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Structural stability of subsonic steady states to the hydrodynamic model for semiconductors with sonic boundary

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Abstract

The hydrodynamic model for semiconductors with sonic boundary, represented by Euler–Poisson equations, possesses the various physical steady states including interior-subsonic/interior-supersonic/shock-transonic/C^1-smooth-transonic steady states. Since these physical steady states result in some serious singularities at the sonic boundary (their gradients are infinity), this makes that the structural stability for these physical solutions is more difficult and challenging, and has remained open as we know. In this paper, we investigate the structural stability of interior subsonic steady states. Namely, when the doping profiles are as small perturbations, the differences between the corresponding subsonic solutions are also small. To overcome the singularities at the sonic boundary, we propose a novel approach, which combines the weighted multiplier technique, local singularity analysis, monotonicity argument and squeezing skill. Both the result itself and the technique developed here will give us some truly enlightening insights into our follow-up study on the structural stability of the remaining types of solutions. A number of numerical

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approximations are also carried out, which intuitively confirm our theoretical results.

Keywords: Euler–Poisson equations, semiconductor effect, sonic boundary, interior subsonic solutions, structural stability

Mathematics Subject Classification numbers: 35B35, 35J70, 35L65, 35Q35

1. Introduction

The hydrodynamic model was first derived by Bløtekjær [2] for electrons in a semiconductor. After appropriate simplification the one-dimensional time-dependent system in the isentropic case reads:

$$\begin{cases} n_t + (nu)_x = 0, \\ (nu)_t + \left(nu^2 + p(n)\right)_x = nE - \dfrac{nu}{\tau}, \\ E_x = n - b(x), \end{cases} \tag{1.1}$$

where $n(x,t)$, $u(x,t)$ and $E(x,t)$ denote the electron density, velocity, and electric field respectively. The given function $p = p(n)$ is the pressure-density relation on which a commonly used hypothesis is

$$p(n) = Tn^{\gamma},$$

where $T > 0$ is Boltzmann's constant and $\gamma \geqslant 1$ is the adiabatic exponent. The constant parameter $\tau > 0$ is the momentum relaxation time. The given background density $b(x) > 0$ is called the doping profile standing for a background fixed charge of ions in the semiconductor crystal. The hydrodynamic model (1.1) is also called Euler–Poisson equations with semiconductor effect. For more details we refer to treatises [22, 33] and references therein.

The well-posedness of the physical solutions for the dynamical system (1.1) and their large-time behaviours are always one of significant and hot research spots in this topic as we know, see [10, 15–17, 23, 24, 27, 31] and the references therein. Meanwhile, the asymptotic profiles for the hydrodynamic model (1.1) are expected to be the steady states to the following stationary Euler–Poisson system

$$\begin{cases} J \equiv \text{constant}, \\ \left(\dfrac{J^2}{n} + p(n)\right)_x = nE - \dfrac{J}{\tau}, \\ E_x = n - b(x), \end{cases} \tag{1.2}$$

where $J = nu$ stands for the current density.

However, with different settings on the boundary, the doping profile $b(x)$ and the relaxation time τ, the stationary Euler–Poisson system (1.2) may or may not possess the physical subsonic/supersonic/transonic solutions, or may have totally different regularities [1, 4, 6, 8, 9, 13, 14, 18, 19]. The influence from these physical quantities, in particular, from the doping profile, is essential and important for the structure of solutions. The main purpose of the paper is to investigate the so-called structural stability of the solutions related to the doping profile in the critical sonic boundary case.

For simplicity, let us consider the isothermal case to (1.2) with $p(n) = Tn$. We first recall some terminologies from gas dynamics. We call $c := \sqrt{P'(n)} = \sqrt{T} > 0$ the speed of sound for the fluid dynamical system. The flow is referred to as subsonic, sonic or supersonic provided the velocity satisfies

$$u < c, \quad u = c \quad \text{or} \quad u > c, \quad \text{respectively.} \tag{1.3}$$

For convenience of notation, we introduce

$$\alpha = \frac{1}{\tau}, \text{ the reciprocal of the momentum relaxation time.} \tag{1.4}$$

Without loss of generality, we set

$$T = 1 \text{ and } J = 1, \tag{1.5}$$

thus the system (1.2) is equivalently reduced to the system

$$\begin{cases} \left(1 - \frac{1}{n^2}\right) n_x = nE - \alpha, \\ E_x = n - b(x). \end{cases} \tag{1.6}$$

From (1.3) and (1.5), it is easy to see that the flow is subsonic if $n > 1$, sonic if $n = 1$, or supersonic if $0 < n < 1$. By virtue of (1.4), we call the system (1.6) the Euler–Poisson equations with the semiconductor effect if $\alpha > 0$, and without the semiconductor effect if $\alpha = 0$, respectively. Throughout this paper, we are interested in the system (1.6) in the open interval $(0,1)$, which is subjected to the sonic boundary condition, the critical case of boundary:

$$n(0) = n(1) = 1. \tag{1.7}$$

We also assume that the doping profile $b(x)$ is of class $C[0,1]$, satisfying the subsonic condition $b(x) > 1$ on $[0,1]$. For simplicity of notation, its infimum and supremum over $[0,1]$ are denoted by

$$\underline{b} := \inf_{x\in[0,1]} b(x) \text{ and } \bar{b} := \sup_{x\in[0,1]} b(x).$$

Let us draw the background picture of research in this topic. Over the past three decades, major advances in the mathematical theory of steady-state Euler–Poisson equations with/without the semiconductor effect have been made by many authors. In what follows, we just list several results which are closely linked to the present paper.

For the purely subsonic steady-state flows, in 1990, Degond and Markowich [8] first proved the existence of the subsonic solution to the one-dimensional steady-state Euler–Poisson equations with the semiconductor effect when its boundary states belong to the subsonic region. Subsequently, Degond and Markowich [9] further showed the existence and local uniqueness of irrotational subsonic flows to the three-dimensional steady-state semiconductor hydrodynamic model under a smallness assumptions on the data. Along this line of research, the steady-state subsonic flows with and without the semiconductor effect were investigated in various physical boundary conditions and different dimensions [3, 11, 15, 26]. It is noteworthy that Donatelli *et al* [10] studied the two-fluid Euler–Poisson equations with semiconductor effect on the entire real line and constructed the unique purely subsonic steady state to the corresponding Cauchy problem, and they also proved that this purely subsonic steady state is the best asymptotic profile of the transient subsonic solutions. As for the purely supersonic steady-state flows, Peng and Violet [28] established the existence and uniqueness of the supersonic solutions with the semiconductor effect, which correspond to a large current density.

Regarding transonic steady states, Ascher *et al* [1] first examined the existence of the transonic solution to the one-dimensional isentropic Euler–Poisson equations without and with the semiconductor effect when the doping profile is a supersonic constant. Cordier *et al* [7] further analysed the travelling wave solutions to the two-fluid isothermal Euler–Poisson equations without the semiconductor effect. Along this research direction, Rosini [29] extended the work in [1, 7] to the non-isentropic case by the analysis of phase plane. When the doping profile is non-constant, Gamba [13, 14] investigated the one-dimensional and two-dimensional transonic solutions with shocks, respectively. However, these transonic solutions yield boundary layers because they are constructed as the limits of vanishing viscosity. Luo *et al* [20, 21] further considered the one-dimensional Euler–Poisson equations without the semiconductor effect, under the restriction that boundary data are far from the sonic state and the doping profile is either a subsonic constant or a supersonic constant, a comprehensive analysis on the structure and classification of steady states was carried out in [21] by using the analysis of phase plane. Meanwhile, both structural and dynamical stability of steady transonic shock solutions was obtained in [20].

What if the sonic state appears in the solutions? As we have seen, all the existing work introduced above cannot answer this question. Even the work regarding transonic shocks cannot radically answer it either because the two different phase states are connected by the jump of shocks satisfying the Rankine–Hugoniot condition and entropy condition, avoiding the degeneracy caused by the sonic state. So, it is significant to study the system with the sonic boundary. In fact, as we see, the system (1.2) or (1.6) will be degenerate at the sonic state, thus the study on the transonic solutions and various steady states satisfying the sonic boundary condition becomes very difficult. Recently, Li *et al* [18, 19] systematically explored this critical case of boundary, that is, the one-dimensional semiconductor Euler–Poisson equations with the sonic boundary condition. The existence, nonexistence and classification of all types of physical steady states to this critical boundary value problem was obtained for the subsonic doping profile in [18] and supersonic doping profile in [19]. More precisely, in [18], they proved that the critical boundary value problem admits a unique subsonic solution, at least one supersonic solution, infinitely many transonic shocks if $\alpha \ll 1$, and infinitely many transonic C^1-smooth solutions if $\alpha \gg 1$; in [19], they showed the nonexistence of all types of physical steady states to the critical boundary-value problem assuming that the doping profile is small enough and $\alpha \gg 1$, and they also discussed the existence of supersonic and transonic shock solutions under the hypothesis that the doping profile is close to the sonic state and $\alpha \ll 1$. Inspired by the groundbreaking works [18, 19], there is a series of interesting generalizations into the transonic doping profile case in [4], the case of transonic C^∞-smooth steady states in [32], the multi-dimensional cases in [5, 6], and even the bipolar case [25].

