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Asymptotic Profiles of Solutions for the BBM-Burgers Equation

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1. Introduction and main result

We are concerned with the asymptotic behavior of solutions to the Cauchy problem for the Benjamin-Bona-Mahony-Burgers equation (say also the BBM-Burgers equation, or the BBM-B for simplicity)

$$\begin{cases} u_t - u_{xxt} - u_{xx} + u^p u_x = 0 \\ u|_{t=0} = u_0(x) \to 0, \qquad \text{as } x \to \pm\infty, \end{cases} \tag{1.1}$$

where $x \in \boldsymbol{R}$, $t > 0$, $p \geq 1$ is integer, the initial value $= u_0(x)$ satisfies

$$\int_{-\infty}^{\infty} u_0(x)dx \neq 0. \tag{1.2}$$

On the time-asymptotic behavior of the solution to Eq. (1.1), there are a number of works, see [1, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 17, 18] and the references therein. Among them, many works, cf. [1, 3, 4, 5, 6, 7, 13, 14, 17, 18], are concerned with the asymptotic behavior of the solution as follows: if (1.2) holds, then the solution $u(x,t)$ converges to 0 in the sharp forms

$$\|u(t)\|_{L^\infty} = O(1)(1+t)^{-1/2}, \qquad \|u(t)\|_{L^2} = O(1)(1+t)^{-1/4}.$$

While in [11, 12] the first author showed that, if

$$\int_{-\infty}^{\infty} u_0(x)dx = 0,$$

then the solution $u(x,t)$ converges to 0 much faster as follows

$$\|u(t)\|_{L^\infty} = O(1)(1+t)^{-1}, \qquad \|u(t)\|_{L^2} = O(1)(1+t)^{-3/4}$$

and the L^1-decay rate

$$\|u(t)\|_{L^1} = O(1)(1+t)^{-1/2} \qquad \text{for } p \geq 2,$$

which are sharp too.

In all above mentioned works, 0 is considered as the asymptotic state of the solution $u(x,t)$ for the BBM-Burgers equation. Very recently, Kinami-Mei-Omata [10] found that a good asymptotic profile of the solution $u(x,t)$ is its corresponding diffusion wave but not the 0. Namely, the convergence to the diffusion wave is faster than to the 0. The diffusion wave means the solution of the Burgers equation for $p=1$ or the heat equation equation $p \geq 2$, more precisely, the Burgers solution for $p=1$ to the Burgers equation

$$\begin{cases} \theta_t - \theta_{xx} + \theta\theta_x = 0 \\ \theta(x,0) = \theta_0(x) \to 0 \qquad \text{as } x \to \pm\infty, \end{cases} \tag{1.3}$$

such a solution is called the corresponding nonlinear diffusion wave; or the solution for $p \geq 2$ to the heat equation

$$\begin{cases} \theta_t - \theta_{xx} = 0 \\ \theta(x,0) = \theta_0(x) \to 0 \qquad \text{as } x \to \pm\infty, \end{cases} \tag{1.4}$$

this solution is called the corresponding linear diffusion wave.

When

$$\int_{-\infty}^{\infty} u_0(x)dx = \int_{-\infty}^{\infty} \theta_0(x)dx \neq 0$$

holds, Kinami-Mei-Omata [10] proved the convergence rates as

$$\|(u-\theta)(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(7/8)+\sigma}, & \text{for } p=1 \\ O(1)(1+t)^{-7/8}\sqrt{\ln(2+t)}, & \text{for } p=2 \\ O(1)(1+t)^{-1}, & \text{for } p \geq 3 \end{cases}$$

and

$$\|(u-\theta)(t)\|_{L^2} = \begin{cases} O(1)(1+t)^{-(3/4)+\sigma}, & \text{for } p=1 \\ O(1)(1+t)^{-3/4}\ln(2+t), & \text{for } p=2 \\ O(1)(1+t)^{-3/4}, & \text{for } p \geq 3. \end{cases}$$

A similar problem is also studied by G. Karch in [9], but his convergence rates are weaker than those in [10].

Therefore, in the case $\int_{-\infty}^{\infty} u_0(x)dx \neq 0$, it is clear that the convergences to the diffusion waves are faster than to the 0. This means, comparing with the 0 the corresponding diffusion wave is a better asymptotic profile for the BBM-B solution. Now our questions are, what is the optimal asymptotic profile for the BBM-B equation and what is the convergence rate? These are our main interests in the present paper. By means of the variable scaling method, we discover that the real asymptotic profile is neither the 0 nor the linear diffusion wave (for the case $p \geq 2$), but should be the solution of the corresponding nonlinear parabolic equation

$$\begin{cases} \theta_t - \theta_{xx} + \theta^p \theta_x = 0 \\ \theta|_{t=0} = \theta_0(x) \to 0, \qquad \text{as } x \to \pm\infty, \end{cases} \tag{1.5}$$

where the initial datum $\theta_0(x)$ is asked to satisfy

$$\int_{-\infty}^{\infty} [u_0(x) - \theta_0(x)]dx = 0. \tag{1.6}$$

We shall further show some much faster convergence rates as follows

$$\|(u-\theta)(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(7/8)+\sigma}, & \text{for } p = 1 \\ O(1)(1+t)^{-1}, & \text{for } p \geq 2 \end{cases}$$

and

$$\|(u-\theta)(t)\|_{L^2} = \begin{cases} O(1)(1+t)^{-(3/4)+\sigma}, & \text{for } p = 1 \\ O(1)(1+t)^{-3/4}, & \text{for } p \geq 2, \end{cases}$$

where $\sigma > 0$ is any given constant.

Now let us make an analysis on what optimal asymptotic profile is for the BBM-Burgers equation (1.1). Setting the following scalings to our variables

$$t \to t/\varepsilon^2, \qquad x \to x/\varepsilon, \qquad u \to \varepsilon^m u \tag{1.7}$$

for some positive constants m and ε, where $\varepsilon \ll 1$, we then scale the BBM-Burgers equation (1.1) to

$$u_t - \varepsilon^2 u_{xxt} - u_{xx} + \varepsilon^{mp-1} u^p u_x = 0. \tag{1.8}$$

Balancing the leading order terms, we choose $m = 1/p$, and Eq. (1.8) is reduced to

$$u_t - \varepsilon^2 u_{xxt} - u_{xx} + u^p u_x = 0. \tag{1.9}$$

For $\varepsilon \ll 1$, we neglect the small term $-\varepsilon^2 u_{xxt}$, and conduct the asymptotic state equation of the BBM-B equation (1.1) as follows

$$u_t - u_{xx} + u^p u_x = 0. \tag{1.10}$$

This parabolic solution should be the optimal asymptotic profile of Eq. (1.1).

