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Stationary solutions for a new hybrid quantum model for semiconductors with discontinuous pressure functional and relaxation time

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Abstract
In this paper we propose a generalization of the hybrid model for semiconductors already discussed by Chiarelli et al.and Di Michele et al., including a non-constant pressure functional and relaxation time. Roughly speaking, we assume that the normalized electron temperature and the relaxation time in the classical and quantum domains are different from each other. We derive the model heuristically, introducing a generalization of the stress tensor, which accounts for an *interface contribute*, and afterwards we prove the existence and uniqueness of weak solutions for such a new hybrid model. We apply the approach proposed by Di Michele et al. to obtain the stationary solutions to our problem, namely we prove the existence of the solution for a regularized problem, then we achieve the existence of a weak solution for the hybrid problem as a proper limit of the regular solution previously obtained.

Keywords
Hybrid quantum hydrodynamic model, hybrid quantum drift-diffusion model, discontinuous pressure, one-dimensional stationary solutions, existence, uniqueness

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1. Introduction

The fast development of the semiconductor industry raises a series of interesting issues, both for theoretical and numerical researchers. One of the major challenges is how to properly model the quantum effect inside a device, reducing the numerical costs. The *hybrid strategy* is one of the most promising methods to reach such an objective. Simply put, the word *hybrid* indicates a theoretical and/or numerical approach requiring the partition of the device domain into classical and quantum parts, and the usage of a classical model in the largest part of the domain, reducing significantly the numerical costs. However, this approach poses a further problem: which kind of transmission conditions must be selected at the interface between the classical and quantum domains? Many solutions have been proposed in recent decades. For example, in the pioneering work of Ben Abdallah [1] the author presents, in the one-dimensional setting, a set of interface conditions obtained from an asymptotic analysis of the Wigner transform. These conditions manage to connect the Boltzmann equation and a set of Schrödinger equations, modeling respectively, the classical and the quantum domains.

In order to reduce the computational costs, macroscopic equations, such as the drift diffusion and hydrodynamical ones, can be employed to model the classical and quantum zones [2–7]. A similar strategy was also employed in Di Michele et al. [8] with application to electrolyte-oxide-semiconductor (EOS) devices. In the present paper, instead of using two different models in the classical and in the quantum domain, we adopt the fully hybrid approach firstly introduced in Chiarelli et al. [9] and then developed in Di Michele et al. [10–12]. In particular, we improve the results in Di Michele et al. [10] allowing some parameters of the problem, namely the temperature T and the relaxation time τ, to be non-constant, assuming different values within the classical and quantum domains.

This work is divided into four sections. In the first one we recall the model introduced in Chiarelli et al. [9] for a quantum drift–diffusion model, and derive the new hybrid model taking into account the discontinuous pressure functional and the relaxation time. In the second section we summarize the main results, then in the third we discuss the existence of smooth solutions to the regularized problem called the $\mathrm{H}-\mathrm{Q}_q\mathrm{HD}$ model. Finally we perform the limit $q \to 0$ obtaining, under a suitable set of assumptions, the existence of the weak solutions for the hybrid quantum hydrodynamic (H-QHD) model.

Finally let us recall, for the sake of completeness, that many important results have been obtained concerning non-hybrid models for semiconductors [13–24, 27].

2. An H-QHD model with discontinuous pressure functional and relaxation time

In this section we present a new hybrid quantum hydrodynamic model accounting for a discontinuous pressure functional and the relaxation time. The hybrid model has been basically derived using the approach proposed in Ancona and Iafrate [25] for the standard fully quantum system, and recently used in Chiarelli et al. [9] Michele et al. [10–12] in the context of quantum hybrid models for semiconductors. In the present paper we introduce and study the one-dimensional case on the bounded domain $\Omega = [0, 1]$. However, the model can be extended to the multi-dimensional case.

We recall the quantum effect function $Q : \Omega \to [0, 1]$, introduced in Chiarelli et al. [9] and Di Michele et al. [10,11], which is a smooth function that indicates where the internal energy depends on the gradient of the charge density.

Let us introduce the stress tensor function as follows

$$\sigma := -P - n\left(n\frac{\partial e_Q}{\partial n_x}\right)_x - \frac{\varepsilon^2}{2}\int_0^x Q'\frac{n_x^2}{n},$$

where

$$P := n\left(n\frac{\partial e_Q}{\partial n} + n_x\frac{\partial e_Q}{\partial n_x}\right) \tag{1}$$

is the pressure functional (according to Ancona and Iafrate [25]) and

$$e_Q := T_Q \ \ln\ n - Q\frac{\varepsilon^2}{2}\frac{n_x^2}{n^2} \tag{2}$$

is the internal energy. As usual, n is the electron density and ε is the scaled Plank constant. The last term in equation (1) models the interface contribution to the stress functional. It represents a sort of localized contribution needed to transform a 'classical' electron into a 'quantum' one and vice versa. We remark that this new term only acts on the interface region where $Q' \neq 0$. Then, in the special case in which $Q(x)$ is a regularization of a step function (or a composition of step functions) and x_0 is an *interface point* (or one of the interface points) where $Q' \rightarrow \delta(x_0)$, interface pressure acts on the isolated point x_0 of our interval (or on a set of isolated points) and this new term does not appear in the standard formulation of the hydrodynamical equation (4).

Moreover, $T_Q : \Omega \rightarrow [T_c, T_q]$ and it is defined as $T_Q = T_c + \Delta T Q$, where $\Delta T = T_q - T_c$. T_c and T_q represent the electron temperatures in the classical and quantum domains, respectively. We assume $T_c > 0$ and that there exists two positive constants T_M and T_m such that $T_m \leqslant T_Q \leqslant T_M$. We only remark that a priori ΔT can be positive, negative or zero. In comparison with the hybrid model presented in Chiarelli et al. [9] and Di Michele et al. [10,12], in the present paper the electron temperature T can be different in the quantum and classical sub-domains. In particular, we have $Q = 0$ and $T_Q = T_c$ in the classical region, whereas in the quantum region we have $Q = 1$ and $T_Q = T_q$. As explained in Di Michele et al. [10], the transition region between the classical and quantum domains should be of order of magnitude approximately ε. This allows a strong coupling between the classical and quantum domains.

In view of equations (1) and (2), the stress function equation (1) can be rewritten as

$$\begin{aligned} \sigma = &-T_Q n + Q\varepsilon^2 n_{xx} - Q\varepsilon^2 \frac{n_x^2}{n} + Q'\varepsilon^2 n_x \\ &-\frac{\varepsilon^2}{2}\int_0^x Q' \frac{n_x^2}{n}. \end{aligned} \tag{3}$$

We recall that the QHD system in the one-dimensional case reads as follows (see for example Gardner [16]):

$$\begin{cases} n_t + J_x = 0 \\ J_t + \left(\dfrac{J^2}{n} - \sigma\right)_x = nV_x - \dfrac{J}{\tau_Q} \\ \lambda^2 V_{xx} = n - C(x). \end{cases} \tag{4}$$

Here V is the electrical potential, J is the current density (assumed to be strictly positive) and $\tau_Q(x) > 0$ is the scaled relaxation time, which is assumed to be a continuous, non-constant function. In particular, we set $\tau_Q(x) = \tau_c + \Delta\tau Q$, where $\Delta\tau = \tau_q - \tau_c$, and $\tau_c > 0$ and $\tau_q > 0$ are the relaxation times in the classical and quantum domains, respectively. In what follows we assume there exists two positive constants τ_m and τ_M, such that $\tau_m \leqslant \tau_Q \leqslant \tau_M$. In this work we do not consider the evolution of the internal energy, but we focus our study on the steady states of the one-dimensional system above, assuming isothermal conditions both in the quantum and classical domains. A jump of the scaled electron temperature is allowed at the interface between the classical and quantum domains.

