}$$

where

$$F_i := F_{i1} + F_{i2} + F_{i3}, \tag{3.67}$$

$$F_{i1} := \Big(p(\phi_{ix} + \hat{n}_i + \bar{n}) - p(\bar{n}) - p'(n_+)\phi_{ix}\Big)_x, \tag{3.68}$$

$$F_{i2} := (-1)^i(\phi_{ix} + \hat{n}_i + \bar{n})\mathcal{H}, \tag{3.69}$$

$$F_{i3} := (-1)^i f_i + g_{ix} - p(\bar{n})_{xt}. \tag{3.70}$$

From (2.5), then (3.66) can be expressed in the integral form for $i = 1, 2$:

$$\begin{aligned} \phi_i(x,t) = & \int_0^\infty [K_0(x-y,t) - K_0(x+y,t)]\phi_{i0}(y)dy \\ & - \int_0^\infty [K_1(x-y,t) - K_1(x+y,t)]\psi_{i0}(y)dy \\ & + \int_0^t \int_0^\infty [K_1(x-y,t-s) - K_1(x+y,t-s)]F_i(y,s)dyds. \end{aligned} \tag{3.71}$$

By applying Lemma 2.2, we immediately have the following estimates.

LEMMA 3.5. *Let $\phi_{i0} \in L^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ and $\psi_{i0} \in L^1(\mathbb{R}_+) \times H^1(\mathbb{R}_+)$. Then for $i = 1, 2$,*

$$\begin{aligned} & \left\| \partial_t^j \partial_x^k \int_0^\infty \Big[K_0(x-y,t) - K_0(x+y,t)\Big]\phi_{i0}(y)dy \right\|_{L^2(\mathbb{R}_+)} \\ & \quad \le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|\phi_{i0}\|_{L^1(\mathbb{R}_+)} + \|\phi_{i0}\|_{H^1(\mathbb{R}_+)}\Big], \quad 0 \le j+k \le 2, \end{aligned} \tag{3.72}$$

$$\begin{aligned} & \left\| \partial_t^j \partial_x^k \int_0^\infty \Big[K_1(x-y,t) - K_1(x+y,t)\Big]\psi_{i0}(y)dy \right\|_{L^2(\mathbb{R}_+)} \\ & \quad \le C(1+t)^{-j-\frac{2k+1}{4}}\Big[\|\psi_{i0}\|_{L^1(\mathbb{R}_+)} + \|\psi_{i0}\|_{H^1(\mathbb{R}_+)}\Big], \quad 0 \le j+k \le 1, \end{aligned} \tag{3.73}$$

$$\begin{aligned} & \left\| \partial_t^j \partial_x^k \int_0^\infty \Big[K_0(x-y,t) - K_0(x+y,t)\Big]\phi_{i0}(y)dy \right\|_{L^\infty(\mathbb{R}_+)} \\ & \quad \le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|\phi_{i0}\|_{L^1(\mathbb{R}_+)} + \|\phi_{i0}\|_{H^2(\mathbb{R}_+)}\Big], \quad 0 \le j+k \le 2, \end{aligned} \tag{3.74}$$

$$\begin{aligned} & \left\| \partial_t^j \partial_x^k \int_0^\infty \Big[K_1(x-y,t) - K_1(x+y,t)\Big]\psi_{i0}(y)dy \right\|_{L^\infty(\mathbb{R}_+)} \\ & \quad \le C(1+t)^{-j-\frac{k+1}{2}}\Big[\|\psi_{i0}\|_{L^1(\mathbb{R}_+)} + \|\psi_{i0}\|_{H^1(\mathbb{R}_+)}\Big], \quad 0 \le j+k \le 1. \end{aligned} \tag{3.75}$$

Next we establish the decay energy estimates for the nonlinear term.

LEMMA 3.6. *Let* $(\phi_i, \psi_i, \mathcal{H})(x,t) \in X_2(0,T)$. *Then*

$$\begin{aligned}&\int_0^t \left\|\partial_t^j\partial_x^k \int_0^\infty \Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]F_i(y,s)dy\right\|_{L^2(\mathbb{R}_+)} ds \\ &\le C[\delta+N_2(T)](1+t)^{-j-\frac{2k+1}{4}}, \quad 0\le 2j+k\le 2,\end{aligned} \tag{3.76}$$

$$\begin{aligned}&\int_0^t \left\|\partial_t^j\partial_x^k \int_0^\infty \Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]F_i(y,s)dy\right\|_{L^\infty(\mathbb{R}_+)} ds \\ &\le C[\delta+N_2(T)](1+t)^{-j-\frac{k+1}{2}}, \quad 0\le j+k\le 1.\end{aligned} \tag{3.77}$$

Proof. By Taylor's expansion, we have

$$\begin{aligned}|F_i| \approx O(1)\Big(&|\hat{n}_i+\phi_{ix}||\hat{n}_{ix}+\phi_{ixx}|+|\hat{n}_{ix}|+|\bar{n}_x\hat{n}_i| \\ &+|\bar{n}-n_+||\phi_{ixx}|+|\bar{n}_x\phi_{ix}|+|\phi_{ix}+\hat{n}_i+\bar{n}||\mathcal{H}|+|\bar{n}_{xt}| \\ &+|\phi_{ix}+\hat{n}_i+(\bar{n}-n_+)||\hat{E}|+|\phi_{it}|+|\hat{J}_i|+|\bar{J}|\Big).\end{aligned}$$