By other choice of m we derive other asymptotic states considered previously.

Let m be chosen as $mp < 1$, we then rewrite Eq. (1.8) as

$$\varepsilon^{1-mp}(u_t - \varepsilon^2 u_{xxt} - u_{xx}) + u^p u_x = 0. \tag{1.11}$$

For $\varepsilon \ll 1$, the term $\varepsilon^{1-mp}(u_t - \varepsilon^2 u_{xxt} - u_{xx})$ may be neglected so that the asymptotic equation of Eq. (1.11) is

$$u^p u_x = 0, \qquad i.e., \qquad \left(\frac{u^{p+1}}{p+1}\right)_x = 0.$$

Integrating it over $(\pm\infty, x]$ and noting that, formally $u(\pm\infty, t) = 0$, we obtain

$$u^{p+1}(x,t) = u^{p+1}(\pm\infty, t) = 0, \qquad \text{namely,} \qquad u(x,t) = 0.$$

This means that the 0 is the asymptotic state of the solution $u(x,t)$ of the BBM-B equation. This case has been well studied by many people, for example see [1, 3, 4, 5, 6, 7, 11, 12, 13, 14, 17].

Now let $m = 1$. Then Eq. (1.8) becomes

$$u_t - \varepsilon^2 u_{xxt} - u_{xx} + \varepsilon^{p-1} u^p u_x = 0. \tag{1.12}$$

When $p = 1$, dropping the small term $-\varepsilon^2 u_{xxt}$ due to $\varepsilon \ll 1$, we see easily from (1.12) that the asymptotic state equation is the Burgers equation

$$u_t - u_{xx} + u u_x = 0. \tag{1.13}$$

When $p \geq 2$, we neglect the small terms $-\varepsilon^2 u_{xxt}$ and $\varepsilon^{p-1} u^p u_x$, and obtain the heat equation

$$u_t - u_{xx} = 0. \tag{1.14}$$

The solutions for the Cauchy problem to Eqs. (1.13) and (1.14) are the so-called nonlinear and linear diffusion waves. Such diffusion waves are considered as the asymptotic profiles in each case $p = 1$, $p = 2$ and $p \geq 3$ by Karch [9] and

Kinami-Mei-Omata [10] very recently, and the convergence rates to them are given too.

Now let us state our main results. Firstly, we note that, by using the Green function method together with the basic energy estimates, when

$$\theta_0(x) \in L^1(\mathbf{R}) \cap H^2(\mathbf{R}),$$

Jeffrey and Zhao [8] prove that the solution $\theta(x,t)$ of Eq. (1.5) exists uniquely and globally in time, and has the decay rates as in (1.15) below. In the same fashion, we can further prove that, if

$$\theta_0(x) \in L^1(\mathbf{R}) \cap H^4(\mathbf{R}),$$

then

$$\|\partial_x^j \theta_t(t)\|_{L^1} = O(1)(1+t)^{-1-(j/2)}, \qquad j = 0,1,2.$$

We list all of them as follows, and say that they are contributed by Jeffrey and Zhao [8], although the last part on L^1 estimates (1.16) is excluded there.

Theorem 1.1 ([8]). *Suppose that* $\theta_0(x) \in H^2(\mathbf{R}) \cap L^1(\mathbf{R})$ *holds. Then there exists a positive constant* δ_0 *such that when* $\|\theta_0\|_{L^1} + \|\theta_0\|_{H^2} \le \delta_0$, *then the Cauchy problem* (1.5) *has a unique global solution* $\theta(x,t)$

$$\theta(x,t) \in C(0,\infty; H^2(\mathbf{R}) \cap L^1(\mathbf{R})) \cap L^2(0,\infty; H^1(\mathbf{R})),$$

and satisfies

$$\|\partial_x^j \theta(t)\|_{L^q} = O(1)(\|\theta_0\|_{L^1} + \|\theta_0\|_{H^2})(1+t)^{-((j+1)q-1)/(2q)}, \quad 1 \le q \le \infty, \qquad j = 0,1,2. \tag{1.15}$$

Furthermore, if

$$\theta_0 \in L^1(\mathbf{R}) \cap H^4(\mathbf{R}),$$

then

$$\|\partial_x^j \theta_t(t)\|_{L^1} = O(1)(\|\theta_0\|_{L^1} + \|\theta_0\|_{H^4})(1+t)^{-1-(j/2)}, \qquad j = 0,1,2. \tag{1.16}$$

Now our main results are as follows.

Theorem 1.2. *Suppose that* (1.6),

$$v_0(x) := \int_{-\infty}^{x} [u_0(y) - \theta_0(y)]dy \in W^{3,1}(\mathbf{R}), \tag{1.17}$$

and $\theta_0(x) \in L^1(\mathbf{R}) \cap H^4(\mathbf{R})$ *hold. Let* $\eta := \|\theta_0\|_{L^1} + \|\theta_0\|_{H^4}$. *Then there exists a positive constant* δ_0 *such that when* $\|v_0\|_{W^{3,1}} + \eta \le \delta_0$, *then the Cauchy problem* (1.1) *has a unique global solution* $u(x,t)$

$$u(x,t) - \theta(x,t) \in C(0,\infty; H^1(\mathbf{R})),$$

and satisfies that:
(i) *If* $p = 1$, *for any* $\sigma > 0$, *then the following estimates hold*

$$\|(u-\theta)(t)\|_{L^2} = O(1)(1+t)^{-(3/4)+\sigma}, \tag{1.18}$$

$$\|(u-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-1+\sigma}, \tag{1.19}$$

$$\|(u-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-(7/8)+\sigma}. \tag{1.20}$$

(ii) *If* $p \ge 2$, *the convergence rates are much faster as follows*

$$\|(u-\theta)(t)\|_{L^2} = O(1)(1+t)^{-3/4}, \tag{1.21}$$

$$\|(u-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-5/4}, \tag{1.22}$$

$$\|(u-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-1}. \tag{1.23}$$

Using L^2, L^∞-results in Theorem 1.2 and the interposing inequality

$$\|f\|_{L^q} \le \|f\|_{L^\infty}^{(q-2)/q} \|f\|_{L^2}^{2/q}, \qquad \text{for } 2 \le q \le \infty,$$

we can obtain immediately L^q-decay rates as follows.