Observing that

$$\begin{aligned} &\varepsilon^2\left(Qn_{xx} - Q\frac{n_x^2}{n} + Q'n_x - \frac{\varepsilon^2}{2}\int_0^x Q'\frac{n_x^2}{n}\right)_x \\ &= 2n\varepsilon^2\left(Q\frac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q'\frac{(\sqrt{n})_x}{\sqrt{n}}\right)_x, \end{aligned} \tag{5}$$

we can write the hybrid stationary hydrodynamic system as follows

$$\begin{cases} 2n\varepsilon^2\left(Q\dfrac{\sqrt{n}_{xx}}{\sqrt{n}}+Q'\dfrac{\sqrt{n}_x}{\sqrt{n}}\right)_x - \\ \left(T_Q n+\dfrac{J^2}{n}\right)_x + nV_x = \dfrac{J}{\tau_Q}, \\ J = constant, \\ \lambda^2 V_{xx} = n - C, \end{cases} \tag{6}$$

where the last equation is the usual self-consistent Poisson equation, which models the effects of the electric potential V. The parameter $\lambda > 0$ is the scaled Debye length. The function $C(x)$, appearing in the Poisson equation, models the doping profile, which is the background fixed charge of ions in the semiconductor crystal. From the mathematical point of view, we assume $C(x) \in L^2(0,1)$ and $C(x) \geqslant C_0 > 0$ for all x in $[0, 1]$.

The main purpose of this paper is to prove the existence of a weak solution to the following problem

$$(S)\begin{cases} 2\varepsilon^2\left(Q\dfrac{(\sqrt{n})_{xx}}{\sqrt{n}}+Q'\dfrac{(\sqrt{n})_x}{\sqrt{n}}\right)_x - \\ \left(T_Q+\dfrac{J^2}{2n^2}\right)_x - T_Q(\ln n)_x + V_x = \dfrac{J}{\tau_Q n}, \\ \lambda^2 V_{xx} = n - C, \\ n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \\ V(0) = V_0, \quad J = J_0, \end{cases}$$

where $(\mathrm{S})_1$ has been obtained by dividing equation $(6)_1$ by n, and $(\mathrm{S})_2$ is the Poisson equation. The choice of the boundary condition is often employed in these kinds of problems and will be tackled at the beginning of the next section. Here we only observe that there are many other possible choices, which still make the problem well posed.

3. Working systems and main theorems

In this section we summarize the results presented in the whole paper. Let us differentiate $(\mathrm{S})_1$ with respect to x. Then, in view of the Poisson equation, we derive the following fourth-order differential equation for electronic density n, with the associated boundary conditions:

$$(P)\begin{cases} 2\varepsilon^2\left(Q\dfrac{(\sqrt{n})_{xx}}{\sqrt{n}}+Q'\dfrac{(\sqrt{n})_x}{\sqrt{n}}\right)_{xx} - \\ \left((T_Q-\dfrac{J^2}{n^2})\dfrac{n_x}{n}\right)_x + \dfrac{1}{\lambda^2}(n-C) \\ \qquad = T_Q'' + \left(\dfrac{J}{\tau_Q n}\right)_x, \\ n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \\ V(0) = V_0, \quad J = J_0. \end{cases}$$

The first equation is a fourth–order differential equation, which for any fixed $J = J_0$, is well posed assuming $n(0) = n(1) = 1,\ \ n_x(0) = n_x(1) = 0$ as in $(\mathrm{P})_2$. Moreover, the behavior of the electrical potential V can be reconstructed starting from the Poisson equation in view of the conditions in $(\mathrm{P})_3$ and

$$V_0 = -2\varepsilon^2 Q(0)(\sqrt{n})_{xx}(0) + \frac{J^2}{2} + T_Q(0), \tag{7}$$

then, we get

$$V(x) = -2\varepsilon^2 Q\frac{(\sqrt{n})_{xx}}{\sqrt{n}} - 2\varepsilon^2 Q'\frac{(\sqrt{n})_x}{\sqrt{n}} + \frac{J^2}{2n^2} + T_Q - \int_0^x \frac{J}{\tau_Q n}\mathrm{d}x + \int_0^x T_Q(\ln n)_x \mathrm{d}x. \tag{8}$$

Therefore, the value of the electrical potential at $x = 1$ can be derived from equation (8) assuming $(\mathrm{P})_2$:

$$V_1 = -2\varepsilon^2 Q(1)(\sqrt{n})_{xx}(1) + \frac{J^2}{2} + T_Q(1) - J\int_0^1 \frac{1}{\tau_Q n}\mathrm{d}x + \int_0^1 T_Q(\ln n)_x \mathrm{d}x = V(1). \tag{9}$$

This implies that it is completely equivalent to fix $V(1)$ or J.

Remark 3.1. *We say that the flow is sub-sonic when*

$$\frac{|J|}{n} < \sqrt{p'(n)} = \sqrt{T_Q}, \tag{10}$$

where $p(n) = nT_Q$ is the pressure.

Here we need a stronger sub-sonic condition, namely

$$\frac{|J|}{n} < \frac{\sqrt{T_Q}}{\left(1 + \frac{1}{\tau_Q^2}\right)} \leqslant \frac{\sqrt{T_M}}{\left(1 + \frac{1}{\tau_M^2}\right)}. \tag{11}$$

Clearly equation (11) *implies the uniform ellipticity of the operator* $\left(\left(\frac{T_Q}{n} - \frac{J^2}{n^3}\right)n_x\right)_x$.

By the boundary conditions $n(0) = n(1) = 1$, the following compatibility condition for J_0 and $\sqrt{T_M}$ needs to be satisfied:

$$|J_0| < \frac{\sqrt{T_M}}{\left(1 + \frac{1}{\tau_m^2}\right)}. \tag{12}$$

As usual, we introduce the new variable $w = \sqrt{n}$, then (P) can be rewritten as

$$(P_w)\begin{cases} 2\varepsilon^2\left(Q\dfrac{w_{xx}}{w} + Q'\dfrac{w_x}{w}\right)_{xx} - \\ 2\left(\left(T_Q - \dfrac{J^2}{w^4}\right)\dfrac{w_x}{w}\right)_x + \dfrac{1}{\lambda^2}(w^2 - C) \\ \quad = \left(\dfrac{J}{\tau_Q w^2}\right)_x + T_Q'', \\ \lambda^2 V_{xx} = w^2 - C, \\ w(0) = w(1) = 1, \quad w_x(0) = w_x(1) = 0, \\ V(0) = V_0, \quad J = J_0. \end{cases}$$

Here and after, we will mainly focus on the above system. Notice that the fourth-order elliptic equations in (P) and (P_w) are regionally degenerate where $Q = 0$, and this can make it difficult to prove the existence of the solutions to this kind of boundary value problem (BVP). Therefore, we first look for the solution $(w_q, V_q)(x)$ to (P_w) where, instead of Q, we consider a strictly positive function $Q_q \geqslant q > 0$, such that $Q_q \to Q$ when $q \to 0$. Clearly, even the functions T_Q and τ_Q are automatically regularized by definition and we call them T_{Q_q} and τ_{Q_q}, respectively. Then, by taking the hybrid limit $q \to 0$, we expect that the solution $(w_q, V_q)(x)$ of the BVP (P_w) converges to the really hybrid solution $(w, V)(x)$ in the weak sense. Finally, we show that $(n = w^2, V)$ is the weak solution to the BVP (P). The proof of these results requires two steps: firstly, we introduce and study the mollified problem that we will call (P_q) (see below), then we refocus on the original problems, namely (P_w) and (P).

