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Asymptotic behavior of solutions to Euler–Poisson equations for bipolar hydrodynamic model of semiconductors

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ABSTRACT

In this paper we study the Cauchy problem for 1-D Euler–Poisson system, which represents a physically relevant hydrodynamic model but also a challenging case for a bipolar semiconductor device by considering two different pressure functions and a non-flat doping profile. Different from the previous studies (Gasser et al., 2003 [7], Huang et al., 2011 [12], Huang et al., 2012 [13]) for the case with two identical pressure functions and zero doping profile, we realize that the asymptotic profiles of this more physical model are their corresponding stationary waves (steady-state solutions) rather than the diffusion waves. Furthermore, we prove that, when the flow is fully subsonic, by means of a technical energy method with some new development, the smooth solutions of the system are unique, exist globally and time-algebraically converge to the corresponding stationary solutions. The optimal algebraic convergence rates are obtained.

1. Introduction

Hydrodynamic models are usually used in the description of the charged fluid particles such as electrons and holes in semiconductor devices and positively and negatively charged ions in plasma [2,16,23,31], and are presented as Euler–Poisson equations. For unipolar hydrodynamic model, the studies on the existence of solutions and their large time behavior as well as relaxation-time limit have been extensively carried out, for example, see [1,3–6,8,9,14,15,17–20,22,24,27,29,30,33,34] and

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the references therein. However, for the bipolar hydrodynamic models, the related research on this topic now becomes more and more attractive [7,10–13,26,28,32], but, due to complexity and difficulty of the system itself, the research is still little and quite incomplete.

In this paper, continuing our recent studies in [12,13,26], we consider the 1-D bipolar hydrodynamic model of semiconductor devices, the Euler–Poisson system with the physically relevant assumptions of non-flat doping profile and two different pressure functions:

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n)\right)_x = nE - J, \\ h_t + K_x = 0, \\ K_t + \left(\dfrac{K^2}{h} + q(h)\right)_x = -hE - K, \\ E_x = n - h - D(x), \end{cases} \tag{1.1}$$

with the initial-value condition

$$\begin{cases} n(x,0) = n_0(x) \to n_\pm \quad \text{as } x \to \pm\infty, \\ h(x,0) = h_0(x) \to h_\pm \quad \text{as } x \to \pm\infty, \\ J(x,0) = J_0(x) \to J_\pm \quad \text{as } x \to \pm\infty, \\ K(x,0) = K_0(x) \to K_\pm \quad \text{as } x \to \pm\infty, \\ E(-\infty,t) = E_-. \end{cases} \tag{1.2}$$

Here, $n = n(x,t) > 0$, $h = h(x,t) > 0$, $J = J(x,t)$ and $K = K(x,t)$ represent the density of electrons, the density of holes, the current of electrons, and the current of holes, respectively, and $E = E(x,t)$ is the electrical field. The nonlinear functions $p(n)$ and $q(h)$ denote the pressures of the electrons and the holes, respectively, which are usually different (more physical case) and satisfy:

$$p, q \in C^3(0,+\infty), \quad \text{with } s^2 p'(s) > 0 \text{ and } s^2 q'(s) > 0 \text{ strictly increasing for } s > 0. \tag{1.3}$$

$D(x) \neq 0$ is the doping profile standing for the density of impurities in semiconductor devices. $n_\pm$, $h_\pm$, $J_\pm$, $K_\pm$ and E_- are the state constants for the quantities at far fields.

In the special case, when the doping profile is completely flat $D(x) \equiv 0$ (the flat doping profile means $|D'(x)| \ll 1$), the two pressure functions are completely identical $p(s) \equiv q(s)$, and $E_- = 0$, the system (1.1) reduces to

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n)\right)_x = nE - J, \\ h_t + K_x = 0, \\ K_t + \left(\dfrac{K^2}{h} + p(h)\right)_x = -hE - K, \\ E_x = n - h, \end{cases} \tag{1.4}$$

and the asymptotic behavior of the solution was intensively studied in [7,11–13]. All these previous studies consider the diffusion waves as the asymptotic profiles for the original solutions. Here, the so-called diffusion waves for (1.4) are the self-similar solutions to the corresponding porous media equations

$$\begin{cases} \bar{n}_t = p(\bar{n})_{xx}, \\ \bar{J} = -p(\bar{n})_x, \\ (\bar{n}, \bar{J}) = (\bar{n}, \bar{J})\left(\dfrac{x}{\sqrt{1+t}}\right) \to (n_\pm, 0) \quad \text{as } x \to \pm\infty. \end{cases} \tag{1.5}$$

In the switch-off case (the device has no global voltage, i.e., $E(-\infty, t) - E(+\infty, t) = 0$) but a pioneering work on the study of asymptotic behavior of solutions to the bipolar semiconductor model, Gasser, Hsiao and Li [7] first proved that the smooth solutions of the initial-value problem to the bipolar hydrodynamic system (1.4) converge to the diffusion waves (1.5), precisely,

$$\sup_{x\in R}\left|(n, J, h, K, E)(x,t) - (\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)\left(\frac{x}{\sqrt{1+t}}\right)\right| = O(1)\left(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, e^{-\nu_0 t}\right) \tag{1.6}$$

for the initial-perturbation in the sense of L^2, while

$$\sup_{x\in R}\left|(n, J, h, K, E)(x,t) - (\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)\left(\frac{x}{\sqrt{1+t}}\right)\right| = O(1)\left(t^{-1}, t^{-\frac{3}{2}}, t^{-1}, t^{-\frac{3}{2}}, e^{-\nu_0 t}\right) \tag{1.7}$$

for the initial perturbation in L^1-sense. See also the corresponding convergence in weak sense in [11]. For the switch-on case, there exist some L^2-gaps between the original solutions and the corresponding diffusion waves at far field $x = \pm\infty$, and for this reason the convergence to the diffusion waves was an open problem for many years. By heuristically analyzing what are those exact gaps, Huang, Mei and Wang [12] technically constructed some correction functions to fill in the L^2-gaps and proved the L^∞-convergence (1.6) and (1.7) by the energy method. Furthermore, in [13] they obtained the L^∞-stability of diffusion waves to the case with boundary effect.

When the two pressure functions are different, $p(s) \neq q(s)$, both densities of electrons and holes, $n(x,t)$ and $h(x,t)$, should have different asymptotic profiles, so do the currents of electrons and holes, $J(x,t)$ and $K(x,t)$, and when $D(x) \neq 0$ is non-flat, and $E_- \neq 0$, thus, 0 will not be the asymptotic profile for the electrical field $E(x,t)$. Based on such an observation, obviously the above-mentioned diffusion waves $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x/\sqrt{1+t})$ are no longer the asymptotic profiles of the original solutions $(n, J, h, K, E)(x,t)$. A natural but important question is what will be the really asymptotic profiles for the system (1.1) and (1.2) in this really physical case, and how to derive the optimal convergence rates. These will be the main targets considered in the present paper.

Inspired by the study on unipolar hydrodynamic model, and by the variable scaling method, as we pointed out in [26] for the bounded domain case, the better asymptotic profiles for the system (1.1) should be its corresponding steady-state system

$$\begin{cases} \mathcal{J}_x = 0, \\ \left(\dfrac{\mathcal{J}^2}{\mathcal{N}} + p(\mathcal{N})\right)_x = \mathcal{N}\mathcal{E} - \mathcal{J}, \\ \mathcal{K}_x = 0, \\ \left(\dfrac{\mathcal{K}^2}{\mathcal{H}} + q(\mathcal{H})\right)_x = -\mathcal{H}\mathcal{E} - \mathcal{K}, \\ \mathcal{E}_x = \mathcal{N} - \mathcal{H} - D(x), \\ (\mathcal{N}, \mathcal{H})(\pm\infty) = (n_\pm, h_\pm), \\ \mathcal{E}(-\infty) = E_-. \end{cases}$$

These steady-state solutions are also called stationary waves. In this paper, when the flow is fully subsonic, even if the system is in the switch-on case (global voltage exists in the device, i.e., $E(-\infty,t) \neq E(+\infty,t)$) and the case of $p(s) \neq q(s)$, $|D'(x)| \not\ll 1$ and $E_- \neq 0$, we will prove that the solutions $(n, J, h, K, E)(x,t)$ of the system (1.1) and (1.2) are unique, exist globally and converge to the stationary waves $(\mathcal{N}, \mathcal{J}, \mathcal{H}, \mathcal{K}, \mathcal{E})(x)$ time-algebraically in the form of

$$\sup_{x\in R}\big|(n, J, h, K, E)(x,t) - (\mathcal{N}, \mathcal{J}, \mathcal{H}, \mathcal{K}, \mathcal{E})(x)\big| = O(1)\big(t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}, t^{-\frac{5}{4}}, t^{-\frac{3}{4}}\big)$$

for the initial-perturbation in the sense of L^2. In order to get such optimal decay rates in the sense of L^2 initial perturbation, here we have to face two technical difficulties:

i) $p(s) \neq q(s)$. Indeed when $p(s) = q(s)$, $D(x) = 0$ and $E_- = 0$, after perturbation around the diffusion waves, the main working equation for the perturbation of the electrical field function derived in [7,12] is the single Klein–Gordon equation

$$\chi_{tt} + \chi_t - \big(p'(\bar{n})\chi_x\big)_x + 2\bar{n}\chi = f,$$

which can be standardly proved to be time-exponentially decaying. While, when $p(s) \neq q(s)$, $D(x) \neq 0$ and $E_- \neq 0$, the governing equations are the strongly coupled system of damped wave equations (see (4.1))

$$\begin{cases} \phi_{tt} + \phi_t - \big(p'(\mathcal{N})\phi_x\big)_x + \mathcal{N}\chi = f, \\ \psi_{tt} + \psi_t - \big(q'(\mathcal{H})\psi_x\big)_x + \mathcal{H}\chi = g, \\ \chi = \phi - \psi, \end{cases}$$

which will be more complicated and more difficult to treat than the single Klein–Gordon equation. In fact, this system possesses only algebraic decay (see also the special case of constant equilibria studied in [21]);

ii) Different from the bounded domain case studied in our previous work [26], where we can establish the Poincaré inequality which then guarantees an exponential decay for the solution, here we can obtain only algebraic decay rates, and to get the desired energy estimates is more technical than the case of bounded domain (see Lemma 4.3 and Lemma 4.4 later).