Corollary 1.1. *Under the assumptions in Theorem* 1.2, *then for* $2 \le q \le \infty$ *it follows*

$$\|(u-\theta)(t)\|_{L^q} = \begin{cases} O(1)(1+t)^{-(7/8)+(1/4q)+\sigma}, & \text{for } p = 1 \\ O(1)(1+t)^{-1+(1/2q)}, & \text{for } p \ge 2. \end{cases} \tag{1.24}$$

Remark 1.1. 1. The rates in Case $p = 1$ of Theorem 1.2 have been shown in [10], we list here again only for completeness.

2. In Case $p = 2$, our convergence rates are much better than that to the diffusion wave in [9, 10]. This means that the solution $\theta(x,t)$ of the nonlinear parabolic equation (1.5) is a better asymptotic profile of the BBM-B solution $u(x,t)$ than the linear diffusion wave (1.4) is. However, in Case $p = 1$, it seems that the rates are not sharp. Is it possible to improve them to be optimal ones like those in our Case $p \ge 2$?

3. As shown in the Appendix below, the BBM-B solution $u(x,t)$ of Eq. (1.1) exists globally for any *large* initial datum, and so does the parabolic solution $\theta(x,t)$ of Eq. (1.5). This fact inform us that it should be possible to have the convergence $u(x,t) \to \theta(x,t)$ for the *large* initial perturbation problem. Therefore, it is quite interesting to study such an asymptotic convergence $u(x,t) \to \theta(x,t)$ with some decay rates for a *large* initial perturbation! However, this question in theoretical proof is difficult and remains still open. We expect to have more contributions on it in the future.

Notations. We now introduce some notation for simplicity. C always denotes some positive constants without confusion. $\partial_x^k f := \partial^k f/\partial x^k$. L^p presents the Lebesque integral space with the norm $\|\cdot\|_{L^p}$. Especially, L^2 is the square integral space with the norm $\|\cdot\|_{L^2}$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_{L^\infty}$. $H^k =$ and $W^{k,p}$ denote the usual Sobolev spaces with the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W^{k,p}}$, respectively. Suppose that $f \in L^1 \cap L^2(\boldsymbol{R})$, we define the Fourier transforms of f as follows:

$$F[f](\xi) \equiv \hat{f}(\xi) = \int_{\boldsymbol{R}} f(x)e^{-ix\xi}\,dx.$$

Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T;B)$ $(k \ge 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $[0,T]$. The corresponding spaces of B-valued functions on $[0,\infty)$ are defined similarly.

2. Reformulation of the original problem

From Eqs. (1.1) and (1.5), we have

$$(u-\theta)_t - u_{xxt} - \alpha(u-\theta)_{xx} + \left(\frac{u^{p+1}}{p+1} - \frac{\theta^{p+1}}{p+1}\right)_x = 0. \tag{2.1}$$

Since $\theta(\pm\infty,t) = 0$, and we expect $u(\pm\infty,t) = 0$, $u_x(\pm\infty,t) = 0$, then after integrating (2.1) over $(-\infty,\infty)$, we have formally

$$\frac{d}{dt}\int_{-\infty}^{\infty} [u(x,t) - \theta(x,t)]dx = 0. \tag{2.2}$$

Thanks to the essential assumption (1.6), we integrate (2.2) over $[0,t]$ with respect to t to have

$$\int_{-\infty}^{\infty} [u(x,t) - \theta(x,t)]dx = \int_{-\infty}^{\infty} [u_0(x) - \theta_0(x)]dx = 0. \tag{2.3}$$

Therefore, it is natural to introduce

$$v(x,t) = \int_{-\infty}^{x} [u(y,t) - \theta(y,t)]dy, \qquad i.e., \quad v_x(x,t) = u(x,t) - \theta(x,t), \tag{2.4}$$

which satisfies

$$v_{xt} - v_{xxxt} - \theta_{xxt} - \alpha v_{xxx} + \left(\frac{(\theta + v_x)^{p+1}}{p+1} - \frac{u^{p+1}}{p+1}\right)_x = 0. \tag{2.5}$$

Integrating it over $(-\infty, x]$ with respect to x, and noting $u(\pm\infty, t) = 0$, $\theta(\pm\infty, t) = 0$, we obtain

$$\begin{cases} v_t - v_{xxt} - \alpha v_{xx} - \theta_{xt} + \dfrac{(\theta + v_x)^{p+1}}{p+1} - \dfrac{u^{p+1}}{p+1} = 0, \\ v|_{t=0} = \displaystyle\int_{-\infty}^{x} [u_0(y) - \theta_0(y)]dy = v_0(x). \end{cases} \tag{2.6}$$

That is,

$$\begin{cases} v_t - v_{xxt} - \alpha v_{xx} = F_p, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.7}$$

where

$$F_p = \theta_{xt} - \frac{1}{p+1}[(\theta+v_x)^{p+1} - \theta^{p+1}] = \theta_{xt} - \frac{1}{p+1}\sum_{i=0}^{p} a_i \theta^i v_x^{p+1-i}, \qquad p \geq 1 \tag{2.8}$$

for some positive constants $a_i = C_{p+1}^i$.

Theorem 2.1. *Under the assumptions of Theorem* 1.2, *there exists a positive constant* δ_1, *such that, when* $\|v_0\|_{W^{3,1}} + \eta < \delta_1$, *then the Cauchy problem* (2.7) *has a unique global solution* $v(x,t)$ *satisfying*

$$v(x,t) \in C(0,\infty; H^2(\boldsymbol{R})).$$

Furthermore, we have the following estimates.

1. *When* $p = 1$, *for any given* $\sigma > 0$, *the solution* $v(x,t)$ *of* (2.7) *satisfies*

$$\sum_{j=0}^{1} (1+t)^{((2j+1)/4)-\sigma} \|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma} \|v_{xx}(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \eta). \tag{2.9}$$

2. *When* $p \geq 2$, *the solution* $v(x,t)$ *of* (2.7) *satisfies*

$$\sum_{j=0}^{2}(1+t)^{(2j+1)/4}\|\partial_x^j v(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}}+\eta). \tag{2.10}$$

By Theorem 2.1 and the well-known Sobolev inequalities as, if $f \in H^1$, then

$$\|f\|_{L^\infty} \leq \sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}, \tag{2.11}$$

we can obtain the decay rates for $\|v(t)\|_{L^\infty}$ and $\|v_x(t)\|_{L^\infty}$ as follows.