Step 1

First of all, we consider a modified (P_w) system where we replace $Q(x)$ by the strictly positive function $Q_q(x)$ as explained above. Let $(w_q, V_q)(x)$ be the solutions to the following mollified problem

$$(P_q)\begin{cases} 2\varepsilon^2\left(Q_q\dfrac{(w_q)_{xx}}{w_q} + Q_q'\dfrac{(w_q)_x}{w_q}\right)_{xx} - \\ 2\left(\left(T_{Q_q} - \dfrac{J^2}{w_q^4}\right)\dfrac{(w_q)_x}{w_q}\right)_x + \\ \dfrac{1}{\lambda^2}(w_q^2 - C) = T_Q'' + \left(\dfrac{J}{Q_{\tau,q}w_q^2}\right)_x, \\ \lambda^2 (V_q)_{xx} = w_q^2 - C, \\ w_q(0) = w_q(1) = 1, \quad (w_q)_x(0) = (w_q)_x(1) = 0, \\ V_q(0) = V_0, \quad J = J_0. \end{cases}$$

Let $\tilde{T}_m$ be a strictly positive constant such that

$$T_{Q_q} = T_c + \Delta T Q_q > \tilde{T}_m.$$

The following theorem establishes the existence of the solutions for (P_q).

Theorem 3.2. (Existence of solutions to problem (P_w). *Under the sub-sonic conditions, equations* (11) *and* (12), *assume that $Q_q(x)$ is a non-negative, smooth, and bounded function defined on $\Omega = [0, 1]$ such that*

$$\begin{aligned} &0 < q \leqslant Q_q \leqslant 1 \\ &\alpha := \max\left(\|Q_q'\|_\infty, \|Q''_q\|_\infty\right) < \infty \quad \textit{for all} \quad x \in \Omega, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \varepsilon^2 \max_{x\in\Omega}\frac{|Q_q'|^2}{Q_q} &< 4\left(T_Q - \left(1 + \frac{1}{\tau_Q^2}\right)\frac{J_0^2}{n^2}\right) \\ &< 4\left(T_M - \left(1 + \frac{1}{\tau_M^2}\right)\frac{J_0^2}{n^2}\right). \end{aligned} \tag{14}$$

Then (P_q) admits one solution at least $(w_q, V_q) \in H^4(\Omega) \times H^2(\Omega)$.

The following theorem states the uniqueness of the solution just obtained.

Theorem 3.3. Uniqueness of solutions to H$-$Q$_q$HD. *Assume equations* (11), (13) *and* (14), *then for ε and J small enough and independent of q, the BVP*

$$
\begin{cases}
\varepsilon^2\left(Q_q\left((u_q)_{xx}+\dfrac{(u_q)_x^2}{2}\right)+Q_q'(u_q)_x\right)_{xx}+ \\
(J^2e^{-2u_q}(u_q)_x)_x-(T_{Q_q}(u_q)_x)_x+ \\
\dfrac{e^{u_q}-C(x)}{\lambda^2}-\left(\dfrac{J}{\tau_{Q_q}}e^{-u_q}\right)_x-T''_{Q_q}=0, \\
u_q(0)=u_q(1)=0, \quad (u_q)_x(0)=(u_q)_x(1)=0
\end{cases}
\tag{15}
$$

has a unique solution.

Step 2

As a second step, using a vanish viscosity approach, we show the existence of a solution for (P_q), proving the following theorem.

Theorem 3.4. Hybrid limits and existence of solutions to (P_q). *Assume the sub-sonic conditions, equations* (11) *and* (12). *For any given hybrid quantum effect function $Q(x)\in C^1(0,1)$ with $0\leqslant Q(x)\leqslant 1$, let us construct a sequence $\{Q_q\}$ satisfying the following properties for all $x\in[0,1]$*

$$
\begin{cases}
\{Q_q, Q'_q\} \underset{q\to 0}{\rightarrow} \{Q, Q'\} \ \textit{uniformly in } \Omega, \\
\alpha := \max\left(\|Q_q'\|_\infty, \ \|Q_q''\|_\infty\right)<\infty \ \textit{for all} \quad q \\
\varepsilon^2 \max\limits_{x\in\Omega}\dfrac{|Q_q'|^2}{Q_q}<4\left(T_M-\left(1+\dfrac{1}{\tau_m^2}\right)\dfrac{J_0^2}{n^2}\right).
\end{cases}
\tag{16}
$$

Let $(w_q, V_q)(x)$ be the sequence of solutions to the BVP (P_q) corresponding to these selected approximating functions Q_q. Then, there exists a pair of functions $(w,V)(x)$ such that the sequence $(w_q,V_q)(x)$ converges to (w,V) as follows

$$
\begin{pmatrix}
w_q \rightharpoonup w \ \textit{in } H^1(\Omega), & \\
w_q \rightarrow w \ \textit{in } C^0(\Omega), & \textit{as } q\to 0. \\
V_q \rightharpoonup V \ \textit{in } L^2(\Omega), &
\end{pmatrix}
\tag{17}
$$

In particular, such a pair of limits $(w,V)(x)$ is the weak solution to the BVP (P_w).

Finally we prove the following theorem.

Theorem 3.5. Existence of solutions to problems (P) and (S). *Under the assumptions of Theorem 3.4 the problem (P) admits one solution $(n,V)(x)$ at least such that $n\in H^1(\Omega)$ and $V\in L^2(\Omega)$. The same pair of functions also solves the system (S).*

3.1. Mollified quantum function

Since it is very important to properly define the regularized sequence $\{Q_q\}, q\in\mathbb{R}_+$, in this section we present an example for the hybrid quantum effect function Q, $0\leqslant Q\leqslant 1$, and for the approximating sequence $\{Q_q\}$.

Remark 3.6. *As noted in Di Michele et al.* [10], *we remark that equation* $(16)_3$ *implies that $|Q_q'|^2/Q_q$ can be bounded also if $Q_q\to 0$. A class of function that verifies this constrain is $|x-x_0|^m$, for all $m\geqslant 2$, when $x\to x_0$. One can show that the Q_q-function proposed in Di Michele et al.* [10], *verifies equation* (16). *Here, we present another example. Take*

$$
Q(x)=\begin{cases}
0 & 0\leqslant x\leqslant \frac{1}{2}, \\
4\left(x-\dfrac{1}{2}\right)^2 & \dfrac{1}{2}<x\leqslant 1,
\end{cases}
\tag{18}
$$

clearly $0 \leqslant Q \leqslant 1$. *Then, we may construct the approximating sequence* $\{Q_q\}$ *as*

$$Q_q(x) = \begin{cases} q & 0 \leqslant x \leqslant \frac{1}{2}, \\ q + 4(1-q)\left(x - \dfrac{1}{2}\right)^2 & \dfrac{1}{2} < x \leqslant 1, \end{cases} \tag{19}$$

where $0 < q < 1$.