As showed in [4, 18, 19], the structure of the physical steady states are heavily dependent on the doping profile. When the doping profile $b(x)$ is subsonic, the system (1.2) has many physical solutions (one subsonic solution, one supersonic solution, infinitely many transonic solutions); and when the doping profile is supersonic, basically there is no any physical solutions. So, it is interesting to explore the structural stability of the solutions with respect to the doping profile. Namely, with a small perturbation of the doping profile, we expect the difference of the corresponding physical steady states to be also small, depending on the small perturbation of doping profiles. The study in this critical boundary case is never related due to its difficulty and singularity. Note that, when the boundary data are in different subsonic and supersonic regions separated by the sonic line, Luo *et al* [20] first studied the structural stability of shock transonic steady states, where the perturbed system around the shock transonic steady state does not yield the singularity at the sonic line, because the shock steady state jumps the sonic line. For the smooth C^1-transonic steady states cross the sonic line, there are some singularities for the system. Recently, Feng *et al* [12] demonstrated the structural stability of these smooth transonic steady states by the local singularity analysis. However, in the sonic boundary case, these physical (subsonic/supersonic/transonic) steady states produce the essential singularities at the sonic boundary, because their gradients at the sonic boundary are negative infinity. This makes the study of structural stability for these physical steady states more difficult. To thoroughly solve this problem is full of challenges, owing to the boundary degeneracy and singularity. In this paper, we focus on the subsonic steady states, and prove them to be structurally stable. In order to overcome the singularities of solutions at the sonic boundary, a new method is proposed by combining the weighted multiplier technique, local singularity analysis, monotonicity argument and squeezing skill. This intends to shed new light on this problem.

This paper is organized as follows. Some necessary preliminaries and the main result are stated in section 2. The proof of the main result, theorem 2.1, is given in section 3. Section 4 is devoted to the numerical simulations in order to better understand our theoretical results.

2. Preliminaries and the main result

In this section we shall present the main result. Before proceeding, we first give the important preliminaries from the foregoing research [18]. First of all, we recall the definition of the interior subsonic solution.

Definition 2.1. We say a pair of functions $(n,E)(x)$ is an interior subsonic solution of the boundary value problem (1.6) and (1.7) provided (i) $(n-1)^2 \in H_0^1(0,1)$, (ii) $n(x) > 1$, for all $x \in (0,1)$, (iii) $n(0) = n(1) = 1$, (iv) the following equality holds for all test functions $\varphi \in H_0^1(0,1)$,

$$\int_0^1 \left(\frac{1}{n} - \frac{1}{n^3}\right) n_x \varphi_x \mathrm{d}x + \alpha \int_0^1 \frac{\varphi_x}{n} \mathrm{d}x + \int_0^1 (n-b)\varphi \,\mathrm{d}x = 0, \tag{2.1}$$

and (v) $E(x)$ is given by

$$E(x) = E(0) + \int_0^x (n(y) - b(y))\,\mathrm{d}y. \tag{2.2}$$

In addition, we continue to recall the existence and uniqueness of interior subsonic solutions, which is excerpted from the first part of theorem 1.3 in [18].

Proposition 2.1 (existence [18]). *Suppose that the doping profile $b \in L^\infty(0,1)$ is subsonic such that $\underline{b} > 1$. Then for any $\alpha \in [0,\infty)$ the boundary value problem (1.6) and (1.7) admits a unique interior subsonic solution $(n,E) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ satisfying the boundedness*

$$1 + m\sin(\pi x) \leqslant n(x) \leqslant \bar{b}, \quad x \in [0,1]. \tag{2.3}$$

Furthermore, if $b \in C[0,1]$ *and* $\alpha \geqslant \sqrt{8(b(0)-1)}$*, then the boundary behaviour of solutions at endpoints is the following*

$$E(0) = \alpha, \quad E(1) < \alpha, \tag{2.4}$$

$$\begin{cases} C_1 (1-x)^{\frac{1}{2}} \leqslant n(x) - 1 \leqslant C_2 (1-x)^{\frac{1}{2}}, \\ -C_3 (1-x)^{-\frac{1}{2}} \leqslant n_x(x) \leqslant -C_4 (1-x)^{-\frac{1}{2}}, \end{cases} \quad \text{for } x \text{ near } 1, \tag{2.5}$$

where $m = m(\alpha, \underline{b}) > 0$*,* $C_2 > C_1 > 0$ *and* $C_3 > C_4 > 0$ *are certain uniform estimate constants.*

Remark 2.1. Note that the degeneracy of the boundary value problem (1.6) and (1.7) occurs merely on the boundary. Thus, if we assume that the doping profile has relatively higher-order regularity, say $b \in C[0,1]$, then by virtue of the standard theory for elliptic interior regularity and Sobolev's embedding theorem, the corresponding interior subsonic solution (n, E) is actually of class $\left(C^1(0,1) \cap C^{\frac{1}{2}}[0,1]\right) \times C^1[0,1]$. This fact will be tacitly exploited hereafter.

We are now in a position to formulate the main result in the present paper.

Theorem 2.1 (structural stability). *Assume that doping profiles* $b_1, b_2 \in C[0,1]$ *are subsonic such that* $b_1(x), b_2(x) > 1$ *for all* $x \in [0,1]$*, and that* $\alpha \geqslant 2\sqrt{2}\max\{\sqrt{b_1(0)-1}, \sqrt{b_2(0)-1}\}$*. Let* $(n_1, E_1)(x)$ *and* $(n_2, E_2)(x)$ *be interior subsonic solutions corresponding to their separate doping profiles* $b_1(x)$ *and* $b_2(x)$*. Then the two interior subsonic solutions are structurally stable to one another in the sense that*

$$\|n_1 - n_2\|_{C[0,1]} + \|(1-x)^{\frac{1}{2}} (n_1 - n_2)_x\|_{C[0,1]} + \|E_1 - E_2\|_{C^1[0,1]} \leqslant C\|b_1 - b_2\|_{C[0,1]}, \tag{2.6}$$

where $C > 0$ *is a certain constant independent of* $\|b_1 - b_2\|_{C[0,1]}$.

We conclude this section with a brief sketch of the strategy that underlies the proof of our main result. Due to the boundary degeneracy, the study of the globally structural stability of interior subsonic solutions over the entire interval $[0,1]$ becomes sophisticated and challenging. To overcome this difficulty, we first deal with the monotone case in which we assume that $b_1(x) \geqslant b_2(x)$ for all $x \in [0,1]$ (see lemma 3.8). Based on this building block, we can further remove the extra hypothesis that $b_1(x) \geqslant b_2(x)$ by the squeezing skill (see equations (3.64) and (3.65)).

In the monotone case, we shall have to engage with the boundary singularities of interior subsonic solutions. Therefore, a natural idea is to divide the whole interval $[0,1]$ into three domains as follows:

$$[0,1] = [0,\delta) \cup [\delta, 1-\delta] \cup (1-\delta, 1],$$

where the intrinsic segmentation constant $\delta > 0$ would be appropriately determined (see lemma 3.6), and we will also have to establish structural stability estimates separately on their respective domains in the following order: (i) near the left endpoint $x = 0$; (ii) near the right endpoint $x = 1$; (iii) on the middle domain. This strategy is feasible because we have discovered the following facts:

(1) the local singularity analysis reveals that the plausible singularity at the left endpoint $x = 0$ is removable (see lemma 3.2); based on this, we are able to establish the local structural stability estimate on an intrinsic neighbourhood $[0, \delta_0)$ by the monotonicity argument. The main point is that both the radius δ_0 and the positive estimate constant are independent of

$\|b_1 - b_2\|_{C[0,1]}$ (see lemma 3.3, and this sort of tacit convention will be used throughout the present paper).

(2) the local weighted singularity analysis discloses that the genuine singularity at the right endpoint $x = 1$ can be well controlled by the $(1-x)^{\frac{1}{2}}$-weight (see lemma 3.4); thus, the monotonicity argument further ensures that the local weighted structural stability holds on an intrinsic neighbourhood $(1-\delta_1, 1]$ (see lemma 3.5).

(3) the remaining part constitutes the middle domain, which is regular as to the structural stability (see lemma 3.6).

It is worth mentioning that the monotonicity argument has been playing a crucial role in establishing structural stability estimates near both endpoints. The principle behind the monotonicity argument is given by lemma 3.1. Moreover, lemma 3.1 is also capable of guaranteeing the validity of the squeezing skill, which can help us further to get rid of the additional assumption that $b_1(x) \geqslant b_2(x)$.

