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Zero-Relaxation Limits of the Non-Isentropic Euler–Maxwell System for Well/Ill-Prepared Initial Data

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Abstract

This paper is concerned with the zero-relaxation limits for periodic smooth solutions of the non-isentropic Euler–Maxwell system in a three-dimensional torus prescribing the well/ill-prepared initial data. The non-isentropic Euler–Maxwell system can be reduced to a quasi-linear symmetric hyperbolic system of one order. By observing a special structure of the non-isentropic Euler–Maxwell system, we are able to decouple the system and develop a technique to achieve the a priori H^s estimates, which guarantees the limit for the non-isentropic Euler–Maxwell system as the relaxation time $\tau \to 0$. We realize that the convergence rate of the temperature is the same as the other unknowns in the $L^\infty(0, T_1; H^s)$, but the convergence rate of the temperature is slower than the velocity in $L^2(0, T_1; H^s)$. The zero-relaxation limit presented here is the transport equation coupled with the drift–diffusion system. However, the limit of the isentropic Euler–Maxwell system is the classical drift–diffusion system.

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This shows the essential difference between the isentropic and non-isentropic Euler–Maxwell systems.

Keywords The non-isentropic Euler–Maxwell system · Initial layer problem · Zero-relaxation limits

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1 Introduction and Main Results

1.1 Preliminary

On a three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi)^3$, the following nonlinear system, called the non-isentropic Euler–Maxwell system,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p = -\rho(E + \gamma v \times B) - \frac{\rho v}{\tau}, \\ \partial_t \mathcal{E} + \operatorname{div}(\mathcal{E} v + p v) = -\rho v E - \frac{\rho |v|^2}{\tau} - \tau \mathcal{E}, \\ \gamma \lambda^2 \partial_t E - \nabla \times B = \gamma \rho v, \quad \lambda^2 \operatorname{div} E = b - \rho, \\ \gamma \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{1.1}$$

models the dynamics of electrons and ions under the influence of their self-consistent electromagnetic field (Besse et al. 2004; Brenier et al. 2003; Chen 1984; Kawashima 1984; Markowich et al. 1990; Rishbeth and Garriott 1969; Ueda et al. 2012; Villani 2009). The unknowns are the density $\rho > 0$, the velocity $v \in \mathbb{R}^3$, the absolute temperature $\theta > 0$, the internal energy $e = \dfrac{3}{2} K_B \theta$, the total energy $\mathcal{E} = \rho \left(\dfrac{v^2}{2} + e \right)$, the pressure function $p = \dfrac{2}{3} \rho e$, the electric field $E \in \mathbb{R}^3$, and the magnetic field $B \in \mathbb{R}^3$. The constants $\lambda > 0$, $K_B > 0$, $\dfrac{1}{\gamma} = c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$, ε_0 and μ_0 are the scaled Debye length, Boltzmann constant, speed of light, vacuum permittivity and permeability, respectively. Moreover, $\tau \in (0, 1)$ and $b = b(x) > 0$ stand for the relaxation time and positively charged background ions, respectively. Throughout this paper, we set $\lambda = K_B = \gamma = 1$ for simplicity, because these parameters are not essential for the zero-relaxation limits. Thus, $\rho > 0$ and $\theta > 0$, system (1.1) is reduced to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} \nabla(\rho \theta) = -(E + v \times B) - \frac{v}{\tau}, \\ \partial_t \theta + v \cdot \nabla \theta + \frac{2}{3} \theta \operatorname{div} v = -\frac{\tau}{3} |v|^2 - \tau \theta, \\ \partial_t E - \nabla \times B = \rho v, \quad \operatorname{div} E = b - \rho, \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{1.2}$$

subject to the initial condition

$$(\rho, v, \theta, E, B)\,|_{t=0} = \left(\rho_0^\tau, v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau\right), \quad x \in \mathbb{T}^3, \tag{1.3}$$

satisfying the compatibility condition

$$\operatorname{div} E_0^\tau = b - \rho_0^\tau, \quad \operatorname{div} B_0^\tau = 0, \quad x \in \mathbb{T}^3. \tag{1.4}$$

Euler–Maxwell equations have been one of important typical systems in fluid dynamics, and also extensively studied. In 2000, by using the fractional Godunov scheme as well as the compensated compactness argument, Chen et al. (2000) proved global existence of weak solutions to the initial-boundary value problem in one space dimension for arbitrarily large initial data in L^1. Then, Jerome (2003) provided a local smooth solution theory for the Cauchy problem over $\mathbb{R}^3$ by adapting the classical semigroup-resolvent approach of Kato (1975), and Peng and Wang (2008) established convergence of the compressible Euler–Maxwell system to the incompressible Euler system for well-prepared smooth initial data. Late then, Wu (2016) investigated the initial layer and relaxation limit of non-isentropic compressible Euler equations. See also the other significant contributions in different cases, for example, the asymptotic limits on small physical parameters (Li et al. 2021; Wasiolek 2016; Yang and Wang 2011; Yang and Hu 2019), the existence of global smooth irrotational flow (Germain and Masmoudi 2014; Deng et al. 2017; Guo et al. 2016), the asymptotic behavior of global solutions near a constant equilibrium state (Peng et al. 2011; Duan 2011; Ueda et al. 2012; Xu 2011; Feng et al. 2021, 2014), the large time-decay rates of small non-constant steady-state solutions (Liu and Zhu 2013; Wang and Xu 2016), the stability of large non-constant equilibrium solutions (Peng 2015) and the instability of WKB solution (Dumas et al. 2001).

The zero-relaxation limits for the Euler–Poisson system have been extensively studied recently (Alì and Jüngel 2003; Jüngel and Peng 1999; Junca and Rascle 2000; Lattanzio 2000; Li et al. 2021; Luo et al. 2019; Marcati and Natalini 1995; Yong 2004). In this paper, inspired by Hajjej and Peng (2012) for the initial layer problem to the isentropic Euler–Maxwell system, we consider the zero-relaxation limit of the non-isentropic Euler–Maxwell system (1.1). The usual time scaling for studying the limit $\tau \to 0$ is $t_* = \tau t$. Since $t = 0$ is equivalent to $t_* = 0$, this change of scaling does not affect the initial condition (1.3). Rewriting still t_* by t, system (1.2) becomes

$$\begin{cases} \partial_t \rho + \frac{1}{\tau}\operatorname{div}(\rho v) = 0, \\ \partial_t v + \frac{1}{\tau}\,(v\cdot\nabla)\,v + \frac{1}{\tau}\frac{\nabla(\rho\theta)}{\rho} = -\frac{1}{\tau}(E + v\times B) - \frac{v}{\tau^2}, \\ \partial_t\theta + \frac{1}{\tau} v\cdot\nabla\theta + \frac{2}{3\tau}\theta\operatorname{div} v = -\frac{1}{3}|v|^2 - \theta, \\ \partial_t E - \frac{1}{\tau}\nabla\times B = \frac{1}{\tau}\rho v, \quad \operatorname{div} E = b - \rho, \\ \partial_t B + \frac{1}{\tau}\nabla\times E = 0, \quad \operatorname{div} B = 0, \quad (t,x)\in[0,+\infty)\times\mathbb{T}^3. \end{cases} \tag{1.5}$$

The local existence of (1.5) is shown in Kato (1975), Majda (1984).

Proposition 1.1 (Local existence Kato 1975; Majda 1984) *Let $s \geq 3$ be an integer. Suppose $(\rho_0^\tau, v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau) \in H^s(\mathbb{T}^3)$ with $\rho_0^\tau, \theta_0^\tau \geq 2\kappa$ for some given constant*

$\kappa > 0$, *independent of* τ. *Then, there exists* $T_1^\tau > 0$ *such that problem* (1.3)–(1.5) *has a unique smooth local solution which satisfies* $\rho^\tau, \theta^\tau \geq \kappa$ *in* $[0, T_1^\tau] \times \mathbb{T}^3$ *and*

$$(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau) \in C^1([0, T_1^\tau]; H^{s-1}(\mathbb{T}^3)) \cap C([0, T_1^\tau]; H^s(\mathbb{T}^3)).$$

Remark 1.1 Note that the time scaling $t_* = \tau t$ can reveal the large time behavior of solutions. Obviously, $t = t_* \tau^{-1} = \mathcal{O}(\tau^{-1})$ for fixed $t_* > 0$. Then for a fixed time $T_* > 0$, a local-in-time convergence for system (1.5) on the interval $[0, T_*]$ means the convergence for system (1.2) on a larger interval $[0, T_* \tau^{-1}]$. On the other hand, as $\tau \to 0$, a convergence error $\mathcal{O}(\tau^m)$ with $m > 0$ implies a rate $\mathcal{O}(\tau^{-m})$ of the large time behaviors.

Notations For two quantities a and b, $a \sim b$ means $ca \leq b \leq \frac{1}{c} a$ for a generic constant $0 < c \leq 1$. For an integer $s > 0$, we denote by H^s, L^2 and L^∞ the usual Sobolev spaces $H^s(\mathbb{T}^3)$, $L^2(\mathbb{T}^3))$ and $L^\infty(\mathbb{T}^3))$, and by $\| \cdot \|_s$, $\| \cdot \|$ and $\| \cdot \|_{L^\infty}$ the corresponding norms, respectively. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space L^2, i.e.,

$$\langle f, g \rangle = \int_{\mathbb{T}^3} f(x) g(x) \mathrm{d}x, \quad \forall f = f(x),\ g = g(x) \in L^2.$$

In addition, for a multi-index, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, $\beta \leq \alpha$ stands for $\beta_j \leq \alpha_j$ for $j = 1, 2, 3$, and $\beta < \alpha$ stands for $\beta \leq \alpha$ and $\beta \neq \alpha$.

Lemma 1.1 (Moser-type calculus inequalities Klainerman and Majda 1981; Majda 1984) *Let* $s \geq 3$ *be an integer. Suppose* $u \in H^s$, $\nabla u \in L^\infty$, $v \in H^{s-1} \cap L^\infty$ *and* f *is a smooth function. Then for all multi-index* α *with* $1 \leq |\alpha| \leq s$, *one has* $\partial^\alpha(uv) - u\partial^\alpha v \in L^2$ *and*

$$\begin{aligned} &\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C\big(\|\nabla u\|_{L^\infty} \|D^{|\alpha|-1} v\| + \|D^{|\alpha|} u\| \|v\|_{L^\infty}\big), \\ &\left\|\partial^\alpha f(u)\right\| \leq C(1 + \|u\|_{H^s})^{s-1} \|u\|_{H^s}, \end{aligned}$$

where the constant C *may depend on* $\|u\|_{L^\infty}$ *and* s, *and*

$$\|D^{s'} u\| = \sum_{|\alpha| = s'} \|\partial^\alpha u\|.$$

Lemma 1.2 (Peng and Wang 2007) *Let* $s \geq 0$ *be an integer. Suppose* $f \in H^s$ *and* $g \in H^s$. *Then, problem,*

$$\nabla \times B = f, \quad \mathrm{div} B = g, \quad \mathrm{div} f = 0, \quad \mathfrak{M}(g) = 0, \tag{1.6}$$

has a unique solution $B \in H^{s+1}$ *in the class* $\mathfrak{M}(B) = 0$, *where*

$$\mathfrak{M}(B) = \int_{\mathbb{T}^3} B \mathrm{d}x.$$

1.2 Main Results

In the following, we state the main results of this paper.