$Q'_q(x)$ *and* $Q''_q(x)$ *read, respectively as*

$$Q'_q(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{2}, \\ 8(1-q)\left(x - \dfrac{1}{2}\right) & \dfrac{1}{2} < x \leqslant 1, \end{cases} \tag{20}$$

and

$$Q''_q(x) = \begin{cases} 0 & 0 \leqslant x \leqslant \frac{1}{2}, \\ 8(1-q) & \dfrac{1}{2} < x \leqslant 1. \end{cases} \tag{21}$$

It easy to see that equation $(16)_1$ *is verified. Moreover,* $\| Q'_q \|_\infty = 4(1-q) < 4$ *and* $\| Q'_q \|_\infty = 8(1-q) < 8$. *Therefore, even the second condition in equation* (16) *is verified for all* $\alpha > 8$. *Finally, the last condition in equation* (16) *implies that*

$$|J_0| < n_m \sqrt{\left(T_M - \varepsilon^2 \max_{x \in \Omega} \frac{|Q_{q'}|^2}{4Q_q}\right) \frac{1}{\left(1 + \frac{1}{\tau_m^2}\right)}},$$

where $n_m = w_m^2$, *introduced in Lemma 4.2.*

4. Existence and uniqueness of $\mathrm{H}-\mathrm{Q}_q\mathrm{HD}$ solution and H-QHD

4.1. Step 1: Existence and uniqueness of solution to the BVP (P_q)

In this section we discuss the existence and the uniqueness of the solution of the BVP (P_q). In order to prove Theorem 3.2, we need the following a priori estimates.

Lemma 4.1. a priori estimates. *Assume the sub-sonic conditions, equations* (11) *and* (12), *and that the sequence* Q_q *satisfies equations* (13) *and* (14). *Then, the solution* w_q *to problem* (P_q) *is bounded from above by*

$$||w_q||_{L^\infty(\Omega)} \leqslant w_M. \tag{22}$$

Moreover,

$$\varepsilon^2 \underline{c_1} \int_0^1 (w_q)_{xx}^2 \mathrm{d}x + \underline{c_2} \int_0^1 (w_q)_x^2 \mathrm{d}x \leqslant K, \tag{23}$$

where $\underline{c}_1$, $\underline{c}_2$, *and K are strictly positive constants and therefore* $w_q \in H^2(\Omega)$.

Proof. Multiplying $(P_q)_1$ by $(w_q - 1) \in H_0^1(\Omega)$ and integrating it over the whole domain, we have

$$
\begin{aligned}
&2\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\mathrm{d}x + 2\int_0^1 \left(T_{Q_q} - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\mathrm{d}x \\
&+ 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\mathrm{d}x \\
&= \frac{1}{\lambda^2}\int_0^1 (C - 1 + \lambda^2 T''_{Q_q})(w_q - 1)\mathrm{d}x \\
&- \frac{1}{\lambda^2}\int_0^1 (w_q^2 - 1)(w_q - 1)\mathrm{d}x \\
&- \int_0^1 \frac{J}{\tau_{Q_q} w_q^2}(w_q)_x \mathrm{d}x \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{24}
$$

Proceeding as in Di Michele et al. [10], in view of equation $(16)_3$, we have

$$
\begin{aligned}
I_3 &\leqslant \frac{1}{8\tilde{c}}\int_0^1 \left(\frac{1}{\tau_{Q_q}}\right)^2 \omega_q \mathrm{d}x + 2\tilde{c}\int_0^1 \frac{J^2}{w_q^4}\frac{(w_q)_x^2}{w_q}\mathrm{d}x \\
&= \frac{1}{8\tilde{c}}\int_0^1 \left(\frac{1}{\tau_{Q_q}}\right)^2 (\omega_q - 1)\mathrm{d}x \\
&+ \frac{1}{8\tilde{c}}\int_0^1 \left(\frac{1}{\tau_m}\right)^2 \mathrm{d}x + 2\tilde{c}\int_0^1 \frac{J^2}{w_q^4}\frac{(w_q)_x^2}{w_q}\mathrm{d}x
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
I_1 + I_2 + I_3 &= -\frac{1}{\lambda^2}\int_0^1 (w_q^2 - 1)(w_q - 1)\mathrm{d}x \\
&+ \frac{1}{8\tilde{c}}\int_0^1 \left(\frac{1}{\tau_m}\right)^2 \mathrm{d}x + 2\tilde{c}\int_0^1 \frac{J^2}{w_q^4}\frac{(w_q)_x^2}{w_q}\mathrm{d}x \\
&+ \frac{1}{\lambda^2}\int_0^1 \left(C - 1 + \lambda^2 T''_{Q_q} + \frac{\lambda^2}{8\tilde{c}}\left(\frac{1}{\tau_{Q_q}}\right)^2\right)(w_q - 1)\mathrm{d}x \\
&\leqslant -\frac{1}{\lambda^2}\int_0^1 (w_q - 1)^2 (w_q + 1)\mathrm{d}x \\
&+ \frac{1}{2\lambda^2}\int_0^1 (w_q - 1)^2 \mathrm{d}x \\
&+ \frac{1}{2\lambda^2}\int_0^1 \left(C - 1 + \lambda^2 T''_{Q_q} + \frac{\lambda^2}{8\tilde{c}}\left(\frac{1}{\tau_{Q_q}}\right)^2\right)^2 \mathrm{d}x \\
&+ \frac{1}{8\tilde{c}}\int_0^1 \left(\frac{1}{\tau_m}\right)^2 \mathrm{d}x + 2\tilde{c}\int_0^1 \frac{J^2}{w_q^4}\frac{(w_q)_x^2}{w_q}\mathrm{d}x.
\end{aligned} \tag{26}
$$

Accounting for equations (24) and (26), by setting $\tilde{c} = 1/\tau_m^2$ we get

$$
\begin{aligned}
&2\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\,\mathrm{d}x + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\,\mathrm{d}x \\
&\quad + 2\int_0^1 \left(T_{Q_q} - \left(1 + \frac{1}{\tau_m^2}\right)\frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\,\mathrm{d}x \\
&\quad + \frac{1}{\lambda^2}\int_0^1 (w_q - 1)^2\left(w_q + \frac{1}{2}\right)\mathrm{d}x \\
&\quad \leqslant \frac{1}{8} + \frac{1}{2\lambda^2}\int_0^1 \left(C + 1 + \lambda^2\alpha + \frac{\lambda^2}{8}\right)^2 \mathrm{d}x.
\end{aligned} \tag{27}
$$

We observe that the third term of the left-hand side in equation (27) is strictly positive in view of equation (11), then we construct a quadratic form using the first three terms of the left-hand side in equation (27). Therefore we have

$$
\begin{aligned}
&\int_0^1 \left[2\varepsilon^2 Q_q \frac{(w_q)_{xx}^2}{w_q} + 2\varepsilon^2 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\right. \\
&\quad \left. + 2\left(T_{Q_q} - \left(1 + \frac{1}{\tau_m}\right)\frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\right]\mathrm{d}x \\
&=: \int_0^1 \left(\mathcal{A}_1 \frac{(w_q)_{xx}^2}{w_q} + \mathcal{B}_1 \frac{(w_q)_x (w_q)_{xx}}{w_q} + \mathcal{C}_1 \frac{(w_q)_x^2}{w_q}\right)\mathrm{d}x.
\end{aligned} \tag{28}
$$

The quadratic form we have obtained:

$$
\begin{aligned}
&\mathcal{B}_1^2 - 4\mathcal{A}_1\mathcal{C}_1 \\
&\quad = 4\varepsilon^2\left[\varepsilon^2 |Q_{q'}|^2 - 4Q_q\left(T_{Q_q} - \left(1 + \frac{1}{\tau_{Q_q}^2}\right)\frac{J^2}{w_q^4}\right)\right],
\end{aligned}
$$

is strictly negative defined, in view of the condition equation (14).