3. Proof of theorem 2.1

This section is devoted to proving our main result. In order to make the line of reasoning accessible to the reader, the proof will be divided into a sequence of lemmas.

We let $(n_i, E_i)(x)$ denote the interior subsonic solution corresponding to the subsonic doping profile $b_i(x) > 1$, satisfying the sonic boundary value problem

$$\begin{cases} \left(1 - \dfrac{1}{n_i^2}\right) n_{ix} = n_i E_i - \alpha, \\ E_{ix} = n_i - b_i(x), \quad x \in (0,1), \\ n_i(0) = n_i(1) = 1, \end{cases} \quad \text{for } i = 1,2, \text{ respectively.} \tag{3.1}$$

First of all, we adapt the comparison principle in [18](lemma 2.2, P4773) for use with two doping profiles and their corresponding interior subsonic solutions. This new version of comparison principle will be stated in the following lemma, which is the basis of both the monotonicity argument and the squeezing skill.

Lemma 3.1 (comparison principle). *Let the doping profiles* $b_1, b_2 \in C[0,1]$. *If* $b_1(x) \geqslant b_2(x) > 1$ *on* $[0,1]$. *Then*

$$n_1(x) \geqslant n_2(x), \ on \ [0,1]. \tag{3.2}$$

Proof. According to the relevant arguments from [18] (equation (17), P4773), for $i = 1,2$, since (n_i, E_i) is the interior subsonic solution, thereby having the approximate solution sequence $\{n_{ij}\}_{0<j<1} \subset C^1[0,1]$ satisfying the weak form

$$\int_0^1 A(n_{ij}, n_{ijx}) \varphi_x \mathrm{d}x + \int_0^1 (n_{ij} - b_i) \varphi \mathrm{d}x = 0, \quad \forall \varphi \in H_0^1(0,1), \tag{3.3}$$

where

$$A(z,p) := \left(\frac{1}{z} - \frac{j^2}{z^3}\right) p + \alpha \frac{j}{z}.$$

Subtracting (3.3)$|_{i=1}$ from (3.3)$|_{i=2}$, for all nonnegative test functions $\varphi \in H_0^1(0,1)$, we have

$$\int_0^1 \left(A\left(n_{2j}, n_{2jx}\right) - A\left(n_{1j}, n_{1jx}\right)\right) \varphi_x \mathrm{d}x + \int_0^1 \left(n_{2j} - n_{1j}\right) \varphi \,\mathrm{d}x = \int_0^1 \left(b_2 - b_1\right) \varphi \,\mathrm{d}x \leqslant 0, \qquad (3.4)$$

where we have used the assumption that $b_1(x) \geqslant b_2(x)$ on $[0,1]$ in the last inequality. This is exactly the crucial equation (19) in [18], the same result therefore applies to (3.4) provided we simply imitate the remaining arguments in lemma 2.2 of [18]. That is,

$$n_{1j}(x) \geqslant n_{2j}(x), \text{ on } [0,1], \text{ for } 0 < j < 1. \qquad (3.5)$$

Now the monotonicity relation (3.1) follows after a passage to the limit as $j \to 1^-$ on both sides of the inequality (3.5).

□

In order to get around the difficulty caused by singularities, we first investigate the monotone case. We assume, unless otherwise stated, that the doping profiles satisfy the monotonicity condition

$$b_1(x) \geqslant b_2(x), \quad \forall x \in [0,1]. \qquad (3.6)$$

For $i = 1,2$, we now set about analysing the boundary behaviour of the first-order derivative of $n_i(x)$ at the left endpoint $x = 0$. It seems plausible that the singularity should have appeared there, as a matter of fact the singularity at $x = 0$ is removable because of $E_i(0) = \alpha$.

Lemma 3.2. *Suppose that $b_i, i = 1,2$ satisfy the same conditions in lemma 3.1, and $\alpha \geqslant 2\sqrt{2}\max\{\sqrt{b_1(0)-1}, \sqrt{b_2(0)-1}\}$. Then*

$$\lim_{x\to 0^+} n_{ix}(x) = \frac{1}{4}\left(\alpha - \sqrt{\alpha^2 - 8\left(b_i(0) - 1\right)}\right) =: A_i > 0, \quad i = 1,2. \qquad (3.7)$$

Proof. In much the same way as in [18] (theorem 5.6, P4802), owing to $n_i(0) = 1$ and $E_i(0) = \alpha$, it is easy to see that $\lim_{x\to 0^+} n_{ix}(x)$ exists by the monotone convergence argument. Then from the first equation of (3.1), we have

$$n_{ix} = \frac{E_i n_i^2}{n_i + 1} + \frac{(E_i - \alpha) n_i^2}{(n_i - 1)(n_i + 1)}, \quad \text{in } (0,1).$$

Noting that $n_i(0) = 1$ and $E_i(0) = \alpha$, it follows from the L'Hospital Rule that

$$\begin{aligned} A_i &= \lim_{x\to 0^+} n_{ix}(x) = \lim_{x\to 0^+} \frac{E_i n_i^2}{n_i + 1} + \lim_{x\to 0^+} \frac{(E_i - \alpha) n_i^2}{(n_i - 1)(n_i + 1)} \\ &= \frac{\alpha}{2} + \frac{1}{2}\lim_{x\to 0^+} \frac{(E_i - \alpha)_x}{(n_i - 1)_x} = \frac{\alpha}{2} + \frac{1}{2}\lim_{x\to 0^+} \frac{E_{ix}}{n_{ix}} \\ &= \frac{\alpha}{2} + \frac{1}{2}\lim_{x\to 0^+} \frac{n_i(x) - b_i(x)}{n_{ix}} \\ &= \frac{\alpha}{2} + \frac{1 - b_i(0)}{2A_i}, \end{aligned}$$

which in turn implies that

$$A_i = \frac{1}{4}\left(\alpha - \sqrt{\alpha^2 - 8\,(b_i\,(0) - 1)}\right) = \frac{2\,(b_i\,(0) - 1)}{\alpha + \sqrt{\alpha^2 - 8\,(b_i\,(0) - 1)}} = O\left(\frac{1}{\alpha}\right),$$

or

$$A_i = \frac{1}{4}\left(\alpha + \sqrt{\alpha^2 - 8\,(b_i\,(0) - 1)}\right) = O\,(\alpha)\,.$$

According to the local singularity analysis in [18] (lemma 5.3, P4796), we know that in a small neighbourhood of $x = 0$, the drastic change of the density component $n_i(x)$ of the interior subsonic solution is impossible when α is suitably large. Therefore, we have to choose the former root as the limit value of $\lim_{x\to 0^+} n_{ix}(x)$, and the latter one is the extraneous root. □

Based on proposition 2.1, lemmas 3.1 and 3.2, we are now preparing to establish the local structural stability of interior subsonic solutions to the boundary value problem (1.6) and (1.7) on an intrinsic neighbourhood of the left endpoint $x = 0$.

Lemma 3.3 (local structural stability estimate near $\mathbf{x = 0}$). *Under the same conditions in lemma 3.2. There exist two positive constants $\delta_0 \in (0, \frac{1}{2})$ and $C > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that*

$$\|n_1 - n_2\|_{C^1[0,\delta_0)} + \|E_1 - E_2\|_{C^1[0,\delta_0)} \leqslant C\|b_1 - b_2\|_{C[0,1]}. \tag{3.8}$$

Proof. Firstly, in light of lemma 3.1, it is clear that the following monotonicity relation holds,

$$\frac{n_1^3}{n_1 + 1} \geqslant \frac{n_2^3}{n_2 + 1}, \quad \forall x \in [0, 1]\,. \tag{3.9}$$

Next, for simplicity, we set $\tilde{E}_i := E_i - \frac{\alpha}{n_i}$. Multiplying equation $(3.1)_1$ by $\frac{n_i^2}{n_i^2 - 1}$, we have

$$n_{ix} = \frac{\tilde{E}_i n_i^3}{n_i^2 - 1}, \quad i = 1, 2. \tag{3.10}$$

Taking the difference of equations $(3.10)|_{i=1}$ and $(3.10)|_{i=2}$, near $x = 0$, we compute together with the monotonicity relation (3.9) that

$$\begin{aligned}
(n_1 - n_2)_x &= \frac{\tilde{E}_1 n_1^3}{n_1^2 - 1} - \frac{\tilde{E}_2 n_2^3}{n_2^2 - 1} \\
&= \frac{n_1^3}{n_1 + 1}\frac{\tilde{E}_1}{n_1 - 1} - \frac{n_2^3}{n_2 + 1}\frac{\tilde{E}_1}{n_1 - 1} + \frac{n_2^3}{n_2 + 1}\frac{\tilde{E}_1}{n_1 - 1} - \frac{n_2^3}{n_2 + 1}\frac{\tilde{E}_2}{n_2 - 1} \\
&= \frac{\tilde{E}_1}{n_1 - 1}\left(\frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1}\right) + \frac{n_2^3}{n_2 + 1}\left(\frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1}\right) \\
&\leqslant M_0\alpha\left(\frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1}\right) + \frac{n_2^3}{n_2 + 1}M_0\|b_1 - b_2\|_{C[0,1]}, \\
&\leqslant C(n_1 - n_2) + C\|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0),
\end{aligned} \tag{3.11}$$

where we have used the fact that there exist two positive constants $\delta_0 \in (0, \frac{1}{2})$ and $M_0 > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

$$\frac{\tilde{E}_1}{n_1 - 1}(x) \leqslant M_0\alpha, \quad \text{and} \quad \left(\frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1}\right)(x) \leqslant M_0\|b_1 - b_2\|_{C[0,1]}, \quad x \in [0, \delta_0). \tag{3.12}$$