A straightforward but tedious computation with the help of the decay estimates (2.1)–(2.3) and the a priori assumption of the solution within $X_2(0,T)$ with the norm $N_2(T)$ gives

$$\|F_i(t)\|_{L^1(\mathbb{R}_+)} \le C(\delta+N_2(T))(1+t)^{-\frac{3}{2}}, \tag{3.78}$$

$$\|F_i(t)\|_{H^1(\mathbb{R}_+)} \le C(\delta+N_2(T))(1+t)^{-\frac{3}{2}}. \tag{3.79}$$

Thus, by using Lemma 2.2 and Lemma 2.4 and noting $\frac{3}{2} > j+\frac{2k+1}{4}$ for $0\le k+2j\le 2$, we have

$$\begin{aligned}&\int_0^t \left\|\partial_t^j\partial_x^k \int_0^\infty \Big[K_1(x-y,t-s)-K_1(x+y,t-s)\Big]F_i(y,s)dy\right\|_{L^2(\mathbb{R}_+)} ds \\ &\le C\int_0^t (1+t-s)^{-j-\frac{2k+1}{4}}\Big[\|F_i(s)\|_{L^1(\mathbb{R}_+)}+\|F_i(s)\|_{H^1(\mathbb{R}_+)}\Big]ds \\ &\le C[\delta+N_2(T)]\int_0^t (1+t-s)^{-j-\frac{2k+1}{4}}\Big[(1+s)^{-\frac{3}{2}}+(1+s)^{-\frac{3}{2}}\Big]ds \\ &\le C[\delta+N_2(T)](1+t)^{-j-\frac{2k+1}{4}}, \quad 0\le 2j+k\le 2.\end{aligned} \tag{3.80}$$

Similarly, (3.77) can be proved and the details are omitted. Thus, the proof is complete. ∎

Combining Lemmas 3.5 and 3.6, we prove the next lemma.

LEMMA 3.7. *It holds that for* $i = 1, 2$,

$$\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-j-\frac{2k+1}{4}}, \quad 0\le 2j+k\le 2, \tag{3.81}$$

$$\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-j-\frac{k+1}{2}}, \quad 0\le j+k\le 1, \tag{3.82}$$

provided $\delta + N_2(T) \ll 1$.

Remark 3. The diffusion wave decays as $\|\bar{n}_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{3}{2}}$ (see (2.3) in Lemma 2.1), which implies that the decay of F_i is $\|F_i(t)\|_{L^1(\mathbb{R}_+)} = O(1)(1+t)^{-\frac{3}{2}}$ only. As shown in (3.80), we can obtain the optimal decay rates $\|\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)} =$

$O(1)(1+t)^{-\frac{2k+1}{4}}$ for $k=0,1,2$ only. For $k\geq 3$, notice that $\frac{2k+1}{4}>\frac{3}{2}$, and by Lemma 2.4 we can obtain the higher order ($k\geq 3$) energy estimates as $\|\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)}=O(1)(1+t)^{-\frac{3}{2}}$. So, in order to get the optimal decay rates, we consider only $k=0,1,2$. And also, different from Lemma 3.5, where we may allow $0\leq j+k\leq 2$, we need $0\leq 2j+k\leq 2$ in Lemma 3.6.

Step 3: *Proof of* (3.55). Let the initial perturbation be $(\phi_{i0},\psi_{i0})\in L^{1,\gamma}(\mathbb{R}_+)\cap(H^3(\mathbb{R}_+)\times H^2(\mathbb{R}_+))$ with $0\leq\gamma\leq\frac{1}{4}$. (We will show later that the selection of $\gamma=\frac{1}{4}$ is the optimal.) We define the norm $N_3(T)$ for the solution space $X_2(0,T)$ as

$$N_3(T)\\ =\sup_{0\leq t\leq T}\left\{\sum_{2j+k=0}^{2}(1+t)^{j+\frac{2k+1}{4}+\frac{\gamma}{2}}\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^2}+\sum_{k=0}^{1}(1+t)^{\frac{k+1}{2}+\frac{\gamma}{2}}\|\partial_x^k\phi_i(t)\|_{L^\infty}\right\}.$$

As shown in [28] for the linear damping case (see Theorem 2.2, (2.20) in [28]), we can similarly prove that for $\gamma\in[0,\frac{1}{2}]$,

$$\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^{2,\gamma}}\leq C(1+t)^{-(j+k)/2},\quad 0\leq j+k\leq 2. \tag{3.83}$$

Furthermore, we can also prove

$$\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^{2,\gamma}(\mathbb{R}_+)}\leq C(1+t)^{-j-\frac{2k+1}{4}+\frac{\gamma}{4}},\quad 0\leq j+k\leq 2, \tag{3.84}$$

if $\|\partial_t^j\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)}\leq C(1+t)^{-j-\frac{2k+1}{4}}$.