Theorem 1.1 (For well-prepared initial data) *Let* $s \geq 3$ *be an integer and* (1.4) *holds. Assume* $(\rho_j, v_j, \theta_j, E_j, B_j) \in H^{s+1}$ *for* $j \geq 0$ *in the sense of*

$$(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)(0, x) = \sum_{j \geq 0} \tau^{2j} (\rho_j, \tau v_j, \tau^2 \theta_j, E_j, \tau B_j)(x), \quad x \in \mathbb{T}^3, \tag{1.7}$$

with $\rho_0, \theta_0 \geq const. > 0$ *in* $\mathbb{T}^3$, *and satisfy the compatibility conditions* (2.11)–(2.12) *and* (2.19)–(2.22) *for* $j \geq 1$. *Suppose* $m \geq 0$ *to be any fixed integers and*

$$\left\| \left(\rho_0^\tau, v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau\right) - \sum_{j=0}^{m} \tau^{2j} (\rho_j, \tau v_j, \tau^2 \theta_j, E_j, \tau B_j) \right\|_s \leq C_1 \tau^{2(m+1)}, \tag{1.8}$$

where $C_1 > 0$ *is a constant independent of* τ. *Then, there exist* $T_1 > 0$ *and a constant* $C_2 > 0$, *independent of* τ, *such that as* $\tau \to 0$, *we have* $T_1^\tau \geq T_1$ *and the solution* $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ *to the periodic problem* (1.3)–(1.5) *satisfies*

$$\left\| \left(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau\right)(t) - \left(\rho_\tau^m, v_\tau^m, \theta_\tau^m, E_\tau^m, B_\tau^m\right)(t) \right\|_s \leq C_2 \tau^{2(m+1)}, \quad \forall t \in [0, T_1].$$

Moreover,

$$\left\| v^\tau - v_\tau^m \right\|_{L^2(0,T_1;H^s(\mathbb{T}^3))} \leq C_2 \tau^{2m+3}, \quad \left\| \theta^\tau - \theta_\tau^m \right\|_{L^2(0,T_1;H^s)} \leq C_2 \tau^{2(m+1)}.$$

Theorem 1.2 (For ill-prepared initial data) *Suppose* $s \geq 3$ *to be a fixed integer and* $(\rho_0, v_0, \theta_0, E_0, B_0) \in H^{s+1}$ *with* $\rho_0, \theta_0 \geq const. > 0$ *in* $\mathbb{T}^3$. *Assume that*

$$\left\| \left(\rho_0^\tau, v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau\right) - (\rho_0, \tau v_0, \tau^2 \theta_0, E_0, \tau B_0) \right\|_s \leq C_1 \tau^2, \tag{1.9}$$

where $C_1 > 0$ *is a constant independent of* τ. *Then there exists a constant* $C_2 > 0$, *independent of* τ, *such that as* $\tau \to 0$, *we have* $T_1^\tau \geq T_1$, *and the solution* $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ *to the periodic problem* (1.3)–(1.5) *satisfies*

$$\left\| \left(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau\right)(t) - (\rho^0, v_{\tau,I}, \theta_{\tau,I}, E^0, \tau B^0) \right\|_s \leq C_2 \tau^2, \quad \forall t \in [0, T_1].$$

Furthermore,

$$\left\|v^\tau - v_{\tau,I}\right\|_{L^2(0,T_1;H^s(\mathbb{T}^3))} \leq C_2\tau^3, \quad \left\|\theta^\tau - \theta_{\tau,I}\right\|_{L^2(0,T_1;H^s(\mathbb{T}^3))} \leq C_2\tau^2,$$

where $v_{\tau,I}$ and $\theta_{\tau,I}$ are defined by (3.18).

1.3 Features and Difficulties

As we see later, the proof of Theorems 1.1 and 1.2 for the non-isentropic Euler–Maxwell system contains all the difficulties appeared in the isentropic Euler–Maxwell system. Besides, it also includes the other troubles caused by the absolute temperature θ. For the well-prepared initial data, we improve Hajjej-Peng's asymptotic expansion in Hajjej and Peng (2012) by adding the temperature expansion of the following form:

$$(\rho_\tau^m, v_\tau^m, \theta_\tau^m, E_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^{2j}(\rho^j, \tau v^j, \tau^2\theta^j, E^j, \tau B^j), \quad m \geq 0. \tag{1.10}$$

With the help of this expansion, we overcome the difficulty generated from the energy equation which contains the absolute temperature variable θ and prove the convergence of the solution $(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)$ of the non-isentropic Euler–Maxwell system (1.5) to $(\rho_\tau^m, v_\tau^m, \theta_\tau^m, E_\tau^m, B_\tau^m)$ with the order $\mathcal{O}(\tau^{2(m+1)})$ when the initial data are well-prepared, and the initial error has the same order. In order to prove it, we have to treat the order of the remainder for B. Indeed, there is a loss of one order for $R_B^{\tau,m}$ in comparison with those for variables ρ, v, θ and E. This can be overcome by introducing a correction term into E_τ^m so that the new remainder $R_B^{\tau,m} = 0$ without changing the order of the other remainders (see Sect. 2 for details). On the other hand, for the ill-prepared initial data, the results in Theorem 1.1 are not valid because the approximate solution does not satisfy the initial conditions (2.11)–(2.12) and (2.19)–(2.22). In Sect. 3, we construct initial layer corrections with exponential decay to zero and prove the convergence of the first order asymptotic expansion. The analysis shows that there are no first-order initial layers on unknowns ρ, θ, E and B. Then, we have to consider the second-order initial layer corrections to obtain the desired order of remainders.

Now, let us discuss the vital difference between the isentropic and the non-isentropic Euler–Maxwell system. Firstly, the structure of the non-isentropic Euler–Maxwell system is much more complex than that of the isentropic Euler–Maxwell system. During the process of asymptotic expansion, in order to decouple the non-isentropic system (1.5), we find the term $\frac{1}{\rho}\nabla(\rho\theta)$ has a special structure which can be expanded as follows:

$$\sum_{j\geq 1}\tau^{2(j+1)}\left(\frac{\theta^0}{\rho^0}\nabla\rho^j + \left(\frac{\theta^j}{\rho^0} - \frac{\theta^0\rho^j}{|\rho^0|^2}\right)\nabla\rho^0 + \nabla\theta^j + f^{j-1}\left(\left(\rho^k, \theta^k\right)_{0\leq k\leq j-1}\right)\right)$$
$$+ \tau^2\left(\frac{\theta^0}{\rho^0}\nabla\rho^0 + \nabla\theta^0\right).$$

Then, by using this expansion, we can overcome the difficulties caused by the complex structure of the non-isentropic Euler–Maxwell system. Secondly, the matrix form of the non-isentropic Euler–Maxwell system (see (4.9)) for V_I^τ is very different from that of the isentropic Euler–Maxwell system. During the estimates for V_I^τ, we choose a new symmetrizer

$$\mathcal{A}_0^I\left(\rho^\tau, \theta^\tau\right) = \text{Diag}\left(\frac{\theta^\tau}{\rho^\tau}, 1, 1, 1, \frac{3\rho^\tau}{2\theta^\tau}\right),$$

which implies that

$$\mathcal{A}_0^I\left(\rho^\tau, \theta^\tau\right)\mathcal{A}_i^I\left(\rho^\tau, v^\tau, \theta^\tau\right) = v_i^\tau \mathcal{A}_0^I\left(\rho^\tau, \theta^\tau\right) + \mathcal{Q}_i^I\left(\rho^\tau, \theta^\tau\right), \quad i = 1, 2, 3$$

is symmetric, where

$$\mathcal{Q}_i^I\left(\rho^\tau, \theta^\tau\right) = \begin{pmatrix} 0 & \theta^\tau e_i^t & 0 \\ \theta^\tau e_i & 0 & \rho^\tau e_i \\ 0 & \rho^\tau e_i^t & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

Here, $\mathcal{Q}_i^I\left(\rho^\tau, \theta^\tau\right)$ is not yet a constant matrix (which is a constant matrix for the isentropic Euler–Maxwell system). Therefore, we have to deal with the difficulty caused by this property (see details in Sect. 4). The limit equations of the non-isentropic Euler–Maxwell system are the transport equation coupled with the drift–diffusion system (see Proposition 2.1). However, the limit system of the isentropic Euler–Maxwell system is the classical drift–diffusion system (see Hajjej and Peng (2012)).

The paper is organized as follows. In Sect. 2, we derive asymptotic expansions of solutions and state the convergence result to problem (1.3)–(1.5) for the well-prepared initial data. In Sect. 3, we study the asymptotic expansions in the case of ill-prepared initial data by constructing initial layer corrections which exponentially decay to zero. In the last section, we give the rigorous justification of the both two asymptotic expansions and prove Theorems 1.1 and 1.2.

2 Problem (1.3)–(1.5) with Well-Prepared Initial Data

2.1 Asymptotic Expansions

In the following, for well-prepared initial data, we investigate the zero-relaxation limit $\tau \to 0$ of problem (1.3)–(1.5). Based on the discussion on the asymptotic expansion, we make the ansatz (1.7) for initial data and the following ansatz for the approximate solution

$$(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)(t, x) = \sum_{j \geq 0} \tau^{2j}(\rho^j, \tau v^j, \tau^2\theta^j, E^j, \tau B^j)(t, x),$$
$$(t, x) \in [0, +\infty) \times \mathbb{T}^3, \tag{2.1}$$

where $(\rho_j, v_j, \theta_j, E_j, B_j)_{j\geq 0}$ are given smooth data with $\rho_0, \theta_0 \geq const. > 0$ in $\mathbb{T}^3$. The motivation of this expansion is the following consideration. If we replace v by τv, θ by $\tau^2\theta$ and B by τB, then system (1.5) becomes

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \tau^2 \left(\partial_t v + (v \cdot \nabla) v + \frac{\nabla(\rho\theta)}{\rho} \right) = -E - \tau^2 v \times B - v, \\ \partial_t \theta + v \cdot \nabla\theta + \frac{2}{3}\theta \mathrm{div} v = -\frac{1}{3}|v|^2 - \theta, \\ \partial_t E - \nabla \times B = \rho v, \quad \mathrm{div} E = b - \rho, \\ \tau^2 \partial_t B + \nabla \times E = 0, \quad \mathrm{div} B = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{cases}$$

in which the only small parameter is τ^2.

For the term $\frac{1}{\rho}\nabla(\rho\theta)$ in the second equation of system (1.5), we introduce a new expansion

$$\begin{aligned} \frac{1}{\rho}\nabla(\rho\theta) = & \sum_{j\geq 1} \tau^{2(j+1)} \left(\frac{\theta^0}{\rho^0}\nabla\rho^j + \left(\frac{\theta^j}{\rho^0} - \frac{\theta^0 \rho^{j-1}}{|\rho^0|^2} \right) \nabla\rho^0 + \nabla\theta^j + f^{j-1}\left(\left(\rho^k, \theta^k\right)_{0\leq k\leq j-1} \right) \right) \\ & + \tau^2 \left(\frac{\theta^0}{\rho^0}\nabla\rho^0 + \nabla\theta^0 \right), \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{aligned} \tag{2.2}$$

where f^{j-1} for $j \geq 1$ is a function depending only on $(\rho_k, u_k)_{0\leq k\leq j-1}$.

Next, we determine the profiles $(\rho^j, v^j, \theta^j, E^j, B^j)$ for all $j \geq 0$. Plugging expressions (2.1) and (2.2) into system (1.5) and identifying the coefficients in powers of τ, we find that $(\rho^j, v^j, \theta^j, E^j, B^j)_{j\geq 0}$ satisfies

$$\begin{cases} \partial_t \rho^0 + \mathrm{div}(\rho^0 v^0) = 0, \\ E^0 + v^0 = 0, \\ \partial_t \theta^0 + v^0 \cdot \nabla\theta^0 + \frac{2}{3}\theta^0 \mathrm{div} v^0 = -\frac{1}{3}\left|v^0\right|^2 - \theta^0, \\ \nabla \times E^0 = 0, \quad \mathrm{div} E^0 = b - \rho^0, \\ \nabla \times B^0 = \partial_t E^0 - \rho^0 v^0, \quad \mathrm{div} B^0 = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{2.3}$$

and for $j \geq 1$,

$$\begin{cases} \partial_t \rho^j + \sum_{k=0}^{j} \mathrm{div}(\rho^k v^{j-k}) = 0, \\ \partial_t v^{j-1} + \sum_{k=0}^{j-1} (v^k \cdot \nabla) v^{j-1-k} + \left(\frac{\theta^0}{\rho^0}\nabla\rho^{j-1} + \left(\frac{\theta^{j-1}}{\rho^0} - \frac{\theta^0\rho^{j-1}}{|\rho^0|^2} \right) \nabla\rho^0 + \nabla\theta^{j-1} \right) \\ \qquad = -f^{j-2}\left(\left(\rho^k, \theta^k\right)_{0\leq k\leq j-2} \right) - E^j - \sum_{k=0}^{j-1} v^k \times B^{j-1-k} - v^j, \\ \partial_t \theta^j + \sum_{k=0}^{j} v^k \cdot \nabla\theta^{j-k} + \frac{2}{3} \sum_{k=0}^{j} \theta^k \mathrm{div} v^{j-k} = -\frac{1}{3} \sum_{k=0}^{j} v^k v^{j-k} - \theta^j, \end{cases} \tag{2.4}$$

$$\begin{cases} \nabla \times E^j = -\partial_t B^{j-1}, \quad \mathrm{div} E^j = -\rho^j, \\ \nabla \times B^j = \partial_t E^j - \sum_{k=0}^{j} \rho^k v^{j-k}, \quad \mathrm{div} B^j = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{T}^3. \end{cases} \tag{2.5}$$

In (2.3), equation $\nabla \times E^0 = 0$ implies the existence of a potential χ^0 such that $E^0 = -\nabla\chi^0$. Then, (v^0, χ^0) satisfies a drift–diffusion system

$$\begin{cases} \partial_t \rho^0 + \mathrm{div}(\rho^0 \nabla \chi^0) = 0, \\ -\Delta \chi^0 = b - \rho^0, \quad (t,x) \in [0,+\infty) \times \mathbb{T}^3, \end{cases} \tag{2.6}$$

with the initial condition

$$\rho^0(0,x) = \rho_0, \quad x \in \mathbb{T}^3. \tag{2.7}$$

By the similar procedure in Hajjej and Peng (2012), Markowich et al. (1990), we obtain the existence of smooth solutions to problem (2.6)–(2.7), at least locally in time. The solution χ^0 is unique in the class $\mathfrak{M}(\chi^0) = 0$. Then, (v^0, E^0) are given by

$$v^0 = \nabla\chi^0, \quad E^0 = -\nabla\chi^0. \tag{2.8}$$

Moreover, (2.8) together with the third equation of (2.3) imply that θ^0 solves a linear transport equation

$$\partial_t\theta^0 + \nabla\chi^0 \cdot \nabla\theta^0 + \left(\frac{2\Delta\chi^0}{3} + 1\right)\theta^0 + \frac{1}{3}\left|\nabla\chi^0\right|^2 = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{T}^3, \tag{2.9}$$

with the initial condition

$$\theta^0(0,x) = \theta_0, \quad x \in \mathbb{T}^3. \tag{2.10}$$

It admits a unique global smooth solution (see Jüngel (2009)).