The paper is organized as follows. In Section 2, we first investigate the corresponding steady-state equations, and prove the unique existence of the steady-state solutions (called also stationary waves). In Section 3, we analyze what are the exact L^∞-gaps between the original solutions and the stationary waves at far fields $x = \pm\infty$, and use the technique we recently developed in [12] to construct some correction functions to delete those gaps, such that the perturbation around the stationary waves filling with these correction functions are in $L^2(R)$, then we state our convergence theorem. Finally, in Section 4, we give the proof of convergence theorem. Here the crucial step is to establish the *a priori* energy estimates with some new development.

At the end of this section, we introduce some notations. Throughout this paper, the stationary waves are denoted by $(\mathcal{N}, \mathcal{J}, \mathcal{H}, \mathcal{K}, \mathcal{E})(x)$, and the correction functions are denoted by $(\hat{n}, \hat{J}, \hat{h}, \hat{K}, \hat{E})(x,t)$. C_0, C_i, *etc.* always denote some specific positive constants, and C denotes the generic positive constant. $L^2(\mathbb{R})$ is the space of square integrable real valued functions defined on $\mathbb{R}$ with the norm $\|\cdot\|$, and $H^k(\mathbb{R})$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$. We also denote

$$\big\|(f_1, f_2, \ldots, f_m)\big\|^2 := \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_m\|^2.$$

Let $T > 0$ be a number and $\mathbf{B}$ be a Banach space. We denote by $C^0([0,T], \mathbf{B})$ the space of the $\mathbf{B}$-valued continuous functions on $[0,T]$, $L^2([0,T], \mathbf{B})$ as the space of the $\mathbf{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathbf{B}$-valued functions on $[0,\infty)$ are defined similarly.

2. Stationary waves

In this section, we investigate the existence of stationary solutions to the 1-D steady-state equations of (1.1), namely, the following system of equations

$$\begin{cases} \mathcal{J}_x = 0, \\ \left(\dfrac{\mathcal{J}^2}{\mathcal{N}} + p(\mathcal{N})\right)_x = \mathcal{N}\mathcal{E} - \mathcal{J}, \\ \mathcal{K}_x = 0, \\ \left(\dfrac{\mathcal{K}^2}{\mathcal{H}} + q(\mathcal{H})\right)_x = -\mathcal{H}\mathcal{E} - \mathcal{K}, \\ \mathcal{E}_x = \mathcal{N} - \mathcal{H} - D(x), \\ (\mathcal{N}, \mathcal{H})(\pm\infty) = (n_\pm, h_\pm), \\ \mathcal{E}(-\infty) = E_-. \end{cases} \tag{2.1}$$

Let

$$\mathcal{E}(+\infty) = E_+.$$

Since we expect $\mathcal{N}_x(\pm\infty) = 0$, $\mathcal{H}_x(\pm\infty) = 0$ and $\mathcal{E}_x(\pm\infty) = 0$, from (2.1) and (1.2) we immediately obtain

$$\mathcal{J} = n_\pm E_\pm, \qquad \mathcal{K} = -h_\pm E_\pm, \qquad D_\pm := D(\pm\infty) = n_\pm - h_\pm, \tag{2.2}$$

which implies the following compatibility conditions that we need to assume throughout this paper

$$E_+ = \frac{n_- E_-}{n_+} = \frac{h_- E_-}{h_+}.$$

Denote

$$D^* = \sup_{x\in\mathbb{R}} D(x), \qquad D_* = \inf_{x\in\mathbb{R}} D(x). \tag{2.3}$$

We also assume that the doping profile satisfies

$$D(x) \in C^0(\mathbb{R}), \quad \text{and} \quad \left|D(x) - D_\pm\right| \leqslant O(1)\left|D^* - D_*\right| e^{-\eta_\pm |x|} \quad \text{as } x \to \pm\infty, \tag{2.4}$$

where $\eta_\pm > 0$ are two constants. Dividing the second and the fourth equation of (2.1) by $\mathcal{N}$ and $\mathcal{H}$ and differentiating them with respect to x, and substituting $(2.1)_5$ to the resultant equations, respectively, we have

$$\begin{cases} \left(\left(\dfrac{p'(\mathcal{N})}{\mathcal{N}} - \dfrac{\mathcal{J}^2}{\mathcal{N}^3}\right)\mathcal{N}_x\right)_x = \mathcal{N} - \mathcal{H} - D(x) + \mathcal{J}\dfrac{\mathcal{N}_x}{\mathcal{N}^2}, \\ \left(\left(\dfrac{q'(\mathcal{H})}{\mathcal{H}} - \dfrac{\mathcal{K}^2}{\mathcal{H}^3}\right)\mathcal{H}_x\right)_x = \mathcal{H} - \mathcal{N} + D(x) + \mathcal{K}\dfrac{\mathcal{H}_x}{\mathcal{H}^2}, \\ (\mathcal{N}, \mathcal{H})(\pm\infty) = (n_\pm, h_\pm). \end{cases} \tag{2.5}$$

In order to keep the ellipticity of the system (2.5), we need

$$\frac{1}{\mathcal{N}}p'(\mathcal{N}) - \frac{\mathcal{J}^2}{\mathcal{N}^3} > 0 \iff \mathcal{N}^2 p'(\mathcal{N}) > \mathcal{J}^2,$$
$$\frac{1}{\mathcal{H}}q'(\mathcal{H}) - \frac{\mathcal{K}^2}{\mathcal{H}^3} > 0 \iff \mathcal{H}^2 q'(\mathcal{H}) > \mathcal{K}^2,$$

which imply that the velocities of electrons and holes must satisfy

$$\left|(\mathcal{U},\mathcal{V})\right| := \left|\left(\frac{\mathcal{J}}{\mathcal{N}},\frac{\mathcal{K}}{\mathcal{H}}\right)\right| < \left|\left(\sqrt{p'(\mathcal{N})},\sqrt{q'(\mathcal{H})}\,\right)\right| =: c(\mathcal{N},\mathcal{H}) \quad \text{(the speed of sound)},$$

namely, the system describes a fully subsonic flow. Since both $s^2p'(s)$ and $s^2q'(s)$ are increasing for $s>0$ (see (1.3)), we can conclude that there are minima values $n_*, h_* > 0$ such that

$$s^2p'(s) > \mathcal{J}^2 \quad \text{for } s > n_*, \quad \text{and} \quad s^2q'(s) > \mathcal{K}^2 \quad \text{for } s > h_*.$$

So, in order to keep the system to be uniformly elliptic, or equivalently, the flow to be fully subsonic, we need to restrict

$$n_\pm > n_*, \qquad h_\pm > h_*. \tag{2.6}$$

Now we prove existence and uniqueness of stationary solutions to (2.1), even for the non-flat doping profile $D(x)$, namely we may allow $D'(x)$ to be large.

Theorem 2.1 *(Existence and uniqueness of stationary waves). Assume that (1.3), (2.4) and (2.6) hold. There exists a constant $\delta_0 > 0$, such that when*

$$\eta := |n_+ - n_-| + |h_+ - h_-| + |E_+| + |E_-| + \left|D^* - D_*\right| \leqslant \delta_0, \tag{2.7}$$

then the system (2.1) possesses a unique classical solution $(\mathcal{N},\mathcal{J},\mathcal{H},\mathcal{K},\mathcal{E})(x)$ which satisfies

$$\mathcal{N}(x) > n_* \quad \textit{and} \quad \mathcal{H}(x) > h_* \quad \textit{for all } x \in R, \tag{2.8}$$
$$p'(\mathcal{N}) - \frac{\mathcal{J}^2}{\mathcal{N}^2} > 0, \qquad q'(\mathcal{H}) - \frac{\mathcal{K}^2}{\mathcal{H}^2} > 0 \quad (\textit{uniform ellipticity}), \tag{2.9}$$
$$\left|\mathcal{N}(x) - n_-\right| + \left|\mathcal{H}(x) - h_-\right| + \left|\mathcal{E}(x) - E_-\right| \leqslant C\eta e^{-\eta_-|x|} \quad \textit{for } x \leqslant 0, \tag{2.10}$$
$$\left|\mathcal{N}(x) - n_+\right| + \left|\mathcal{H}(x) - h_+\right| + \left|\mathcal{E}(x) - E_+\right| \leqslant C\eta e^{-\eta_+|x|} \quad \textit{for } x \geqslant 0, \tag{2.11}$$
$$\left|\mathcal{N}_x(x)\right| + \left|\mathcal{H}_x(x)\right| + \left|\mathcal{E}_x(x)\right| \leqslant C\eta e^{-\eta_-|x|} \quad \textit{for } x \leqslant 0, \tag{2.12}$$
$$\left|\mathcal{N}_x(x)\right| + \left|\mathcal{H}_x(x)\right| + \left|\mathcal{E}_x(x)\right| \leqslant C\eta e^{-\eta_+|x|} \quad \textit{for } x \geqslant 0, \tag{2.13}$$

where $\eta_\pm$ are defined in (2.4).