Then, there exists two strictly positive constants k_1 and k_2 such that the following inequality holds

$$
\begin{aligned}
&k_1 \int_0^1 \frac{(w_q)_{xx}^2}{w_q}\,\mathrm{d}x + k_2 \int_0^1 \frac{(w_q)_x^2}{w_q}\,\mathrm{d}x \\
&\quad + \frac{1}{\lambda^2}\int_0^1 (w_q - 1)^2\left(w_q + \frac{1}{2}\right)\mathrm{d}x \\
&\quad \leqslant \frac{1}{8} + \frac{1}{2\lambda^2}\int_0^1 \left(C + 1 + \lambda^2\alpha + \frac{\lambda^2}{8}\right)^2 \mathrm{d}x \\
&=: K_0.
\end{aligned} \tag{29}
$$

This inequality implies, proceeding as in Di Michele et al. [10], that $\| \sqrt{w_q} - 1\|_\infty \leqslant \sqrt{\frac{K_0}{k_2}}$. Moreover, setting $w_M = \left(1 + \sqrt{\frac{K_0}{k_2}}\right)^2$, we obtain equation (22), and then equation (23), which follows from equation (29).

Finally, we have to show that the solution w_q is bounded from below for all $x \in \Omega$.

Lemma 4.2. *Under the assumption of Lemma 4.1, provided that*

$$
2\left(\frac{J}{\tau_M \underline{n}}\right) + \int_0^1 \frac{C(x) - \underline{n}}{\lambda^2}\,\mathrm{d}x \geqslant \alpha\left(\frac{J}{\tau_m^2 \underline{n}} + 1\right), \tag{30}
$$

there exists a constant $w_m > 0$ such that

$$\| w_q\|_\infty \geqslant w_m. \tag{31}$$

Proof. The approach used in Di Michele et al. [10] to prove the strictly positiveness of w_q is not immediately applicable due to the non-constant temperature and the relaxation time. Therefore, in the spirit of Gamba and Jüngel [26], in the following we introduce a truncate form of problem P_q

$$\begin{cases} 2\varepsilon^2\left(Q_q\dfrac{(w_q)_{xx}}{w_q}+Q_q'\dfrac{(w_q)_x}{w_q}\right)_{xx} \\ -2\left(\left(T_{Q_q}-\dfrac{J^2}{w_q^4}\right)\dfrac{(w_q)_x}{w_q}\right)_x+\dfrac{1}{\lambda^2}(w_q^2-C) \\ =T''_Q+\left(\dfrac{J}{Q_{\tau,q}t_r^2}\right)_x, \\ \lambda^2(V_q)_{xx}=w_q^2-C, \\ w_q(0)=w_q(1)=1, \quad (w_q)_x(0)=(w_q)_x(1)=0, \\ V_q(0)=V_0, \quad J=J_0, \end{cases} \tag{32}$$

where $t_r = \max(r(x), w)$ and $r(x)=\sqrt{\underline{n}}(2-x)$ for all $x \in [0,1]$. Then, we multiply equation (32) by $(w_q-\sqrt{\underline{n}})^- := \min(0, w_q-\sqrt{\underline{n}})$, observing that, by equation (23), $(w_q-\sqrt{\underline{n}})^- \in H_0^1(\Omega)$. We integrate by parts and after some manipulation (see Di Michele et al. [10] for more details) we get

$$\begin{aligned} &2\varepsilon^2\int_0^1 Q_q\frac{((w_q-\sqrt{\underline{n}})^-)_{xx}^2}{w_q}\mathrm{d}x \\ &\quad+\int_0^1\left(T_Q-\frac{J^2}{w_q^4}\right)\frac{((w_q-\sqrt{\underline{n}})^-)_x^2}{w_q}\mathrm{d}x \\ &\quad+2\varepsilon^2\int_0^1 Q_{q'}\frac{((w_q-\sqrt{\underline{n}})^-)_x((w_q-\sqrt{\underline{n}})^-)_{xx}}{w_q}\mathrm{d}x \\ &=-\frac{1}{\lambda^2}\int_0^1((w_q-\sqrt{\underline{n}})^-)(w_q^2-\sqrt{\underline{n}}^2)\mathrm{d}x \\ &\quad+\frac{1}{\lambda^2}\int_0^1(C(x)-\underline{n})(w_q-\sqrt{\underline{n}})^-\mathrm{d}x \\ &\quad+\int_0^1 T_Q''((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x \\ &\quad+\int_0^1\left(\frac{J}{\tau_Q t_r(x)^2}\right)_x((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x. \end{aligned} \tag{33}$$

Moreover, in view of the uniform upper bound for T_Q'' we have

$$I_1 \leqslant \alpha\int_0^1((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x, \tag{34}$$

whereas the last term can be estimated as follows

$$
\begin{aligned}
I_1 =& -\int_0^1 \frac{J\tau'_Q}{\tau_Q^2 t_r^2}((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x \\
& -\int_0^1 \frac{2Jt'_r}{\tau_Q^2 t_r^3}((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x \\
=& -\int_0^1 \frac{J\tau'_Q}{\tau_Q^2 t_r^2}((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x \\
& + \int_0^1 \frac{2J\sqrt{\underline{n}}}{\tau_Q^2 t_r^3}((w_q-\sqrt{\underline{n}})^-)\mathrm{d}x.
\end{aligned} \tag{35}
$$

Proceeding as in the proof of Lemma 4.1, we can rearrange the three terms on the left-hand side as a quadratic form which is strictly negative defined in view of the assumption equation (14). Then, accounting for the estimates, equations (34) and (35), we get

$$
\begin{aligned}
& k_1\int_0^1 ((w_q-\sqrt{\underline{n}})^-)^2_{xx}\mathrm{d}x + k_2\int_0^1 ((w_q-\sqrt{\underline{n}})^-)^2_x\mathrm{d}x \\
& + k_3\int_0^1 ((w_q-\sqrt{\underline{n}})^-)^2(w_q+\sqrt{\underline{n}})\mathrm{d}x \\
& \leqslant \int_0^1 f(\underline{n},x)(w_q-\sqrt{\underline{n}})^-\mathrm{d}x,
\end{aligned} \tag{36}
$$

where k_1 and k_2 are two strictly positive constants and

$$
f(\underline{n},x) = \left(\frac{C(x)-\underline{n}}{\lambda^2} + \frac{J}{\tau_Q r^2}\left(\frac{2\sqrt{\underline{n}}}{r} - \frac{\tau'_Q}{\tau_Q}\right) + T''_Q\right).
$$

Therefore, under the assumption, equation (30), the inequality equation (36) implies $(w_q-\sqrt{\underline{n}})^- = 0$ for all x in $[0, 1]$, namely, $w_q \geqslant \sqrt{\underline{n}} > 0$ for all $x \in [0, 1]$, that is equation (31).

In the following Lemma we derive an a priori estimate for the variable $u_q = \ln n_q$.

Lemma 4.3. *Under the assumption of Lemmas 4.1 and 4.2, the variable u_q, defined as $u_q = \ln n_q$, verifies the following estimate*

$$
k_1 \| (u_q)_{xx}\|_{L^2(\Omega)} + k_2 \| (u_q)_x\|_{L^2(\Omega)} \leqslant K_0, \tag{37}
$$

and there exists a constant $u_M > 0$ such that

$$
\| u_q\|_\infty \leqslant u_M. \tag{38}
$$

Proof. The proof follows by using the definition of u as a direct consequence of Lemmas 4.1 and 4.2.

Now we can prove Theorem 3.2, namely the existence of one solution, at least, to problem (P_q).