Due to the fact that $(\rho^0, v^0, \theta^0, E^0)$ is achieved, B^0 solves the linear system of curl-div equations of type (1.6) in the class $\mathfrak{M}(B^0) = 0$.

It follows from the last equation in (2.3) and (2.8)–(2.9) that the first-order compatibility conditions are

$$\begin{aligned} &v_0 = -E_0 = \nabla\chi_0, \quad \partial_t\theta^0(0,x) + v_0 \cdot \nabla\theta_0 + \left(\frac{2}{3}\mathrm{div} v_0 + 1\right)\theta_0 + \frac{1}{3}|v_0|^2 = 0, \\ &B_0 = B^0(0,x), \end{aligned} \tag{2.11}$$

where χ_0 satisfies

$$-\Delta\chi_0 = b - \rho_0, \quad x \in \mathbb{T}^3, \quad \text{and} \quad \mathfrak{M}(\chi_0) = 0. \tag{2.12}$$

For $j \geq 1$, we can get the profiles $(\rho^j, v^j, \theta^j, E^j, B^j)$ by induction in j. Suppose that $(\rho^j, v^j, \theta^j, E^j, B^j)_{0 \leq k \leq j-1}$ have already been obtained in steps above. Equations for B^j are of curl-div type (1.6) and admit a unique smooth solution B^j in the class $\mathfrak{M}(B^j) = 0$. In addition, it follows from $\mathrm{div} B^j = 0$ that there exists a vector ω^j

such that $B^j = -\nabla \times \omega^j$. Hence, equation $\nabla \times E^j = -\partial_t B^{j-1}$ in (2.5) turns into $\nabla \times (E^j - \partial_t \omega^{j-1}) = 0$, which implies that there exists a potential function χ^j such that

$$E^j = \partial_t \omega^{j-1} - \nabla \chi^j, \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3. \tag{2.13}$$

In view of (2.4), we have

$$\begin{aligned} v^j = \nabla \chi^j &- \left(\frac{\theta^0}{\rho^0} \nabla \rho^{j-1} + \left(\frac{\theta^{j-1}}{\rho^0} - \frac{\theta^0 \rho^{j-1}}{|n^0|^2} \right) \nabla \rho^0 + \nabla \theta^{j-1} \right) - f^{j-2} \left(\left(\rho^k, \theta^k \right)_{0 \le k \le j-2} \right) \\ &- \left(\partial_t v^{j-1} + \partial_t \omega^{j-1} + \sum_{k=0}^{j-1} (v^k \cdot \nabla) v^{j-1-k} + \sum_{k=0}^{j-1} v^k \times B^{j-1-k} \right). \end{aligned} \tag{2.14}$$

Then, in the class, $\mathfrak{M}(\chi^j) = 0$, (ρ^j, χ^j) satisfies the following linearized system:

$$\begin{cases} \partial_t \rho^j - \operatorname{div} \left(\theta^0 \nabla \rho^{j-1} + \left(\theta^{j-1} - \frac{\theta^0 \rho^{j-1}}{\rho^0} \right) \nabla \rho^0 + \rho^0 \nabla \theta^{j-1} - \rho^0 \nabla \chi^j \right) + \operatorname{div} \left(\rho^j v^0 \right) \\ \quad = h^j \left(\left(W^k, \partial_t W^k, \partial_x W^k, \partial_t \partial_x W^k, \partial_x^2 W^k \right)_{0 \le k \le j-1} \right) + \operatorname{div} \left(\rho^0 \partial_t \omega^{j-1} \right), \\ \Delta \chi^j = \rho^j + \partial_t \left(\operatorname{div} \omega^{j-1} \right), \quad (t, x) \in [0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{2.15}$$

with the initial condition

$$\rho^j (0, x) = \rho_j(x), \quad x \in \mathbb{T}^3, \tag{2.16}$$

where h^j is a given smooth function and $W^k = (\rho^k, v^k, \theta^k, \omega^k)$. Linear problem (2.15)–(2.16) admits a unique global smooth solution (ρ^j, χ^j). Then, (v^j, E^j) follows by (2.13)–(2.14). On the other hand, the existence of v^j together with the third equation of (2.3) indicates that θ^j solves the following linear transport equation:

$$\partial_t \theta^j + \sum_{k=0}^{j} v^k \cdot \nabla \theta^{j-k} + \frac{2}{3} \sum_{k=0}^{j} \operatorname{div} v^{j-k} \theta^k = -\frac{1}{3} \sum_{k=0}^{j} v^k v^{j-k} - \theta^j, \tag{2.17}$$

with the initial condition

$$\theta^j (0, x) = \theta_j(x), \quad x \in \mathbb{T}^3. \tag{2.18}$$

It admits a unique global smooth solution θ^j.

Therefore, for $j \ge 1$, we obtain the high-order compatibility conditions

$$\begin{aligned} v_j = \nabla \chi_j &- \left(\frac{\theta_0}{\rho_0} \nabla \rho_{j-1} + \left(\frac{\theta^{j-1}}{\rho^0} - \frac{\theta_0 \rho_{j-1}}{|\rho_0|^2} \right) \nabla \rho_0 + \nabla \theta_{j-1} \right) - f^{j-2} \left((\rho_k, \theta_k)_{0 \le k \le j-2} \right) \\ &- \left(\partial_t v^{j-1} (0, x) + \partial_t \omega^{j-1} (0, x) + \sum_{k=0}^{j-1} (v_k \cdot \nabla) v_{j-1-k} + \sum_{k=0}^{j-1} v_k \times B_{j-1-k} \right), \quad x \in \mathbb{T}^3, \end{aligned} \tag{2.19}$$

$$\partial_t \theta^j(0,x) + \sum_{k=0}^{j} v_k \cdot \nabla \theta_{j-k} + \frac{2}{3} \sum_{k=0}^{j} \mathrm{div} v_{j-k} \theta_k = -\frac{1}{3} \sum_{k=0}^{j} v_k v_{j-k} - \theta_j, \quad x \in \mathbb{T}^3. \tag{2.20}$$

$$E_j = \partial_t \omega^{j-1}(0,x) - \nabla \chi_j, \quad B_j = B^j(0,x), \quad x \in \mathbb{T}^3, \tag{2.21}$$

where χ_j satisfies

$$\Delta \chi_j = \rho_j + \partial_t \left(\mathrm{div} \omega^{j-1}\right)(0,x), \quad x \in \mathbb{T}^3, \quad \text{and} \quad \mathfrak{M}\left(\chi_j\right) = 0. \tag{2.22}$$

Summarizing what mentioned in the above, we get the existence of an approximate solution as follows.

Proposition 2.1 *Let integer $s \geq 3$. Suppose $(\rho_j, v_j, \theta_j, E_j, B_j) \in H^{s+1}$ for $j \geq 0$, with $\rho_0, \theta_0 \geq const. > 0$ in $\mathbb{T}^3$, and satisfy the compatibility conditions (2.11)–(2.12) and (2.19)–(2.22) for $j \geq 1$. Then, there is a unique asymptotic expansion up to any order of the form (2.1), i.e., there are $T_1 > 0$ and a unique smooth solution $(\rho^j, v^j, \theta^j, E^j, B^j)_{j \geq 0}$ in the time interval $[0, T_1]$ for problems (2.6)–(2.10) and (2.13)–(2.18) for $j \geq 1$. Furthermore, it holds $\rho^0, \theta^0 \geq const. > 0$ in $[0, T_1] \times \mathbb{T}^3$ and*

$$(\rho^j, v^j, \theta^j, E^j, B^j) \in C^1([0, T_1]; H^s) \cap C([0, T_1]; H^{s+1}), \quad \forall j \geq 0.$$

In particular, the formal zero-relaxation limit $\tau \to 0$ of the non-isentropic Euler–Maxwell system (1.5) is the transport equation (2.9) coupled with the drift–diffusion system (2.6) and (2.8).

2.2 Convergence

Let $m \geq 0$ be an integer and denote

$$(\rho_\tau^m, v_\tau^m, \theta_\tau^m, \tilde{E}_\tau^m, B_\tau^m) = \sum_{j=0}^{m} \tau^{2j} (\rho^j, \tau v^j, \tau^2 \theta^j, E^j, \tau B^j), \quad (t,x) \in [0, +\infty) \times \mathbb{T}^3, \tag{2.23}$$

as an approximate solution of order m, where $(\rho^j, v^j, \theta^j, E^j, B^j)_{0 \leq j \leq m}$ are constructed in the section above. It follows from the construction of the approximate solution that

$$\mathrm{div} \tilde{E}_\tau^m = b - \rho_\tau^m, \quad \mathrm{div} B_\tau^m = 0, \quad (t,x) \in [0, +\infty) \times \mathbb{T}^3. \tag{2.24}$$

Let us define the remainders $R_\rho^{\tau,m}$, $R_v^{\tau,m}$, $R_\theta^{\tau,m}$, $R_E^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$ by

$$\begin{cases} \partial_t \rho_\tau^m + \frac{1}{\tau}\mathrm{div}(\rho_\tau^m v_\tau^m) = R_\rho^{\tau,m}, \\ \partial_t v_\tau^m + \frac{1}{\tau}\left(v_\tau^m \cdot \nabla\right) v_\tau^m + \frac{1}{\tau}\frac{\nabla(\rho_\tau^m \theta_\tau^m)}{\rho_\tau^m} = -\frac{1}{\tau}(E_\tau^m + v_\tau^m \times B_\tau^m) - \frac{v_\tau^m}{\tau^2} + R_v^{\tau,m}, \\ \partial_t \theta_\tau^m + \frac{1}{\tau} v_\tau^m \cdot \nabla \theta_\tau^m + \frac{2}{3\tau}\theta_\tau^m \mathrm{div} v_\tau^m = -\frac{1}{3}\left|v_\tau^m\right|^2 - \theta_\tau^m + R_\theta^{\tau,m}, \\ \partial_t \tilde{E}_\tau^m - \frac{1}{\tau}\nabla \times B_\tau^m = \frac{1}{\tau}\rho_\tau^m v_\tau^m + R_E^{\tau,m}, \\ \partial_t B_\tau^m + \frac{1}{\tau}\nabla \times \tilde{E}_\tau^m = \tilde{R}_B^{\tau,m}, \quad (t,x) \in [0,+\infty) \times \mathbb{T}^3. \end{cases} \tag{2.25}$$

Obviously, the convergence rate depends on the order of the remainders with respect to τ. Due to the fact that the last equation in the non-isentropic Euler–Maxwell equations is the same as that in the isentropic Euler–Maxwell equations (see Hajjej and Peng (2012)), for sufficiently smooth profiles $(\rho^j, v^j, \theta^j, E^j, B^j)_{j\geq 0}$, it follows from (1.5), (2.1), (2.5) and (2.23) that

$$\begin{aligned} \tilde{R}_B^{\tau,m} &= -\tau^{2m+1}\nabla \times E^{m+1} - \sum_{j\geq m+2} \tau^{2j-1}\nabla \times E^j - \sum_{j\geq m+1} \tau^{2j+1}\partial_t B^j \\ &= \tau^{2m+1}\partial_t B^m - \sum_{j\geq m+1} \tau^{2j+1}(\partial_t B^j + \nabla \times E^{j+1}) \\ &= \tau^{2m+1}\partial_t B^m. \end{aligned} \tag{2.26}$$

Similarly, we also get

$$\begin{aligned} &R_\rho^{\tau,m} = \mathcal{O}\left(\tau^{2(m+1)}\right), \quad R_\theta^{\tau,m} = \mathcal{O}\left(\tau^{2(m+1)}\right), \quad R_E^{\tau,m} = \mathcal{O}\left(\tau^{2(m+1)}\right), \\ &R_v^{\tau,m} = \mathcal{O}\left(\tau^{2m+1}\right). \end{aligned} \tag{2.27}$$

In (2.26)–(2.27), there is a loss of one order for the remainders $R_v^{\tau,m}$ and $\tilde{R}_B^{\tau,m}$. For $R_v^{\tau,m}$, this loss will be recovered in the error estimate of convergence due to the dissipation term for v. However, the situation is different for $\tilde{R}_B^{\tau,m}$ since the equation for B is not dissipative. Inspired by Hajjej and Peng (2012), we remedy this by introducing a correction term into $\tilde{E}_\tau^m$ so that

$$E_\tau^m = \tilde{E}_\tau^m + \tau^{2(m+1)}E_c^{m+1} = \sum_{j=0}^{m} \tau^{2j}E^j + \tau^{2(m+1)}E_c^{m+1}, \tag{2.28}$$

where E_c^{m+1} is defined by

$$\nabla \times E_c^{m+1} = -\partial_t B^m, \quad \mathrm{div} E_c^{m+1} = 0, \quad \mathfrak{M}\left(E_c^{m+1}\right) = 0. \tag{2.29}$$

Then, we obtain that the new remainder $R_B^{\tau,m}$ of B satisfies

$$R_B^{\tau,m} \triangleq \partial_t B_\tau^m + \frac{1}{\tau}\nabla \times E_\tau^m = 0, \tag{2.30}$$

and we also get

$$\mathrm{div}E_\tau^m = b - \rho_\tau^m, \quad \mathrm{div}B_\tau^m = 0. \tag{2.31}$$

The orders of the remainders $R_\rho^{\tau,m}$, $R_v^{\tau,m}$, $R_\theta^{\tau,m}$ and $R_E^{\tau,m}$ are not changed due to the fact that the correction term is of order $\mathcal{O}\left(\tau^{2(m+1)}\right)$. Furthermore, the correction term does not affect assumption (1.8). We conclude the above discussion with the following result.