Now let $\phi(x,t)\in X_2(0,T)$ with the norm $N_3(t)$. From (3.71), and by using integration by parts and Hölder inequality, as in [26, 30], we can similarly derive for $\gamma\in[0,1)$,

$$\|F_i(t)\|_{L^{1,\gamma}(\mathbb{R}_+)}\leq C[\delta+N_3(T)](1+t)^{-\frac{3}{2}+\frac{\gamma}{2}},\quad i=1,2. \tag{3.85}$$

Thus, from (3.71) and noticing Lemma 2.3 and (3.85), we have

$$\begin{aligned}
&\|\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)}\\
&\quad\leq\left\|\partial_x^k\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]\phi_{i0}(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\qquad+\left\|\partial_x^k\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]\psi_{i0}(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\qquad+\int_0^t\left\|\partial_x^k\int_0^\infty[K_1(x-y,t-s)-K_1(x+y,t-s)]F_i(y,s)dy\right\|_{L^2(\mathbb{R}_+)}ds\\
&\quad\leq C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)})\\
&\qquad+C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|\psi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)})\\
&\qquad+C\int_0^t(1+t-s)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}[\|F_i(s)\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|F_i(s)\|_{H^1(\mathbb{R}_+)}]ds\\
&\quad\leq C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}(\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)})\\
&\qquad+C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}(\mathbb{R}_+)}(\|\psi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)})\\
&\qquad+C\eta[1+N_3(T)]\int_0^t(1+t-s)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}[(1+s)^{-\frac{3}{2}+\frac{\gamma}{2}}+(1+s)^{-\frac{3}{2}}]ds,\\
&\qquad k=0,1,2.
\end{aligned} \tag{3.86}$$

Let us balance the order of rates in the last term of (3.86):

$$-\frac{2k+1}{4}-\frac{\gamma}{2}=-\frac{3}{2}+\frac{\gamma}{2}\quad \text{for the largest case } k=2,$$

which gives

$$\gamma=\frac{1}{4}. \tag{3.87}$$

So, we have

$$\frac{2k+1}{4}+\frac{\gamma}{2}\leq\frac{3}{2}-\frac{\gamma}{2}\quad \text{for } k=0,1,2 \text{ and } 0\leq\gamma\leq\frac{1}{4}.$$

Applying Lemma 2.4 to (3.86), we obtain

$$\|\partial_x^k\phi_i(t)\|_{L^2(\mathbb{R}_+)}\leq C\bar{\delta}(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}},\quad k=0,1,2,\quad 0\leq\gamma\leq\frac{1}{4}, \tag{3.88}$$

where

$$\bar{\delta}:=\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|U_0\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)}+[\delta+N_3(T)].$$

Similarly, we have

$$\begin{aligned}
&\|\phi_{it}(t)\|_{L^2(\mathbb{R}_+)}\\
&\quad\leq\left\|\partial_t\int_0^\infty[K_0(x-y,t)-K_0(x+y,t)]\phi_{i0}(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\qquad+\left\|\partial_t\int_0^\infty[K_1(x-y,t)-K_1(x+y,t)]\psi_{i0}(y)dy\right\|_{L^2(\mathbb{R}_+)}\\
&\qquad+\int_0^t\left\|\partial_t\int_0^\infty[K_1(x-y,t-s)-K_1(x+y,t-s)]F_i(y,s)dy\right\|_{L^2(\mathbb{R}_+)}ds\\
&\quad\leq C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)})\\
&\qquad+C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\psi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)})\\
&\qquad+C\int_0^t(1+t-s)^{-\frac{5}{4}-\frac{\gamma}{2}}[\|F_i(s)\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|F_i(s)\|_{H^1(\mathbb{R}_+)}]ds\\
&\quad\leq C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)})\\
&\qquad+C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\psi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)})\\
&\qquad+C\delta[1+N_3(T)]\int_0^t(1+t-s)^{-\frac{5}{4}-\frac{\gamma}{2}}[(1+s)^{-\frac{3}{2}+\gamma}+(1+s)^{-\frac{3}{2}}]ds\\
&\quad\leq C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\phi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\phi_{i0}\|_{H^2(\mathbb{R}_+)})\\
&\qquad+C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}(\|\psi_{i0}\|_{L^{1,\gamma}(\mathbb{R}_+)}+\|\psi_{i0}\|_{H^1(\mathbb{R}_+)})\\
&\qquad+C\delta[1+N_3(T)](1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}+C[\delta+N_3(T)](1+t)^{-\frac{3}{2}}\\
&\quad\leq C[\delta+N_3(T)](1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}.
\end{aligned} \tag{3.89}$$

We may also prove L^∞-estimates similarly:

$$\|\partial_x^k\phi_i(t)\|_{L^\infty(\mathbb{R}_+)}\leq C[\delta+N_3(T)](1+t)^{-\frac{k+1}{2}-\frac{\gamma}{2}},\ k=0,1,\ i=1,2. \tag{3.90}$$

However, since the decay of the source term likes $\|F_i(t)\|_{L^1(\mathbb{R}_+)} = O(t^{-3/2})$ due to the decay of the diffusion wave $\|\bar{n}_{xt}(t)\|_{L^1(\mathbb{R}_+)} = O(t^{-3/2})$, the decay rate for $\partial_t\phi_i$ is $\|\phi_{it}(t)\|_{L^1(\mathbb{R}_+)} = O(t^{-3/2})$ rather than $t^{-\frac{3}{2}-\frac{\gamma}{2}}$. So we have only

$$\|\phi_{it}(t)\|_{L^1(\mathbb{R}_+)} \le C[\delta + N_3(T)](1+t)^{-3/2}. \tag{3.91}$$

Combining (3.88), (3.89), (3.90), and (3.91), we prove the following lemma.

LEMMA 3.8. *It holds that for $i = 1, 2$ and $0 \le \gamma \le \frac{1}{4}$,*

$$\|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-\frac{2k+1}{4}-\frac{\gamma}{2}}, \quad k = 0, 1, 2, \tag{3.92}$$

$$\|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{k+1}{2}-\frac{\gamma}{2}}, \quad k = 0, 1, \tag{3.93}$$

$$\|\phi_{1t}(t)\|_{L^2(\mathbb{R}_+)} \le C(1+t)^{-\frac{5}{4}-\frac{\gamma}{2}}, \tag{3.94}$$

$$\|\phi_{it}(t)\|_{L^\infty(\mathbb{R}_+)} \le C(1+t)^{-\frac{3}{2}}, \tag{3.95}$$

provided $\delta + N_3(T) \ll 1$.