Proof. Define the solution space by

$$X = \left\{(n,h,\varepsilon)(x) \,\middle|\, n,h,\varepsilon \in C^1(\mathbb{R}) \text{ with } n > n_* \text{ and } h > h_*\right\}$$

equipped with the norm

$$\begin{aligned}\big\|(n,h,\varepsilon)\big\|_X = {}& \sup_{x\in\mathbb{R}_-} e^{\eta_-|x|}\big\{\big|n(x)-n_-\big|+\big|h(x)-h_-\big|+\big|\varepsilon(x)-E_-\big|\big\}\\ &+\sup_{x\in\mathbb{R}_+} e^{\eta_+|x|}\big\{\big|n(x)-n_+\big|+\big|h(x)-h_+\big|+\big|\varepsilon(x)-E_+\big|\big\}\\ &+\sup_{x\in\mathbb{R}_-} e^{\eta_-|x|}\big\{\big|n_x(x)\big|+\big|h_x(x)\big|+\big|\varepsilon_x(x)\big|\big\}\\ &+\sup_{x\in\mathbb{R}_+} e^{\eta_+|x|}\big\{\big|n_x(x)\big|+\big|h_x(x)\big|+\big|\varepsilon_x(x)\big|\big\}.\end{aligned}$$

Notice that (2.1) is equivalent to

$$\begin{cases}\mathcal{J}=n_\pm E_\pm,\\ \left(\dfrac{p'(\mathcal{N})}{\mathcal{N}}-\dfrac{\mathcal{J}^2}{\mathcal{N}^3}\right)\mathcal{N}_x=\mathcal{E}-\dfrac{\mathcal{J}}{\mathcal{N}},\\ \mathcal{K}=-h_\pm E_\pm,\\ \left(\dfrac{q'(\mathcal{H})}{\mathcal{H}}-\dfrac{\mathcal{K}^2}{\mathcal{H}^3}\right)\mathcal{H}_x=-\mathcal{E}-\dfrac{\mathcal{K}}{\mathcal{H}},\\ \mathcal{E}_x=\mathcal{N}-\mathcal{H}-D(x),\\ (\mathcal{N},\mathcal{H})(\pm\infty)=(n_\pm,h_\pm),\\ \mathcal{E}(-\infty)=E_-.\end{cases}\tag{2.14}$$

Letting $(n,h,\varepsilon)\in X$, one has

$$\begin{aligned}&\big|n(x)-n_-\big|\leqslant O(1)|n_+-n_-|e^{-\eta_-|x|}\quad \text{for } x\leqslant 0,\\ &\big|n(x)-n_+\big|\leqslant O(1)|n_+-n_-|e^{-\eta_+|x|}\quad \text{for } x\geqslant 0,\\ &\big|h(x)-h_-\big|\leqslant O(1)|h_+-h_-|e^{-\eta_-|x|}\quad \text{for } x\leqslant 0,\\ &\big|h(x)-n_+\big|\leqslant O(1)|h_+-h_-|e^{-\eta_+|x|}\quad \text{for } x\geqslant 0,\\ &\big|\varepsilon(x)-E_-\big|\leqslant O(1)|E_+-E_-|e^{-\eta_-|x|}\quad \text{for } x\leqslant 0,\\ &\big|\varepsilon(x)-E_+\big|\leqslant O(1)|E_+-E_-|e^{-\eta_+|x|}\quad \text{for } x\geqslant 0.\end{aligned}$$

Now we linearize Eqs. (2.14) around $(n,h,\varepsilon)\in X$ as follows

$$\begin{cases}\mathcal{P}'(n)\mathcal{N}_x=\varepsilon-\dfrac{\mathcal{J}}{n},\\ \mathcal{Q}'(h)\mathcal{H}_x=-\varepsilon-\dfrac{\mathcal{K}}{h},\\ \mathcal{E}_x=n-h-D(x),\\ (\mathcal{N},\mathcal{H})(\pm\infty)=(n_\pm,h_\pm),\\ \mathcal{E}(-\infty)=E_-,\end{cases}$$

where $\mathcal{P}(s)$ and $\mathcal{Q}(s)$ are the positive and increasing functions defined by

$$\mathcal{P}'(s):=\frac{p'(s)}{s}-\frac{\mathcal{J}^2}{s^3}>0,\qquad \mathcal{Q}'(s):=\frac{q'(s)}{s}-\frac{\mathcal{K}^2}{s^3}>0.$$

We can write

$$\begin{cases} \mathcal{N}_x = \left(\varepsilon - \dfrac{\mathcal{J}}{n}\right)/\mathcal{P}'(n), \\ \mathcal{H}_x = \left(-\varepsilon - \dfrac{\mathcal{K}}{h}\right)/\mathcal{Q}'(h), \\ \mathcal{E}_x = n - h - D(x), \\ (\mathcal{N}, \mathcal{H})(\pm\infty) = (n_\pm, h_\pm), \\ \mathcal{E}(-\infty) = E_-, \end{cases} \tag{2.15}$$

which defines the operator $P(n, h, \varepsilon)$ as follows

$$P(n, h, \varepsilon) = (\mathcal{N}, \mathcal{H}, \mathcal{E}).$$

Let us integrate (2.15)$_1$ over $(-\infty, x]$ for $x \leqslant 0$. Note that $\mathcal{J} = n_- E_-$, $\mathcal{P}'(n) \geqslant C_1 > 0$ with $n > n_*$ and $\mathcal{Q}'(h) \geqslant C_2 > 0$ with $h > h_*$ for some positive constants C_1 and C_2. Let $n, \varepsilon \in X$, then we obtain

$$\begin{aligned} \left|\mathcal{N}(x) - n_-\right| &= \left| \int_{-\infty}^{x} \left(\varepsilon(y) - \frac{\mathcal{J}}{n(y)}\right)/\mathcal{P}'\big(n(y)\big)\, dy \right| \\ &= \left| \int_{-\infty}^{x} \left(\big(\varepsilon(y) - E_-\big) - \left(\frac{\mathcal{J}}{n(y)} - \frac{\mathcal{J}}{n_-}\right)\right)/\mathcal{P}'\big(n(y)\big)\, dy \right| \\ &\leqslant C \int_{-\infty}^{x} \big[\left|\varepsilon(y) - E_-\right| + \left|n(y) - n_-\right|\big]\, dy \\ &\leqslant C \int_{-\infty}^{x} \big[|E_+ - E_-| + |n_+ - n_-|\big] e^{-\eta_- |y|}\, dy \\ &\leqslant C\big[|E_+ - E_-| + |n_+ - n_-|\big] e^{-\eta_- |x|}, \quad \text{for } x \leqslant 0. \end{aligned} \tag{2.16}$$

On the other hand, integrating (2.15)$_1$ over $[x, \infty)$ for $x \geqslant 0$, and noting that $\mathcal{J} = n_+ E_+$, we can similarly obtain

$$\left|n_+ - \mathcal{N}(x)\right| \leqslant C\big[|E_+ - E_-| + |n_+ - n_-|\big] e^{-\eta_+ |x|}, \quad \text{for } x \geqslant 0.$$

In the same fashion, we can also prove

$$\begin{aligned} \left|\mathcal{H}(x) - h_-\right| &\leqslant C\big[|E_+ - E_-| + |h_+ - h_-|\big] e^{-\eta_- |x|}, \quad \text{for } x \leqslant 0, \\ \left|h_+ - \mathcal{H}(x)\right| &\leqslant C\big[|E_+ - E_-| + |h_+ - h_-|\big] e^{-\eta_+ |x|}, \quad \text{for } x \geqslant 0. \end{aligned}$$

Integrating (2.15)$_3$ over $(-\infty, x]$ with $x \leqslant 0$, and noting that $D_- = n_- - h_-$ (see (2.2)), we have

$$\begin{aligned}
\left|\mathcal{E}(x)-E_-\right| &= \left|\int_{-\infty}^{x}\left[n(y)-h(y)-D(y)\right]dy\right| \\
&= \left|\int_{-\infty}^{x}\left[\left(n(y)-n_-\right)-\left(h(y)-h_-\right)-\left(D(y)-D_-\right)\right]dy\right| \\
&\leqslant C\int_{-\infty}^{x}\left[|n_+-n_-|+|h_+-h_-|+\left|D^*-D_*\right|\right]e^{-\eta_-|y|}\,dy \\
&\leqslant C\left[|n_+-n_-|+|h_+-h_-|+\left|D^*-D_*\right|\right]e^{-\eta_-|x|}, \quad \text{for } x\leqslant 0.
\end{aligned}$$

Integrating (2.15)$_3$ over $[x,\infty)$ with $x\geqslant 0$, and noting that $D_+=n_+-h_+$, we further have

$$\left|E_+-\mathcal{E}(x)\right|\leqslant C\left[|n_+-n_-|+|h_+-h_-|+\left|D^*-D_*\right|\right]e^{-\eta_+|x|}, \quad \text{for } x\leqslant 0.$$

Similarly, estimating Eqs. (2.15) directly, we can prove

$$|\mathcal{N}_x|\leqslant C\left(|E_+-E_-|+|n_+-n_-|\right)e^{-\eta_-|x|} \quad \text{for } x\leqslant 0, \tag{2.17}$$

$$|\mathcal{N}_x|\leqslant C\left(|E_+-E_-|+|n_+-n_-|\right)e^{-\eta_+|x|} \quad \text{for } x\geqslant 0, \tag{2.18}$$

$$|\mathcal{H}_x|\leqslant C\left(|E_+-E_-|+|h_+-h_-|\right)e^{-\eta_-|x|} \quad \text{for } x\leqslant 0, \tag{2.19}$$

$$|\mathcal{H}_x|\leqslant C\left(|E_+-E_-|+|h_+-h_-|\right)e^{-\eta_+|x|} \quad \text{for } x\geqslant 0, \tag{2.20}$$

$$|\mathcal{E}_x|\leqslant C\left(|n_+-n_-|+|h_+-h_-|+\left|D^*-D_*\right|\right)e^{-\eta_-|x|} \quad \text{for } x\leqslant 0, \tag{2.21}$$

$$|\mathcal{E}_x|\leqslant C\left(|n_+-n_-|+|h_+-h_-|+\left|D^*-D_*\right|\right)e^{-\eta_+|x|} \quad \text{for } x\geqslant 0. \tag{2.22}$$

Combining (2.16)–(2.22), we then get

$$\left\|(\mathcal{N},\mathcal{H},\mathcal{E})\right\|_X\leqslant C_3\left(|n_+-n_-|+|h_+-h_-|+|E_+-E_-|+\left|D^*-D_*\right|\right)$$

for some positive constant C_3, and

$$\begin{aligned}
\mathcal{N}(x) &\geqslant \min\left\{n_- - O(1)\left(|E_+-E_-|+|n_+-n_-|\right)e^{-\eta_-|x|},\right. \\
&\qquad \left. n_+ - O(1)\left(|E_+-E_-|+|n_+-n_-|\right)e^{-\eta_+|x|}\right\} \\
&\geqslant n_* \quad \left(\text{see } (2.6)\right) \\
\mathcal{H}(x) &\geqslant \min\left\{h_- - O(1)\left(|E_+-E_-|+|h_+-h_-|\right)e^{-\eta_-|x|},\right. \\
&\qquad \left. h_+ - O(1)\left(|E_+-E_-|+|h_+-h_-|\right)e^{-\eta_+|x|}\right\} \\
&\geqslant h_* \quad \left(\text{see } (2.6)\right)
\end{aligned}$$

provided with $|n_+-n_-|+|h_+-h_-|+|E_+-E_-|+|D^*-D_*|\ll 1$.