Proposition 2.2 *Let the assumptions of Proposition 2.1 hold. For all integers $m \geq 0$ and $s \geq 3$, the remainders $R_\rho^{\tau,m}$, $R_v^{\tau,m}$, $R_\theta^{\tau,m}$, $R_E^{\tau,m}$ and $R_B^{\tau,m}$ satisfy (2.30) and*

$$\sup_{0\le t\le T_1} \left\| \left(R_\rho^{\tau,m}, R_\theta^{\tau,m}, R_E^{\tau,m}\right)(t,\cdot)\right\| \le C_m \tau^{2(m+1)}, \quad \sup_{0\le t\le T_1} \left\| R_v^{\tau,m}(t,\cdot)\right\| \le C_m \tau^{2m+1}, \tag{2.32}$$

where $C_m > 0$ is a constant independent of τ.

3 Problem (1.3)–(1.5) with Ill-Prepared Initial Data

In this section, we study the asymptotic expansions in the case of ill-prepared initial data by constructing initial layer corrections which exponentially decay to zero.

3.1 Initial Layer

In Theorem 1.1, we introduce compatibility conditions on the initial data, namely the initial profiles $(v^j, \theta^j, E^j, B^j)(0,\cdot)$ are determined through the resolution of the problems (2.3)–(2.5) for $(\rho^j, v^j, \theta^j, E^j, B^j)$. Therefore, we cannot show $(v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau)$ explicitly. The phenomenon of initial layers must appear as long as these conditions are not satisfied. In the following, we investigate problem (1.3)–(1.5) with ill-prepared initial data. Similarly as that for the isentropic Euler–Maxwell equations considered in Hajjej and Peng (2012), we also look for the simplest possible form of an asymptotic expansion with initial layer corrections such that its remainders are at least of order $\mathcal{O}(\tau)$ for variable u.

Assume that the initial data of an approximate solution $(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)$ has an asymptotic expansion of the form

$$(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)\,|_{t=0} = \left(\rho_0, \tau v_0, \tau^2\theta_0, E_0, \tau B_0\right) + O\left(\tau^2\right), \tag{3.1}$$

where the given function $(\rho_0, v_0, \theta_0, E_0, B_0)$ is smooth. Regarding the expansion for the well-prepared initial data, we give an asymptotic expansion including initial layer corrections as

$$\begin{aligned}(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)(t,x) &= \left(\rho^0, \tau v^0, \tau^2\theta^0, E^0 + \tau^2 E_c^1, \tau B^0\right)(t,x) + \left(\rho_I^0, \tau v_I^0, \tau^2\theta_I^0, E_I^0, \tau B_I^0\right)(z,x) \\ &\quad + \tau^2\left(\rho_I^1, \tau v_I^1, \tau^2\theta_I^1, E_I^1, \tau B_I^1\right)(z,x) + O\left(\tau^2\right),\end{aligned} \tag{3.2}$$

where $z = \dfrac{t}{\tau^2} \in \mathbb{R}$, and E_c^1 is the correction term defined by (2.29) with $m = 0$.

Remark 3.1 It should be pointed out that the expansion (3.2) is enough to give the remainders at least of order $\mathcal{O}(\tau)$ for variable u, which is the case of well-prepared initial data for $m = 0$.

It is easy to see that $(\rho^0, v^0, \theta^0, E^0, B^0)$ always satisfies system (2.3). The rest is to determine the initial layer profiles $(\rho_I^0, v_I^0, \theta_I^0, E_I^0, B_I^0)$ and $(\rho_I^1, v_I^1, \theta_I^1, E_I^1, B_I^1)$. Plugging (3.2) into system (1.5) and using (2.3), we have

$$\partial_z \rho_I^0 = 0, \quad \partial_z \theta_I^0 = 0, \quad \partial_z E_I^0 = 0, \quad \partial_z B_I^0 + \nabla \times E_I^0 = 0, \tag{3.3}$$

and

$$\partial_z v_I^0 + v_I^0 = 0. \tag{3.4}$$

It follows from (3.3) that there are no first-order initial layers for variables ρ, θ, E and B. Therefore, up to a constant for variable B, we may take

$$\rho^0(0, x) = \rho_0(x), \quad \theta^0(0, x) = \theta_0(x), \quad E^0(0, x) = E_0(x), \quad B^0(0, x) = B_0(x). \tag{3.5}$$

Furthermore, by (3.1) and (3.2), we obtain

$$v_I^0(0, x) = v_0(x) - v^0(0, x), \tag{3.6}$$

where $v^0(0, x)$ is given by (2.11)–(2.12). And then, by (3.4), we get

$$v_I^0(z, x) = v_I^0(0, x)e^{-z} = \Big(v_0(x) - v^0(0, x)\Big)\, e^{-z}. \tag{3.7}$$

In a similar way, we find that the second-order initial layer functions satisfy

$$v_I^1 = 0, \quad \theta_I^1 = 0, \tag{3.8}$$

$$\partial_z \rho_I^1(z, x) + \operatorname{div}\Big(\rho^0(0, x) v_I^0(z, x)\Big) = 0, \tag{3.9}$$

$$\partial_z E_I^1(z, x) = \rho^0(0, x) v_I^0(z, x), \tag{3.10}$$

and

$$\partial_z B_I^1(z, x) + \nabla \times E_I^1(z, x) = 0. \tag{3.11}$$

Suppose (ρ_1, E_1, B_1) to be smooth functions such that

$$E_1(x) = -\rho^0(0, x)\Big(v_0(x) - v^0(0, x)\Big), \tag{3.12}$$

and

$$\rho_1 = \operatorname{div} E_1, \quad \operatorname{div} B_1 = 0. \tag{3.13}$$

We also set

$$\left(\rho_I^1, E_I^1, B_I^1\right)(0,x) = (\rho_1, E_1, B_1)(x).$$

Then together with (3.7) and (3.9)–(3.12), we easily obtain

$$\rho_I^1(z,x) = \rho_1(x) - \operatorname{div}\left(\rho^0(0,x)\left(v_0(x) - v^0(0,x)\right)\right)\left(1 - e^{-z}\right), \tag{3.14}$$

$$E_I^1(z,x) = -\rho^0(0,x)\left(v_0(x) - v^0(0,x)\right)e^{-z} \tag{3.15}$$

and

$$B_I^1(z,x) = B_1(x) + \nabla \times \left(\rho^0(0,x)\left(v_0(x) - v^0(0,x)\right)\right)\left(1 - e^{-z}\right). \tag{3.16}$$

Finally, it follows from (3.13) that

$$0 = \operatorname{div} E_I^1 + \rho_I^1, \quad \operatorname{div} B_I^1 = 0. \tag{3.17}$$

Therefore, the initial layer functions $(\rho_I^0, v_I^0, \theta_I^0, E_I^0, B_I^0)$ and $(\rho_I^1, v_I^1, \theta_I^1, E_I^1, B_I^1)$ are constructed by (3.3), (3.7)–(3.8) and (3.14)–(3.16). These functions are bounded with respect to z.

3.2 Convergence

In view of the previous asymptotic expansions, we introduce

$$\begin{cases} \rho_{\tau,I}(t,x) = \rho^0(t,x) + \tau^2 \rho_I^1(t/\tau^2, x), \\ v_{\tau,I}(t,x) = \tau\left(v^0(t,x) + v_I^0(t/\tau^2, x)\right), \\ \theta_{\tau,I}(t,x) = \tau^2\theta^0(t,x), \\ E_{\tau,I}(t,x) = E^0(t,x) + \tau^2\left(E_c^1(t,x) + E_I^1(t/\tau^2, x)\right), \\ B_{\tau,I}(t,x) = \tau\left(B^0(t,x) + \tau^2 B_I^1(t/\tau^2, x)\right). \end{cases} \tag{3.18}$$

It follows that

$$\begin{aligned} &(\rho_{\tau,I}, v_{\tau,I}, \theta_{\tau,I}, E_{\tau,I}, B_{\tau,I})(0,x) \\ &\quad = (\rho_0, \tau v_0, \tau^2\theta_0, E_0, \tau B_0) + \tau^2(\rho_1, 0, 0, E_1 + E_c^1(0,x), \tau B_1). \end{aligned} \tag{3.19}$$

On the other hand, by (2.3), (2.31) and (3.17), we obtain

$$\operatorname{div} E_{\tau,I} = b - \rho_{\tau,I}, \quad \operatorname{div} B_{\tau,I} = 0. \tag{3.20}$$

The remainders $R_\rho^{\tau,I}$, $R_v^{\tau,I}$, $R_\theta^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ are defined by

$$\begin{cases} \partial_t \rho_{\tau,I} + \frac{1}{\tau}\mathrm{div}(\rho_{\tau,I} v_{\tau,I}) = R_\rho^{\tau,I}, \\ \partial_t v_{\tau,I} + \frac{1}{\tau}\left(v_{\tau,I}\cdot\nabla\right) v_{\tau,I} + \frac{1}{\tau}\frac{\nabla(\rho_{\tau,I}\theta_{\tau,I})}{\rho_{\tau,I}} = -\frac{1}{\tau}(E_{\tau,I} + v_{\tau,I}\times B_{\tau,I}) - \frac{v_{\tau,I}}{\tau^2} + R_v^{\tau,I}, \\ \partial_t\theta_{\tau,I} + \frac{1}{\tau} v_{\tau,I}\cdot\nabla\theta_{\tau,I} + \frac{2}{3\tau}\theta_{\tau,I}\mathrm{div} v_{\tau,I} = -\frac{1}{3}\left|v_{\tau,I}\right|^2 - \theta_{\tau,I} + R_\theta^{\tau,I}, \\ \partial_t E_{\tau,I} - \frac{1}{\tau}\nabla\times B_{\tau,I} = \frac{1}{\tau}\rho_{\tau,I}v_{\tau,I} + R_E^{\tau,I}, \\ \partial_t B_{\tau,I} + \frac{1}{\tau}\nabla\times E_{\tau,I} = R_B^{\tau,I}. \end{cases} \tag{3.21}$$

With the help of (2.3), (2.29), (3.3)–(3.4) and (3.8)–(3.11), we get

$$\begin{aligned} R_\rho^{\tau,I} &= \mathrm{div}\left(\left(\rho^0(t,x) - \rho^0(0,x)\right) v_I^0(z,x)\right) + \tau^2\mathrm{div}\left(\rho_I^1(v^0 + v_I^0)\right) \\ &= \tau^2 z\mathrm{div}\left(\partial_t\rho^0(\lambda,x) v_I^0(z,x)\right) + \tau^2\mathrm{div}\left(\rho_I^1(v^0+v_I^0)\right) \\ &= -\tau^2 z e^{-z}\mathrm{div}\left(\mathrm{div}\left(\rho^0 v^0\right)(\lambda,x)\left(v_0 - v^0(0,x)\right)\right) + \tau^2\mathrm{div}\left(\rho_I^1(v^0+v_I^0)\right) \\ &= O\left(\tau^2\right), \end{aligned}$$

where $\lambda\in[0,t]\subset[0,T_1]$, and we have also used the fact that the function $z\mapsto ze^{-z}$ is bounded for $z\geq 0$. And similarly, we have

$$\begin{aligned} R_v^{\tau,I} &= \frac{1}{\tau}\left(\partial_z v_I^0 + v_I^0\right) + \left(v_I^0\cdot\nabla\right)\left(\tau(v^0+v_I^0)\right) + v^0\cdot\nabla\left(\tau v_I^0\right) + \tau E_I^1 + \tau^3 v_I^0\times B_I^1 \\ &\quad + v_I^0\times\left(\tau B^0 + \tau^3 B_I^1\right) + \frac{1}{\tau}\left(\frac{\nabla\left(\left(\rho^0+\tau^2\rho_I^1\right)\tau^2\theta^0\right)}{\rho^0+\tau^2\rho_I^1} - \frac{\nabla\left(\rho^0\tau^2\theta^0\right)}{\rho^0}\right) \\ &= O(\tau), \\ R_\theta^{\tau,I} &= \tau^2 v_I^0\cdot\nabla\theta^0 + \frac{2}{3}\tau^2\theta^0\mathrm{div}v^0 + \frac{1}{3}\tau^2\left(\left|v_I^0\right|^2 + 2v^0 v_I^0\right) = O\left(\tau^2\right), \\ R_E^{\tau,I} &= \left(\rho^0(t,x) - \rho^0(0,x)\right)v_I^0(z,x) + \tau^2\left(\rho_I^1(v^0+v_I^0) + \partial_t E_c^1 - \nabla\times B_I^1\right) \\ &= \tau^2 z\partial_t\rho^0(\lambda,x)v_I^0(z,x) + \tau^2\left(\rho_I^1(v^0+v_I^0) + \partial_t E_c^1 - \nabla\times B_I^1\right) \\ &\quad - \tau^2 z e^{-z}\mathrm{div}\left(\rho^0 v^0\right)(\lambda,x)\left(v_0 - v^0(0,x)\right) + \tau^2\left(\rho_I^1(v^0+v_I^0) + \partial_t E_c^1 - \nabla\times B_I^1\right) \\ &= O\left(\tau^2\right), \end{aligned}$$

and

$$R_B^{\tau,I} = 0.$$

The above discussions about the remainders yield the following error estimates.