4. Neumann IBVP.

In this section, we consider the Euler–Poisson system (1.1) with the initial data (1.2) and the Neumann boundary (1.4). We will first study the solutions to (1.1), (1.2), and (1.4) at the far field so that we know the exact difference between the original Neumann IBVP solution and the corresponding asymptotic profile (1.14) at the far field. Then we construct some suitable correction functions and show how to select the initial data $\bar{n}_0(x)$ in (1.14) such that we can find the suitable asymptotic profile. Finally, we will prove the convergence of the solution of the IBVP (1.1), (1.2), and (1.4) to the chosen diffusion wave (1.14).

4.1. Correction functions.

Let us study the solution of (1.1) at far field $x = +\infty$. Taking limits in (1.1) as $x \to +\infty$, and denoting $(n_1^+, J_1^+, n_2^+, E^+)(t) = \lim_{x\to\infty}(n_1, J_1, n_2, J_2, E)(x,t)$, where $E^+(t) = a(t)$ as the boundary condition given in (1.4), then we have

$$\begin{cases} \frac{d}{dt} n_i^+(t) = 0, \\ \frac{d}{dt} J_1^+(t) = n_1^+(t)E^+(t) - J_1^+(t), \\ \frac{d}{dt} J_2^+(t) = -n_2^+(t)E^+(t) - J_2^+(t), \end{cases} \quad \text{with} \begin{cases} n_i^+|_{t=0} = n_+, \\ J_i^+|_{t=0} = J_{i+}, \end{cases} \quad i = 1, 2,$$

which gives

$$\begin{cases} n_i^+(t) = n_+, \\ \frac{d}{dt} J_1^+(t) = n_+E^+(t) - J_1^+(t), \\ \frac{d}{dt} J_2^+(t) = -n_+E^+(t) - J_2^+(t), \end{cases} \quad \text{with} \quad J_i^+|_{t=0} = J_{i+}, i = 1, 2, \tag{4.1}$$

namely,

$$J_1^+(t) = J_{1+}e^{-t} + n_+ \int_0^t e^{-(t-s)} a(s) ds, \tag{4.2}$$

$$J_1^+(t) = J_{1+}e^{-t} - n_+ \int_0^t e^{-(t-s)} a(s) ds. \tag{4.3}$$

Since (see (1.5) and (1.6))

$$|E(\infty,t)| = |E^+(t)| = |a(t)| = \begin{cases} O(e^{-\eta t}),\ \eta>0,\ \text{or}, \\ O(t^{-\theta}),\ \theta>\frac{3}{2}, \end{cases}$$

we immediately obtain

$$|J_i^+(t)| \le \begin{cases} Ce^{-\nu_3 t}, & 0<\nu_3<\min\{\eta,\frac{1}{2}\}, \\ C(1+t)^{-\theta}, & \theta>\frac{3}{2}, \end{cases} \quad i=1,2. \tag{4.4}$$

Notice that the expected asymptotic profile of the IBVP (1.14) behaves like

$$\bar{n}(+\infty,t)=n_+,\quad \bar{J}_i(\infty,t)=0,\quad \bar{E}(+\infty,t)=0, \tag{4.5}$$

so there are some difference between $J_i^+(t)$ and $\bar{J}(\infty,t)$:

$$|E^+(t)-0| = |a(t)| \neq 0 \ \text{ and } \ |J_i^+(t)-\bar{J}| = O(1)t^{-3/2}\neq 0,\quad i=1,2. \tag{4.6}$$

To overcome this, we need to construct some correction functions. As shown in section 3, we can similarly construct the correction functions $(\hat{n}_1,\hat{J}_1,\hat{n}_2,\hat{J}_2,\hat{E})(x,t)$ such that

$$\begin{cases} \hat{n}_{1t}+\hat{J}_{1x}=0, \\ \hat{J}_{1t}=n_+\hat{E}-\hat{J}_1, \\ \hat{n}_{2t}+\hat{J}_{2x}=0, \\ \hat{J}_{2t}=-n_+\hat{E}-\hat{J}_2, \\ \hat{E}_x=\hat{n}_1-\hat{n}_2, \\ \hat{J}_i|_{x=\infty}=J_i^+(t), \\ \hat{E}|_{x=\infty}=a(t). \end{cases} \tag{4.7}$$

These correction functions can be technically constructed as follows.

Case 1. When $1-8n_+=0$, then

$$\hat{n}_1(x,t) = -\frac{1}{2}m_0(x)\big\{-(J_{1+}+J_{2+})e^{-t}+te^{-\frac{t}{2}}(J_{1+}-J_{2+})\big\}, \tag{4.8}$$

$$\hat{n}_2(x,t) = -\frac{1}{2}m_0(x)\big\{-(J_{1+}+J_{2+})e^{-t}-te^{-\frac{t}{2}}(J_{1+}-J_{2+})\big\}, \tag{4.9}$$