To prove that the crucial estimate (3.12) on a certain intrinsic neighbourhood $[0, \delta_0)$ holds, we assume for the sake of contradiction that for any $\delta \in (0, \frac{1}{2})$ and $M > 0$, there exists $x_\delta \in [0, \delta)$ such that

$$\frac{\tilde{E}_1}{n_1 - 1}(x_\delta) > M\alpha, \quad \text{or} \quad \left(\frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1}\right)(x_\delta) > M\|b_1 - b_2\|_{C[0,1]}. \tag{3.13}$$

Particularly, we take $\delta = \frac{1}{k}, k = 3, 4, 5, \ldots$, for any $M > 0$, there is $x_k \in [0, \frac{1}{k})$ such that

$$\frac{\tilde{E}_1}{n_1 - 1}(x_k) > M\alpha, \quad \text{or} \quad \left(\frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1}\right)(x_k) > M\|b_1 - b_2\|_{C[0,1]},$$

which implies that

$$\varliminf_{x_k \to 0^+} \frac{\tilde{E}_1}{n_1 - 1}(x_k) \geqslant M\alpha, \tag{3.14}$$

or

$$\varliminf_{x_k \to 0^+} \left(\frac{\tilde{E}_1}{n_1 - 1} - \frac{\tilde{E}_2}{n_2 - 1}\right)(x_k) \geqslant M\|b_1 - b_2\|_{C[0,1]}. \tag{3.15}$$

Combining the boundary behaviour (2.4), the L'Hospital Rule, equation $(3.1)_2$ and lemma 3.2, we calculate

$$\begin{aligned}
\lim_{x \to 0^+} \frac{\tilde{E}_i}{n_i - 1}(x) &= \lim_{x \to 0^+} \frac{E_i - \frac{\alpha}{n_i}}{n_i - 1}(x) \\
&= \lim_{x \to 0^+} \frac{E_i(x) - E_i(0) + \alpha - \frac{\alpha}{n_i(x)}}{n_i(x) - 1} \\
&= \lim_{x \to 0^+} \frac{E_i(x) - E_i(0)}{n_i(x) - 1} + \lim_{x \to 0^+} \frac{\alpha}{n_i(x)} \\
&= \lim_{x \to 0^+} \frac{n_i - b_i}{n_{ix}} + \alpha \\
&= \frac{1 - b_i(0)}{A_i} + \alpha < \alpha, \quad i = 1, 2,
\end{aligned} \tag{3.16}$$

and

$$\lim_{x\to 0+}\left(\frac{\tilde{E}_1}{n_1-1}-\frac{\tilde{E}_2}{n_2-1}\right)(x)=\frac{b_2(0)-1}{A_2}-\frac{b_1(0)-1}{A_1}$$
$$=\frac{1}{2}\left(\sqrt{\alpha^2-8(b_2(0)-1)}-\sqrt{\alpha^2-8(b_1(0)-1)}\right)$$
$$=\frac{2}{\sqrt{\alpha^2-8(\eta-1)}}(b_1(0)-b_2(0))$$
$$\leqslant \tilde{C}_0\|b_1-b_2\|_{C[0,1]}, \tag{3.17}$$

where $\eta\in(b_2(0),b_1(0))$. Furthermore, we note that the constant M in (3.14) and (3.15) can be chosen arbitrarily. Consequently, if we take $M=2$ in (3.14), together with (3.16), we obtain the contradiction that $2\alpha<\alpha$; if we take $M=2\tilde{C}_0$ in (3.15), combined with (3.17), we have the contradiction $2\leqslant 1$.

Based on the local estimate (3.11), we continue establishing the structural stability locally on the intrinsic neighbourhood $[0,\delta_0)$. To this end, we multiply through the inequality (3.11) by n_1-n_2 and calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}(n_1-n_2)^2(x)\leqslant C(n_1-n_2)^2(x)+C\|b_1-b_2\|^2_{C[0,1]},\quad x\in[0,\delta_0), \tag{3.18}$$

where we have used lemma 3.1 and Cauchy's inequality. By Gronwall's inequality and the sonic boundary condition $n_1(0)=n_2(0)=1$, we get

$$(n_1-n_2)^2(x)\leqslant C\|b_1-b_2\|^2_{C[0,1]},\quad x\in[0,\delta_0), \tag{3.19}$$

which in turn implies that

$$|n_1-n_2|(x)+|(n_1-n_2)_x|(x)\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in[0,\delta_0). \tag{3.20}$$

with the aid of the foregoing local estimate (3.11) again.

Finally, from equation (2.2) in definition 2.1 and the boundary behaviour (2.4), we have

$$E_i(x)=\alpha+\int_0^x(n_i(y)-b_i(y))\,\mathrm{d}y,\quad i=1,2. \tag{3.21}$$

Taking the difference of $(3.21)|_{i=1}$ and $(3.21)|_{i=2}$, we compute that

$$|E_1-E_2|(x)\leqslant\int_0^x|n_1-n_2|(y)\mathrm{d}y+\int_0^x|b_1-b_2|(y)|\mathrm{d}y$$
$$\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in[0,\delta_0), \tag{3.22}$$

and

$$|(E_1-E_2)_x|(x)=|n_1-n_2-(b_1-b_2)|(x)$$
$$\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in[0,\delta_0). \tag{3.23}$$

Hence, the local structural stability estimate (3.8) follows immediately from equations (3.20), (3.22) and (3.23). □

We now turn to analysing the refined boundary behaviour of the first-order derivative of $n_i(x)$ at the right endpoint $x = 1$. From the boundary estimate displayed in the second line of (2.5), we know that $\lim_{x\to 1^-} n_{ix}(x) = -\infty$. This means the genuine singularity will occur at the right endpoint $x = 1$. Inspired by (2.5), we are able to implement the local weighted singularity analysis. The result is summarized as follows.

Lemma 3.4. *Assume that $b_i, i = 1,2$ satisfy the same conditions in lemma 3.1. Then*

$$\lim_{x\to 1^-} (1-x)^{\frac{1}{2}} n_{ix}(x) = -\frac{1}{2}\sqrt{\int_0^1 (b_i - n_i)\,\mathrm{d}x} =: B_i < 0, \quad i = 1,2. \tag{3.24}$$

Proof. For $i = 1,2$, from the boundary estimate (2.5), we know that the coefficient $1-\frac{1}{n_i^2}$ in the degenerate principal part of equation $(3.1)_1$ is comparable to $(1-x)^{\frac{1}{2}}$ near the right endpoint $x = 1$. Thus the regularity theory of boundary-degenerate elliptic equations in one dimension (e.g. [30]) ensures that $(1-x)^{\frac{1}{2}} n_{ix}(x)$ is continuous up to the right endpoint $x = 1$.

We now proceed to calculate the exact limit value of $\lim_{x\to 1-} (1-x)^{\frac{1}{2}} n_{ix}(x)$. For convenience, we set

$$B_i := \lim_{x\to 1-} (1-x)^{\frac{1}{2}} n_{ix}(x).$$

Thereupon, multiplying through equation $(3.1)_1$ by $(1-x)^{\frac{1}{2}} \frac{n_i^2}{n_i^2-1}$, we have

$$(1-x)^{\frac{1}{2}} n_{ix} = \frac{n_i^3}{n_i+1}\left(E_i - \frac{\alpha}{n_i}\right)\frac{(1-x)^{\frac{1}{2}}}{n_i - 1}.$$

By virtue of the sonic boundary condition $n_i(1) = 1$, the known boundary behaviour (2.4) and the L'Hospital Rule, we compute

$$\begin{aligned} B_i &= \lim_{x\to 1^-} (1-x)^{\frac{1}{2}} n_{ix} \\ &= \lim_{x\to 1^-} \frac{n_i^3}{n_i+1} \lim_{x\to 1^-}\left(E_i - \frac{\alpha}{n_i}\right) \lim_{x\to 1^-} \frac{(1-x)^{\frac{1}{2}}}{n_i - 1} \\ &= \frac{1}{2}(E_i(1) - \alpha) \lim_{x\to 1^-} \frac{-\frac{1}{2}(1-x)^{-\frac{1}{2}}}{n_{ix}} \\ &= \frac{1}{4}(\alpha - E_i(1)) \lim_{x\to 1^-} \frac{1}{n_{ix}(1-x)^{\frac{1}{2}}} \\ &= \frac{1}{4}(E_i(0) - E_i(1))\frac{1}{B_i}, \end{aligned} \tag{3.25}$$

which implies from equation $(3.1)_2$ that

$$B_i = -\frac{1}{2}\sqrt{E_i(0) - E_i(1)} = -\frac{1}{2}\sqrt{\int_0^1 (b_i - n_i)\,\mathrm{d}x} < 0,$$

where the boundary estimate $(2.5)_2$ has been employed to uniquely determine the value of B_i, which is strictly negative. □

Proposition 2.1 alongside lemmas 3.1 and 3.4 now enable us to demonstrate the local weighted structural stability of interior subsonic solutions to the boundary value problem (1.6) and (1.7) on an intrinsic neighbourhood of the right endpoint $x = 1$.