Proposition 3.1 *Let integer $s \geq 3$. For given smooth data, the remainders $R_\rho^{\tau,I}$, $R_v^{\tau,I}$, $R_\theta^{\tau,I}$, $R_E^{\tau,I}$ and $R_B^{\tau,I}$ satisfy*

$$\sup_{0\leq t\leq T_1}\left\|\left(R_\rho^{\tau,I}, R_\theta^{\tau,I}, R_E^{\tau,I}\right)(t,\cdot)\right\| \leq C\tau^2, \quad \sup_{0\leq t\leq T_1}\left\|R_v^{\tau,I}(t,\cdot)\right\| \leq C\tau, \quad R_B^{\tau,I}=0. \tag{3.22}$$

4 Proof of Theorems 1.1 and 1.2

In this section, we give the rigorous justification of the both two asymptotic expansions and prove Theorems 1.1 and 1.2. To this end, we establish a more general convergence theorem which implies the convergence of the both expansions.

4.1 General Convergence Theorem

In the following, we justify rigorously the asymptotic expansions of solutions $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ to the periodic problem (1.3)–(1.5) developed in Sects. 2–3. We establish a more usual convergence result which yields both Theorems 1.1 and 1.2. As a result, we acquire the existence of exact solutions $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ in a time interval independent of τ. In order to justify the asymptotic expansions (2.1) and (3.18), we should establish the uniform estimates of solutions to (1.5) with respect to τ.

Assume that $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ is the exact solution to (1.5) with initial data $(\rho_0^\tau, v_0^\tau, \theta_0^\tau, E_0^\tau, B_0^\tau)$ and $(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)$ is an approximate periodic solution defined on $[0, T_1]$, with

$$(\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau) \in C^1\left([0,T_1]; H^{s-1}(\mathbb{T}^3)\right) \cap C\left([0,T_1]; H^s(\mathbb{T}^3)\right).$$

The remainders of the approximate solution are defined by

$$\begin{cases} R_\rho^\tau = \partial_t\rho_\tau + \frac{\operatorname{div}(\rho_\tau v_\tau)}{\tau}, \\ R_v^\tau = \partial_t v_\tau + \frac{(v_\tau\cdot\nabla)v_\tau}{\tau} + \frac{\nabla(\rho_\tau\theta_\tau)}{\tau\rho_\tau} + \frac{E_\tau + v_\tau\times B_\tau}{\tau} + \frac{v_\tau}{\tau^2}, \\ R_\theta^\tau = \partial_t\theta_\tau + \frac{v_\tau\cdot\nabla\theta_\tau}{\tau} + \frac{2\theta_\tau\operatorname{div}v_\tau}{3\tau} + \frac{|v_\tau|^2}{3} + \theta_\tau, \\ R_E^\tau = \partial_t E_\tau - \frac{\nabla\times B_\tau}{\tau} - \frac{\rho_\tau v_\tau}{\tau}, \\ R_B^\tau = \partial_t B_\tau + \frac{\nabla\times E_\tau}{\tau}. \end{cases} \tag{4.1}$$

We assume that

$$\operatorname{div}E_\tau = b - \rho_\tau, \quad \operatorname{div}B_\tau = 0. \tag{4.2}$$

$$\sup_{0\leq t\leq T_1}\|(\rho_\tau, \theta_\tau, E_\tau, B_\tau)(t,\cdot)\|_s \leq C_1, \quad \sup_{0\leq t\leq T_1}\|v_\tau(t,\cdot)\|_s \leq C_1\tau, \tag{4.3}$$

$$\left\|\left(\rho_0^\tau - \rho_\tau(0,\cdot), v_0^\tau - v_\tau(0,\cdot), \theta_0^\tau - \theta_\tau(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\right)\right\|_s \leq C_1\tau^{r+1}, \tag{4.4}$$

$$\sup_{0\le t\le T_1}\left\|\left(R_\rho^\tau, R_\theta^\tau, R_E^\tau\right)(t,\cdot)\right\| \le C_1\tau^{r+1}, \quad \sup_{0\le t\le T_1}\left\|R_v^\tau(t,\cdot)\right\| \le C_1\tau^r, \quad R_B^\tau = 0, \tag{4.5}$$

where $r \ge 0$ and $C_1 > 0$ are constants independent of τ.

Theorem 4.1 *Let $s \ge 3$ be an integer and $r \ge 0$. Under the above assumptions, there exists a constant $C_2 > 0$, independent of τ, such that as $\tau \to 0$ we have $T_1^\tau \ge T_1$ and the solution $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ of the periodic problem (1.3)–(1.5) satisfies*

$$\left\|\left(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau\right)(t) - (\rho_\tau, v_\tau, \theta_\tau, E_\tau, B_\tau)(t)\right\|_s \le C_2\tau^{r+1}, \quad \forall t \in [0, T_1]. \tag{4.6}$$

Moreover,

$$\left\|v^\tau - v_\tau\right\|_{L^2(0,T_1;H^s)} \le C_2\tau^{r+2}, \quad \left\|\theta^\tau - \theta_\tau\right\|_{L^2(0,T_1;H^s)} \le C_2\tau^{r+1}. \tag{4.7}$$

4.2 Proof of the Main Result

It follows from Proposition 1.1 that the exact solution $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$ is defined in a time interval $[0, T_1^\tau]$ with $T_1^\tau > 0$. Since $\rho^\tau, \theta^\tau \in C([0, T_1^\tau], H^s)$ and $H^s \hookrightarrow C(\mathbb{T}^3)$ is continuous, we have $\rho^\tau, \theta^\tau \in C([0, T_1^\tau] \times \mathbb{T}^3)$. From (4.3)–(4.4) and assumption $\rho_0^\tau, \theta_0^\tau \ge 2\kappa > 0$, we obtain that there is $T_2^\tau \in (0, T_1^\tau]$ and a constant $C_* > 0$, independent of τ, such that

$$\kappa \le \rho^\tau(t,x), \theta^\tau(t,x) \le C_*, \quad \forall (t,x) \in [0, T_2^\tau] \times \mathbb{T}^3.$$

Similarly, the function $t \mapsto \|(\rho^\tau(t,\cdot), v^\tau(t,\cdot), \theta^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot))\|_s$ is continuous in $C[0, T_2^\tau]$. From (4.3), the sequence $\left(\|(\rho^\tau(0,\cdot), v^\tau(0,\cdot), \theta^\tau(0,\cdot), E^\tau(0,\cdot), B^\tau(0,\cdot))\|_s\right)_{\tau>0}$ is bounded. Then, there is $T_3^\tau \in (0, T_2^\tau]$ and a constant, still denoted by C_* such that

$$\left\|\left(\rho^\tau(t,\cdot), v^\tau(t,\cdot), \theta^\tau(t,\cdot), E^\tau(t,\cdot), B^\tau(t,\cdot)\right)\right\|_s \le C_*, \quad \forall t \in (0, T_3^\tau].$$

Set $T^\tau = \min\{T_1, T_3^\tau\} > 0$ and introduce the perturbation variable as follows:

$$\left(\zeta^\tau, \mathscr{V}^\tau, \Theta^\tau, F^\tau, G^\tau\right) = \left(\rho^\tau - \rho_\tau, v^\tau - v_\tau, \theta^\tau - \theta_\tau, E^\tau - E_\tau, B^\tau - B_\tau\right), \quad (t,x) \in [0, T^\tau] \times \mathbb{T}^3. \tag{4.8}$$

Obviously, $(\zeta^\tau, \mathscr{V}^\tau, \Theta^\tau, F^\tau, G^\tau)$ satisfies the following system

$$\begin{cases} \partial_t\zeta^\tau + \frac{1}{\tau}\left(v^\tau\cdot\nabla\zeta^\tau + \rho^\tau \mathrm{div}\mathscr{V}^\tau\right) = \frac{1}{\tau}\left(\mathscr{V}^\tau\cdot\nabla\zeta^\tau - \zeta^\tau\mathrm{div}v_\tau - \mathscr{V}^\tau\cdot\nabla\rho_\tau\right) - R_\rho^\tau, \\ \partial_t\mathscr{V}^\tau + \frac{1}{\tau}\left((v^\tau\cdot\nabla)\mathscr{V}^\tau + \frac{\theta^\tau}{\rho^\tau}\nabla\zeta^\tau + \nabla\Theta^\tau\right) = -\frac{1}{\tau}\left((\mathscr{V}^\tau\cdot\nabla)v_\tau + \left(\frac{\theta^\tau}{\rho^\tau} - \frac{\theta_\tau}{\rho_\tau}\right)\nabla\rho_\tau\right) - \frac{\mathscr{V}^\tau}{\tau^2} \\ \qquad -\frac{1}{\tau}\left(F^\tau + (\mathscr{V}^\tau + v_\tau)\times G^\tau + \mathscr{V}^\tau\times B_\tau\right) - R_v^\tau, \\ \partial_t\Theta^\tau + \frac{2}{3\tau}\theta^\tau\mathrm{div}\mathscr{V}^\tau + \frac{1}{\tau}v^\tau\cdot\nabla\Theta^\tau = \frac{1}{\tau}\mathscr{V}^\tau\cdot\nabla\theta_\tau \\ \qquad -\frac{2}{3\tau}\Theta^\tau\mathrm{div}v_\tau - \frac{1}{3}\mathscr{V}^\tau\left(2v_\tau + \mathscr{V}^\tau\right) - \Theta^\tau - R_\theta^\tau, \\ \partial_t F^\tau - \frac{1}{\tau}\nabla\times G^\tau = \frac{1}{\tau}\left(\zeta^\tau\mathscr{V}^\tau + \zeta^\tau v_\tau + \rho_\tau\mathscr{V}^\tau\right) - R_E^\tau, \quad \mathrm{div}F^\tau = -\zeta^\tau, \\ \partial_t G^\tau + \frac{1}{\tau}\nabla\times F^\tau = 0, \quad \mathrm{div}G^\tau = 0, \quad (t,x)\in[0,T^\tau]\times\mathbb{T}^3, \end{cases} \tag{4.9}$$

with the initial condition

$$
\begin{aligned}
&\left(\zeta^\tau, \mathscr{V}^\tau, \Theta^\tau, F^\tau, G^\tau\right)|_{t=0} \\
&= \left(\rho_0^\tau - \rho_\tau(0,\cdot), v_0^\tau - v_\tau(0,\cdot), \theta_0^\tau - \theta_\tau(0,\cdot), E_0^\tau - E_\tau(0,\cdot), B_0^\tau - B_\tau(0,\cdot)\right), \quad x \in \mathbb{T}^3.
\end{aligned} \tag{4.10}
$$