Corollary 2.1. *Under the assumptions in Theorem* 1.2, *it follows*

$$\|v(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(1/2)+\sigma}, & \text{for } p=1 \\ O(1)(1+t)^{-1/2}, & \text{for } p\geq 2 \end{cases} \tag{2.12}$$

and

$$\|v_x(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(7/8)+\sigma}, & \text{for } p=1 \\ O(1)(1+t)^{-1}, & \text{for } p\geq 2. \end{cases} \tag{2.13}$$

Once Theorem 2.1 is proved, we then very easily obtain Theorem 1.2 by $u(x,t)-\theta(x,t)=v_x(x,t)$. Therefore, to prove Theorem 2.1 is our main goal in the following.

We now define the solution spaces as follows, for any $T>0$ and given $\delta>0$,

$$X_p(0,T)=\{v\in C(0,\infty;H^2(\boldsymbol{R}))\mid M_p(T)\leq\delta\}, \qquad p\geq 1,$$

where

$$M_1(T)=\sup_{0\leq t\leq T}\left\{\sum_{j=0}^{1}(1+t)^{((2j+1)/4)-\sigma}\|\partial_x^j v(t)\|_{L^2}+(1+t)^{1-\sigma}\|v_{xx}(t)\|_{L^2}\right\} \tag{2.14}$$

$$M_p(T)=\sup_{0\leq t\leq T}\sum_{j=0}^{2}(1+t)^{(2j+1)/4}\|\partial_x^j v(t)\|_{L^2}, \qquad \text{for } p\geq 2. \tag{2.15}$$

We are going to state the local existence and the *a priori* estimates.

Proposition 2.1 (Local Existence). *Suppose that* $v_0\in H^2(\boldsymbol{R})$ *holds, then there exists a positive constant* T_0 *such that the Cauchy problem* (2.7) *has a unique solution* $v(x,t)\in X_p(0,T_0)$ *satisfying* $M_p(T_0)\leq 2M_p(0)$ *for all* $p\geq 1$.

Proposition 2.2 (A Priori Estimate). *Let T be a positive constant, and $v(x,t) \in X_p(0,T)$ $(p \geq 1)$ be a solution of the Cauchy problem* (2.7). *Suppose that the assumptions in Theorem* 1.2 *hold, then there exist positive constants δ_3 and C independent of T such that if $M_p \leq \delta_3$, then for $t \in [0,T]$ the following estimates hold*:

1. *When $p = 1$, for any $\sigma > 0$*

$$\sum_{j=0}^{1}(1+t)^{((2j+1/4)-\sigma}\|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma}\|v_{xx}(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \eta). \tag{2.16}$$

2. *When $p \geq 2$,*

$$\sum_{j=0}^{2}(1+t)^{(2j+1)/4}\|\partial_x^j v(t)\|_{L^2} \leq C(\|v_0|_{W^{3,1}} + \eta). \tag{2.17}$$

Using the continuation argument based on Propositions 2.1 and 2.2, we can prove Theorem 2.1. So, to prove the above two propositions is our goal. Since Proposition 2.1 can be proved in the standard way, our main effort will be made on the proof of Proposition 2.2 in the next section.

3. A priori estimates

Since the case $p = 1$ has been proved in [10], we focus only on the case $p \geq 2$. As in [12, 10], taking the Fourier transform to Eq. (2.7), we have

$$\hat{v}_t - (i\xi)^2\hat{v}_t - \alpha(i\xi)^2\hat{v} = \hat{F}_p,$$

namely,

$$\hat{v}_t + \frac{\alpha\xi^2}{1+\xi^2}\hat{v} = \frac{\hat{F}_p}{1+\xi^2},$$

with the solution

$$\hat{v}(\xi,t) = e^{-B(\xi)t}\hat{v}_0(\xi) + \int_0^t e^{-B(\xi)(t-s)}\frac{\hat{F}_p(\xi,s)}{1+\xi^2},$$

where

$$B(\xi) = \frac{\alpha\xi^2}{1+\xi^2}. \tag{3.1}$$

Then taking the inverse Fourier transform to the above resultant equation, we have

$$(3.2)\quad v(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi x}e^{-B(\xi)t}\hat{v}_0(\xi)d\xi+\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}e^{i\xi x}e^{-B(\xi)(t-s)}\frac{\hat{F}_p(\xi,s)}{1+\xi^2}\,d\xi ds.$$

Differentiating it with respect to x, we have

$$(3.3)\quad \begin{aligned}\partial_x^j v_x(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{i\xi x}e^{-B(\xi)t}\hat{v}_0(\xi)d\xi\\ &-\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}(i\xi)^j e^{i\xi x}e^{-B(\xi)(t-s)}\frac{\hat{F}_p(\xi,s)}{1+\xi^2}\,d\xi ds.\end{aligned}$$

We first have two lemmas which were proved by the first author in [12]. The second lemma is to estimate the initial perturbation.

Lemma 3.1 ([12]). *It holds*

$$(3.4)\quad \int_{-\infty}^{\infty}\frac{|\xi|^j e^{-CB(\xi)t}}{(1+\xi^2)(1+|\xi|)^j}\,d\xi\le C(1+t)^{-(j+1)/2}.$$

Lemma 3.2 ([12]). *If* $v_0\in W^{3,1}(\boldsymbol{R})$, *then*

$$(3.5)\quad \left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{ix\xi}e^{-B(\xi)t}\hat{v}_0(\xi)d\xi\right\|_{L^2}\le C\|v_0\|_{W^{j+1,1}}(1+t)^{-(2j+1)/4}$$

for $j=0,1,2$.

We are going to estimate the nonlinear term.

Lemma 3.3. *Let* $v(x,t)\in X_p(0,T)$ $(p\ge 2)$, *then*

$$(3.6)\quad \begin{aligned}&\int_0^t\left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{i\xi x}\frac{e^{-B(\xi)(t-s)}}{1+\xi^2}\hat{F}_p(\xi,s)d\xi\right\|_{L^2}ds\\ &\qquad\le C[\eta+(\eta+M_p(T))^{p+1}](1+t)^{-(1+2j)/4}\end{aligned}$$

for $j=0,1,2$.