4.1.1. Proof of Theorem 3.2. Let $u_q = \ln n_q$, then the BVP (P_q) becomes equation (15):

$$
\begin{cases}
\varepsilon^2\left(Q_q\left((u_q)_{xx} + \frac{(u_q)_x^2}{2}\right) + Q'_q(u_q)_x\right)_{xx} \\
\quad + (J^2 e^{-2u_q}(u_q)_x)_x - (T_{Q_q}(u_q)_x)_x \\
\quad + \frac{e^{u_q}-C(x)}{\lambda^2} - \left(\frac{J}{\tau_{Q_q}}e^{-u_q}\right)_x - T''_{Q_q} = 0, \\
u_q(0) = u_q(1) = 0, \qquad (u_q)_x(0) = (u_q)_x(1) = 0.
\end{cases}
$$

Since $Q_q \geqslant q > 0$, equation (15) basically is a QHD model, therefore we can apply the standard approaches based on the Leray–Shauder fixed point theorem [18,20,21]. Let us take $\nu \in X = C^{0,1}(\Omega)$ and consider the following linear problem

$$\begin{aligned}
&\varepsilon^2\Big(Q_q\Big((u_q)_{xx} + \frac{\sigma}{2}\nu_x^2\Big) + Q_q'(u_q)_x\Big)_{xx} + \sigma J^2\big(e^{-2\nu}\nu_x\big)_x \\
&-(T_{Q_q}(u_q)_x)_x + \frac{\sigma}{\lambda^2}\left(\frac{e^{\nu}-1}{\nu}u_q + 1 - C\right) \\
&-\sigma\frac{J}{\tau_{Q_q}}(e^{-\nu})_x - \sigma\frac{J}{(\tau_{Q_q})_x}(e^{-\nu}) = 0,
\end{aligned} \tag{39}$$

coupled with the boundary conditions equation $(15)_2$, where $\sigma \in [0,1]$. For each $u_q, \phi \in H^2(\Omega)$, the following bilinear form is continuous and coercive in $H^2(\Omega)$ for $\phi \in H^2(\Omega)$:

$$\begin{aligned}
a(u_q,\phi) = &\int_0^1 \Big(\varepsilon^2\Big(Q_q(u_q)_{xx} + Q_q'(u_q)_x\Big)\phi_{xx}\Big)\mathrm{d}x \\
&+ \int_0^1 \left(T_{Q_q}(u_q)_x)\phi_x + \frac{\sigma}{\lambda^2}\frac{e^{u_q}-1}{\nu}u_q\phi\right)\mathrm{d}x
\end{aligned}$$

and the functional F defined as

$$\begin{aligned}
F(\phi) = &-\int_0^1 \left(\sigma\frac{J}{\tau_{Q_q}}e^{-\nu_k}\phi_x\right)\mathrm{d}x + \int_0^1 T'_{Q_q}\phi_x \mathrm{d}x \\
&\int_0^1 -Q\frac{\varepsilon^2\sigma}{2}\nu_x^2\phi_{xx} + \sigma J^2 e^{-2\nu_k}\nu_x\phi_x + \frac{\sigma}{\lambda^2}(C-1)\phi\mathrm{d}x
\end{aligned}$$

is linear and continuous in $H^2(\Omega)$ for $\phi \in H^2(\Omega)$. By using the Lax–Milgram Lemma, we get the existence of a unique solution $u \in H^2(\Omega)$ to the BVP equation (39)–$(15)_2$. By doing so, we have defined a continuous and compact fixed point operator on $X \equiv H^2$

$$S : X \times [0,1] \to X, \ (\nu,\sigma) \to u_q \tag{40}$$

verifying

- $S(\nu, 0) = 0$ for all $\nu \in X$,
- there is a constant $c > 0$ such that

$$\| u\|_X \leqslant c, \tag{41}$$

for all $(u_q, \sigma) \in X \times [0,1]$ satisfying $S(u_q, \sigma) = u_q$.

For $\sigma = 1$ the inequality equation (41) follows from the a priori estimates already discussed, whereas for $0 < \sigma < 1$ it can be obtained proceeding in a similar way.

The existence of a fixed point u_q follows applying the Leray–Schauder fixed point theorem.

Now we prove the uniqueness of the sub-sonic solution to equation (15) for sufficiently small values of the current density J.

4.1.2. Proof of Theorem 3.3. As in Di Michele et al. [10], we prove the theorem by contradiction. Following Brezzi et al. [14] and Gyi and Jüngel [18], let $u_q, v_q \in H^2(\Omega)$ be two solutions to equation (15), then the difference between the two corresponding equations gives

$$
\begin{aligned}
&\varepsilon^2\big(Q_q(u_q-v_q)_{xx}\big)_{xx}+\varepsilon^2 Q_q\left(\frac{(u_q)_x^2}{2}-\frac{(v_q)_x^2}{2}\right)_{xx}\\
&+\varepsilon^2\big(Q_{q'}(u_q-v_q)_x\big)_{xx}\\
&-\frac{J^2}{2}\big((e^{-2u_q}-e^{-2v_q})(u_q)_x\big)_x\\
&+\frac{J^2}{2}\big(e^{-2v_q}(u_q-v_q)_x\big)_x-\big(T_{Q_q}(u_q-v_q)_x\big)_x\\
&+\frac{e^{u_q}-e^{v_q}}{\lambda^2}-\left(\frac{J}{\tau_{Q_q}}(e^{-u_q}-e^{-v_q})\right)_x=0
\end{aligned}
\tag{42}
$$

coupled with the following boundary conditions

$$
\begin{aligned}
&(u_q-v_q)(0)=(u_q-v_q)(1)=0\\
&(u_q-v_q)_x(0)=(u_q-v_q)_x(1)=0.
\end{aligned}
\tag{43}
$$

Multiplying equation (42) by $(u_q-v_q)\in H_0^2(\Omega)$ and integrating it by parts over the whole domain, we get

$$
\begin{aligned}
&\varepsilon^2\int_0^1 Q_q(u_q-v_q)_{xx}^2\mathrm{d}x+\int_0^1 T_{Q_q}(u_q-v_q)_x^2\mathrm{d}x\\
&+\frac{\varepsilon^2}{2}\int_0^1 Q_q(u_q+v_q)_x(u_q-v_q)_x(u_q-v_q)_{xx}\mathrm{d}x\\
&+\frac{1}{\lambda^2}\int_0^1 (e^{u_q}-e^{v_q})(u_q-v_q)\mathrm{d}x\\
&=\varepsilon^2\int_0^1 Q''_q(x)(u_q-v_q)_x^2\mathrm{d}x\\
&+\varepsilon^2\int_0^1 Q'_q(x)(u_q-v_q)_{xx}(u_q-v_q)_x\mathrm{d}x\\
&+\frac{J^2}{2}\int_0^1 (e^{-2u_q}-e^{-2v_q})(u_q)_x(u_q-v_q)_x\mathrm{d}x\\
&+\frac{J^2}{2}\int_0^1 e^{-2v_q}(u_q-v_q)_x^2\mathrm{d}x\\
&-\int_0^1 \frac{J}{\tau_{Q_q}}(e^{-u_q}-e^{-v_q})(u_q-v_q)_x\mathrm{d}x.
\end{aligned}
\tag{44}
$$

We observe that

$$
\frac{1}{\lambda^2}\int_0^1 (e^{u_q}-e^{v_q})(u_q-v_q)\mathrm{d}x>0
$$

and, from the boundedness results (Lemmas 4.1, 4.3 and 4.2), we obtain uniform estimates for $u_q, v_q, (u_q)_x, (v_q)_x$ in L^∞. Moreover, by equation (43) the Poincaré inequality $||(u_q-v_q)||\leqslant c_p||(u_q-v_q)_x||$ holds. Standard computations, by using also the mean value theorem, lead to the following estimates:

$$
\begin{aligned}
& J^2 \int_0^1 (e^{-2u_q} - e^{-2v_q})(u_q)_x (u_q - v_q)_x \mathrm{d}x \\
& \leqslant C_1 J^2 \int_0^1 |(u_q - v_q)_x|^2 \mathrm{d}x
\end{aligned}
$$

and

$$
J^2 \int_0^1 e^{-2v_q}(u_q - v_q)_x^2 \mathrm{d}x \leqslant C_2 J^2 \int_0^1 |(u_q - v_q)_x|^2 \mathrm{d}x.
$$