Let us define

$$
V_I^\tau = \begin{pmatrix} \zeta^\tau \\ \mathscr{V}^\tau \\ \Theta^\tau \end{pmatrix}, \quad V_{II}^\tau = \begin{pmatrix} F^\tau \\ G^\tau \end{pmatrix}, \quad V^\tau = \begin{pmatrix} V_I^\tau \\ V_{II}^\tau \end{pmatrix},
$$

$$
\mathcal{A}_i^I\left(\rho^\tau, v^\tau, \theta^\tau\right) = \begin{pmatrix} v_i^\tau & \rho^\tau e_i^t & 0 \\ \dfrac{\theta^\tau}{\rho^\tau} e_i & v_i^\tau \mathrm{I}_3 & e_i \\ 0 & \dfrac{2}{3}\theta^\tau e_i^t & v_i^\tau \end{pmatrix}, \quad i = 1,2,3, \quad \mathcal{R}^\tau = \begin{pmatrix} R_\rho^\tau \\ R_v^\tau \\ R_\theta^\tau \end{pmatrix},
$$

$$
\mathcal{H}_1\left(V_I^\tau\right) = -\begin{pmatrix} \zeta^\tau \mathrm{div} v_\tau + \mathscr{V}^\tau \cdot \nabla \rho_\tau \\ \left(\mathscr{V}^\tau \cdot \nabla\right) v_\tau + \left(\dfrac{\theta^\tau}{\rho^\tau} - \dfrac{\theta_\tau}{\rho_\tau}\right) \nabla \rho_\tau \\ \dfrac{2}{3}\Theta^\tau \mathrm{div} v_\tau - \mathscr{V}^\tau \cdot \nabla \theta_\tau \end{pmatrix}, \quad \mathcal{H}_2\left(V_I^\tau\right) = -\begin{pmatrix} 0 \\ \dfrac{\mathscr{V}^\tau}{\tau^2} \\ \Theta^\tau \end{pmatrix},
$$

$$
\mathcal{H}_3\left(V_I^\tau\right) = \begin{pmatrix} \mathscr{V}^\tau \cdot \nabla \zeta^\tau \\ -F^\tau - \left(\mathscr{V}^\tau + v_\tau\right) \times G^\tau - \mathscr{V}^\tau \times B_\tau \\ 0 \end{pmatrix},
$$

$$
\mathcal{H}_4\left(V_I^\tau\right) = -\begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{3}\mathscr{V}^\tau\left(2v_\tau + \mathscr{V}^\tau\right) \end{pmatrix},
$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, v_i^τ denotes the ith component of $v^\tau \in \mathbb{R}^3$ and I_3 is the 3×3 unit matrix. Then, we rewrite system (4.9) for V_I^τ in the matrix form as

$$
\partial_t V_I^\tau + \frac{1}{\tau}\sum_{i=1}^{3} \mathcal{A}_i^\tau(\rho^\tau, v^\tau, \theta^\tau)\partial_i V_I^\tau = \frac{1}{\tau}\left(\mathcal{H}_1\left(V_I^\tau\right) + \mathcal{H}_3\left(V_I^\tau\right)\right) + \mathcal{H}_2\left(V_I^\tau\right) + \mathcal{H}_4\left(V_I^\tau\right) - \mathcal{R}^\tau. \tag{4.11}
$$

It is symmetrizable hyperbolic with symmetrizer

$$
\mathcal{A}_0^I\left(\rho^\tau, \theta^\tau\right) = \begin{pmatrix} \dfrac{\theta^\tau}{\rho^\tau} & 0 & 0 \\ 0 & \rho^\tau \mathrm{I}_3 & 0 \\ 0 & 0 & \dfrac{3\rho^\tau}{2\theta^\tau} \end{pmatrix},
$$

which is a positive definite matrix when $0<\kappa\le\rho^\tau=\zeta^\tau+\rho_\tau, \theta^\tau=\Theta^\tau+\theta_\tau\le C_*$. Moreover,

$$\tilde{\mathcal{A}}_i^I\left(\rho^\tau,v^\tau,\theta^\tau\right)=\mathcal{A}_0^I\left(\rho^\tau,\theta^\tau\right)\mathcal{A}_i^I\left(\rho^\tau,v^\tau,\theta^\tau\right)=v_i^\tau\mathcal{A}_0^I\left(\rho^\tau,\theta^\tau\right)+\mathcal{Q}_i^I\left(\rho^\tau,\theta^\tau\right), \quad i=1,2,3 \tag{4.12}$$

is symmetric, where the matrix $\mathcal{Q}_i^I(\rho^\tau,\theta^\tau)$ is defined by

$$\mathcal{Q}_i^I\left(\rho^\tau,\theta^\tau\right)=\begin{pmatrix}0 & \theta^\tau e_i^t & 0\\ \theta^\tau e_i & 0 & \rho^\tau e_i\\ 0 & \rho^\tau e_i^t & 0\end{pmatrix}, \quad i=1,2,3.$$

It is easy to know that the existence and uniqueness of smooth solutions to system (1.5) with (1.3)–(1.4) is equivalent to that of (4.9)–(4.10). Therefore, to prove Theorem 4.1, we should obtain uniform estimates of V^τ with respect to τ. In the sequence, we denote by $C>0$ various constants independent of τ and for $\alpha\in\mathbb{N}^3$, $(V_{I\alpha}^\tau, V_{II\alpha}^\tau)=\partial^\alpha(V_I^\tau,V_{II}^\tau)$, etc. The main estimates are included in the following two lemmas for V_I^τ and V_{II}^τ, respectively.

We first consider the estimate for V_{II}^τ. Since the Maxwell equations in the non-isentropic Euler–Maxwell equations are the same as that in the isentropic Euler–Maxwell equations, we can obtain the estimate for V_{II}^τ by following the similar way as that in Hajjej and Peng (2012). Here, we only list the results without details for simplicity.

Lemma 4.1 (Hajjej and Peng 2012) *Under the assumptions of Theorem 4.1, for all $t\in(0,T^\tau]$, as $\tau\to0$, we have*

$$\left\|V_{II}^\tau(t)\right\|_s^2\le\int_0^t\left(\frac{1}{2\tau^2}\left\|\mathscr{V}^\tau(\xi)\right\|_s^2+C\left\|V^\tau(\xi)\right\|_s^2+C\left\|V^\tau(\xi)\right\|_s^4\right)\mathrm{d}\xi+C\tau^{2(r+1)}. \tag{4.13}$$

Next, let us establish the estimate for V_I^τ.

Lemma 4.2 *Under the assumptions of Theorem 4.1, for all $t\in(0,T^\tau]$, as $\tau\to0$, we have*

$$\begin{aligned}&\left\|V_I^\tau(t)\right\|_s^2+\int_0^t\left(\frac{1}{\tau^2}\left\|\mathscr{V}^\tau(\xi)\right\|_s^2+\left\|\Theta^\tau(\xi)\right\|_s^2\right)\mathrm{d}\xi\\ &\le C\int_0^t\left(\left\|V^\tau(\xi)\right\|_s^2+\left\|V^\tau(\xi)\right\|_s^4\right)\mathrm{d}\xi+C\tau^{2(r+1)}.\end{aligned} \tag{4.14}$$

Proof For $\alpha\in\mathbb{N}^3$ with $|\alpha|\le s$, applying ∂^α to (4.11), we have

$$\begin{aligned}&\partial_tV_{I\alpha}^\tau+\frac{1}{\tau}\sum_{i=1}^3\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial_iV_{I\alpha}^\tau\\ &=\frac{1}{\tau}\left(\partial^\alpha\mathcal{H}_1\left(V_I^\tau\right)+\partial^\alpha\mathcal{H}_3\left(V_I^\tau\right)\right)+\partial^\alpha\mathcal{H}_2\left(V_I^\tau\right)+\partial^\alpha\mathcal{H}_4\left(V_I^\tau\right)-\partial^\alpha\mathcal{R}^\tau\end{aligned}$$

$$+\frac{1}{\tau}\sum_{i=1}^{3}\left(\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial_i V_{I\alpha}^\tau-\partial^\alpha\left(\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial_i V_I^\tau\right)\right). \tag{4.15}$$

By multiplying (4.15) by $\mathcal{A}_0^I(\rho^\tau,\theta^\tau)$ and taking the inner product of the resulting equations with $2V_{I\alpha}^\tau$ in L^2, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)V_{I\alpha}^\tau,V_{I\alpha}^\tau\right\rangle-2\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_2\left(V_I^\tau\right),V_{I\alpha}^\tau\right\rangle\\
&\quad=\frac{2}{\tau}\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\left(\partial^\alpha\mathcal{H}_1\left(V_I^\tau\right)+\partial^\alpha\mathcal{H}_3\left(V_I^\tau\right)\right),V_{I\alpha}^\tau\right\rangle+2\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_4\left(V_I^\tau\right),V_{I\alpha}^\tau\right\rangle\\
&\qquad+\frac{2}{\tau}\left\langle\mathcal{J}_\alpha^\tau,V_{I\alpha}^\tau\right\rangle+\left\langle\mathrm{div}\mathcal{A}_\tau^I(\rho^\tau,v^\tau,\theta^\tau)V_{I\alpha}^\tau,V_{I\alpha}^\tau\right\rangle-2\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{R}^\tau,V_{I\alpha}^\tau\right\rangle,
\end{aligned} \tag{4.16}$$

where

$$\mathcal{J}_\alpha^\tau=-\sum_{i=1}^{3}\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\left(\partial^\alpha\left(\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial_i V_I^\tau\right)-\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial^\alpha\partial_i V_I^\tau\right)$$

and

$$\mathrm{div}\mathcal{A}_\tau^I(\rho^\tau,v^\tau,\theta^\tau)=\partial_t\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau)+\frac{1}{\tau}\sum_{i=1}^{3}\partial_i\tilde{\mathcal{A}}_i^I(\rho^\tau,v^\tau,\theta^\tau). \tag{4.17}$$

We estimate every term on both side of (4.16). A straightforward computation yields

$$\begin{aligned}
-\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_2\left(V_I^\tau\right),V_{I\alpha}^\tau\right\rangle&=\left\langle\frac{\rho^\tau}{\tau^2}\mathscr{V}_\alpha^\tau,\mathscr{V}_\alpha^\tau\right\rangle+\left\langle\frac{3\rho^\tau}{2\theta^\tau}\Theta_\alpha^\tau,\Theta_\alpha^\tau\right\rangle\\
&\geq C^{-1}\left(\frac{\left\|\mathscr{V}_\alpha^\tau\right\|^2}{\tau^2}+\left\|\Theta_\alpha^\tau\right\|^2\right).
\end{aligned} \tag{4.18}$$

Next, due to the fact that matrix $\mathcal{A}_0^I(\rho^\tau,\theta^\tau)$ is positive definite, we have

$$\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)V_{I\alpha}^\tau,V_{I\alpha}^\tau\right\rangle\geq C^{-1}\left\|V_{I\alpha}^\tau\right\|^2. \tag{4.19}$$

Furthermore, in view of the expression of $\mathcal{H}_I(V_I^\tau)$, we get

$$\begin{aligned}
&\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_1\left(V_I^\tau\right),V_{I\alpha}^\tau\right\rangle\\
&\quad=\frac{3}{2}\left\langle\frac{\rho^\tau}{\theta^\tau}\Theta_\alpha^\tau,\partial^\alpha\left(\mathscr{V}^\tau\cdot\nabla\theta_\tau-\frac{2}{3}\Theta^\tau\mathrm{div}v_\tau\right)\right\rangle-\left\langle\frac{\theta^\tau}{\rho^\tau}\zeta_\alpha^\tau,\partial^\alpha\left(\zeta^\tau\mathrm{div}v_\tau+\mathscr{V}^\tau\cdot\nabla\rho_\tau\right)\right\rangle\\
&\qquad-\left\langle\rho^\tau\mathscr{V}_\alpha^\tau,\partial^\alpha\left(\left(\mathscr{V}^\tau\cdot\nabla\right)v_\tau+\left(\frac{\Theta^\tau+\theta_\tau}{\zeta^\tau+\rho_\tau}-\frac{\theta_\tau}{\rho_\tau}\right)\nabla\rho_\tau\right)\right\rangle.
\end{aligned}$$

Then by (4.3) and Lemma 1.1, we have

$$\begin{aligned}
&\frac{1}{\tau}\left\langle \mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_1\left(V_I^\tau\right), V_{I\alpha}^\tau\right\rangle \\
&\quad\le \frac{C}{\tau}\left(\|\zeta^\tau\|_s\|\mathscr{V}^\tau\|_s+\tau\|\zeta^\tau\|_s^2+\tau\|\mathscr{V}^\tau\|_s^2+\tau\|\Theta^\tau\|_s^2+(1+\tau)\|\Theta^\tau\|_s\|\mathscr{V}^\tau\|_s\right) \\
&\quad\le \frac{\varepsilon}{\tau^2}\|\mathscr{V}^\tau\|_s^2+C_\varepsilon\|V^\tau\|_s^2, \qquad (4.20)
\end{aligned}$$

where and in the sequence, ε denotes a small constant independent of τ and $C_\varepsilon > 0$ denotes a constant depending only on ε.