Lemma 3.5 (local weighted structural stability estimate near $\mathbf{x} = 1$). *Under the same conditions in lemma 3.2. There exist two positive constants $\delta_1 \in (0, \frac{1}{2})$ and $C > 0$ independent of $\|b_1 - b_2\|_{C[0,1]}$ such that*

$$\left\|(1-x)^{-\frac{1}{2}}(n_1 - n_2)\right\|_{C(1-\delta_1,1]} + \left\|(1-x)^{\frac{1}{2}}(n_{1x} - n_{2x})\right\|_{C(1-\delta_1,1]} + \|E_1 - E_2\|_{C^1(1-\delta_1,1]} \leqslant C\|b_1 - b_2\|_{C[0,1]}. \tag{3.26}$$

Proof. For $i = 1, 2$, from $(2.5)_1$, we have known that $\frac{n_i - 1}{(1-x)^{1/2}}$ possesses the uniform positive upper and lower bounds near $x = 1$, and so does its reciprocal $\frac{(1-x)^{1/2}}{n_i - 1}$. This property will be used repeatedly hereafter.

Owing to the fact that $n_{ix}(x)$ has the genuine singularity at $x = 1$, we are compelled to establish the structural stability estimate near $x = 1$ only in the weighted manner as follows.

Firstly, multiplying through equation $(3.1)_1$ by $(1-x)^{\frac{1}{2}} \frac{n_i^2}{n_i^2 - 1}$ and taking the difference of resultant equations for $i = 1, 2$, we calculate that

$$\begin{aligned}
&(1-x)^{\frac{1}{2}}(n_{1x} - n_{2x}) \\
&= \frac{n_1^2}{n_1+1}(n_1E_1 - \alpha)\frac{(1-x)^{\frac{1}{2}}}{n_1 - 1} - \frac{n_2^2}{n_2+1}(n_2E_2 - \alpha)\frac{(1-x)^{\frac{1}{2}}}{n_2 - 1} \\
&= \frac{n_1^2}{n_1+1}(n_1E_1 - \alpha)\left(\frac{(1-x)^{\frac{1}{2}}}{n_1 - 1} - \frac{(1-x)^{\frac{1}{2}}}{n_2 - 1}\right) \\
&\quad + \left(\frac{n_1^2}{n_1+1}(n_1E_1 - \alpha) - \frac{n_2^2}{n_2+1}(n_2E_2 - \alpha)\right)\frac{(1-x)^{\frac{1}{2}}}{n_2 - 1} \\
&= h(n_1, E_1)\frac{(1-x)^{\frac{1}{2}}}{n_1 - 1}\frac{(1-x)^{\frac{1}{2}}}{n_2 - 1}\frac{n_2 - n_1}{(1-x)^{\frac{1}{2}}} + (h(n_1, E_1) - h(n_2, E_2))\frac{(1-x)^{\frac{1}{2}}}{n_2 - 1} \\
&=: I_1 + I_2,
\end{aligned} \tag{3.27}$$

where

$$h(n_i, E_i) := \frac{n_i^2}{n_i + 1}(n_iE_i - \alpha), \quad i = 1, 2.$$

In what follows, near $x = 1$, we shall estimate I_1 and I_2, respectively. But first, we claim that the following estimates

$$|E_i(x)| \leqslant \alpha + 2\bar{b}_i, \quad x \in [0, 1], \quad i = 1, 2, \tag{3.28}$$

$$|E_1(1) - E_2(1)| \leqslant C\|b_1 - b_2\|_{C[0,1]} \tag{3.29}$$

hold, where the estimate constant $C > 0$ is independent of $\|b_1 - b_2\|_{C[0,1]}$, and the proof of which is deferred to lemma 3.7 at the end of this paper.

As for I_1, it is clear from (2.5)$_1$ and (3.28) that

$$|I_1| = \left| h(n_1, E_1) \frac{(1-x)^{\frac{1}{2}}}{n_1 - 1} \frac{(1-x)^{\frac{1}{2}}}{n_2 - 1} \frac{n_2 - n_1}{(1-x)^{\frac{1}{2}}} \right| \leqslant C \frac{|n_1 - n_2|}{(1-x)^{\frac{1}{2}}}. \tag{3.30}$$

However, as far as I_2 is concerned, the situation becomes more complicated because of the factor $h(n_1, E_1) - h(n_2, E_2)$. Next, we are taking it step by step. Precisely, a straightforward computation gives

$$\begin{aligned} h(n_1, E_1) - h(n_2, E_2) &= \frac{n_1^2}{n_1 + 1}(n_1 E_1 - \alpha) - \frac{n_2^2}{n_2 + 1}(n_2 E_2 - \alpha) \\ &= \alpha \left(\frac{n_2^2}{n_2 + 1} - \frac{n_1^2}{n_1 + 1} \right) + \left(\frac{n_1^3 E_1}{n_1 + 1} - \frac{n_2^3 E_2}{n_2 + 1} \right) \\ &=: R_1 + R_2. \end{aligned} \tag{3.31}$$

From (2.3) and lemma 3.1, we know that

$$1 \leqslant 1 + m(\alpha, \underline{b}_2) \sin(\pi x) \leqslant n_2(x) \leqslant n_1(x) \leqslant \overline{b}_1, \quad x \in [0, 1]. \tag{3.32}$$

Consequently, it follows from the mean-value theorem of differentials that

$$|R_1| = \left| \alpha \left(\frac{n_2^2}{n_2 + 1} - \frac{n_1^2}{n_1 + 1} \right) \right| \leqslant C|n_1 - n_2| \leqslant C \frac{|n_1 - n_2|(x)}{(1-x)^{\frac{1}{2}}}. \tag{3.33}$$

We now turn to estimating R_2 near $x = 1$. Combining (3.32), (3.28) and (3.29), the mean-value theorem of differentials, and the mean-value theorem of integrals, we have

$$\begin{aligned} |R_2| &= \left| \frac{n_1^3 E_1}{n_1 + 1} - \frac{n_2^3 E_2}{n_2 + 1} \right| \\ &= \left| E_1 \left(\frac{n_1^3}{n_1 + 1} - \frac{n_2^3}{n_2 + 1} \right) + \frac{n_2^3}{n_2 + 1}(E_1 - E_2) \right| \\ &\leqslant C|n_1 - n_2|(x) + C \left| \left[(E_1(1) - E_2(1)) - \left(\int_x^1 (n_1 - n_2) - (b_1 - b_2) \, dy \right) \right] \right| \\ &\leqslant C|n_1 - n_2|(x) + C|E_1(1) - E_2(1)| + C \int_x^1 |n_1 - n_2| dy + C\|b_1 - b_2\|_{C[0,1]} \\ &\leqslant C\|b_1 - b_2\|_{C[0,1]} + C \left(\frac{|n_1 - n_2|(x)}{(1-x)^{\frac{1}{2}}} + \frac{|n_1 - n_2|(\xi)}{(1-\xi)^{\frac{1}{2}}} \right), \quad \exists \xi \in [x, 1], \end{aligned} \tag{3.34}$$

where we have used the formula

$$E_i(x) = E_i(1) - \int_x^1 (n_i - b_i)(y) \, dy, \quad i = 1, 2. \tag{3.35}$$

Substituting (3.33) and (3.34) into (3.31), we have

$$|h(n_1,E_1)-h(n_2,E_2)|\leqslant C\|b_1-b_2\|_{C[0,1]}+C\left(\frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}}+\frac{|n_1-n_2|(\xi)}{(1-\xi)^{\frac{1}{2}}}\right),\quad \exists\xi\in[x,1],$$

which further implies that

$$\begin{aligned}|I_2|&=\left|(h(n_1,E_1)-h(n_2,E_2))\frac{(1-x)^{\frac{1}{2}}}{n_2-1}\right|\\&\leqslant C\|b_1-b_2\|_{C[0,1]}+C\left(\frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}}+\frac{|n_1-n_2|(\xi)}{(1-\xi)^{\frac{1}{2}}}\right),\quad \exists\xi\in[x,1].\end{aligned}\tag{3.36}$$