$$\begin{aligned} \hat{J}_1(x,t) = J_1^+(t) + \frac{1}{2}\bigg\{ &-(J_{1+}+J_{2+})e^{-t} \\ &+ e^{-\frac{t}{2}}\Big[-(J_{1+}-J_{2+})+\frac{t}{2}(J_{1+}-J_{2+})\Big]\bigg\}\int_x^{+\infty} m_0(y)dy, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \hat{J}_2(x,t) = J_2^+(t) + \frac{1}{2}\bigg\{ &-(J_{1+}+J_{2+})e^{-t} \\ &- e^{-\frac{t}{2}}\Big[-(J_{1+}-J_{2+})+\frac{t}{2}(J_{1+}-J_{2+})\Big]\bigg\}\int_x^{+\infty} m_0(y)dy, \end{aligned} \tag{4.11}$$

$$\hat{E}(x,t) = E^+(t)+te^{-\frac{t}{2}}(J_{1+}-J_{2+})\int_x^{+\infty} m_0(y)dy, \tag{4.12}$$

where the nonnegative and smooth function $m_0(x)$ is selected as

$$m_0(x) \geq 0, \quad m_0 \in C_0^\infty(\mathbb{R}^+), \quad \int_0^{+\infty} m_0(y)dy = 1. \tag{4.13}$$

Case 2. When $1 - 8n_+ < 0$, then

$$\hat{n}_1(x,t) = \left(\frac{1}{2}(J_{1+} + J_{2+})e^{-t} - \frac{J_{1+} - J_{2+}}{\sqrt{8n_+ - 1}} e^{-\frac{t}{2}} \sin \frac{\sqrt{8n_+ - 1}}{2} t\right) m_0(x), \tag{4.14}$$

$$\hat{n}_2(x,t) = \left(\frac{1}{2}(J_{1+} + J_{2+})e^{-t} + \frac{J_{1+} - J_{2+}}{\sqrt{8n_+ - 1}} e^{-\frac{t}{2}} \sin \frac{\sqrt{8n_+ - 1}}{2} t\right) m_0(x), \tag{4.15}$$

$$\begin{aligned} \hat{J}_1(x,t) = J_1^+(t) + \bigg\{ &- \frac{1}{2}(J_{1+} + J_{2+})e^{-t} - \frac{1}{2} e^{-\frac{t}{2}} (J_{1+} - J_{2+}) \cos \frac{\sqrt{8n_+ - 1}}{2} t \\ &+ \frac{1}{2} e^{-\frac{t}{2}} \frac{J_{1+} - J_{2+}}{\sqrt{8n_+ - 1}} \sin \frac{\sqrt{8n_+ - 1}}{2} t \bigg\} \int_x^{+\infty} m_0(y)dy, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \hat{J}_2(x,t) = J_2^+(t) + \bigg\{ &- \frac{1}{2}(J_{1+} + J_{2+})e^{-t} + \frac{1}{2} e^{-\frac{t}{2}} (J_{1+} - J_{2+}) \cos \frac{\sqrt{8n_+ - 1}}{2} t \\ &- \frac{1}{2} e^{-\frac{t}{2}} \frac{J_{1+} - J_{2+}}{\sqrt{8n_+ - 1}} \sin \frac{\sqrt{8n_+ - 1}}{2} t \bigg\} \int_x^{+\infty} m_0(y)dy, \end{aligned} \tag{4.17}$$

$$\hat{E}(x,t) = E^+(t) + \left(2 \frac{J_{1+} - J_{2+}}{\sqrt{8n_+ - 1}} e^{-\frac{t}{2}} \sin \frac{\sqrt{8n_+ - 1}}{2} t\right) \int_x^{+\infty} m_0(y)dy. \tag{4.18}$$

Case 3. When $1 - 8n_+ > 0$, then

$$\hat{n}_1(x,t) = \left\{\frac{1}{2}(J_{1+} + J_{2+})e^{-t} - \frac{1}{2}\left(\lambda_1 A_2 e^{\lambda_1 t} + \lambda_2 B_2 e^{\lambda_2 t}\right)\right\} m_0(x), \tag{4.19}$$

$$\hat{n}_2(x,t) = \left\{\frac{1}{2}(J_{1+} + J_{2+})e^{-t} + \frac{1}{2}\left(\lambda_1 A_2 e^{\lambda_1 t} + \lambda_2 B_2 e^{\lambda_2 t}\right)\right\} m_0(x), \tag{4.20}$$

$$\begin{aligned} \hat{J}_1(x,t) &= J_1^+(t) \\ &\quad - \frac{1}{2}\left\{(J_{1+} + J_{2+})e^{-t} + \lambda_1 A_2 e^{\lambda_1 t} + \lambda_2 B_2 e^{\lambda_2 t}\right\} \int_x^{+\infty} m_0(y)dy, \end{aligned} \tag{4.21}$$

$$\hat{J}_2(x,t) = J_2^+(t) - \frac{1}{2}\left\{(J_{1+} + J_{2+})e^{-t}\lambda_1 A_2 e^{\lambda_1 t} - \lambda_2 B_2 e^{\lambda_2 t}\right\} \int_x^{+\infty} m_0(y)dy, \tag{4.22}$$

$$\hat{E}(x,t) = E^+(t) + \left(\lambda_1 A_2 e^{\lambda_1 t} + \lambda_2 B_2 e^{\lambda_2 t}\right) \int_x^{+\infty} m_0(y)dy \tag{4.23}$$

with

$$A_2 = \frac{J_{1+} - J_{2+}}{\sqrt{1 - 8n_+}} \quad \text{and} \quad B_2 = -\frac{J_{1+} - J_{2+}}{\sqrt{1 - 8n_+}}.$$

These correction functions have the following properties.