On the other hand, for the terms containing $\mathcal{H}_3$ and $\mathcal{H}_4$, by an integration by parts, we obtain

$$\begin{aligned}
&\frac{2}{\tau}\left\langle \mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_3\left(V_I^\tau\right), V_{I\alpha}^\tau\right\rangle \\
&\quad=\frac{2}{\tau}\left\langle\frac{\theta^\tau}{\rho^\tau}\partial^\alpha\left(\mathscr{V}^\tau\cdot\nabla\zeta^\tau\right),\zeta_\alpha^\tau\right\rangle-\frac{2}{\tau}\left\langle\rho^\tau\mathscr{V}_\alpha^\tau,\partial^\alpha\left(F^\tau+(\mathscr{V}^\tau+v_\tau)\times G^\tau+\mathscr{V}^\tau\times B_\tau\right)\right\rangle \\
&\quad\le\frac{\varepsilon}{\tau^2}\|\mathscr{V}_\alpha^\tau\|^2+C_\varepsilon\left\|\partial^\alpha\left(F^\tau+(\mathscr{V}^\tau+v_\tau)\times G^\tau+\mathscr{V}^\tau\times B_\tau\right)\right\|^2+C_\varepsilon\|\zeta^\tau\|_s^4 \\
&\quad\le\frac{\varepsilon}{\tau^2}\|\mathscr{V}_\alpha^\tau\|^2+C_\varepsilon\left(\|V^\tau\|_s^2+\|V^\tau\|_s^4\right), \qquad (4.21)
\end{aligned}$$

and

$$\begin{aligned}
2\left|\left\langle \mathcal{A}_0^I(\rho^\tau,\theta^\tau)\partial^\alpha\mathcal{H}_4\left(V_I^\tau\right), V_{I\alpha}^\tau\right\rangle\right| &= \left|\left\langle\frac{\rho^\tau}{\theta^\tau}\partial^\alpha\left(\mathscr{V}^\tau\left(2v_\tau+\mathscr{V}^\tau\right)\right),\Theta_\alpha^\tau\right\rangle\right| \\
&\le C\left(\tau+\|\mathscr{V}^\tau\|_s\right)\|\mathscr{V}^\tau\|_s\|\Theta^\tau\|_s \\
&\le\frac{\varepsilon}{\tau^2}\|\mathscr{V}_\alpha^\tau\|^2+C_\varepsilon\left(\|V^\tau\|_s^2+\|V^\tau\|_s^4\right). \qquad (4.22)
\end{aligned}$$

Next, we consider the estimate for the term containing $\mathcal{J}_\alpha^\tau$. By the definition of $\mathcal{A}_i^I(\rho^\tau, v^\tau, \theta^\tau)$ and $\mathcal{A}_0^I(\rho^\tau,\theta^\tau)$, it follows

$$\left\langle \mathcal{A}_0^I(\rho^\tau,\theta^\tau)\left(\partial^\alpha\left(\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial_i V_I^\tau\right)-\mathcal{A}_i^\tau(\rho^\tau,v^\tau,\theta^\tau)\partial^\alpha\partial_i V_I^\tau\right), V_{I\alpha}^\tau\right\rangle \triangleq \sum_{j=1}^6\mathcal{J}_{ij},$$

where

$$\begin{aligned}
\mathcal{J}_{i1}&=\left\langle\frac{\theta^\tau}{\rho^\tau}\left(\partial^\alpha\left(\left(\mathscr{V}^\tau+v_\tau\right)_i\partial_i\zeta^\tau\right)-\left(\mathscr{V}^\tau+v_\tau\right)_i\partial^\alpha\partial_i N^\tau\right),\zeta_\alpha^\tau\right\rangle, \\
\mathcal{J}_{i2}&=\left\langle\frac{\theta^\tau}{\rho^\tau}\left(\partial^\alpha\left(\left(\zeta^\tau+\rho_\tau\right)e_i^t\partial_i\mathscr{V}^\tau\right)-\left(\zeta^\tau+\rho_\tau\right)e_i^t\partial^\alpha\partial_i\mathscr{V}^\tau\right),\zeta_\alpha^\tau\right\rangle, \\
\mathcal{J}_{i3}&=\left\langle\rho^\tau\left(\partial^\alpha\left(\frac{\Theta^\tau+\theta_\tau}{\zeta^\tau+\rho_\tau}\partial_i\zeta^\tau e_i\right)-\frac{\Theta^\tau+\theta_\tau}{\zeta^\tau+\rho_\tau}\partial^\alpha\partial_i\zeta^\tau e_i\right),\mathscr{V}_\alpha^\tau\right\rangle,
\end{aligned}$$

$$\mathcal{J}_{i4} = \left\langle \rho^\tau \left(\partial^\alpha \left(\left(\mathscr{V}^\tau + v_\tau \right)_i \partial_i \mathscr{V}^\tau \right) - \left(\mathscr{V}^\tau + v_\tau \right)_i \partial^\alpha \partial_i \mathscr{V}^\tau \right), \mathscr{V}_\alpha^\tau \right\rangle,$$
$$\mathcal{J}_{i5} = \left\langle \frac{\rho^\tau}{\theta^\tau} \left(\partial^\alpha \left(\left(\Theta^\tau + \theta_\tau \right) \partial_i \mathscr{V}^\tau \cdot e_i^t \right) - \left(\Theta^\tau + \theta_\tau \right) \partial^\alpha \partial_i \mathscr{V}^\tau \cdot e_i^t \right), \Theta_\alpha^\tau \right\rangle$$

and

$$\mathcal{J}_{i6} = \left\langle \frac{\rho^\tau}{\theta^\tau} \left(\partial^\alpha \left(\left(\mathscr{V}^\tau + v_\tau \right)_i \partial_i \Theta^\tau \right) - \left(\mathscr{V}^\tau + v_\tau \right)_i \partial^\alpha \partial_i \Theta^\tau \right), \Theta_\alpha^\tau \right\rangle.$$

It follows from (4.3) and Lemma 1.1 that

$$|\mathcal{J}_{i1}| + |\mathcal{J}_{i4}| + |\mathcal{J}_{i6}| \leq \left(\tau + \left\| \mathscr{V}^\tau \right\|_s \right) \left\| V_I^\tau \right\|_s^2 \leq \frac{\varepsilon}{\tau} \left\| \mathscr{V}_\alpha^\tau \right\|^2 + C_\varepsilon \tau \left(\left\| V^\tau \right\|_s^2 + \left\| V^\tau \right\|_s^4 \right),$$

and

$$\begin{aligned} |\mathcal{J}_{i2}| + |\mathcal{J}_{i3}| + |\mathcal{J}_{i5}| &\leq \left(1 + \left\| \zeta^\tau \right\|_s + \left\| \Theta^\tau \right\|_s \right) \left(\left\| \zeta^\tau \right\|_s + \left\| \Theta^\tau \right\|_s \right) \left\| \mathscr{V}^\tau \right\|_s \\ &\leq \frac{\varepsilon}{\tau} \left\| \mathscr{V}_\alpha^\tau \right\|^2 + C_\varepsilon \tau \left(\left\| V^\tau \right\|_s^2 + \left\| V^\tau \right\|_s^4 \right). \end{aligned}$$

Then, in view of the expression of $\mathcal{J}_\alpha^\tau$, we have

$$\frac{2}{\tau} \left\langle \mathcal{J}_\alpha^\tau, V_{I\alpha}^\tau \right\rangle \leq \frac{\varepsilon}{\tau^2} \left\| \mathscr{V}_\alpha^\tau \right\|^2 + C_\varepsilon \left(\left\| V^\tau \right\|_s^2 + \left\| V^\tau \right\|_s^4 \right). \tag{4.23}$$

Noticing the expression of $\mathcal{A}_0^I(\rho^\tau, \theta^\tau)$, we easily get

$$-2 \left\langle \mathcal{A}_0^I(\rho^\tau, \theta^\tau) \partial^\alpha \mathcal{R}^\tau, V_{I\alpha}^\tau \right\rangle = -2 \left\langle \frac{\theta^\tau}{\rho^\tau} \zeta_\alpha^\tau, \partial^\alpha R_\rho^\tau \right\rangle - 2 \left\langle \rho^\tau \mathscr{V}_\alpha^\tau, \partial^\alpha R_v^\tau \right\rangle - 3 \left\langle \frac{\rho^\tau}{\theta^\tau} \Theta_\alpha^\tau, \partial^\alpha R_\theta^\tau \right\rangle.$$

In view of (4.5), we obtain

$$-2 \left\langle \mathcal{A}_0^I(\rho^\tau, \theta^\tau) \partial^\alpha \mathcal{R}^\tau, V_{I\alpha}^\tau \right\rangle \leq C \left\| V^\tau \right\|_s^2 + \frac{\varepsilon}{\tau^2} \left\| \mathscr{V}^\tau \right\|_s^2 + C_\varepsilon \tau^{2(r+1)}. \tag{4.24}$$

In the end, for $i = 1, 2, 3$, it follows from (4.12) and (4.17) that

$$\begin{aligned} &\mathrm{div} \mathcal{A}_\tau^I(\rho^\tau, v^\tau, \theta^\tau) \\ &= \partial_{\rho^\tau} \mathcal{A}_0^\tau(\rho^\tau, \theta^\tau) \left(\partial_t \rho^\tau + \frac{1}{\tau} \nabla \rho^\tau \cdot v^\tau \right) + \frac{1}{\tau} \mathcal{A}_0^I \left(\rho^\tau, \theta^\tau \right) \mathrm{div} v^\tau \\ &\quad + \partial_{\theta^\tau} \mathcal{A}_0^\tau(\rho^\tau, \theta^\tau) \left(\partial_t \theta^\tau + \frac{1}{\tau} \nabla \theta^\tau \cdot v^\tau \right) + \frac{1}{\tau} \sum_{i=1}^3 \partial_i \left(\mathcal{Q}_i^I \left(\rho^\tau, \theta^\tau \right) \right) \\ &= \frac{\mathrm{div} u^\tau}{\tau} \left(\mathcal{A}_0^I \left(\rho^\tau, \theta^\tau \right) - \rho^\tau \partial_{\rho^\tau} \mathcal{A}_0^\tau(\rho^\tau, \theta^\tau) \right) + \frac{1}{\tau} \sum_{i=1}^3 \partial_i \left(\mathcal{Q}_i^I \left(\rho^\tau, \theta^\tau \right) \right) \\ &\quad + \partial_{\theta^\tau} \mathcal{A}_0^\tau(\rho^\tau, \theta^\tau) \left(\frac{1}{3} \left| \mathscr{V}^\tau + v^\tau \right|^2 - (\Theta^\tau + \theta_\tau) - \frac{2}{3\tau} \theta^\tau \mathrm{div} v^\tau \right) \end{aligned}$$

$$= \frac{\mathrm{div} v^\tau}{\tau}\left(\mathcal{A}_0^I\left(\rho^\tau,\theta^\tau\right) - \rho^\tau\partial_{\rho^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau) - \frac{2}{3}\theta^\tau\partial_{\theta^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau)\right)$$
$$+\partial_{\theta^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau)\left(\frac{1}{3}\left|\mathscr{V}^\tau + v^\tau\right|^2 - (\Theta^\tau+\theta_\tau)\right) + \frac{1}{\tau}\sum_{i=1}^3 \partial_i\left(\mathcal{Q}_i^I\left(\rho^\tau,\theta^\tau\right)\right). \tag{4.25}$$

In view of $\kappa \le \rho^\tau = \zeta^\tau + \rho_\tau, \theta^\tau = \Theta^\tau + \theta_\tau \le C_*, v^\tau = \mathscr{V}^\tau + v_\tau$ and $v_\tau = O(\tau)$, we get

$$\left\|\frac{\mathrm{div} v^\tau}{\tau}\left(\mathcal{A}_0^I\left(\rho^\tau,\theta^\tau\right) - \rho^\tau\partial_{\rho^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau) - \frac{2}{3}\theta^\tau\partial_{\theta^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau)\right)\right\|_{L^\infty}$$
$$\le C\left\|\frac{\mathrm{div} v^\tau}{\tau}\right\|_{L^\infty} \le C\left(1+\frac{1}{\tau}\|\mathscr{V}^\tau\|_s\right),$$

and

$$\left\|\partial_{\theta^\tau}\mathcal{A}_0^\tau(\rho^\tau,\theta^\tau)\left(\frac{1}{3}\left|\mathscr{V}^\tau + v^\tau\right|^2 - (\Theta^\tau+\theta_\tau)\right)\right\|_{L^\infty} \le C\left(1+\|\mathscr{V}^\tau\|_s^2\right).$$

On the other hand, by noticing the expression of $\mathcal{Q}_i^I(\rho^\tau,\theta^\tau)$, we obtain

$$\frac{1}{\tau}\sum_{i=1}^3\left\langle\partial_i\left(\mathcal{Q}_i^I\left(\rho^\tau,\theta^\tau\right)\right)V_{I\alpha}^\tau, V_{I\alpha}^\tau\right\rangle$$
$$= \frac{1}{\tau}\left\langle\nabla\theta^\tau\mathscr{V}_\alpha^\tau,\zeta_\alpha^\tau\right\rangle + \frac{1}{\tau}\left\langle\zeta_\alpha^\tau(\nabla\theta^\tau)^t + \Theta_\alpha^\tau(\nabla\rho^\tau)^t,\mathscr{V}_\alpha^\tau\right\rangle + \frac{1}{\tau}\left\langle\mathscr{V}_\alpha^\tau\nabla\rho^\tau,\Theta_\alpha^\tau\right\rangle$$
$$\le \frac{C}{\tau}\|\mathscr{V}^\tau\|_s\|\zeta^\tau\|_s\left(1+\|\Theta^\tau\|_s\right) + \frac{C}{\tau}\|\mathscr{V}^\tau\|_s\|\Theta^\tau\|_s\left(1+\|\zeta^\tau\|_s\right)$$
$$\le \frac{\varepsilon}{\tau^2}\|\mathscr{V}^\tau\|_s^2 + C_\varepsilon\left(\|V_I^\tau\|_s^2 + \|V_I^\tau\|_s^4\right).$$