Inserting (3.30) and (3.36) into (3.27), near $x=1$, we obtain

$$\begin{aligned}&(1-x)^{\frac{1}{2}}|(n_1-n_2)_x|(x)\\&\quad\leqslant C\left(\frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}}+\frac{|n_1-n_2|(\xi)}{(1-\xi)^{\frac{1}{2}}}\right)+C\|b_1-b_2\|_{C[0,1]},\quad \exists\xi\in[x,1].\end{aligned}\tag{3.37}$$

It is worth mentioning that the generic constant $C>0$ in (3.37) is independent of $\|b_1-b_2\|_{C[0,1]}$. Moreover, the term $\frac{|n_1-n_2|(x)}{(1-x)^{1/2}}+\frac{|n_1-n_2|(\xi)}{(1-\xi)^{1/2}}$ on the right-hand side of (3.37) can be bounded by an appropriate constant multiple of $\|b_1-b_2\|_{C[0,1]}$ in an intrinsic neighbourhood of the right endpoint $x=1$. Precisely, we claim that there exist two positive constants $0<\delta_1<\frac{1}{2}$ and $M_1>0$ independent of $\|b_1-b_2\|_{C[0,1]}$ such that

$$\frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}}\leqslant M_1\|b_1-b_2\|_{C[0,1]},\quad x\in(1-\delta_1,1].\tag{3.38}$$

Aiming for a contradiction, suppose that for any $\delta\in(0,\frac{1}{2})$ and $M>0$, there is $x_\delta\in(1-\delta,1]$ such that

$$\frac{|n_1-n_2|(x_\delta)}{(1-x_\delta)^{\frac{1}{2}}}>M\|b_1-b_2\|_{C[0,1]}.\tag{3.39}$$

By the arbitrariness, we could take $\delta=\frac{1}{k},k=3,4,5,\cdots$, for arbitrary $M>0$, there exists $x_k\in(1-\frac{1}{k},1]$ such that

$$\frac{|n_1-n_2|(x_k)}{(1-x_k)^{\frac{1}{2}}}>M\|b_1-b_2\|_{C[0,1]},\tag{3.40}$$

which implies that

$$\varliminf_{x_k\to1^-}\frac{|n_1-n_2|(x_k)}{(1-x_k)^{\frac{1}{2}}}\geqslant M\|b_1-b_2\|_{C[0,1]}.\tag{3.41}$$

Besides, combining lemma 3.1, the L'Hospital Rule and lemma 3.4, we calculate that

$$\begin{aligned}
\lim_{x\to 1^-}\frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}} &= \lim_{x\to 1^-}\frac{(n_1-n_2)(x)}{(1-x)^{\frac{1}{2}}} = \lim_{x\to 1^-}\frac{(n_1-1)(x)}{(1-x)^{\frac{1}{2}}} - \lim_{x\to 1^-}\frac{(n_2-1)(x)}{(1-x)^{\frac{1}{2}}} \\
&= -2\lim_{x\to 1^-}(1-x)^{\frac{1}{2}}n_{1x} + 2\lim_{x\to 1^-}(1-x)^{\frac{1}{2}}n_{2x} \\
&= \sqrt{\int_0^1 (b_1-n_1)\,\mathrm{d}x} - \sqrt{\int_0^1 (b_2-n_2)\,\mathrm{d}x} \\
&= \frac{\int_0^1 (b_1-b_2)\,\mathrm{d}x - \int_0^1 (n_1-n_2)\,\mathrm{d}x}{\sqrt{\int_0^1 (b_1-n_1)\,\mathrm{d}x} + \sqrt{\int_0^1 (b_2-n_2)\,\mathrm{d}x}} \\
&\leqslant \frac{\int_0^1 (b_1-b_2)\,\mathrm{d}x}{\sqrt{\int_0^1 (b_1-n_1)\,\mathrm{d}x} + \sqrt{\int_0^1 (b_2-n_2)\,\mathrm{d}x}} \\
&\leqslant \tilde{C}_1\|b_1-b_2\|_{C[0,1]}. \qquad (3.42)
\end{aligned}$$

Moreover, we note that the constant $M>0$ in (3.41) is arbitrary. Therefore, together with (3.42), taking $M=2\tilde{C}_1$ in (3.41) leads to the contradiction that $2\leqslant 1$.

Applying (3.38) to (3.37), we have

$$(1-x)^{\frac{1}{2}}|(n_1-n_2)_x|(x)\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in(1-\delta_1,1]. \qquad (3.43)$$

Similarly to (3.34), we are able to compute that

$$\begin{aligned}
|E_1-E_2|(x) &\leqslant C\|b_1-b_2\|_{C[0,1]} + \frac{|n_1-n_2|(\xi)}{(1-\xi)^{\frac{1}{2}}},\quad \exists\xi\in[x,1] \\
&\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in(1-\delta_1,1], \qquad (3.44)
\end{aligned}$$

and

$$\begin{aligned}
|(E_1-E_2)_x|(x) &= |n_1-n_2-(b_1-b_2)|(x)\leqslant |n_1-n_2|(x)+|b_1-b_2|(x) \\
&\leqslant \frac{|n_1-n_2|(x)}{(1-x)^{\frac{1}{2}}} + \|b_1-b_2\|_{C[0,1]} \\
&\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in(1-\delta_1,1]. \qquad (3.45)
\end{aligned}$$

Finally, putting results (3.38), (3.43), (3.44) and (3.45) together, we obtain the desired local weighted estimate (3.26). □

Up to now, we have obtained two intrinsic small domains $[0,\delta_0)$ and $(1-\delta_1,1]$ distributed around the two endpoints $x=0$ and $x=1$, respectively. This fact enables us to establish the structural stability estimate on a certain regular domain $[\delta,1-\delta]$, where $0<\delta:=\min\{\delta_0,\delta_1\}<1/2$.

Lemma 3.6. *Under the same conditions in lemma 3.2. Let $\delta:=\min\{\delta_0,\delta_1\}$. Then there is a positive constant $C>0$ independent of $\|b_1-b_2\|_{C[0,1]}$ such that*

$$\|n_1-n_2\|_{C^1[\delta,1-\delta]} + \|E_1-E_2\|_{C^1[\delta,1-\delta]} \leqslant C\|b_1-b_2\|_{C[0,1]}. \qquad (3.46)$$

Proof. We are now able to work with equations (3.1) on a regular closed interval $[\delta, 1-\delta]$ away from singularities, where δ has been defined in the hypothesis of the present lemma.

Firstly, we rewrite the estimate (3.32) on the regular interval as follows.

$$1 < l := 1 + m(\alpha, \underline{b}_2)\sin(\pi\delta) \leqslant n_2(x) \leqslant n_1(x) \leqslant \bar{b}_1, \quad x \in [\delta, 1-\delta]. \tag{3.47}$$

Secondly, subtracting $(3.1)|_{i=2}$ from $(3.1)|_{i=1}$, for $x \in [\delta, 1-\delta]$, we thus get

$$\begin{aligned}(n_1 - n_2)_x &= \frac{n_1^3 E_1 - \alpha n_1^2}{n_1^2 - 1} - \frac{n_2^3 E_2 - \alpha n_2^2}{n_2^2 - 1} \\ &= E_1\left(f(n_1) - f(n_2)\right) + f(n_2)(E_1 - E_2) - \alpha\left(g(n_1) - g(n_2)\right) \\ &= \left(E_1 f'(\bar{\eta}) - \alpha g'(\tilde{\eta})\right)(n_1 - n_2) + f(n_2)(E_1 - E_2), \quad \exists \bar{\eta}, \tilde{\eta} \in (n_2, n_1),\end{aligned} \tag{3.48}$$

and

$$(E_1 - E_2)_x = (n_1 - n_2) - (b_1 - b_2), \tag{3.49}$$

where

$$f(n) := \frac{n^3}{n^2 - 1}, \quad g(n) := \frac{n^2}{n^2 - 1}, \quad \forall n \in \left[l, \bar{b}_1\right],$$

and we have used the mean-value theorem of differentials in the third line of equation (3.48).