Proof. Let $v\in X_p(0,T)$ $(p\ge 2)$. Due to the definition of $X_p(0,T)$ and the Sobolev inequality (2.11) it is easy to obtain

$$(3.7)\quad \|v(t)\|_{L^\infty}\le\sqrt{2}\|v(t)\|_{L^2}^{1/2}\|v_x(t)\|_{L^2}^{1/2}\le\sqrt{2}M_p(T)(1+t)^{-1/2}$$

and

$$(3.8)\quad \|v_x(t)\|_{L^\infty}\le\sqrt{2}\|v_x(t)\|_{L^2}^{1/2}\|v_{xx}(t)\|_{L^2}^{1/2}\le\sqrt{2}M_p(T)(1+t)^{-1}.$$

On the other hand, thanks to (2.8) we get

$$|F_p| \le |\theta_{xt}| + \frac{1}{p+1}\sum_{j=0}^{p} a_j|\theta^j v_x^{p+1-j}| \tag{3.9}$$

and

$$\begin{aligned} |\partial_x F_p| \le\ & |\theta_{xxt}| + |v_x^p v_{xx}| \\ & + \frac{1}{p+1}\sum_{j=1}^{p} a_j[j|\theta^{j-1}\theta_x v_x^{p+1-j}| + (p+1-j)|\theta^j v_x^{p-j} v_{xx}|]. \end{aligned} \tag{3.10}$$

Applying (3.7), (3.8) and Theorem 1.1 to (3.9) and (3.10), we then have

$$\begin{aligned} \sup_{\xi\in R}|\hat{F}_p(\xi,s)| &\le \int_{-\infty}^{\infty} |F_p(x,s)|dx \\ &\le \int_{-\infty}^{\infty}\left\{|\theta_{xt}| + \frac{1}{p+1}\sum_{j=0}^{p} a_j|\theta^j v_x^{p+1-j}|\right\}dx \\ &\le \|\theta_{xt}(s)\|_{L^1} + \frac{1}{p+1}a_p\|\theta(s)\|_{L^\infty}^{p-1}\int_{-\infty}^{\infty}|\theta v_x|dx \\ &\quad + \frac{1}{p+1}\sum_{j=0}^{p-1} a_j\|\theta(s)\|_{L^\infty}^{j}\|v_x(s)\|_{L^\infty}^{p-1-j}\int_{-\infty}^{\infty}|v_x|^2dx \\ &\le \|\theta_{xt}(s)\|_{L^1} + \frac{1}{p+1}a_p\|\theta(s)\|_{L^\infty}^{p-1}\|\theta(s)\|_{L^2}\|v_x(s)\|_{L^2} \\ &\quad + \frac{1}{p+1}\sum_{j=0}^{p-1} a_j\|\theta(s)|_{L^\infty}^{j}\|v_x(s)\|_{L^\infty}^{p-1-j}\|v_x(s)\|_{L^2}^2 \\ &\le C\left\{\eta(1+s)^{-3/2} + a_p\eta^p M_p(T)(1+s)^{-(p+1)/2}\right. \\ &\quad \left. + \sum_{j=0}^{p-1} a_j\eta^j M_p(T)^{p+1-j}(1+s)^{-(2p-j+1)/2}\right\} \\ &\le C[\eta(1+s)^{-3/2} + (\eta+M_p(t))^{p+1}(1+s)^{-(p+1)/2}] \\ &\le C[\eta + (\eta+M_p(T))^{p+1}](1+s)^{-3/2}, \end{aligned} \tag{3.11}$$

where we used $p \ge 2$, namely, $(p+1)/2 \ge 3/2$ and $(2p-j+1)/2 > (p+1)/2$ for $0 \le j \le p-1$, which implies $(1+s)^{-(2p-j+1)/2} < (1+s)^{-(p+1)/2} \le (1+s)^{-3/2}$; and we have

$$
\begin{aligned}
(3.12)\quad \sup_{\xi\in R}|\xi||\hat{F}_p(\xi,s)| &\le \int_{-\infty}^{\infty}|\partial_x F_p(x,s)|dx \\
&\le \int_{-\infty}^{\infty}\Big\{|\theta_{xxt}|+|v_{xx}||v_x|^p+\frac{1}{p+1} \\
&\quad\times\sum_{j=1}^{p}a_j[j|\theta^{j-1}\theta_x v_x^{p+1-j}|+(p+1-j)|\theta^j v_x^{p-j}v_{xx}|]\Big\}dx \\
&\le \|\theta_{xxt}(s)\|_{L^1}+\|v_x(s)\|_{L^2}^{p-1}\int_{-\infty}^{\infty}|v_x v_{xx}|dx \\
&\quad+\frac{1}{p+1}\sum_{j=1}^{p}a_j\Big[j\|\theta(s)\|_{L^\infty}^{j-1}\|v_x(s)\|_{L^\infty}^{p-j}\int_{-\infty}^{\infty}|\theta_x v_x|dx \\
&\quad+(p+1-j)\|\theta(s)\|_{L^\infty}^{j-1}\|v_x(s)\|_{L^\infty}^{p-j}\int_{-\infty}^{\infty}|\theta v_{xx}|dx\Big] \\
&\le \|\theta_{xxt}(s)\|_{L^1}+\|v_x(s)\|_{L^2}^{p-1}\|v_x(s)\|_{L^2}\|v_{xx}(s)\|_{L^2} \\
&\quad+\frac{1}{p+1}\sum_{j=1}^{p}a_j[j\|\theta(s)\|_{L^\infty}^{j-1}\|v_x(s)\|_{L^\infty}^{p-j}\|\theta_x(s)\|_{L^2}\|v_x(s)\|_{L^2} \\
&\quad+(p+1-j)\|\theta(s)\|_{L^\infty}^{j-1}\|v_x(s)\|_{L^\infty}^{p-j}\|\theta(s)\|_{L^2}\|v_{xx}(s)\|_{L^2}] \\
&\le C\Big\{\eta(1+s)^{-2}+M_p(T)^{p+1}(1+s)^{-(p+1)} \\
&\quad+\frac{1}{p+1}\sum_{j=1}^{p}a_j[j\eta^j M_p(T)^{p-j+1}(1+s)^{-(2p-j+2)/2} \\
&\quad+(p+1-j)\eta^j M_p(T)^{p-j+1}(1+s)^{-(2p-j+2)/2}]\Big\} \\
&\le C\Big\{\eta(1+s)^{-2}+M_p(T)^{p+1}(1+s)^{-(p+1)} \\
&\quad+\sum_{j=1}^{p}a_j\eta^j M_p(T)^{p-j+1}(1+s)^{-(p+2)/2} \\
&\le C[\eta+(\eta+M_p(T))^{p+1}](1+s)^{-2},
\end{aligned}
$$