Thus, we have proved that $(\mathcal{N},\mathcal{H},\mathcal{E})(x)$ is uniformly bounded in X, and we can prove, with the same arguments, that the following iteration

$$\left(\mathcal{N}^{(l)},\mathcal{H}^{(l)},\mathcal{E}^{(l)}\right)=P\left(\mathcal{N}^{(l-1)},\mathcal{H}^{(l-1)},\mathcal{E}^{(l-1)}\right), \quad l\geqslant 1$$

defines a Cauchy sequence, where $(\mathcal{N}^{(0)}, \mathcal{H}^{(0)}, \mathcal{E}^{(0)})(x)$ is selected in X. Therefore, there exists a subsequence, still denoted as $(\mathcal{N}^{(l)}, \mathcal{H}^{(l)}, \mathcal{E}^{(l)})$, such that

$$\lim_{l\to+\infty}\big(\mathcal{N}^{(l)}, \mathcal{H}^{(l)}, \mathcal{E}^{(l)}\big) = (\mathcal{N}, \mathcal{H}, \mathcal{E}), \quad \text{and} \quad (\mathcal{N}, \mathcal{H}, \mathcal{E}) = P(\mathcal{N}, \mathcal{H}, \mathcal{E}),$$

namely, $(\mathcal{N}, \mathcal{H}, \mathcal{E})$ is the solution of (2.1).

Next, we prove the uniqueness of the solution. In fact, let $(\mathcal{N}_i, \mathcal{H}_i, \mathcal{E}_i) \in X$ for $i = 1, 2$ be two solutions of (2.1), or equivalently, of (2.5), which can be also written as

$$\begin{cases} \big(\mathcal{P}(\mathcal{N}_i)\big)_{xx} = \mathcal{N}_i - \mathcal{H}_i - D(x) - \mathcal{J}\left(\dfrac{1}{\mathcal{N}_i}\right)_x, \\ \big(\mathcal{Q}(\mathcal{H}_i)\big)_{xx} = \mathcal{H}_i - \mathcal{N}_i + D(x) - \mathcal{K}\left(\dfrac{1}{\mathcal{H}_i}\right)_x, \\ (\mathcal{N}_i, \mathcal{H}_i)(\pm\infty) = (n_\pm, h_\pm), \end{cases} \quad i = 1, 2.$$

Considering the difference of $\mathcal{N}_1 - \mathcal{N}_2$ and $\mathcal{H}_1 - \mathcal{H}_2$, we have

$$\begin{cases} \big(\mathcal{P}(\mathcal{N}_1) - \mathcal{P}(\mathcal{N}_2)\big)_{xx} = (\mathcal{N}_1 - \mathcal{N}_2) - (\mathcal{H}_1 - \mathcal{H}_2) - \mathcal{J}\left(\dfrac{1}{\mathcal{N}_1} - \dfrac{1}{\mathcal{N}_2}\right)_x, \\ \big(\mathcal{Q}(\mathcal{H}_1) - \mathcal{Q}(\mathcal{H}_2)\big)_{xx} = (\mathcal{H}_1 - \mathcal{H}_2) - (\mathcal{N}_1 - \mathcal{N}_2) - \mathcal{K}\left(\dfrac{1}{\mathcal{H}_1} - \dfrac{1}{\mathcal{H}_2}\right)_x, \\ (\mathcal{N}_1 - \mathcal{N}_2, \mathcal{H}_1 - \mathcal{H}_2)(\pm\infty) = (0, 0). \end{cases} \tag{2.23}$$

Taking

$$\int_{\mathbb{R}} \big((2.23)_1 \cdot (\mathcal{N}_1 - \mathcal{N}_2) + (2.23)_2 \cdot (\mathcal{H}_1 - \mathcal{H}_2)\big)\, dx,$$

and integrating it by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \big(\mathcal{P}(\mathcal{N}_1) - \mathcal{P}(\mathcal{N}_2)\big)_x (\mathcal{N}_1 - \mathcal{N}_2)_x\, dx + \int_{\mathbb{R}} \big(\mathcal{Q}(\mathcal{H}_1) - \mathcal{Q}(\mathcal{H}_2)\big)_x (\mathcal{H}_1 - \mathcal{H}_2)_x\, dx \\ &\qquad + \int_{\mathbb{R}} \big((\mathcal{N}_1 - \mathcal{N}_2) - (\mathcal{H}_1 - \mathcal{H}_2)\big)^2\, dx \\ &\quad = -\mathcal{J}\int_{\mathbb{R}} \left(\frac{1}{\mathcal{N}_1} - \frac{1}{\mathcal{N}_2}\right)(\mathcal{N}_1 - \mathcal{N}_2)_x\, dx - \mathcal{K}\int_{\mathbb{R}} \left(\frac{1}{\mathcal{H}_1} - \frac{1}{\mathcal{H}_2}\right)(\mathcal{H}_1 - \mathcal{H}_2)_x\, dx. \end{aligned} \tag{2.24}$$

Notice that, by the Hölder inequality,

$$\begin{aligned} &\int_{\mathbb{R}} \big(\mathcal{P}(\mathcal{N}_1) - \mathcal{P}(\mathcal{N}_2)\big)_x (\mathcal{N}_1 - \mathcal{N}_2)_x\, dx \\ &\quad = \int_{\mathbb{R}} \mathcal{P}'(\mathcal{N}_1)\big|(\mathcal{N}_1 - \mathcal{N}_2)_x\big|^2\, dx + \int_{\mathbb{R}} \big(\mathcal{P}'(\mathcal{N}_1) - \mathcal{P}'(\mathcal{N}_2)\big)\mathcal{N}_{2x}(\mathcal{N}_1 - \mathcal{N}_2)_x\, dx \end{aligned}$$

$$\begin{aligned}&\geqslant C_1\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\|^2 - O(1)\|\mathcal{N}_{2x}\|_{L^\infty}\|\mathcal{N}_1-\mathcal{N}_2\|\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\| \\ &\geqslant C_1\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\|^2 - O(1)\eta\|\mathcal{N}_1-\mathcal{N}_2\|\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\|, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} \big(\mathcal{Q}(\mathcal{H}_1)-\mathcal{Q}(\mathcal{H}_2)\big)_x(\mathcal{H}_1-\mathcal{H}_2)_x\,dx \\ &\quad \geqslant C_2\|\mathcal{H}_{1x}-\mathcal{H}_{2x}\|^2 - O(1)\eta\|\mathcal{H}_1-\mathcal{H}_2\|\|\mathcal{H}_{1x}-\mathcal{H}_{2x}\|, \end{aligned} \tag{2.26}$$

where $|\mathcal{N}_{2x}| \leqslant C(|n_+-n_-|+|h_+-h_-|+|E_+|+|E_-|+|D^*-D_*|) = C\eta$ (see (2.17)–(2.22)). Notice also that

$$\left|\mathcal{J}\int_{\mathbb{R}}\left(\frac{1}{\mathcal{N}_1}-\frac{1}{\mathcal{N}_2}\right)(\mathcal{N}_1-\mathcal{N}_2)_x\,dx\right| \leqslant O(1)\eta\|\mathcal{N}_1-\mathcal{N}_2\|\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\|, \tag{2.27}$$

$$\left|\mathcal{K}\int_{\mathbb{R}}\left(\frac{1}{\mathcal{H}_1}-\frac{1}{\mathcal{H}_2}\right)(\mathcal{H}_1-\mathcal{H}_2)_x\,dx\right| \leqslant O(1)\eta\|\mathcal{H}_1-\mathcal{H}_2\|\|\mathcal{H}_{1x}-\mathcal{H}_{2x}\|, \tag{2.28}$$

where we used $|\mathcal{J}| = |n_\pm E_\pm| \leqslant C|E_\pm| \leqslant C\eta$ and $|\mathcal{K}| = |h_\pm E_\pm| \leqslant C\eta$.

By using the estimates (2.25)–(2.28) in (2.24), we prove

$$\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\| + \|\mathcal{H}_{1x}-\mathcal{H}_{2x}\| \leqslant O(1)\eta\big(\|\mathcal{N}_1-\mathcal{N}_2\| + \|\mathcal{H}_1-\mathcal{H}_2\|\big). \tag{2.29}$$

Thus, by the Sobolev inequality $\|f\|_{L^\infty} \leqslant \sqrt{2}\|f\|\|f_x\|$, and using (2.29), we then have

$$\begin{aligned}&\|\mathcal{N}_1-\mathcal{N}_2\|_{L^\infty} + \|\mathcal{H}_1-\mathcal{H}_2\|_{L^\infty} \\ &\quad\leqslant \sqrt{2}\big(\|\mathcal{N}_1-\mathcal{N}_2\|\|\mathcal{N}_{1x}-\mathcal{N}_{2x}\| + \|\mathcal{H}_1-\mathcal{H}_2\|\|\mathcal{H}_{1x}-\mathcal{H}_{2x}\|\big) \\ &\quad\leqslant O(1)\eta\big(\|\mathcal{N}_1-\mathcal{N}_2\|^2 + \|\mathcal{H}_1-\mathcal{H}_2\|^2\big) \\ &\quad\leqslant O(1)\eta\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{L^\infty}\|\mathcal{N}_1-\mathcal{N}_2\|_{L^1} + \|\mathcal{H}_1-\mathcal{H}_2\|_{L^\infty}\|\mathcal{H}_1-\mathcal{H}_2\|_{L^1}\big) \\ &\quad\leqslant O(1)\eta\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{L^\infty} + \|\mathcal{H}_1-\mathcal{H}_2\|_{L^\infty}\big). \end{aligned}$$

Here we used the fact (see (2.10)–(2.11))

$$\begin{aligned}\|\mathcal{N}_1-\mathcal{N}_2\|_{L^1} + \|\mathcal{H}_1-\mathcal{H}_2\|_{L^1} &\leqslant \int_{-\infty}^{0}\big(|\mathcal{N}_1-n_-|+|\mathcal{N}_2-n_-|+|\mathcal{H}_1-h_-|+|\mathcal{H}_2-h_-|\big)\,dx \\ &\quad + \int_{0}^{\infty}\big(|\mathcal{N}_1-n_+|+|\mathcal{N}_2-n_+|+|\mathcal{H}_1-h_+|+|\mathcal{H}_2-h_+|\big)\,dx \\ &\leqslant C. \end{aligned}$$

Therefore, we prove

$$\big(1-O(1)\eta\big)\big(\|\mathcal{N}_1-\mathcal{N}_2\|_{L^\infty} + \|\mathcal{H}_1-\mathcal{H}_2\|_{L^\infty}\big) \leqslant 0,$$

which implies

$$\mathcal{N}_1 = \mathcal{N}_2, \ \mathcal{H}_1 = \mathcal{H}_2,$$

provided $\eta \ll 1$. The proof is complete. □

Remark 2.2. In Theorem 2.1, we proved the existence and uniqueness of the stationary solution for a non-flat doping profile $D(x)$ (the physical case). However we need it to decay exponentially to $D_\pm$ at far fields $x = \pm\infty$. But such a requirement is not necessary when $D(x)$ is almost flat. In fact, if $D(x)$ is almost flat, namely,

$$\left|D'(x)\right| \ll 1, \qquad \int_{-\infty}^{x} \left|D(y) - D_-\right| dy + \int_{x}^{\infty} \left|D(y) - D_+\right| dy \ll 1,$$

then, with a similar calculation in [22], the existence and uniqueness of the stationary solution can be obtained without the assumption of exponential decay of $D(x)$ to $D_\pm$ as $x \to \pm\infty$.