Thirdly, multiplying through (3.48) by $n_1 - n_2$, and using (3.28), (3.47) and Cauchy's inequality together, we have

$$\left((n_1 - n_2)^2\right)_x \leqslant C\left(\alpha, l, \bar{b}_1\right)\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right), \quad x \in [\delta, 1-\delta]. \tag{3.50}$$

Similarly, multiplying through (3.49) by $E_1 - E_2$, and employing Cauchy's inequality, we obtain

$$\left((E_1 - E_2)^2\right)_x \leqslant (n_1 - n_2)^2 + 2(E_1 - E_2)^2 + \|b_1 - b_2\|^2_{C[0,1]}, \quad x \in [\delta, 1-\delta]. \tag{3.51}$$

And then, summing estimates (3.50) and (3.51) gives

$$\begin{aligned}&\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)_x(x) \\ &\quad \leqslant C\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)(x) + \|b_1 - b_2\|^2_{C[0,1]}, \quad x \in [\delta, 1-\delta].\end{aligned} \tag{3.52}$$

Applying the Gronwall inequality to (3.52), we have

$$\begin{aligned}&\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)(x) \\ &\quad \leqslant e^{\int_\delta^x C \mathrm{d}y}\left[\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)(\delta) + \int_\delta^x \|b_1 - b_2\|^2_{C[0,1]} \mathrm{d}y\right] \\ &\quad \leqslant C\left[\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)(\delta) + \|b_1 - b_2\|^2_{C[0,1]}\right], \quad x \in [\delta, 1-\delta].\end{aligned} \tag{3.53}$$

Noting that $\delta \leqslant \delta_0$ and the continuity of the error function pair $(n_1 - n_2, E_1 - E_2)(x)$ at $x = \delta_0$, from lemma 3.3 we see

$$\left((n_1 - n_2)^2 + (E_1 - E_2)^2\right)(\delta) \leqslant C\|b_1 - b_2\|^2_{C[0,1]}, \tag{3.54}$$

which along with (3.53) implies

$$|n_1-n_2|(x)+|E_1-E_2|(x)\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in[\delta,1-\delta]. \tag{3.55}$$

Finally, from equations (3.48) and (3.49), and the estimate (3.55), we directly calculate

$$|(n_1-n_2)_x|(x)+|(E_1-E_2)_x|(x)\leqslant C\|b_1-b_2\|_{C[0,1]},\quad x\in[\delta,1-\delta]. \tag{3.56}$$

Combining estimates (3.55) and (3.56) yields the desired structural stability estimate (3.46) on the regular domain $[\delta,1-\delta]$. □

Last but not least, let us prove the estimates (3.28) and (3.29) in the following lemma.

Lemma 3.7. *Under the same conditions in lemma 3.5. Then there exists a positive constant C independent of $\|b_1-b_2\|_{C[0,1]}$ such that estimates (3.28) and (3.29) hold, that is,*

$$|E_i(x)|\leqslant\alpha+2\bar{b}_i,\quad x\in[0,1],\quad i=1,2,$$

and

$$|E_1(1)-E_2(1)|\leqslant C\|b_1-b_2\|_{C[0,1]},$$

respectively.

Proof. From equation (2.2) in definition 2.1 and the boundary behaviour (2.4), we have

$$E_i(x)=\alpha+\int_0^x(n_i-b_i)(y)\,\mathrm{d}y,\quad\forall x\in[0,1],\quad i=1,2. \tag{3.57}$$

First of all, in light of the lower and upper bounds (2.3) of $n_i(x)$, a straightforward computation gives

$$|E_i(x)|=\left|\alpha+\int_0^x(n_i-b_i)\,\mathrm{d}y\right|\leqslant\alpha+\int_0^1(n_i+b_i)\,\mathrm{d}y\leqslant\alpha+2\bar{b}_i,\quad\forall x\in[0,1],\quad i=1,2. \tag{3.58}$$

Next, taking the value $x=1$ in equation (3.57), we have

$$E_i(1)=\alpha+\int_0^1(n_i-b_i)(y)\,\mathrm{d}y,\quad i=1,2. \tag{3.59}$$

Furthermore, taking the difference of equations $(3.59)|_{i=1}$ and $(3.59)|_{i=2}$, we calculate

$$\begin{aligned}|E_1(1)-E_2(1)|&\leqslant\left|\int_0^1[(n_1-b_1)-(n_2-b_2)]\,\mathrm{d}y\right|\\&\leqslant\int_0^1|n_1-n_2|(y)\,\mathrm{d}y+\|b_1-b_2\|_{C[0,1]}\\&=|n_1-n_2|(\xi)+\|b_1-b_2\|_{C[0,1]},\quad\exists\xi\in[0,1],\end{aligned} \tag{3.60}$$

where we have used the mean-value theorem of integrals in the last line.

Finally, we claim that there is a positive constant C independent of $\|b_1 - b_2\|_{C[0,1]}$ such that

$$|n_1 - n_2|(\xi) \leqslant C\|b_1 - b_2\|_{C[0,1]}, \tag{3.61}$$

wherever the point ξ is located in the whole interval $[0,1]$. In fact, take δ the same as in lemma 3.6, and if $\xi \in [0, 1-\delta]$, it is clear from estimates (3.8) and (3.46) that (3.61) is true; if $\xi \in (1-\delta, 1]$, the intrinsic local estimate (3.38) guarantees that (3.61) is true as well. Consequently, substituting (3.61) into (3.60), we obtain the desired estimate (3.29). □

Obviously, putting all the estimates we have established in lemmas 3.3, 3.5 and 3.6 together, we have the following globally structural stability estimate under the monotonicity condition (3.6).

Lemma 3.8 (structural stability in the monotone case). *Let the doping profiles $b_1, b_2 \in C[0,1]$. If $b_1(x) \geqslant b_2(x) > 1$ on $[0,1]$, and $\alpha \geqslant \sqrt{8(b_1(0)-1)}$. Then there exists a positive constant C independent of $\|b_1 - b_2\|_{C[0,1]}$ such that*

$$\|n_1 - n_2\|_{C[0,1]} + \|(1-x)^{\frac{1}{2}}(n_1 - n_2)_x\|_{C[0,1]} + \|E_1 - E_2\|_{C^1[0,1]} \leqslant C\|b_1 - b_2\|_{C[0,1]}. \tag{3.62}$$

We end this section by proving theorem 2.1. Thanks to lemma 3.8 and the squeezing skill, we can now dispense with the monotonicity condition (3.6).

Proof of theorem 2.1. Let $b_1, b_2 \in C[0,1]$ be any two subsonic doping profiles, there is no need to require the monotonicity condition (3.6). Let $\alpha \geqslant 2\sqrt{2}\max\{\sqrt{b_1(0)-1}, \sqrt{b_2(0)-1}\}$, for $i = 1,2$, let $(n_i, E_i)(x)$ denote the interior subsonic solution corresponding to $b_i(x)$.

Firstly, we define

$$d_1(x) := \max\{b_1(x), b_2(x)\}, \quad d_2(x) := \min\{b_1(x), b_2(x)\}, \quad \forall x \in [0,1], \tag{3.63}$$

thus we have the squeezing relation

$$d_1(x) \geqslant b_i(x) \geqslant d_2(x) > 1, \quad \text{on } [0,1], \quad \text{for } i = 1,2, \tag{3.64}$$

and the squeezing estimate

$$\|b_i - d_j\|_{C[0,1]} \leqslant \|b_1 - b_2\|_{C[0,1]}, \quad \text{for } i,j = 1,2. \tag{3.65}$$

Next, for $j = 1,2$, we denote by $(\rho_j, \mathcal{E}_j)(x)$ the interior subsonic solution corresponding to the subsonic doping profile $d_j(x)$. The squeezing properties (3.64) and (3.65) enable us to choose, be it by $d_1(x)$ or via $d_2(x)$, a background solution $(\rho, \mathcal{E})(x)$ that is common to both doping profiles $b_1(x)$ and $b_2(x)$. For example, we opt for $(\rho, \mathcal{E})(x) = (\rho_1, \mathcal{E}_1)(x)$. Note that $d_1(x) \geqslant b_i(x) > 1$ (see (3.64)), combined with lemma 3.8 gives

$$\|\rho_1 - n_i\|_{C[0,1]} + \|(1-x)^{\frac{1}{2}}(\rho_1 - n_i)_x\|_{C[0,1]} + \|\mathcal{E}_1 - E_i\|_{C^1[0,1]} \leqslant C\|d_1 - b_i\|_{C[0,1]}, \quad \text{for } i = 1,2. \tag{3.66}$$

Finally, it follows from (3.65) and (3.66) that

$$\begin{aligned} &\|n_1 - n_2\|_{C[0,1]} + \|(1-x)^{\frac{1}{2}}(n_1 - n_2)_x\|_{C[0,1]} + \|E_1 - E_2\|_{C^1[0,1]} \\ &\leqslant \sum_{i=1}^{2}\left(\|\rho_1 - n_i\|_{C[0,1]} + \|(1-x)^{\frac{1}{2}}(\rho_1 - n_i)_x\|_{C[0,1]} + \|\mathcal{E}_1 - E_i\|_{C^1[0,1]}\right) \\ &\leqslant C\sum_{i=1}^{2}\|d_1 - b_i\|_{C[0,1]} \\ &\leqslant C\|b_1 - b_2\|_{C[0,1]}. \end{aligned} \tag{3.67}$$

□

4. Numerical simulations

In this section, we engage in the numerical verification of our theoretical results. Due to the boundary degeneracy, we cannot directly perform the numerical simulations of the degenerate problems (1.6) and (1.7). Instead, we make the most of the subsonic-current-approximation problem below: for fixed $0 < j < 1$,

$$\begin{cases} n_x = \dfrac{nE - \alpha j}{1 - j^2/n^2}, \\ E_x = n - b(x), \quad x \in (0,1), \\ n(0) = n(1) = 1. \end{cases} \tag{4.1}$$

For notational convenience, the approximate solutions here are still denoted by $(n,E)(x)$.