where we used $p\ge 2$, $1\le j\le p$, namely, $p+1\ge 3$ and $(2p-j+2)/2\ge (p+2)/2\ge 2$, which imply $(1+s)^{-(p+1)}<(1+s)^{-2}$ and $(1+s)^{-(2p-j+2)/2}\le (1+s)^{-(p+2)/2}\le (1+s)^{-2}$.

Making use of Parseval's equality, (3.11), (3.12), Lemma 3.1, and the well-known inequality (cf. [16, 12])

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds \le C(1+t)^{-\min(a,b)}, \qquad \text{for } a, b > 0 \text{ and } \max(a,b) > 1,$$

we can prove this lemma as follows

$$\begin{aligned}
(3.13)\qquad & \int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\frac{e^{-B(\xi)(t-s)}}{1+\xi^2}\hat{F}_p(\xi,s)d\xi = \right\|_{L^2} ds \\
&= \int_0^t \left\| \frac{e^{-B(\xi)(t-s)}}{1+\xi^2}\hat{F}_p(\xi,s) \right\|_{L^2} ds \\
&= \int_0^t \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\hat{F}_p(\xi,s)|^2 d\xi \right)^{1/2} ds \\
&\le \int_0^t \sup_{\xi\in R} |\hat{F}_p(\xi,s)| \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} d\xi \right)^{1/2} ds \\
&\le C[\eta + (\eta + M_p(T))^{p+1}] \int_0^t (1+s)^{-3/2}(1+t-s)^{-1/4} ds \\
&\le C[\eta + (\eta + M_p(T))^{p+1}](1+t)^{-1/4},
\end{aligned}$$

and

$$\begin{aligned}
(3.14)\qquad & \int_0^t \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{i\xi x}\frac{e^{-B(\xi)(t-s)}}{1+\xi^2}\hat{F}_p(\xi,s)d\xi \right\|_{L^2} ds \\
&= \int_0^t \left\| i\xi \frac{e^{-B(\xi)(t-s)}}{1+\xi^2}\hat{F}_p(\xi,s) \right\|_{L^2} ds \\
&= \int_0^t \left(\int_{-\infty}^{\infty} \frac{|\xi|^2 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\hat{F}_p(\xi,s)|^2 d\xi \right)^{1/2} ds \\
&\le \int_0^t \sup_{\xi\in R} |\hat{F}_p(\xi,s)| \left(\int_{-\infty}^{\infty} \frac{|\xi|^2 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} d\xi \right)^{1/2} ds \\
&\le C[\eta + (\eta + M_p(T))^{p+1}] \int_0^t (1+s)^{-3/2}(1+t-s)^{-3/4} ds \\
&\le C[\eta + (\eta + M_p(T))^{p+1}](1+t)^{-3/4},
\end{aligned}$$

and

$$\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^2 e^{i\xi x} \frac{e^{-B(\xi)(t-s)}}{1+\xi^2} \hat{F}_p(\xi, s) d\xi \right\|_{L^2} ds$$

$$= \int_0^t \left\| (i\xi)^2 \frac{e^{-B(\xi)(t-s)}}{1+\xi^2} \hat{F}_p(\xi, s) \right\|_{L^2} ds$$

$$= \int_0^t \left(\int_{-\infty}^{\infty} \frac{|\xi|^4 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\hat{F}_p(\xi, s)|^2 d\xi \right)^{1/2} ds$$

$$\leq \int_0^t \sup_{\xi \in R}((1+|\xi|)|\hat{F}_p(\xi, s)|) \left(\int_{-\infty}^{\infty} \frac{|\xi|^4 e^{-2B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^4} d\xi \right)^{1/2} ds$$

$$\leq C[\eta + (\eta + M_p(T))^{p+1}]$$

$$\times \int_0^t [(1+s)^{-3/2} + (1+s)^{-2}](1+t-s)^{-5/4} ds$$

$$\leq C[\eta + (\eta + M_p(T))^{p+1}] \int_0^t (1+s)^{-3/2}(1+t-s)^{-5/4}] \tag{3.15}$$

$$\leq C[\eta + (\eta + M_p(T))^{p+1}](1+t)^{-5/4}. \tag{3.16}$$

Thus, we proved (3.6) for $j = 0, 1, 2$. □

Proof of Proposition 2.2 (*A Priori Estimates*). Let $v(x, t) \in X_p(0, T)$ be the unique solution of Eq. (3.2), $p \geq 2$. From (3.3), Lemma 3.2, and Lemma 3.3 we obtain

$$\|\partial_x^j v(t)\|_{L^2} \leq \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} e^{-B(\xi)t} v_0(\delta) d\xi \right\|_{L^2} \tag{3.17}$$

$$+ \int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} \frac{e^{-B(\xi)(t-s)}}{1+\xi^2} \hat{F}_p(\xi, s) d\xi \right\|_{L^2} ds$$

$$\leq C\|v_0\|_{W^{j+1,1}}(1+t)^{-(1+2j)/4}$$

$$+ C[\eta + (\eta + M_p(T))^{p+1}](1+t)^{-(1+2j)/4}$$

for $j = 0, 1, 2$. Multiplying (3.17) on both sides by $(1+t)^{-(1+2j)/4}$ $(j = 0, 1, 2)$, respectively, and adding all of them, we then obtain

$$M_p(T) = \sup_{0<t\leq T} \sum_{j=0}^{2} (1+t)^{(1+2j)/4} \|\partial_x^j v(t)\|_{L^2} \leq C[\|v_0\|_{W^{3,1}} + \eta + (\eta + M_p(T))^{p+1}],$$

that is,

$$M_p(T)\left[1 - C\sum_{j=0}^{p} a_j\eta^j M_p(T)^{p-j}\right] \le C[\|v_0\|_{W^{3,1}} + \eta + \eta^{p+1}]. \tag{3.18}$$