LEMMA 4.1. *It holds that*

$$\|(\hat{n}_1, \hat{n}_2)(t)\|_{L^\infty(\mathbb{R}_+)} \leq C\delta_3 e^{-\nu_0 t}, \tag{4.24}$$

$$\begin{aligned} &\|(\hat{J}_1, \hat{J}_2, \hat{E})(t)\|_{L^\infty(\mathbb{R}_+)} \\ &\quad \leq \begin{cases} C\delta_3 e^{-\nu_3 t} & \text{as } |a(t)| = O(e^{-\eta t}),\ \eta > 0, \\ C\delta_3 (1+t)^{-\theta} & \text{as } |a(t)| = O(t^{-\theta}),\ \theta > \frac{3}{2}, \end{cases} \end{aligned} \tag{4.25}$$

$$(\hat{n}_1, \hat{n}_2)|_{x=0} = (0,0), \quad (\hat{n}_1, \hat{n}_2)|_{x=\infty} = (0,0), \tag{4.26}$$

$$(\hat{J}_1, \hat{J}_2, \hat{E})|_{x=\infty} = (J_1^+(t), J_2^+(t), E^+(t)), \tag{4.27}$$

$$\partial_x(\hat{n}_1, \hat{J}_1, \hat{n}_2, \hat{J}_2, \hat{E})|_{x=0} = (0,0,0,0,0), \tag{4.28}$$

$$\partial_x(\hat{n}_1, \hat{J}_1, \hat{n}_2, \hat{J}_2, \hat{E})|_{x=\infty} = (0,0,0,0,0), \tag{4.29}$$

where $\delta_3 := |J_{1+}| + |J_{2+}|$ *and* $0 < \nu_0 < \frac{1}{2}$.

4.2. Asymptotic profiles. We are going to show how to specify the initial data $\bar{u}_0(x)$ for the IBVP (1.14) for choosing a desired asymptotic profiles for the Neumann IBVP (1.1), (1.2), and (1.4). Let us heuristically consider a perturbation of the IBVP (1.1), (1.2), and (1.4) around a potential asymptotic equations (1.14) by adding the correction functions (4.7),

$$\begin{cases} (n_1 - \hat{n}_1 - \bar{n})_t + (J_1 - \hat{J}_1 - \bar{J})_x = 0, \\ (J_1 - \hat{J}_1 - \bar{J})_t + \left(\frac{J_1^2}{n_1} + p(n_1) - p(\bar{n})\right)_x \\ \qquad = n_1 E - n_+ \hat{E} - (J_1 - \hat{J}_1 - \bar{J}) + p(\bar{n})_{xt}, \\ (n_2 - \hat{n}_2 - \bar{n})_t + (J_2 - \hat{J}_2 - \bar{J})_x = 0, \\ (J_2 - \hat{J}_2 - \bar{J})_t + \left(\frac{J_2^2}{n_2} + p(n_2) - p(\bar{n})\right)_x \\ \qquad = -n_2 E + n_+ \hat{E} - (J_2 - \hat{J}_2 - \bar{J}) + p(\bar{n})_{xt}, \\ (E - \hat{E})_x = (n_1 - \hat{n}_1 - \bar{n}) - (n_2 - \hat{n}_2 - \bar{n}). \end{cases} \tag{4.30}$$

Notice the boundary conditions (1.4), (1.14), and (4.29), i.e., $J_{ix}|_{x=0} = 0$, $\bar{J}_x|_{x=0} = 0$ and $\hat{J}_{ix}|_{x=0} = 0$ for $i = 1, 2$; then $(4.30)_1$ and $(4.30)_2$ give that

$$(n_i - \hat{n}_i - \bar{n})_t|_{x=0} = 0, \quad i = 1, 2,$$

which implies

$$(n_i - \hat{n}_i - \bar{n})(0,t) = n_{i0}(0) - \hat{n}_{i0}(0) - \bar{n}_0(0) = n_{i0}(0) - \bar{n}_0(0) = 0, \tag{4.31}$$

by selecting the initial data $\bar{n}_0(x)$ to the IBVP (1.14) such that

$$\bar{n}_0(+\infty) = 0 \text{ and } \bar{n}_0(0) = n_{i0}(0), \quad i = 1, 2. \tag{4.32}$$

Here, we need

$$n_{10}(0) = n_{20}(0), \text{ or equivalently } E_x|_{x=0} = 0. \tag{4.33}$$

If $n_{10}(0) \neq n_{20}(0)$, then the diffusion wave $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x,t)$ to the IBVP (1.1), (1.2), and (1.4) is unstable. For details, see Remark 4 below.

CLAIM 2. *The asymptotic profile for the original solution* $(n_1, J_1, n_2, J_2, E)(x,t)$ *of the Neumann IBVP* (1.1), (1.2), *and* (1.4) *is* $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x,t)$, *where* $(\bar{n}, \bar{J})(x,t)$ *is the solution of the IBVP* (1.14) *with a specified initial data* $\bar{n}_0(x)$ *satisfying* (4.32).