3. Convergence to stationary waves

In this section, we are going to state our main results, that is, the 1-D solutions $(n, J, h, K, E)(x,t)$ of the bipolar hydrodynamic model (1.1) and (1.2) globally exist, and converge to the steady-state solutions $(\mathcal{N}, \mathcal{J}, \mathcal{H}, \mathcal{K}, \mathcal{E})(x)$ of (2.1) time-algebraically.

3.1. Heuristic analysis at far fields

First of all, let us investigate the behavior of the solutions to (1.1) and (1.2) at far fields $x = \pm\infty$, and see what will be the exact gaps between the original solutions and the corresponding steady-state solutions. Set

$$\left(n^\pm, J^\pm, h^\pm, K^\pm, E^\pm\right)(t) := (n, J, h, K, E)(\pm\infty, t).$$

As shown in [12] (initially inspired by [25]), by solving the corresponding ordinary differential equations of (1.1) as $x \to \pm\infty$:

$$\begin{cases} \dfrac{d}{dt} n^\pm(t) = 0, \quad \text{i.e.,} \quad n^\pm(t) = n_\pm, \\ \dfrac{d}{dt} J^\pm(t) = n_\pm E^\pm(t) - J^\pm(t), \\ \dfrac{d}{dt} h^\pm(t) = 0, \quad \text{i.e.,} \quad h^\pm(t) = h_\pm, \\ \dfrac{d}{dt} K^\pm(t) = -h_\pm E^\pm(t) - K^\pm(t), \\ \left(n^\pm, J^\pm, h^\pm, K^\pm\right)(0) = (n_\pm, J_\pm, h_\pm, K_\pm), \end{cases} \tag{3.1}$$

we have

$$\begin{cases} J^{\pm}(t) = J_{\pm}e^{-t} + n_{\pm} \displaystyle\int_0^t e^{-(t-s)} E^{\pm}(s)\,ds, \\ K^{\pm}(t) = K_{\pm}e^{-t} - h_{\pm} \displaystyle\int_0^t e^{-(t-s)} E^{\pm}(s)\,ds. \end{cases} \tag{3.2}$$

Since

$$E^-(t) = E^-(-\infty, t) = E_-, \tag{3.3}$$

we quickly obtain from (3.2) that

$$\begin{cases} J^-(t) = n_-E_- + (J_- - n_-E_-)e^{-t}, \\ K^-(t) = -h_-E_- + (K_- + h_-E_-)e^{-t}. \end{cases} \tag{3.4}$$

Integrating $(1.1)_5$ over $(-\infty, +\infty)$ with respect to x, and noting (3.3), we get

$$E^+(t) - E_- = \int_{-\infty}^{\infty} \big[n(x,t) - h(x,t) - D(x)\big]\,dx.$$

Setting $t = 0$ in the above equation, we derive the initial condition for $E^+(t)$ as

$$E^+(0) = E_- + \int_{-\infty}^{\infty} \big[n_0(x) - h_0(x) - D(x)\big]\,dx := E_{01}. \tag{3.5}$$

On the other hand, differentiating $(1.1)_5$ with respect to t and using $(1.1)_1$ and $(1.1)_3$, we have

$$E_{xt} = \big(n - h - D(x)\big)_t = -(J - K)_x.$$

Integrating it over $(-\infty, \infty)$ with respect to x, and noting $\frac{d}{dt}E^-(t) = (E_-)' = 0$ (see (3.3)), we further have

$$\frac{d}{dt}E^+(t) = -\big[J^+(t) - K^+(t)\big] + \big[J^-(t) - K^-(t)\big], \tag{3.6}$$

which is equivalent, by using (3.4), to

$$J^+(t) - K^+(t) = -\frac{d}{dt}E^+(t) + (n_- + h_-)E_- + \big[(J_- - n_-E_-) - (K_- + h_-E_-)\big]e^{-t}. \tag{3.7}$$

Taking $t = 0$ in (3.6), we get the initial condition for $\frac{d}{dt}E^+(t)$ as follows

$$\frac{d}{dt}E^+(t)\bigg|_{t=0} = -[J_+ - K_+] + [J_- - K_-] =: E_{02}. \tag{3.8}$$

Subtracting (3.1)$_4$ from (3.1)$_2$ for index "+", we get

$$\frac{d}{dt}\big[J^+(t)-K^+(t)\big]=(n_++h_+)E^+(t)-\big[J^+(t)-K^+(t)\big]. \tag{3.9}$$

Substituting (3.7) to (3.9), and noting $n_+E_+=n_-E_-$ and $h_+E_+=h_-E_-$, and applying the initial conditions (3.5) and (3.8), we obtain the following ODE

$$\begin{cases} \dfrac{d^2}{dt^2}E^+(t)+\dfrac{d}{dt}E^+(t)+(n_++h_+)E^+(t)=(n_++h_+)E_+, \\ E^+\big|_{t=0}=E_{01}, \qquad \dfrac{d}{dt}E^+\Big|_{t=0}=E_{02}. \end{cases} \tag{3.10}$$

Solving (3.10), we have

$$E^+(t)=\begin{cases} E_++A_1e^{-\lambda_1 t}+A_2e^{-\lambda_2 t}, & \text{for } n_++h_+<\frac{1}{4}, \\ E_++A_3e^{-\frac{1}{2}t}+A_4te^{-\frac{1}{2}t}, & \text{for } n_++h_+=\frac{1}{4}, \\ E_++A_5e^{-\frac{1}{2}t}\cos\lambda_3 t+A_6e^{-\frac{1}{2}t}\sin\lambda_3 t, & \text{for } n_++h_+>\frac{1}{4}, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} \lambda_1&=\frac{1-\sqrt{1-4(n_++h_+)}}{2}, \quad \text{for } n_++h_+<\frac{1}{4}, \\ \lambda_2&=\frac{1+\sqrt{1-4(n_++h_+)}}{2}, \quad \text{for } n_++h_+<\frac{1}{4}, \\ \lambda_3&=\frac{\sqrt{4(n_++h_+)-1}}{2}, \quad \text{for } n_++h_+>\frac{1}{4}, \\ A_1&=\frac{\lambda_2(E_{01}-E_+)+E_{02}}{\lambda_2-\lambda_1}, \qquad A_2=\frac{\lambda_1(E_{01}-E_+)+E_{02}}{\lambda_1-\lambda_2}, \\ A_3&=E_{01}-E_+, \qquad A_4=E_{02}+\frac{1}{2}(E_{01}-E_+), \\ A_5&=E_{01}-E_+, \qquad A_6=\left(E_{02}+\frac{1}{2}(E_{01}-E_+)\right)/\lambda_3. \end{aligned}$$

Substituting (3.11) to (3.2), we can solve for $J^+(t)$ and $K^+(t)$ as

$$J^+(t)=\begin{cases} n_+E_++(J_+-n_+E_+)e^{-t}+\frac{n_+A_1}{1-\lambda_1}(e^{-\lambda_1 t}-e^{-t})+\frac{n_+A_2}{1-\lambda_2}(e^{-\lambda_2 t}-e^{-t}), \\ \quad \text{for } n_++h_+<\frac{1}{4}, \\ n_+E_++(J_+-n_+E_+)e^{-t}+2n_+A_3(e^{-\frac{1}{2}t}-e^{-t})+2n_+A_4(te^{-\frac{1}{2}t}-2e^{-\frac{1}{2}t}+2e^{-t}), \\ \quad \text{for } n_++h_+=\frac{1}{4}, \\ n_+E_++(J_+-n_+E_+)e^{-t}+\frac{n_+A_5}{1+4\lambda_3^2}(2e^{-\frac{1}{2}t}\cos\lambda_3 t-2e^{-t}+2\lambda_3e^{-\frac{1}{2}t}\sin\lambda_3 t) \\ \quad +\frac{n_+A_6}{1+4\lambda_3^2}(2e^{-\frac{1}{2}t}\sin\lambda_3 t-4\lambda_3e^{-\frac{1}{2}t}\cos\lambda_3 t+4\lambda_3e^{-t}), \\ \quad \text{for } n_++h_+>\frac{1}{4}, \end{cases} \tag{3.12}$$

and

$$K^+(t) = \begin{cases} -h_+E_+ + (K_+ + h_+E_+)e^{-t} - \frac{h_+A_1}{1-\lambda_1}(e^{-\lambda_1 t} - e^{-t}) - \frac{h_+A_2}{1-\lambda_2}(e^{-\lambda_2 t} - e^{-t}), \\ \quad \text{for } n_+ + h_+ < \frac{1}{4}, \\ -h_+E_+ + (K_+ + h_+E_+)e^{-t} - 2h_+A_3(e^{-\frac{1}{2}t} - e^{-t}) - 2h_+A_4(te^{-\frac{1}{2}t} - 2e^{-\frac{1}{2}t} + 2e^{-t}), \\ \quad \text{for } n_+ + h_+ = \frac{1}{4}, \\ -h_+E_+ + (K_+ + h_+E_+)e^{-t} - \frac{h_+A_5}{1+4\lambda_3^2}(2e^{-\frac{1}{2}t}\cos\lambda_3 t - 2e^{-t} + 2\lambda_3 e^{-\frac{1}{2}t}\sin\lambda_3 t) \\ \quad - \frac{h_+A_6}{1+4\lambda_3^2}(2e^{-\frac{1}{2}t}\sin\lambda_3 t - 4\lambda_3 e^{-\frac{1}{2}t}\cos\lambda_3 t + 4\lambda_3 e^{-t}), \\ \quad \text{for } n_+ + h_+ > \frac{1}{4}. \end{cases} \tag{3.13}$$

Summarizing (3.4) and (3.11)–(3.13), we have

$$\begin{cases} \left|n(\pm\infty, t) - n\pm\right| = 0, \\ \left|h(\pm\infty, t) - h\pm\right| = 0, \\ \left|J(+\infty, t) - n_+E_+\right| = O(1)e^{-\nu_0 t}, \\ \left|J(-\infty, t) - n_-E_-\right| = O(1)e^{-t}, \\ \left|K(+\infty, t) - (-h_+E_+)\right| = O(1)e^{-\nu_0 t}, \\ \left|K(-\infty, t) - (-h_-E_-)\right| = O(1)e^{-t}, \\ \left|E(+\infty, t) - E_+\right| = O(1)e^{-\nu_0 t}, \\ \left|E(-\infty, t) - E_-\right| = 0, \end{cases} \tag{3.14}$$

for some constant $0 < \nu_0 < \frac{1}{2}$.