Since there is some positive constant $C_1 > 0$ such that

$$C\sum_{j=0}^{p} a_j\eta^j M_p(T)^{p-j} \le C_1(\eta + M_p(T))^p,$$

(3.18) is reduced to

$$M_p(T)[1 - C_1(\eta + M_p(T))^p] \le C[\|v_0\|_{W^{3,1}} + \eta + \eta^{p+1}]. \tag{3.19}$$

Let us choose δ_3 in Proposition 2.2 to be

$$\delta_3 < \min\left\{1, \frac{1}{4C_1}, \frac{1}{(4C_1)^{1/p}}\right\},$$

when $\eta < \delta_3$ and $M_p(T) < \delta_3$, from (3.19) we prove

$$M_p(T) \le 2C[\|v_0\|_{W^{3,1}} + \eta + \eta^{p+1}] \le C_2[\|v_0\|_{W^{3,1}} + \eta]$$

for some positive constant C_2. That means

$$\sum_{j=0}^{2}(1+t)^{(1+2j)/4}\|\partial_x^j v(t)\|_{L^2} \le C_2[\|v_0\|_{W^{3,1}} + \eta]$$

holds for all $t \in [0, T]$.

Thus, the proof of the *a priori* estimates is complete. □

Appendix: Global existence for large initial values

As indicated in Remark 1.1, we are going to give a proof on the global existences of the solutions $u(x,t)$ to Eq. (1.1) and $\theta(x,t)$ to Eq. (1.5) for *large* initial data. The method we will adopt is still the elementary energy method.

Theorem 3.1. *For any* $u_0(x) \in H^2(\boldsymbol{R})$, *the Cauchy problem* (1.1) *has a unique global solution*

$$u(x,t) \in C^0(0, +\infty; H^2(\boldsymbol{R})), \qquad u_x(x,t) \in L^2(0, +\infty; H^1(\boldsymbol{R})).$$

Furthermore, $u(x,t)$ satisfies

$$\text{(A.1)} \quad \|u(t)\|_{H^2}^2 + \int_0^t \|u_x(\tau)\|_{H^1}^2 d\tau \le 2\|u_0\|_{H^2}^2 + 2^{p-1}\|u_0\|_{H^1}^{2(p+1)} \quad \text{for all } t \ge 0.$$

Theorem 3.2. *For any $\theta_0(x) \in H^2(\boldsymbol{R})$, the Cauchy problem* (1.5) *has a unique global solution*

$$\theta(x,t) \in C^0(0,+\infty; H^2(\boldsymbol{R})), \qquad \theta_x(x,t) \in L^2(0,+\infty; H^1(\boldsymbol{R})).$$

Furthermore, $\theta(x,t)$ satisfies

$$\text{(A.2)} \quad \|\theta(t)\|_{H^2}^2 + \int_0^t \|\theta_x(\tau)\|_{H^1}^2\, d\tau \le 2\|\theta_0\|_{H^2}^2 + 2^{p-1}\|\theta_0\|_{H^1}^{2(p+1)} \quad \text{for all } t \ge 0.$$

We are going to prove Theorem 3.1. Since Theorem 3.2 can be proved in the quite same way, we shall omit the details. Now let us define the solution space for the Cauchy problem (1.1) by

$$\text{(A.3)} \qquad Y_M(t_1,t_2) = \left\{ u \mid u \in C^0(t_1,t_2; H^2(\boldsymbol{R})),\ u_x \in L^2(t_1,t_2; H^1(\boldsymbol{R})) \text{ with } \sup_{[t_1,t_2]} \|u(t)\|_{H^2} \le M \right\}$$

where t_1, t_2 and M are some positive constants.

We now prove two results as follows. One is the local existence, another is the *a priori* estimates.

Proposition 3.1 (Local Existence). *Consider the Cauchy problem with the initial time τ*

$$\text{(A.4)} \qquad \begin{cases} u_t - u_{xxt} - \alpha u_{xx} + u^p u_x = 0 \\ u|_{t=\tau} = u_\tau(x). \end{cases}$$

Then, if $u_\tau(x) \in H^2(\boldsymbol{R})$ and $\|u_\tau\|_{H^2} \le M$, then there exists $t_0 = t_0(M) > 0$ such that there exists a unique solution $u(x,t)$ to (A.4) *in $Y_{2M}(\tau, \tau + t_0)$.*

The proof of Proposition 3.1 can be done by an iteration method for its integral equation. Since it is standard, we omit the detail.

Proposition 3.2 (A Priori Estimates). *Let M and T be arbitrarily fixed, and $u(x,t) \in Y_{2M}(0,T)$ be a solution of* (1.1). *Then it holds that*

$$\text{(A.5)} \qquad \|u(t)\|_{H^2}^2 + \int_0^t \|u_x(\tau)\|_{H^1}^2 d\tau \le 2\|u_0\|_{H^2}^2 + 2^{p-1}\|u_0\|_{H^1}^{2(p+1)}.$$

Proof. Multiplying (1.1) by u and integrating it over $[0, t] \times \boldsymbol{R}$, we get the first basic energy equality

$$\text{(A.6)} \qquad \|u(t)\|_{H^1}^2 + 2\int_0^t \|u_x(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{H^1}^2.$$

Furthermore, multiplying (1.1) by $-u_{xx}$ and integrating it over $[0, t] \times \boldsymbol{R}$, we then obtain

$$\text{(A.7)} \qquad \|u_x(t)\|_{H^1}^2 + 2\int_0^t \|u_{xx}(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \int_R u_{xx} u^p u_x \, dx d\tau = \|u_{0x}\|_{H^1}^2.$$