4.3. Convergence theorem. Define

$$\begin{cases} \phi_i(x,t) := -\int_x^\infty [n_i(\xi,t) - \hat{n}_i(\xi,t) - \bar{n}(\xi,t)] d\xi, \\ \psi_i(x,t) := J_i(x,t) - \hat{J}_i(x,t) - \bar{J}(x,t), \quad i = 1, 2, \\ \mathcal{H}(x,t) := E(x,t) - \hat{E}(x,t). \end{cases} \tag{4.34}$$

Then we get the following equations for perturbation:

$$
(4.35)\qquad \begin{cases}
\phi_{1t}+\psi_1=0,\\
\psi_{1t}+\left(\frac{(-\psi_{1t}+\hat{J}_1+\bar{J})^2}{\phi_{1x}+\hat{n}_1+\bar{n}}+p(\phi_{1x}+\hat{n}_1+\bar{n})-p(\bar{n})\right)_x\\
\qquad\qquad =(\phi_{1x}+\hat{n}_1+\bar{n})\mathcal{H}+(\phi_{1x}+\hat{n}_1+\bar{n}-n_+)\hat{E}-\psi_1+p(\bar{n})_{xt},\\
\phi_{2t}+\psi_2=0,\\
\psi_{2t}+\left(\frac{(-\psi_{2t}+\hat{J}_2+\bar{J})^2}{\phi_{2x}+\hat{n}_2+\bar{n}}+p(\phi_{2x}+\hat{n}_2+\bar{n})-p(\bar{n})\right)_x\\
\qquad\qquad =-(\phi_{2x}+\hat{n}_2+\bar{n})\mathcal{H}-(\phi_{2x}+\hat{n}_2+\bar{n}-n_+)\hat{E}-\psi_2+p(\bar{n})_{xt},\\
\mathcal{H}=\phi_1-\phi_2,
\end{cases}
$$

with the initial data

$$
(4.36)\qquad \begin{cases}
\phi_{i0}(x):=\phi_i(x,0)=\int_0^x[n_{i0}(\xi)-\hat{n}_i(\xi,0)-\bar{n}_0(\xi)]d\xi,\\
\psi_{i0}(x):=\psi_i(x,0)=J_{i0}(x)-\hat{J}_i(x,0)-\bar{J}(x,t),\quad i=1,2,\\
\mathcal{H}_0(x):=\phi_{10}(x)-\phi_{20}(x)
\end{cases}
$$

and the boundary conditions

$$
(4.37)\qquad \begin{cases}
(\phi_{ix},\mathcal{H}_x)|_{x=0}=0\ \text{(Neumann boundary)},\\
\psi|_{x=0}=0, & i=1,2,\\
(\phi_i,\psi_i,\mathcal{H})|_{x=\infty}=0.
\end{cases}
$$

From the above equations, we have the following IBVP for $\mathcal{H}$:

$$
(4.38)\qquad \begin{cases}
\mathcal{H}_{tt}+\mathcal{H}_t+2\bar{n}\mathcal{H}=h,\\
\mathcal{H}|_{t=0}=:\mathcal{H}_0(x),\\
\mathcal{H}_t|_{t=0}=-[\psi_{10}(x)-\psi_{20}(0)]=:\mathcal{H}_1(x),\\
\mathcal{H}_x|_{x=0}=0,
\end{cases}
$$

where $h:=h_{1x}-h_2-h_3+h_{4x}$ is defined similarly to (3.52) and satisfies

$$
|h|\approx\begin{cases}
Ce^{-\nu_3 t} & \text{as } |a(t)|=O(e^{-\eta t}),\ \eta>0,\\
C(1+t)^{-\theta} & \text{as } |a(t)|=O(t^{-\theta}),\ \theta>\frac{3}{2}.
\end{cases}
$$

The IBVP of damped wave equations for ϕ_i $(i=1,2)$ is

$$
(4.39)\qquad \begin{cases}
\phi_{itt}+\phi_{it}-p'(n_+)\phi_{ixx}=F_i,\\
(\phi_i,\phi_{it})|_{t=0}=(\phi_{i0},-\psi_{i0})(x),\\
\phi_{ix}|_{x=0}=0,
\end{cases}
$$

which can be expressed in the integral form

$$
\begin{aligned}
\phi_i(x,t)=&\int_0^\infty[K_0(x-y,t)+K_0(x+y,t)]\phi_{i0}(y)dy\\
&-\int_0^\infty[K_1(x-y,t)+K_1(x+y,t)]\psi_{i0}(y)dy\\
&+\int_0^t\int_0^\infty[K_1(x-y,t-s)+K_1(x+y,t-s)]F_i(y,s)dyds.
\end{aligned}\tag{4.40}
$$

As shown in Theorem 3.2, we can obtain the following convergence.

THEOREM 4.2. *Let* $n_{10}(0) = n_{20}(0)$ *and* $(\phi_{10}, \phi_{20}, \psi_{10}, \psi_{20}) \in H^3(\mathbb{R}_+)\times H^3(\mathbb{R}_+)\times H^2(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$, $\delta := |J_{1+}| + |J_{1-}| + |J_{2+}| + |J_{2-}| + |E_+| + \max_{x\in\mathbb{R}_+}[|n_{10}(x) - n_+| + |n_{20}(x) - n_+|]$, *and* $\Phi_0 := \|(\phi_{10}, \phi_{20})\|_{H^3(\mathbb{R}_+)} + \|(\psi_{10}, \psi_{20})\|_{H^2(\mathbb{R}_+)}$. *Then there exists* $\delta_0 > 0$ *such that if* $\delta + \Phi_0 \le \delta_0$, *the solution* (n_1, n_2, J_1, J_2, E) *of the Neumann IBVP* (1.1), (1.2), *and* (1.3) *is unique and globally exists, and it satisfies for* $i = 1, 2$,

$$(4.41)\qquad \begin{cases} \|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-k/2}, & k = 0,1,2,3,\\ \|\partial_x^k \psi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(k+2)/2}, & k = 0,1,\\ \|\partial_x^k\partial_t^l \mathcal{H}(t)\|_{L^2(\mathbb{R})} & \\ \qquad = \begin{cases} O(1)e^{-\nu t} & \text{as } |a(t)| = O(e^{-\eta t}), \eta > 0,\\ O(1)(1+t)^{-\theta} & \text{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}, \end{cases} & 0 \le k+l \le 2,\\ \|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, & k = 0,1,2,\\ \|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}, & \\ \|\partial_x^k\partial_t^l \mathcal{H}(t)\|_{L^\infty(\mathbb{R})} & \\ \qquad = \begin{cases} O(1)e^{-\nu t} & \text{as } |a(t)| = O(e^{-\eta t}), \eta > 0,\\ O(1)(1+t)^{-\theta} & \text{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}, \end{cases} & 0 \le k+l \le 1, \end{cases}$$

for $0 < \nu < \nu_3$.