3.2. Correction functions

From (2.1)–(2.2) and (3.14), now it is well-known that the difference between the original solutions $(n, J, h, K, E)(x, t)$ and the stationary solutions $(\mathcal{N}, \mathcal{J}, \mathcal{H}, \mathcal{K}, \mathcal{E})(x)$ at far fields $x = \pm\infty$ is

$$\begin{cases} \left|n(\pm\infty, t) - \mathcal{N}(\pm\infty)\right| = 0, \\ \left|h(\pm\infty, t) - \mathcal{H}(\pm\infty)\right| = 0, \\ \left|J(+\infty, t) - \mathcal{J}\right| = O(1)e^{-\nu_0 t} \neq 0, \\ \left|J(-\infty, t) - \mathcal{J}\right| = O(1)e^{-t} \neq 0, \\ \left|K(+\infty, t) - \mathcal{K}\right| = O(1)e^{-\nu_0 t} \neq 0, \\ \left|K(-\infty, t) - \mathcal{K}\right| = O(1)e^{-t} \neq 0, \\ \left|E(+\infty, t) - \mathcal{E}(+\infty)\right| = O(1)e^{-\nu_0 t} \neq 0, \\ \left|E(-\infty, t) - \mathcal{E}(-\infty)\right| = 0. \end{cases}$$

Clearly, there are some gaps for $J - \mathcal{J}$, $K - \mathcal{K}$ and $E - \mathcal{E}$, which, indeed, are essentially caused by the switch-on condition $E_+ - E_- \neq 0$, such that

$$J - \mathcal{J},\ K - \mathcal{K},\ E - \mathcal{E} \notin L^2(\mathbb{R}).$$

Thus, the usual L^2-energy method cannot be applied in this case to prove the stability of stationary waves in L^2-sense. In order to delete these gaps, we need to technically construct some correction functions. Inspired by [12], we select the correction functions $(\hat{n}, \hat{J}, \hat{h}, \hat{K}, \hat{E})(x,t)$ such that

$$\begin{cases} \hat{n}_t + \hat{J}_x = 0, \\ \hat{J}_t = \breve{n}\hat{E} - \hat{J}, \\ \hat{h}_t + \hat{K}_x = 0, \\ \hat{K}_t = -\breve{h}\hat{E} - \hat{K}, \\ \hat{E}_x = \hat{n} - \hat{h}, \\ \hat{n}(x,t) \to 0 & \text{as } x \to \pm\infty, \\ \hat{J}(x,t) \to J^{\pm}(t) - \mathcal{J} & \text{as } x \to \pm\infty, \\ \hat{h}(x,t) \to 0 & \text{as } x \to \pm\infty, \\ \hat{K}(x,t) \to K^{\pm}(t) - \mathcal{K} & \text{as } x \to \pm\infty, \\ \hat{E}(x,t) \to 0 & \text{as } x \to -\infty, \\ \hat{E}(x,t) \to E^{+}(t) - E_{+} & \text{as } x \to +\infty, \end{cases} \tag{3.15}$$

where $\breve{n}(x)$ and $\breve{h}(x)$ are selected as

$$\breve{n}(x) = n_- + (n_+ - n_-) \int_{-\infty}^{x+2L_0} m_0(y)\,dy$$

and

$$\breve{h}(x) = h_- + (h_+ - h_-) \int_{-\infty}^{x+2L_0} m_0(y)\,dy$$

and $m_0(x)$ is selected as

$$m_0(x) \geqslant 0, \qquad m_0 \in C_0^{\infty}(R), \qquad \operatorname{supp} m_0 \subseteq [-L_0, L_0], \qquad \int_{\mathbb{R}} m_0(y)\,dy = 1,$$

with some constant $L_0 > 0$, and the initial data $\hat{n}(x,0)$, $\hat{h}(x,0)$, $\hat{J}(x,0)$, $\hat{K}(x,0)$ and $\hat{E}(x,0)$ are chosen as follows

$$\hat{n}(x,0) = m_0(x) \int_{-\infty}^{\infty} \big[n_0(x) - \mathcal{N}(x)\big]\,dx =: \bar{n}_0 m_0(x),$$

$$\hat{h}(x,0) = m_0(x) \int_{-\infty}^{\infty} \big[h_0(x) - \mathcal{H}(x)\big]\,dx =: \bar{h}_0 m_0(x),$$

$$\hat{J}(x,0) = \big[J^-(0) - \mathcal{J}\big] + \big[J^+(0) - J^-(0)\big] \int_{-\infty}^{x} m_0(y)\,dy,$$

$$\hat{K}(x,0) = \big[K^-(0) - \mathcal{K}\big] + \big[K^+(0) - K^-(0)\big] \int_{-\infty}^{x} m_0(y)\,dy,$$

$$\hat{E}(x,0) = \big[E^+(0) - E_+\big] \int_{-\infty}^{x} m_0(y)\,dy. \tag{3.16}$$

Here, from (3.15)$_5$, the following compatibility condition

$$\hat{E}_x(x,0) = \hat{n}(x,0) - \hat{h}(x,0)$$

holds. In fact, from (3.16) and (3.5), we have

$$\begin{aligned}
\hat{E}_x(x,0) &= \big[E^+(0) - E_+\big] m_0(x) \\
&= \left(E_- + \int_{-\infty}^{\infty} \big[n_0(x) - h_0(x) - D(x)\big]\,dx - E_+\right) m_0(x) \\
&= \left(\int_{-\infty}^{\infty} \big[n_0(x) - h_0(x) - D(x)\big]\,dx - (E_+ - E_-)\right) m_0(x) \\
&= \left(\int_{-\infty}^{\infty} \big[n_0(x) - h_0(x) - D(x)\big]\,dx - \int_{-\infty}^{\infty} \mathcal{E}_x\,dx\right) m_0(x) \\
&= \left(\int_{-\infty}^{\infty} \big[n_0(x) - h_0(x) - D(x)\big]\,dx - \int_{-\infty}^{\infty} \big[\mathcal{N}(x) - \mathcal{H}(x) - D(x)\big]\,dx\right) m_0(x) \\
&= \left(\int_{-\infty}^{\infty} \big[n_0(x) - \mathcal{N}(x)\big]\,dx - \int_{-\infty}^{\infty} \big[h_0(x) - \mathcal{H}(x)\big]\,dx\right) m_0(x) \\
&= \hat{n}(x,0) - \hat{h}(x,0).
\end{aligned}$$

In the same fashion as in [12] but with a tedious computation, we can solve (more precisely saying, we construct) the above correction functions as follows

$$\hat{n}(x,t) = \left(\bar{n}_0 + \int_0^t \big[J^-(s) - J^+(s)\big]\,ds\right) m_0(x) =: \bar{n}(t) m_0(x),$$

$$\hat{J}(x,t) = J^-(t) - \mathcal{J} + \big[J^+(t) - J^-(t)\big] \int_{-\infty}^{x} m_0(y)\,dy,$$

$$\hat{h}(x,t) = \left(\bar{h}_0 + \int_0^t \big[K^-(s) - K^+(s)\big]\,ds\right) m_0(x) =: \bar{k}(t) m_0(x),$$

$$\hat{K}(x,t) = K^-(t) - \mathcal{K} + \left[K^+(t) - K^-(t)\right] \int_{-\infty}^{x} m_0(y)\,dy,$$

$$\hat{E}(x,t) = \left[E^+(t) - E_+\right] \int_{-\infty}^{x} m_0(y)\,dy,$$

where $\bar{n}(t)$ and $\bar{h}(t)$ are the main part of antiderivatives (with zero constant terms) of $J^-(t) - J^+(t)$ and $K^-(t) - K^+(t)$, respectively, and based on (3.4), (3.12) and (3.13), it holds

$$\left|\bar{n}(t)\right| \leqslant Ce^{-\nu_0 t}, \qquad \left|\bar{h}(t)\right| \leqslant Ce^{-\nu_0 t}$$

for $0 < \nu_0 < \frac{1}{2}$.

On the other hand, noting $(1.1)_1$, $(2.1)_1$ and $(3.15)_1$, i.e.,

$$(n - \mathcal{N} - \hat{n})_t = -(J - \mathcal{J} - \hat{J})_x,$$

after integrating it over $(-\infty, \infty)$ with respect to x, we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left[n(x,t) - \mathcal{N}(x) - \hat{n}(x,t)\right] dx = -(J - \mathcal{J} - \hat{J})\big|_{x=-\infty}^{\infty} = 0,$$

which implies

$$\int_{-\infty}^{\infty} \left[n(x,t) - \mathcal{N}(x) - \hat{n}(x,t)\right] dx = \int_{-\infty}^{\infty} \left[n_0(x) - \mathcal{N}(x) - \hat{n}(x,0)\right] dx = 0.$$

Similarly, we also have

$$\int_{-\infty}^{\infty} \left[h(x,t) - \mathcal{H}(x) - \hat{h}(x,t)\right] dx = \int_{-\infty}^{\infty} \left[h_0(x) - \mathcal{H}(x) - \hat{h}(x,0)\right] dx = 0.$$

Summarizing what we have obtained before, we have the following lemma.