Making use of the Sobolev inequality (2.11), for $f \in H^1(\boldsymbol{R})$,

$$\|f\|_{L^\infty} \le \sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2} \le (\|f\|_{L^2}^2 + \|f_x\|_{L^2}^2)^{1/2},$$

and noting the first basic energy equality (A.6), and the Cauchy inequality $|ab| \le \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ for any $\varepsilon > 0$, we have

$$\begin{aligned}
\text{(A.8)} \quad & 2\int_0^t \int_R u_{xx} u^p u_x \, dx d\tau \\
&\le 2\varepsilon \int_0^t \|u_{xx}\|_{L^2}^2 d\tau + \frac{1}{2\varepsilon}\int_0^t \int_R |u|^{2p} |u_x|^2 dx d\tau \\
&\le 2\varepsilon \int_0^t \|u_{xx}\|_{L^2}^2 d\tau + \frac{1}{2\varepsilon}\|u(t)\|_{L^\infty}^{2p} \int_0^t \|u_x\|_{L^2}^2 d\tau \\
&\le 2\varepsilon \int_0^t \|u_{xx}\|_{L^2}^2 d\tau + \frac{2^{p-1}}{\varepsilon}(\|u(t)\|_{L^2}^2 + \|u_x(t)\|_{L^2}^2)^p \int_0^t \|u_x\|_{L^2}^2 d\tau \\
&\le 2\varepsilon \int_0^t \|u_{xx}\|_{L^2}^2 d\tau + \frac{2^{p-2}}{\varepsilon}\|u_0\|_{H^1}^{2(p+1)}
\end{aligned}$$

Substituting (A.8) into (A.7) and choosing $\varepsilon = 1/2$, we have

$$\text{(A.9)} \qquad \|u_x(t)\|_{H^1}^2 + \int_0^t \|u_{xx}(\tau)\|_{L^2}^2 d\tau \le \|u_{0x}\|_{H^1}^2 + 2^{p-1}\|u_0\|_{H^1}^{2(p+1)}.$$

Thus, combining (A.6) and (A.9), we prove the estimate (A.5). □

Proof of Theorem 3.1. For any given initial value $u_0(x) \in H^2(\boldsymbol{R})$, let M be the constant such that

$$M^2 \ge 2\|u_0\|_{H^2}^2 + 2^{p-1}\|u_0\|_{H^1}^{2(p+1)},$$

then there exists a unique local solution $u(x,t) \in Y_{2M}(0,t_0)$ by Proposition 3.1 with $\tau = 0$. For such a local solution in $[0,t_0]$, by using Proposition 3.2, we have the *a priori* estimate as follows

$$\|u(t)\|_{H^2}^2 + \int_0^t \|u_x(\tau)\|_{H^1}^2 d\tau \le 2\|u_0\|_{H^2}^2 + 2^{p-1}\|u_0\|_{H^1}^{2(p+1)} \le M^2,$$

which implies $u \in Y_M(0,t_0)$ and $\|u(t_0)\|_{H^2}^2 \le M^2$. Now at the "initial" time $\tau = t_0$, Proposition 3.1 gives $u(x,t) \in Y_{2M}(t_0, 2t_0)$. So it extends $u(x,t) \in Y_{2M}(0,2t_0)$. Then Proposition 3.2 further shows the a priori estimate (A.5) for all $t \in [0,2t_0]$ and $u \in Y_M(0,2t_0)$, especially, the solution $u(x,t)$ at $t = 2t_0$ to be bounded as $\|u(2t_0)\|_{H^2} \le M$. Repeating the previous procedure, we will finally prove $u(x,t) \in X_M(0,+\infty)$ and (A.1). □

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References

[1] Amick, C. J., Bona, J. L. and Schonbek, M. E., Decay of solutions of some nonlinear wave equations, J. Differential Equations, **81** (1989) 1–49.

[2] Benjamin, T. B., Bona, J. L. and Mahony, J. J., Model equations for long waves in nonlinear dispersive systems, Philos. Trans. Royal Soc. London A, **272** (1972) 47–78.

[3] Bisognin, V., On the asymptotic behavior of the solutions of a nonlinear dispersive system of Benjamin-Bona-Mahony's type, Boll. Un. Mat. Ital. B(7), **10** (1996) 99–128.

[4] Bisognin, V. and Perla Menzala, G., Decay rates of the solutions of nonlinear dispersive equations, Proc. Roy. Soc. Edinburgh Sect. A, **124** (1994) 1231–1246.

[5] Bona, J. L. and Luo, L., Decay of solutions to nonlinear, dispersive wave equations, Differential and Integral Equations, **6** (1993) 961–980.

[6] Bona, J. L. and Luo, L., More results on the decay of solutions to nonlinear, dispersive wave equations, Discrete and Continuous Dynamics Systems, **1** (1995) 151–193.

[7] Dix, D. B., The dissipation of nonlinear dispersive waves: The case of asymptotically weak nonlinearity, Comm. Partial Differential Equations, **17** (1992) 1665–1693.

[8] Jeffrey, A. and Zhao, H., Global existence and optimal temporal decay estimates for system parabolic conservation laws. I. The one-dimensional case, Applicable Analysis, **70** (1998) 175–193.

[9] Karch, G., Large-time behaviour of solutions to nonlinear wave equations: higher-order asymptotics, Math. Meth. Appl. Sci., **22** (1999) 1671–1697.

[10] Kinami, S., Mei, M. and Omata, S., Convergence to diffusion waves of the solutions for Benjamin-Bona-Mahony-Burgers equations, Applicable Analysis, **75** (2000) 317–340.

[11] Mei M., Large-time behavior of solution for generalized Benjamin-Bona-Mahony-Burgers equations, Nonlinear Anal., **33** (1998) 699–714.

[12] Mei M., L_q-decay rates of solutions for Benjamin-Bona-Mahony-Burgers equations, J. Differential Equations, **158** (1999) 314–340.

[13] Namkin, P. I., Large-time asymptotic behavior of a step for Benjamin-Bona-Mahoney-Burgers equation, Proc. Roy. Soc. Edinburgh, **126A** (1996) 1–18.

[14] Namkin, P. I. and Shishmarov, I. A., Nonlinear nonlocal equations in the theory of waves, Trans. Math. Monographs, **133**, 1994.

[15] Peregrine, D. H., Calculations of the development of an undular bore, J. Fluid Mech., **25** (1966) 321–330.

[16] Segal, I. E., Quantization and dispersion for nonlinear relativistic equations, Mathematical Theory of Elementary Particles, M. I. T. Press, Combridge, Mass., (1966) 79–108.

[17] Zhang, L., Decay of solutions of generalized Benjamin-Bona-Mahony-Burgers equations, Acta Math. Sinica (New Series), **10** (1994) 428–438.

[18] Zhang, L., Decay of solutions of generalized Benjamin-Bona-Mahony-Burgers equations in n-dimensional space, Nonlinear Anal., **25** (1995) 1343–1369.

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