In order to sufficiently reflect the nature of the interior subsonic solution with sonic boundary, we set the approximate current $j = 0.9$ which is in close proximity to 1. Furthermore, the computational interval is $[0,1]$ with 100000 uniform mesh points. We use the `bvp5c` solver in `MATLAB` to numerically study the boundary value problem (4.1), where the relaxation time is set as $\tau = 0.1$, that is $\alpha = 10$. The reasonable choice of the `initial guess` is $[1;\alpha]$ because of $(4.1)_3$ and (2.4).

In what follows, we set about confirming the structural stability estimate (2.6) by using the above parameter settings. To this end, we consider the following doping profiles:

$$3.1 + \sin(\pi x) =: b_1(x) > b_2(x) := 3 + \sin(\pi x),$$

and denote the corresponding numerical solutions to the problem (4.1) by $(n_1, E_1)(x)$ and $(n_2, E_2)(x)$ respectively. The numerical simulations are displayed in figures 1–4.

Firstly, the numerical result in figure 1 demonstrates that the comparison principle in lemma 3.1 is true.

In addition, from figures 1, 2 and 4, one can easily see that the difference of $\|n_1 - n_2\|_{C[0,1]} + \|E_1 - E_2\|_{C^1[0,1]}$ can be controlled by the perturbation $\|b_1 - b_2\|_{C[0,1]}$ of doping profiles. So, the numerical results approximately coincide with theorem 2.1.

Last but not least, figure 3 numerically indicates that the first derivative of the first component of the interior subsonic solution will produce the genuine singularity at the right endpoint

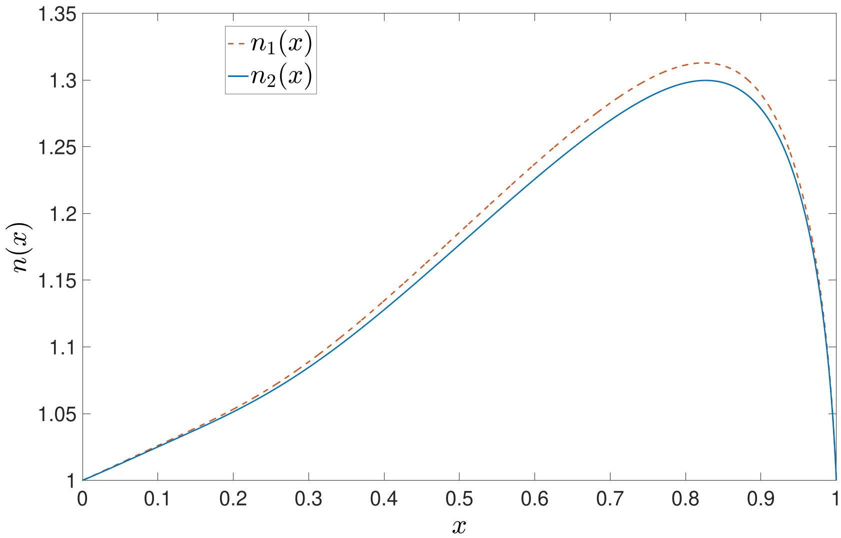

Figure 1. Comparison between $n_1(x)$ and $n_2(x)$.

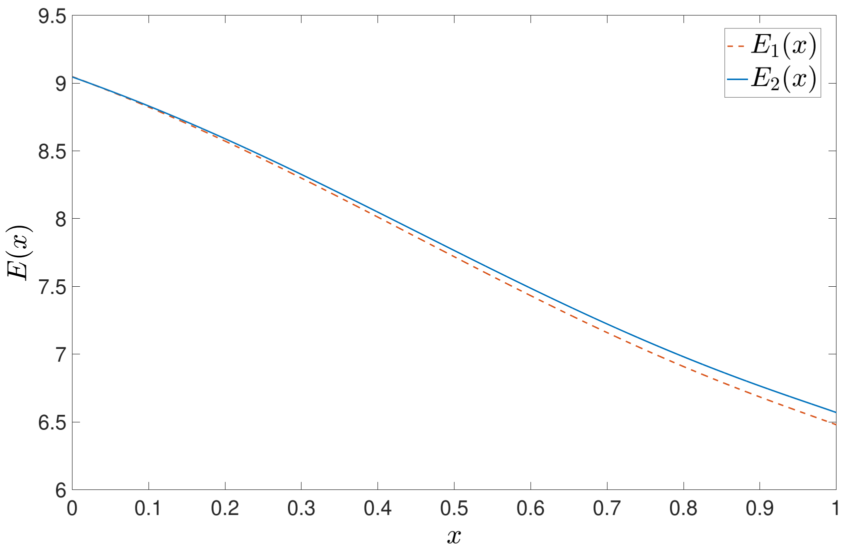

Figure 2. Comparison between $E_1(x)$ and $E_2(x)$.

$x = 1$ (i.e. $\lim_{x\to 1^-} n_x(x) = -\infty$), and the singularity at the left endpoint $x = 0$ is removable. These numerical observations also agree with our local singularity analyses in lemmas 3.2 and 3.4. Also, form figure 3, one can note that the curves of $n_{1,x}(x)$ and $n_{2,x}(x)$ are almost overlapped except near the right endpoint $x = 1$. Therefore, introducing the $(1-x)^{\frac{1}{2}}$-weight in the structural stability estimate (2.6) is necessary.

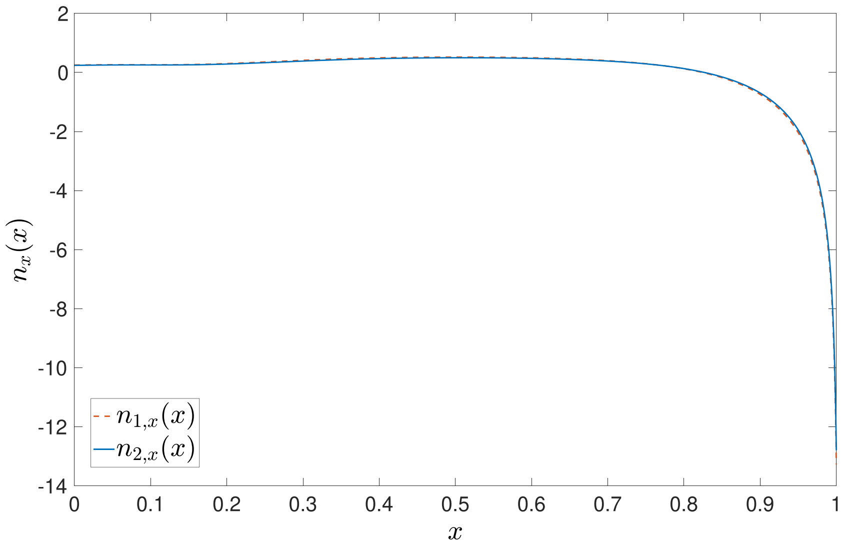

Figure 3. Comparison between first derivatives of $n_1(x)$ and $n_2(x)$.

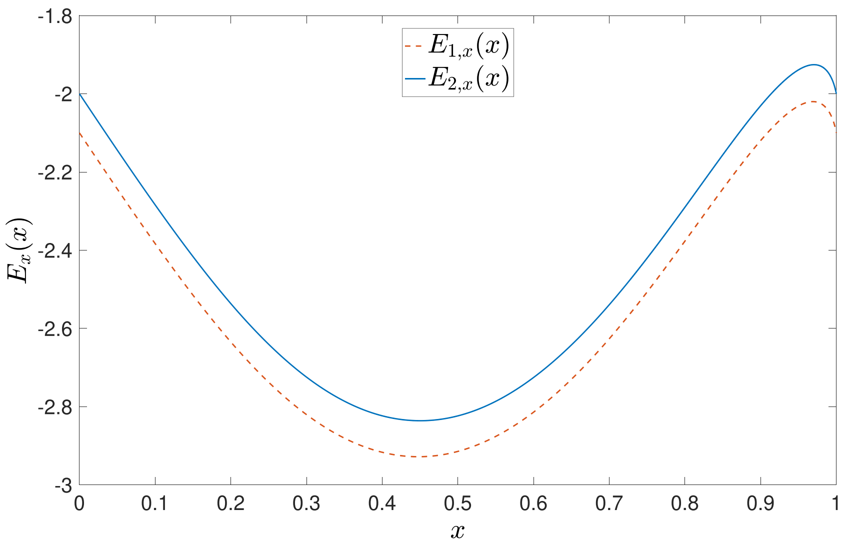

Figure 4. Comparison between first derivatives of $E_1(x)$ and $E_2(x)$.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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