Since $T_{Q_q} > \tau_m > 0$, one has

$$
\begin{aligned}
& \int_0^1 \frac{J}{\tau_{Q_q}} (e^{-u_q} - e^{-v_q})(u_q - v_q)_x \mathrm{d}x \\
& \leqslant \frac{|J|}{\tau_m} C_3 \int_0^1 |(u_q - v_q)_x|^2 \mathrm{d}x.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& \frac{\varepsilon^2}{2} \int_0^1 Q_q (u_q + v_q)_x (u_q - v_q)_x (u_q - v_q)_{xx} \mathrm{d}x \\
& \leqslant \frac{\varepsilon^2}{4} \int_0^1 Q_q |(u_q - v_q)_{xx}|^2 \mathrm{d}x \\
& + C_4 \varepsilon^2 \int_0^1 |(u_q - v_q)_x|^2 \mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
& \varepsilon^2 \int_0^1 Q_q'(x)(u_q - v_q)_{xx}(u_q - v_q)_x \mathrm{d}x \\
& = \frac{\varepsilon^2}{2} \int_0^1 Q_q'((u_q - v_q)_x^2)_x \mathrm{d}x \\
& = -\frac{\varepsilon^2}{2} \int_0^1 Q''_q |(u_q - v_q)_x|^2 \mathrm{d}x.
\end{aligned}
$$

Introducing the previous estimates in equation (44) we obtain

$$
\begin{aligned}
& \varepsilon^2 q \left(1 - \frac{1}{4}\right) \int_0^1 (u_q - v_q)_{xx}^2 \mathrm{d}x \\
& + k \int_0^1 |(u_q - v_q)_x|^2 \mathrm{d}x \leqslant 0,
\end{aligned}
$$

where α has been defined in equation (13) and

$$
k = \left(\tilde{T}_m - C_1 J^2 - C_2 J^2 - C_3 \frac{|J|}{\tau_m} - C_4 \varepsilon^2 - \frac{\varepsilon^2 \alpha}{2}\right).
$$

Then, taking ε and $|J|$ small enough, the uniqueness follows.

Finally, we prove the main Theorem 3.4, following Gyi and Jüngel [18]. Since both functions Q_q and T_{Q_q} are smooth enough, one can show that there exists a solution $u_q \in H^4(\Omega)$ to equation (15).

Consequently, observing that $w_m^2 \leqslant n_q = e^{u_q} \leqslant w_M^2$, the BVP (P_q) admits a unique solution $n_q \in H^4(\Omega)$. Finally, $V_q \in H^2(\Omega)$, thanks to the Poisson equation. This concludes the proof.

4.2. Step 2: Hybrid limit: existence of solution to the BVPs (P_w) and P

Let $(w, V)(x)$ be the weak solutions to the problem (P_w) defined as follows.

Definition 4.4. *$(w, V)(x)$ is said to be a weak solution of (P_w), if for any $\phi \in C_0^\infty(\Omega)$ it holds*

$$\begin{aligned} 2\varepsilon^2 &\int_0^1 \left(Q\frac{w_{xx}}{w} + Q'\frac{w_x}{w}\right)\phi_{xx}\mathrm{d}x \\ &+ 2\int_0^1 \left((T_Q - \frac{J^2}{w^4})\frac{w_x}{w}\right)\phi_x \mathrm{d}x \\ &+ \int_0^1 \frac{1}{\lambda^2}(w^2 - C)\phi \mathrm{d}x \\ &+ \int_0^1 \left(\frac{J}{\tau_Q w^2} - T'_Q\right)\phi_x \mathrm{d}x = 0. \end{aligned} \tag{45}$$

In a similar way, from $(S)_1$, for the electrical potential we get:

$$\begin{aligned} &\int_0^1 V\phi_x \mathrm{d}x = -2\varepsilon^2 \int_0^1 Q(x)\frac{w_{xx}}{w}\phi_x \mathrm{d}x \\ &- 2\varepsilon^2 \int_0^1 Q'(x)\frac{w_x}{w}\phi_x \mathrm{d}x \\ &+ \int_0^1 \frac{J^2}{2w^4}\phi_x \mathrm{d}x + \int_0^1 T_Q \phi_x \mathrm{d}x \\ &+ 2\int_0^1 T_Q \ln w \phi_x \mathrm{d}x \\ &+ 2\int_0^1 T'_Q \ln w \phi \mathrm{d}x + \int_0^1 \frac{J}{\tau_Q}\frac{1}{w^2}\phi_x \mathrm{d}x. \end{aligned} \tag{46}$$

In this section we show that even the limit problem (P_w) admits one solution at least, as explained in Theorem 3.4, and we discuss its regularity. The hybrid problem is obtained from (P_q) for $q \to 0$, namely for the real hybrid case we have $0 \leqslant Q \leqslant 1$. In particular, if $Q = 0$, we obtain the classical quantum hydrodynamical equation. Moreover, the function $T_{Q_q} = T_c + \Delta T Q_q$, for $q \to 0$ reaches the electron temperature T_C in the classical domain and T_Q in the quantum one. The same holds for the function $Q_{\tau,q} = \tau_Q + \Delta\tau Q_q$, namely $Q_{\tau,q} \to \tau_c$ in the classical domain, and $Q_{\tau,q} \to \tau_q$ in the quantum one. A transition limit between the classical and quantum regions is still admitted, indeed $\max(\| Q'\|_\infty, \| Q''\|_\infty) = \alpha$.

4.2.1. Proof of Theorem 3.4.

Let $\{Q_q\}$ be a sequence of approximating functions to the quantum function $Q \in C^1$, satisfying equation (16). Let $(w_q, V_q)(x)$ be the solutions to (P_q) and $\bar{K}$ or $\bar{c}_i$ all q-independent constants. Following the approach already proposed by Di Michele et al. [10], we show that

$$||w_q||_{H^1(\Omega)} \leqslant \bar{K}, \quad ||\sqrt{Q_q} w_{q,xx}||_{L^2(\Omega)} \leqslant \bar{K}. \tag{47}$$

We proceed as in the proof of Lemma 4.1, until we obtain the inequality equation (27). Then, after some calculations, we get

$$\begin{aligned}
&\int_0^1 \left[\frac{\varepsilon^2 Q_q}{w_q} w_{q_{xx}}^2 + \frac{2\varepsilon^2 Q'_q}{w_q} w_{q_x} w_{q_{xx}} \right.\\
&\left. + \left(\frac{T_{Q_q}}{w_q} - \left(1+\frac{1}{\tau_m^2}\right)\frac{J^2}{w_q{}^5}\right) w_{q_x}^2\right] \mathrm{d}x\\
&+ \int_0^1 \left[\frac{\varepsilon^2 Q_q}{w_q} w_{q_{xx}}^2 + \left(\frac{T_{Q_q}}{w_q} - \left(1+\frac{1}{\tau_m^2}\right)\frac{J^2}{w_q{}^5}\right) w_{q_x}^2\right] \mathrm{d}x\\
&=: \int_0^1 (\mathcal{A}_2 w_{q_{xx}}^2 + \mathcal{B}_2 w_{q_x} w_{q_{xx}} + \mathcal{C}_2 w_{q_x}^2) \mathrm{d}x\\
&+ \int_0^1 \frac{\varepsilon^2 Q_q}{w_q} w_{q_{xx}}^2 + \left(\frac{T_{Q_q}}{w_q} - \left(1+\frac{1}{\tau_m^2}\right)\frac{J^2}{w_q{}^5}\right) w_{q_x}^2 \mathrm{d}x.
\end{aligned} \tag{48}$$