Then, (4.25) together with the three estimates above yield

$$\left\langle\mathrm{div}\mathcal{A}_\tau^I(\rho^\tau,v^\tau,\theta^\tau)V_{I\alpha}^\tau, V_{I\alpha}^\tau\right\rangle$$
$$\le C\left(1+\frac{1}{\tau}\|\mathscr{V}^\tau\|_s + \|\mathscr{V}^\tau\|_s^2\right)\|V_{I\alpha}^\tau\|^2 + \frac{\varepsilon}{\tau^2}\|\mathscr{V}^\tau\|_s^2 + C_\varepsilon\left(\|V_I^\tau\|_s^2 + \|V_I^\tau\|_s^4\right)$$
$$\le \frac{\varepsilon}{\tau^2}\|\mathscr{V}^\tau\|_s^2 + C_\varepsilon\left(\|V_I^\tau\|_s^2 + \|V_I^\tau\|_s^4\right). \tag{4.26}$$

Thus, by combining (4.16), (4.18), (4.20)–(4.24) and (4.26), we get, for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\mathcal{A}_0^I(\rho^\tau,\theta^\tau)V_{I\alpha}^\tau, V_{I\alpha}^\tau\right\rangle + C^{-1}\left(\frac{1}{\tau^2}\|\mathscr{V}_\alpha^\tau\|^2 + \|\Theta_\alpha^\tau\|^2\right)$$

$$\leq \frac{\varepsilon}{\tau^2}\left\|\mathscr{V}^\tau\right\|_s^2+C_\varepsilon\left(\left\|V_I^\tau\right\|_s^2+\left\|V_I^\tau\right\|_s^4\right)+C_\varepsilon\tau^{2(r+1)}. \tag{4.27}$$

Integrating (4.27) over $(0, t)$ and taking summation over $|\alpha| \leq s$, choosing $\varepsilon > 0$ small enough such that the term containing $\frac{\varepsilon}{\tau^2}\left\|\mathscr{V}_\alpha^\tau\right\|^2$ can be bounded by the left-hand side. Therefore, (4.14) follows by combining the resulting inequality of (4.27), (4.4) and (4.19). We have finished the proof of Lemma 4.2. □

Proof of Theorems 1.1, 1.2 and 4.1 Once Theorem 4.1 has been proved, we easily find that Theorems 1.1 and 1.2 follow. In fact, $r = 2m + 1$ with $m \geq 0$ in Theorem 1.1 and $m = 1$ in Theorem 1.2, since

$$\left\|\left(\rho_{\tau,I}, \theta_{\tau,I}, E_{\tau,I}, B_{\tau,I}\right)(t)-\left(\rho^0, \tau^2\theta^0, E^0, \tau B^0\right)(t)\right\|_s=\mathcal{O}\left(\tau^2\right),$$

uniformly with respect to τ. Thus, the rest is to prove Theorem 4.1.

Let $\tau \to 0$ be small enough. It follows from Lemmas 4.1–4.2 that

$$\begin{aligned}&\left\|V^\tau(t)\right\|_s^2+\int_0^t\left(\frac{1}{2\tau^2}\left\|\mathscr{V}^\tau(\xi)\right\|_s^2+\left\|\Theta^\tau(\xi)\right\|_s^2\right)\mathrm{d}\xi\\&\quad\leq C\int_0^t\left(\left\|V^\tau(\xi)\right\|_s^2+\left\|V^\tau(\xi)\right\|_s^4\right)\mathrm{d}\xi+C\tau^{2(r+1)},\quad \forall t\in(0,T^\tau].\end{aligned} \tag{4.28}$$

Set

$$\Xi(t)=C\int_0^t\left(\left\|V^\tau(\xi)\right\|_s^2+\left\|V^\tau(\xi)\right\|_s^4\right)\mathrm{d}\xi+C\tau^{2(r+1)}.$$

Then from (4.28), we have

$$\left\|V^\tau(t)\right\|_s^2\leq\Xi(t),\quad \int_0^t\left(\frac{1}{2\tau^2}\left\|U^\tau(\xi)\right\|_s^2+\left\|\Theta^\tau(\xi)\right\|_s^2\right)\mathrm{d}\xi\leq\Xi(t),\quad\forall t\in(0,T^\tau], \tag{4.29}$$

and

$$\Xi'(t)=\left(\left\|V^\tau(t)\right\|_s^2+\left\|V^\tau(t)\right\|_s^4\right)\leq C\left(\Xi(t)+\Xi^2(t)\right)\quad\text{with}\quad\Xi(0)=C\tau^{2(r+1)}.$$

And then we get

$$\Xi(t)\leq C\tau^{2(r+1)}e^{Ct}\leq C\tau^{2(r+1)}e^{CT_1},\quad\forall t\in(0,T^\tau].$$

Hence, the inequality above together with (4.29) yields

$$\begin{aligned}\left\|V^\tau(t)\right\|_s&\leq\Xi^{\frac12}(t)\leq C\tau^{r+1},\\\int_0^t\left\|\Theta^\tau(\xi)\right\|_s^2\mathrm{d}\xi&\leq\Xi(t)\leq C\tau^{2(r+1)},\end{aligned}$$

$$\int_0^t \left\|U^\tau(\xi)\right\|_s^2 \mathrm{d}\xi \leq 2\tau^2 \Xi(t) \leq C\tau^{2(r+2)}, \quad \forall t \in (0, T^\tau].$$

In particular, we obtain that V^τ is bounded in $L^\infty(0, T^\tau; H^s)$, so is $(\rho^\tau, v^\tau, \theta^\tau, E^\tau, B^\tau)$. By the standard argument on the time extension of smooth solutions, we get $T_3^\tau \geq T_1$, which implies that $T^\tau = T_1$. The proof of Theorem 4.1 is completed. □

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Declarations

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References

Alì, G., Jüngel, A.: Global smooth solutions to the multi-dimensional hydrodynamic model for two-carrier plasmas. J. Differ. Equ. **190**, 663–685 (2003)

Besse, C., Claudel, J., Degond, P., Deluzet, F., Gallice, G., Tessieras, C.: A model hierarchy for ionospheric plasma modeling. Math. Models Methods Appl. Sci. **14**, 393–415 (2004)

Brenier, Y., Mauser, N., Puel, M.: Incompressible Euler and e-MHD as scaling limits of the Vlasov–Maxwell system. Commun. Math. Sci. **1**, 437–447 (2003)

Chen, F.: Introduction to Plasma Physics and Controlled Fusion, vol. 1. Plenum Press, New York (1984)

Chen, G.Q., Jerome, J.W., Wang, D.H.: Compressible Euler–Maxwell equations. Transp. Theory Stat. Phys. **29**, 311–331 (2000)

Deng, Y., Ionescu, A.D., Pausader, B.: The Euler–Maxwell system for electrons: global solutions in 2D. Arch. Ration. Mech. Anal. **225**, 771–871 (2017)

Duan, R.J.: Global smooth flows for the compressible Euler–Maxwell system: the relaxation case. J. Hyper. Differ. Equ. **8**, 375–413 (2011)

Dumas, E., Lu, Y., Texier, B.: Space-time resonances and high-frequency Raman instabilities in the two-fluid Euler–Maxwell system. Preprint (2021)

Feng, Y.H., Wang, S., Kawashima, S.: Global existence and asymptotic decay of solutions to the non-isentropic Euler–Maxwell system. Math. Models Methods Appl. Sci. **24**, 2851–2884 (2014)

Feng, Y.H., Li, X., Mei, M., Wang, S.: Asymptotic decay of bipolar isentropic/non-isentropic compressible Navier–Stokes–Maxwell systems. J. Differ. Equ. **301**, 471–542 (2021)

Germain, P., Masmoudi, N.: Global existence for the Euler–Maxwell system. Ann. Sci. Ec. Norm. Super. **47**(4), 469–503 (2014)

Guo, Y., Ionescu, A.D., Pausader, B.: Global solutions of the Euler–Maxwell two fluid system in 3D. Ann. Math. **183**(2), 377–498 (2016)

Hajjej, M.L., Peng, Y.J.: Initial layers and zero-relaxation limits of Euler–Maxwell equations. J. Differ. Equ. **252**, 1441–1465 (2012)

Jerome, J.W.: The Cauchy problem for compressible hydrodynamic-Maxwell systems: a local theory for smooth solutions. Differ. Integral Equ. **16**, 1345–1368 (2003)

Junca, S., Rascle: Relaxation of the isothermal Euler–Poisson system to the drift–diffusion equations. Q. Appl. Math. **58**, 511–521 (2000)

Jüngel, A.: Transport Equations for Semiconductors. Lecture Notes in Physics, vol. 773. Springer, Berlin (2009)

Jüngel, A., Peng, Y.J.: A hierarchy of hydrodynamic models for plasmas: zero-relaxation-time limits. Commun. Partial Differ. Equ. **24**, 1007–1033 (1999)

Kato, T.: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. **58**, 181–205 (1975)

Kawashima, S.: Systems of a hyperbolic-parabolic composite type with applications to the equations of magnetohydrodynamics. Doctoral thesis, Kyoto University (1984)

Klainerman, S., Majda, A.: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. Commun. Pure Appl. Math. **34**, 481–524 (1981)

Lattanzio, C.: On the 3-D bipolar isentropic Euler–Poisson model for semiconductors and the drift–diffusion limit. Math. Models Methods Appl. Sci. **10**, 351–360 (2000)

Li, Y.C., Peng, Y.J., Zhao, L.: Convergence rates in zero-relaxation limits for Euler–Maxwell and Euler–Poisson systems. J. Math. Pures Appl. **154**, 185–211 (2021)

Liu, Q.Q., Zhu, C.J.: Asymptotic stability of stationary solutions to the compressible Euler–Maxwell equations. Indiana Univ. Math. J. **62**, 1203–1235 (2013)

Luo, T., Wang, S., Wang, Y.L.: Initial layer and incompressible limit for Euler–Poisson equation in non-thermal plasma. Math. Models Methods Appl. Sci. **29**(9), 1733–1751 (2019)

Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Springer, New York (1984)

Marcati, P., Natalini, R.: Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift diffusion equations. Arch. Ration. Mech. Anal. **129**, 129–145 (1995)

Markowich, P., Ringhofer, C.A., Schmeiser, C.: Semiconductor Equations. Springer, New York (1990)

Peng, Y.J.: Stability of non-constant equilibrium solutions for Euler–Maxwell equations. J. Math. Pures Appl. **103**, 39–67 (2015)

Peng, Y.J., Wang, S.: Convergence of compressible Euler–Maxwell equations to compressible Euler–Poisson equations. Chin. Ann. Math. Ser. B **28**, 583–602 (2007)

Peng, Y.J., Wang, S.: Convergence of compressible Euler–Maxwell equations to incompressible Euler equations. Commun. Partial Differ. Equ. **33**, 349–376 (2008)

Peng, Y.J., Wang, S., Gu, G.L.: Relaxation limit and global existence of smooth solutions of compressible Euler–Maxwell equations. SIAM J. Math. Anal. **43**, 944–970 (2011)

Rishbeth, H., Garriott, O.K.: Introduction to Ionospheric Physics. Academic Press, Cambridge (1969)

Ueda, Y., Duan, R.J., Kawashima, S.: Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application. Arch. Ration. Mech. Anal. **205**, 239–266 (2012)

Ueda, Y., Wang, S., Kawashima, S.: Dissipative structure of the regularity type and time asymptotic decay of solutions for the Euler–Maxwell system. SIAM J. Math. Anal. **44**, 2002–2017 (2012)

Villani, C.: Hypocoercivity. Mem. Am. Math. Soc. **202** (2009)

Wang, W.K., Xu, X.: Global existence and decay of solution for the nonisentropic Euler-Maxwell system with a nonconstant background density. Z. Angew. Math. Phys. **67**, 55 (2016)

Wasiolek, V.: Uniform global existence and convergence of Euler–Maxwell systems with small parameters. Commun. Pure Appl. Anal. **15**, 2007–2021 (2016)

Wu, F.Z.: Initial layer and relaxation limit of non-isentropic compressible Euler equations with damping. J. Differ. Equ. **260**, 5103–5127 (2016)

Xu, J.: Global classical solutions to the compressible Euler–Maxwell equations. SIAM J. Math. Anal. **43**, 2688–2718 (2011)

Yang, Y.F., Hu, H.F.: Uniform global convergence of non-isentropic Euler–Maxwell system with dissipation. Nonlinear Anal. Real World **47**, 332–347 (2019)

Yang, J.W., Wang, S.: The diffusive relaxation limit of non-isentropic Euler–Maxwell equations for plasmas. J. Math. Anal. Appl. **380**, 343–353 (2011)

Yong, W.A.: Diffusive relaxation limit of multidimensional isentropic hydrodynamical models for semiconductors. SIAM J. Appl. Math. **64**, 1737–1748 (2004)