Moreover, if $(\phi_{i0}, \psi_{i0}) \in L^1(\mathbb{R}_+)$ *for* $i = 1, 2$, *then the optimal* $L^p(\mathbb{R}_+)$ $(2 \le p \le +\infty)$ *decay rates hold,*

$$(4.42)\qquad \begin{cases} \|\partial_x^k \phi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-(2k+1)/4}, & k = 0,1,2,\\ \|\psi_i(t)\|_{L^2(\mathbb{R}_+)} = O(1)(1+t)^{-5/4}, & \\ \|\partial_x^k \phi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-(k+1)/2}, & k = 0,1,\\ \|\psi_i(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(1+t)^{-3/2}. & \end{cases}$$

From the above convergence theorem, and noting the exponential/algebraic decay of the correction functions $(\hat{n}_i, \hat{J}_i, \hat{E})(x,t)$ (see Lemma 4.1), we immediately get the following L^∞-stability of diffusion waves.

COROLLARY 4.3 (convergence to diffusion waves).

1. *When* $(\phi_{i0}, \psi_{i0}) \in H^3(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ *for* $i = 1, 2$, *then*

$$(4.43)\qquad \begin{aligned} &\|(n_i - \bar{n}, J_i - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &\quad = \begin{cases} O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, e^{-\nu t}) & \text{as } |a(t)| = O(e^{-\eta t}), \eta > 0,\\ O(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\theta}) & \text{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}, \end{cases} \end{aligned}$$

$$(4.44)\qquad \begin{aligned} &\|(n_1 - n_2, J_1 - J_2)(t)\|_{L^\infty(\mathbb{R}_+)}\\ &\quad = \begin{cases} O(1)(e^{-\nu t}, e^{-\nu t}) & \text{as } |a(t)| = O(e^{-\eta t}), \eta > 0,\\ O(1)(e^{-\nu t}, t^{-\theta}) & \text{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}. \end{cases} \end{aligned}$$

2. *When* $(\phi_{i0}, \psi_{i0}) \in L^1(\mathbb{R}_+) \cap (H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+))$ *for* $i = 1, 2$, *then*

$$\begin{aligned} &\|(n_i - \bar{n}, J_i - \bar{J}, E)(t)\|_{L^\infty(\mathbb{R}_+)} \\ &= \begin{cases} O(1)(t^{-1}, t^{-\frac{3}{2}}, e^{-\nu t}) & \textit{as } |a(t)| = O(e^{-\eta t}), \eta > 0 \\ O(1)(t^{-1}, t^{-\frac{3}{2}}, t^{-\theta}) & \textit{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}, \end{cases} \end{aligned} \tag{4.45}$$

$$\begin{aligned} &\|(n_1 - n_2, J_1 - J_2)(t)\|_{L^\infty(\mathbb{R}_+)} \\ &= \begin{cases} O(1)(e^{-\nu t}, e^{-\nu t}) & \textit{as } |a(t)| = O(e^{-\eta t}), \eta > 0, \\ O(1)(e^{-\nu t}, t^{-\theta}) & \textit{as } |a(t)| = O(t^{-\theta}), \theta > \frac{3}{2}. \end{cases} \end{aligned} \tag{4.46}$$

Remark 4.

1. Different from the case with Dirichlet boundary condition studied in section 3, whether the initial perturbations $(\phi_{i0}, \psi_{i0})(x)$ for $i = 1, 2$ are in the weighted $L^{1,\gamma}(\mathbb{R}_+)$ or not, like linear damped wave equations [18], we cannot have the improved decay rates like (3.55) for the solutions $\phi_i(x, t)$ and $\psi_i(x, t)$ for the Neumann IBVP (4.35)–(4.37).
2. As mentioned, the condition $n_{10}(0) = n_{20}(0)$ or equivalently $E_x|_{x=0} = 0$ is necessary for the convergence of the solution $(n_1, J_1, n_2, J_2, E)(x, t)$ of the Neumann IBVP (1.1), (1.2), and (1.4) to the corresponding diffusion wave $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x, t)$ of the IBVP (1.14). Otherwise, the diffusion wave $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x, t)$ is unstable, because

$$E_x(0, t) = n_{10} - n_{20}(0) \neq 0$$

implies

$$\partial_x(E)(x, t)|_{x=0} \nrightarrow 0 \quad \text{as } t \to +\infty.$$

3. For the Neumann boundary (1.4), if we replace the boundary condition $E|_{x=\infty} = a(t)$ by $E|_{x=0} = a(t)$, since $(J_1, J_2, E)(+\infty, t)$ is implicitly determined by the IBVP system of (1.1), (1.2), and (1.4), we cannot find the exact difference between J_i and $\bar{J}$ and E and $\bar{E} = 0$ at the far field $x = \infty$. Hence, we cannot construct the corresponding correction functions and the stability of diffusion wave is still not known.

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