We observe that the first term on the right-hand side can be seen as a positive-definite quadratic form since $\mathcal{B}_2^2 - 4\mathcal{A}_2\mathcal{C}_2 < 0$ by equation (16). Moreover, by the same equation (16), the second term on the right-hand side is positive. Therefore, equation (48) simply implies

$$\begin{aligned}
&\int_0^1 \frac{\varepsilon^2 Q_q}{w_q} w_{q_{xx}}^2 \mathrm{d}x\\
&+ \int_0^1 \left(T_{Q_q} - \left(1+\frac{1}{\tau_m^2}\right)\frac{J^2}{w_q{}^5}\right) w_{q_x}^2 \mathrm{d}x \leqslant \bar{K}
\end{aligned} \tag{49}$$

where $\frac{\varepsilon^2 Q_q}{w_q} w_{q_{xx}}^2 \geqslant 0$. In view of equation (11),

$$T_m - \left(1+\frac{1}{\tau_m^2}\right)\frac{J^2}{w_q{}^5} \geqslant \bar{c}_1 > 0,$$

where $\bar{c}_1$ is q-independent. Therefore, as in Lemma 4.1, we get

$$||w_q||_{L^\infty(\Omega)} \leqslant \bar{K}, \tag{50}$$

and

$$\bar{c}_2 \varepsilon^2 \int_0^1 Q_q w_{q_{xx}}^2 \mathrm{d}x + \bar{c}_3 \int_0^1 w_{q_x}^2 \mathrm{d}x \leqslant \bar{K}, \tag{51}$$

where the uniform upper and lower bound for w_q and the assumption $0 < Q_q \leqslant 1$ have been used. The estimate equation (47) follows directly from equations (50) and (51). According to the standard theory, we can show that there exists a $w(x)$ such that

$$w_q \rightharpoonup w \quad \text{in} \quad H^1(\Omega), \tag{52}$$

$$w_q \to w \quad \text{in} \quad C^0(\Omega), \tag{53}$$

for $q \to 0$. The last result can be obtained observing that $H^1(\Omega) \hookrightarrow C^0(\Omega)$. Finally, starting from the general expression for the electrical potential given in equation (8), we obtain

$$\begin{aligned}
V_q(x) = &- 2\varepsilon^2 Q \frac{w_{q_{xx}}}{w_q} - 2\varepsilon^2 Q' \frac{w_{q_x}}{w_q} + \frac{J^2}{2w_q^4} + T_Q\\
&- \int_0^x \frac{J}{\tau_Q} \frac{1}{w_q^2} \mathrm{d}x + 2\int_0^x T_Q (\ln w_q)_x \mathrm{d}x.
\end{aligned} \tag{54}$$

From equations (16) and (47), we get $\| V_q \|_{L^2} \leqslant \bar{K}$. This implies the existence of a function V such that

$$V_q \rightharpoonup V \quad \text{in} \quad L^2(\Omega). \tag{55}$$

Now we have to prove that (w, V) are weak solutions to (P_w) and they satisfy equations (45) and (46), respectively. Let $\phi \in C_0^\infty(\Omega)$ a given test function. We multiply the first equation in P_q by ϕ. After integration by parts, in view of the boundary conditions, we get

$$\begin{aligned}
&2\varepsilon^2 \int_0^1 \left(Q_q \frac{w_{q_{xx}}}{w_q} + Q_q' \frac{w_{q_x}}{w_q} \right) \phi_{xx} \mathrm{d}x \\
&+ 2 \int_0^1 \left(T_{Q_q} - \frac{J^2}{w_q{}^4} \right) \frac{w_{q_x}}{w_q} \phi_x \mathrm{d}x \\
&- \int_0^1 (T_{Q_q}) \phi_{xx} \mathrm{d}x + \int_0^1 \frac{w_q{}^2 - C}{\lambda^2} \phi \mathrm{d}x \\
&+ \int_0^1 \left(\frac{J}{\tau_{Q_q} w_q{}^2} \right) \phi_x \mathrm{d}x = 0.
\end{aligned} \tag{56}$$

We have to show that equation (56) converges in L^2 to the hybrid problem in the weak form, that is

$$\begin{aligned}
&2\varepsilon^2 \int_0^1 \left(Q \frac{w_{xx}}{w} + Q_q' \frac{w_x}{w} \right) \phi_{xx} \mathrm{d}x \\
&+ 2 \int_0^1 \left(Q_T - \frac{J^2}{w^4} \right) \frac{w_x}{w} \phi_x \mathrm{d}x \\
&- \int_0^1 (Q_T) \phi_{xx} \mathrm{d}x + \int_0^1 \frac{w^2 - C}{\lambda^2} \phi \mathrm{d}x \\
&+ \int_0^1 \left(\frac{J}{Q_\tau w^2} \right) \phi_x \mathrm{d}x = 0.
\end{aligned} \tag{57}$$

This convergence can be obtained by means of equations (16), (31) and (47). Therefore, w is the weak solution to (P_w).

We proceed in a similar way to prove that V is a weak solution to (P_w). We multiply equation (54) by ϕ, then after integration on Ω we obtain:

$$\begin{aligned}
&\int_0^1 V_q \phi_x \mathrm{d}x = -2\varepsilon^2 \int_0^1 Q_q \frac{w_{xx}}{w} \phi_x \mathrm{d}x \\
&- 2\varepsilon^2 \int_0^1 Q_q' \frac{w_x}{w} \phi_x \mathrm{d}x \\
&+ \int_0^1 \frac{J^2}{2 w_q{}^4} \phi_x \mathrm{d}x + \int_0^1 T_{Q_q} \phi_x \mathrm{d}x \\
&+ 2 \int_0^1 T_{Q_q} \ln w_q \phi_x \mathrm{d}x \\
&+ 2 \int_0^1 T_{Q_q}' \ln w \phi \mathrm{d}x + \int_0^1 \frac{J}{\tau_{Q_q}} \frac{1}{w_q{}^2} \phi_x \mathrm{d}x.
\end{aligned} \tag{58}$$

Now, as above, we have to show that equation (58) converges in L^2 to the weak form for the potential in the hybrid case, that is

$$
\begin{aligned}
\int_0^1 V\phi_x \mathrm{d}x = & - 2\varepsilon^2 \int_0^1 Q(x)\frac{w_{xx}}{w}\phi_x \mathrm{d}x \\
& - 2\varepsilon^2 \int_0^1 Q'(x)\frac{w_x}{w}\phi_x \mathrm{d}x \\
& + \int_0^1 \frac{J^2}{2w^4}\phi_x \mathrm{d}x + \int_0^1 T_Q\phi_x \mathrm{d}x \\
& + 2\int_0^1 T_Q \ln w\phi_x \mathrm{d}x \\
& + 2\int_0^1 T_Q' \ln w\phi \mathrm{d}x + \int_0^1 \frac{J}{\tau_Q}\frac{1}{w^2}\phi_x \mathrm{d}x.
\end{aligned}
\tag{59}
$$

The thesis follows observing that $V_q \rightharpoonup V$ in L^2 and that the limit potential V is a solution in the weak sense of the Poisson equation.

4.2.2. Proof of Theorem 3.5. By definition, just observing that $w = \sqrt{n}$.

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