Lemma 3.1. *It holds that*

$$\left\|(\hat{n}, \hat{J}, \hat{h}, \hat{K}, \hat{E})(t)\right\|_{L^\infty(\mathbb{R})} \leqslant C\sigma e^{-\nu_0 t} \tag{3.17}$$

and

$$\operatorname{supp} \hat{n} = \operatorname{supp} \hat{h} = \operatorname{supp} m_0 \subseteq [-L_0, L_0] \tag{3.18}$$

for $\sigma := |J_+| + |J_-| + |E_-| + |E_+|$ *and* $0 < \nu_0 < \frac{1}{2}$.

Furthermore, it can be verified

$$\begin{cases} \int_{-\infty}^{\infty} \big[n(x,t) - \hat{n}(x,t) - \mathcal{N}(x)\big]\,dx = \int_{-\infty}^{\infty} \big[n_0(x) - \hat{n}(x,0) - \mathcal{N}(x)\big]\,dx = 0, \\ \int_{-\infty}^{\infty} \big[h(x,t) - \hat{h}(x,t) - \mathcal{H}(x)\big]\,dx = \int_{-\infty}^{\infty} \big[h_0(x) - \hat{h}(x,0) - \mathcal{H}(x)\big]\,dx = 0, \\ J(\pm\infty,t) - \hat{J}(\pm\infty,t) - \mathcal{J} = 0, \\ K(\pm\infty,t) - \hat{K}(\pm\infty,t) - \mathcal{K} = 0, \\ E(\pm\infty,t) - \hat{E}(\pm\infty,t) - \mathcal{E}(\pm\infty) = 0. \end{cases}$$

3.3. Convergence theorems

Now we are going to make the perturbation of (1.1) to the steady-state equations (2.1) corrected by (3.15):

$$\begin{cases} (n - \mathcal{N} - \hat{n})_t + (J - \mathcal{J} - \hat{J})_x = 0, \\ (J - \mathcal{J} - \hat{J})_t + \left(\dfrac{J^2}{n} - \dfrac{\mathcal{J}^2}{\mathcal{N}} + p(n) - p(\mathcal{N})\right)_x = nE - \mathcal{N}\mathcal{E} - \breve{n}\hat{E} - (J - \mathcal{J} - \hat{J}), \\ (h - \mathcal{H} - \hat{h})_t + (K - \mathcal{K} - \hat{K})_x = 0, \\ (K - \mathcal{K} - \hat{K})_t + \left(\dfrac{K^2}{h} - \dfrac{\mathcal{K}^2}{\mathcal{H}} + q(n) - q(\mathcal{H})\right)_x = -(hE - \mathcal{H}\mathcal{E} - \breve{h}\hat{E}) - (K - \mathcal{K} - \hat{K}), \\ (E - \mathcal{E} - \hat{E})_x = (n - \mathcal{N} - \hat{n}) - (h - \mathcal{H} - \hat{h}). \end{cases} \tag{3.19}$$

Let

$$\begin{cases} \phi(x,t) := \int_{-\infty}^{x} \big[n(y,t) - \mathcal{N}(y) - \hat{n}(y,t)\big]\,dy, \\ \psi(x,t) := \int_{-\infty}^{x} \big[h(y,t) - \mathcal{H}(y) - \hat{h}(y,t)\big]\,dy, \\ \theta(x,t) := J(x,t) - \mathcal{J} - \hat{J}(x,t), \\ \vartheta(x,t) := K(x,t) - \mathcal{K} - \hat{K}(x,t), \\ \chi(x,t) := E(x,t) - \mathcal{E}(x) - \hat{E}(x,t), \\ \phi_0(x) := \int_{-\infty}^{x} \big[n_0(y) - \mathcal{N}(y) - \hat{n}(y,0)\big]\,dy, \\ \psi_0(x) := \int_{-\infty}^{x} \big[h_0(y) - \mathcal{H}(y) - \hat{h}(y,0)\big]\,dy, \\ \theta_0(x) := J_0(x) - \mathcal{J} - \hat{J}(x,0), \\ \vartheta_0(x) := K_0(x) - \mathcal{K} - \hat{K}(x,0). \end{cases}$$

Then (3.19) is reduced to

$$\begin{cases} \phi_t + \theta = 0, \\ \theta_t + \big(p'(\mathcal{N})\phi_x\big)_x = \mathcal{N}\chi - \theta - F_{1x} - F_{2x} + F_3, \\ \psi_t + \vartheta = 0, \\ \vartheta_t + \big(q'(\mathcal{H})\psi_x\big)_x = -\mathcal{H}\chi - \vartheta - G_{1x} - G_{2x} + G_3, \\ \chi = \phi - \psi, \\ (\phi, \psi, \theta, \vartheta)\big|_{t=0} = (\phi_0, \psi_0, \theta_0, \vartheta_0)(x), \end{cases} \tag{3.20}$$

where

$$F_1 := p(\mathcal{N} + \hat{n} + \phi_x) - p(\mathcal{N}) - p'(\mathcal{N})\phi_x, \tag{3.21}$$

$$F_2 := \frac{(\mathcal{J} + \hat{J} + \theta)^2}{\mathcal{N} + \hat{n} + \phi_x} - \frac{\mathcal{J}^2}{\mathcal{N}} = \frac{(\mathcal{J} + \hat{J} - \phi_t)^2}{\mathcal{N} + \hat{n} + \phi_x} - \frac{\mathcal{J}^2}{\mathcal{N}},$$

$$F_3 := (\mathcal{N} - \breve{n})\hat{E} + (\hat{n} + \phi_x)(\chi + \mathcal{E} + \hat{E}), \tag{3.22}$$

$$G_1 := q(\mathcal{H} + \hat{h} + \psi_x) - q(\mathcal{H}) - q'(\mathcal{H})\psi_x,$$

$$G_2 := \frac{(\mathcal{K} + \hat{K} + \vartheta)^2}{\mathcal{H} + \hat{h} + \psi_x} - \frac{\mathcal{K}^2}{\mathcal{H}} = \frac{(\mathcal{K} + \hat{K} - \psi_t)^2}{\mathcal{H} + \hat{h} + \psi_x} - \frac{\mathcal{K}^2}{\mathcal{H}},$$

$$G_3 := -(\mathcal{H} - \breve{h})\hat{E} - (\hat{h} + \psi_x)(\chi + \mathcal{E} + \hat{E}). \tag{3.23}$$

Theorem 3.2 *(L^2-convergence). Assume that $(\phi_0, \psi_0) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$, $(\theta_0, \vartheta_0) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$, and (1.3), (2.4) and (2.6) hold. Then there exists a constant $\delta_1 > 0$ such that when $\eta + \Phi_0 < \delta_1$ (η is defined in (2.7)), the solutions $(\phi, \theta, \psi, \vartheta, \chi)(x, t)$ of the IVP (3.20) uniquely and globally exist, and satisfy*

$$(\phi, \theta, \psi, \vartheta, \chi) \in C^0\big(0, +\infty; H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R})\big),$$
$$\chi_t \in C^0\big(0, +\infty; H^1(\mathbb{R})\big)$$

and

$$\sum_{l=0}^{3}(1+t)^l \big\|\partial_x^l(\phi, \psi)(t)\big\|^2 + \sum_{l=0}^{2}(1+t)^{l+2}\big\|\partial_x^l(\theta, \vartheta)(t)\big\|^2 + \sum_{l=0}^{2}(1+t)^{l+1}\big\|\partial_x^l\chi(t)\big\|^2 + \sum_{l=0}^{1}(1+t)^{l+3}\big\|\partial_x^l\chi_t(t)\big\|^2 \leqslant C\big[\eta + \big\|(\phi_0, \psi_0)\big\|_3 + \big\|(\theta_0, \vartheta_0)\big\|_2\big]. \tag{3.24}$$

By using the Sobolev inequality $\|f\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{1/2}\|f_x\|_{L^2(\mathbb{R})}^{1/2}$, and noting the exponential decay for $(\hat{n}, \hat{J}, \hat{h}, \hat{K}, \hat{E})(x, t)$ (see (3.17)), then, from Theorem 3.2, we have the following stability of stationary waves.

Corollary 3.3 *(L^∞-stability of stationary waves). Under the assumption of Theorem 3.2, the solutions $(n, J, h, K, E)(x,t)$ of the IVP (1.1) and (1.2) uniquely and globally exist, and converge to the stationary solutions $(\mathcal{N}(x), \mathcal{J}, \mathcal{H}(x), \mathcal{K}, \mathcal{E}(x))$ of (2.1) in the form of*

$$\begin{cases} \sup_{x\in\mathbb{R}}\big|\big(n(x,t)-\mathcal{N}(x)\big)\big| \leqslant C(1+t)^{-\frac{3}{4}}, \\ \sup_{x\in\mathbb{R}}\big|\big(J(x,t)-\mathcal{J}(x)\big)\big| \leqslant C(1+t)^{-\frac{5}{4}}, \\ \sup_{x\in\mathbb{R}}\big|\big(h(x,t)-\mathcal{H}(x)\big)\big| \leqslant C(1+t)^{-\frac{3}{4}}, \\ \sup_{x\in\mathbb{R}}\big|\big(K(x,t)-\mathcal{K}(x)\big)\big| \leqslant C(1+t)^{-\frac{5}{4}}, \\ \sup_{x\in\mathbb{R}}\big|\big(E(x,t)-\mathcal{E}(x)\big)\big| \leqslant C(1+t)^{-\frac{3}{4}}. \end{cases} \tag{3.25}$$

Remark 3.4.

1. In the previous works [7,11–13], the authors assume $p(s)=g(s)$, $D(x)=0$, and $E_-=0$ and $n_\pm = h_\pm$. But here we allow $p(s)\neq g(s)$, $|D'(x)| \not\ll 1$ (non-flat), and $E_-\neq 0$ and $n_\pm \neq h_\pm$, which is essentially different from the previous studies for the case of unbounded domain.
2. The algebraic decay rates shown in (3.24) and (3.25) are optimal when the initial perturbation is in L^2-sense. If the initial perturbation is further in $L^1(\mathbb{R})$, the better decay rates are expected by constructing some approximation Green functions for the coupled system, but hard to show at this moment. This still remains open for future.
3. Different from the case of bounded domain, where we can establish the Poincaré inequality so then we may further archive the exponential convergence, also there is no restriction on the size of doping profile $|D^*-D_*| = \sup_{x\in R} D(x) - \inf_{x\in R} D(x)$, however, here we can show the algebraic convergence only (actually as shown in [21], the decay rates should be algebraic only even for the special case with constant coefficients), and we still need reasonably to assume $|D^*-D_*| \ll 1$ because of the technical requirement on energy estimates for this unbounded domain case.

4. Proof of convergence theorem

Theorem 3.2 can be proved by the elementary L^2-energy method with continuation argument based on the local existence of the solutions $(\phi,\theta,\psi,\vartheta,\chi)(x,t)$ to (3.20) and the *a priori* energy estimates. The local existence of the solutions $(\phi,\theta,\psi,\vartheta,\chi)(x,t)$ can be obtained by the standard iteration method. The key step is to establish the *a priori* energy estimates, which is our main target in this section.

For $T>0$, we define the solution space as follows

$$Y(0,T) := \big\{(\phi,\psi,\chi)(x,t) \;\big|\; \partial_t^l\phi, \partial_t^l\psi \in C\big(0,T;H^{3-l}(\mathbb{R})\big),\ l=0,1,$$
$$\partial_t^l\chi \in C\big(0,T;H^{2-l}(\mathbb{R})\big),\ l=0,1,\ 0\leqslant t\leqslant T\big\}$$

equipped with the norm

$$M(T)^2 := \sup_{0\leqslant t\leqslant T}\Bigg\{\sum_{l=0}^{3}(1+t)^l\big\|\partial_x^l(\phi,\psi)(t)\big\|^2 + \sum_{l=0}^{2}(1+t)^{l+2}\big\|\partial_x^l(\phi_t,\psi_t)(t)\big\|^2$$
$$+\sum_{l=0}^{2}(1+t)^{l+1}\big\|\partial_x^l\chi(t)\big\|^2 + \sum_{l=0}^{1}(1+t)^{l+3}\big\|\partial_x^l\chi_t(t)\big\|^2\Bigg\}.$$

Substituting $(3.20)_1$ and $(3.20)_3$ into $(3.20)_2$ and $(3.20)_4$, respectively, and applying $(3.20)_5$, we get

$$\begin{cases} \phi_{tt} + \phi_t - \big(p'(\mathcal{N})\phi_x\big)_x + \mathcal{N}(\phi - \psi) = F_{1x} + F_{2x} - F_3, \\ \psi_{tt} + \psi_t - \big(q'(\mathcal{H})\psi_x\big)_x + \mathcal{H}(\psi - \phi) = G_{1x} + G_{2x} - G_3, \\ \chi = \phi - \psi, \\ (\phi, \psi, \phi_t, \psi_t)\big|_{t=0} = (\phi_0, \psi_0, -\theta_0, -\vartheta_0)(x). \end{cases} \tag{4.1}$$

Lemma 4.1 *(Basic energy estimates). It holds*

$$\big\|(\phi, \phi_x, \phi_t, \psi, \psi_x, \psi_t, \chi)(t)\big\|^2 + \int_0^t \big\|(\phi_x, \phi_t, \psi_x, \psi_t, \chi)(s)\big\|^2\, ds$$
$$\leqslant C\big[\big\|(\phi_0, \psi_0)\big\|_1^2 + \big\|(\theta_0, \vartheta_0)\big\|^2 + \eta\big] \tag{4.2}$$

provided with $M(T) + \eta \ll 1$.

Proof. Let's perform the following computation

$$\int_{-\infty}^{\infty} \big((4.1)_1 \times \mathcal{H}(\phi + 2\phi_t) + (4.1)_2 \times \mathcal{N}(\psi + 2\psi_t)\big)\, dx,$$

which implies

$$\frac{d}{dt}\int_{\mathbb{R}} I_1(x,t)\, dx + \int_{\mathbb{R}} I_2(x,t)\, dx = \int_{\mathbb{R}} I_3(x,t)\, dx + \int_{\mathbb{R}} I_4(x,t)\, dx, \tag{4.3}$$

where

$$\begin{aligned} I_1(x,t) &= \mathcal{H}\bigg(\phi\phi_t + \frac{1}{2}\phi^2 + \phi_t^2\bigg) + p'(\mathcal{N})\mathcal{H}\phi_x^2 + \mathcal{N}\bigg(\psi\psi_t + \frac{1}{2}\psi^2 + \psi_t^2\bigg) + q'(\mathcal{H})\mathcal{N}\psi_x^2 \\ &\quad + \mathcal{N}\mathcal{H}(\phi - \psi)^2, \\ I_2(x,t) &= \mathcal{H}\phi_t^2 + \mathcal{H}p'(\mathcal{N})\phi_x^2 + 2\mathcal{H}_x p'(\mathcal{N})\phi_x\phi_t + \mathcal{N}\psi_t^2 + \mathcal{N}q'(\mathcal{H})\psi_x^2 + 2\mathcal{N}_x q'(\mathcal{H})\psi_x\psi_t \\ &\quad + \mathcal{N}\mathcal{H}(\phi - \psi)^2, \\ I_3(x,t) &= -(F_1 + F_2)\big(\mathcal{H}(\phi + 2\phi_t)\big)_x - F_3 \cdot \mathcal{H}(\phi + 2\phi_t), \\ I_4(x,t) &= -(G_1 + G_2)\big(\mathcal{N}(\psi + 2\psi_t)\big)_x - G_3\mathcal{N}(\psi + 2\psi_t). \end{aligned}$$

We first estimate the nonlinear terms in the right-hand side of (4.3). By Taylor's formula

$$F_1 = p'(\mathcal{N})\hat{n} + O(1)(\hat{n} + \phi_x)^2,$$

and using the Sobolev inequality

$$|\phi|, |\phi_x|, |\phi_t| \leqslant CM(T),$$

and the exponential decay of $(\hat{n}, \hat{h})$ shown in Lemma 3.1, we have

$$\left|\int_{\mathbb{R}} F_1 (\mathcal{H}\phi)_x \, dx\right| = \left|\int_{\mathbb{R}} \left[p'(\mathcal{N})\hat{n} + O(1)(\hat{n}+\phi_x)^2\right](\mathcal{H}\phi)_x \, dx\right| \leqslant O(1)\eta e^{-\nu_0 t} + O(1)M(T)\left\|\phi_x(t)\right\|^2. \tag{4.4}$$

Since

$$2\int_{\mathbb{R}} F_1 (\mathcal{H}\phi_t)_x \, dx = 2\int_{\mathbb{R}} F_1 \mathcal{H}_x \phi_t \, dx + 2\int_{\mathbb{R}} F_1 \mathcal{H}\phi_{xt} \, dx,$$

similarly to (4.4), we have

$$\left|2\int_{\mathbb{R}} F_1 \mathcal{H}_x \phi_t \, dx\right| \leqslant O(1)\eta e^{-\nu_0 t} + O(1)M(T)\left\|\phi_x(t)\right\|^2. \tag{4.5}$$

To estimate $2\int_{\mathbb{R}} F_1 \mathcal{H}\phi_{xt} \, dx$, we first note that

$$F_1 \mathcal{H}\phi_{xt} = \frac{\partial}{\partial t}\left\{\mathcal{H}\left[\left(\int_{\mathcal{N}}^{\mathcal{N}+\hat{n}+\phi_x} p(s)\, ds\right) - p(\mathcal{N})\phi_x - \frac{1}{2}p'(\mathcal{N})\phi_x^2\right]\right\} - \mathcal{H}p(\mathcal{N}+\hat{n}+\phi_x)\hat{n}_t. \tag{4.6}$$

Let

$$H(z) := \int_{\mathcal{N}}^{z} p(s)\, ds.$$

Since

$$H(\mathcal{N}) = 0, \qquad H'(z) = p(z), \qquad H''(z) = p'(z),$$

then Taylor's formula gives

$$\int_{\mathcal{N}}^{\mathcal{N}+\hat{n}+\phi_x} p(s)\, ds = H(\mathcal{N}+\hat{n}+\phi_x) = p(\mathcal{N})(\hat{n}+\phi_x) + \frac{1}{2}p'(\mathcal{N})(\hat{n}+\phi_x)^2 + O(1)(\hat{n}+\phi_x)^3.$$

Substituting this to (4.6), we obtain

$$F_1 \mathcal{H}\phi_{xt} = \frac{\partial}{\partial t}\left\{\mathcal{H}\left(p(\mathcal{N})\hat{n}_t + \frac{1}{2}p'(\mathcal{N})\hat{n}(\hat{n}+2\phi_x) + O(1)(\hat{n}+\phi_x)^3\right)\right\} - \mathcal{H}p(\mathcal{N}+\hat{n}+\phi_x)\hat{n}_t.$$

Thus, we have

$$-2\int_{\mathbb{R}} F_1 \mathcal{H}\phi_{xt} \, dx \leqslant -2\frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}\left(p(\mathcal{N})\hat{n}_t + \frac{1}{2}p'(\mathcal{N})\hat{n}(\hat{n}+2\phi_x) + O(1)(\hat{n}+\phi_x)^3\right) dx + O(1)\eta e^{-\nu_0 t}. \tag{4.7}$$

On the other hand, since

$$\begin{aligned}
F_2 &= \frac{(\mathcal{J}+\hat{J}-\phi_t)^2}{\mathcal{N}+\hat{n}+\phi_x} - \frac{\mathcal{J}^2}{\mathcal{N}} \\
&= -\frac{2\mathcal{J}}{\mathcal{N}}\phi_t - \frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x + \frac{2\mathcal{J}+\hat{J}-\phi_t}{\mathcal{N}+\hat{n}+\phi_x}\hat{J} - \frac{\mathcal{J}^2}{\mathcal{N}(\mathcal{N}+\hat{n}+\phi_x)}\hat{n} \\
&\quad + \frac{2\mathcal{J}(\hat{n}+\phi_x)-\mathcal{N}(\hat{J}-\phi_t)}{\mathcal{N}(\mathcal{N}+\hat{n}+\phi_x)}\phi_t + \frac{\mathcal{J}^2(\hat{n}+\phi_x)}{\mathcal{N}^2(\mathcal{N}+\hat{n}+\phi_x)}\phi_x \\
&= -\frac{2\mathcal{J}}{\mathcal{N}}\phi_t - \frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x + O(1)(\hat{J}+\hat{n}) + O(1)(\hat{J}+\hat{n}+\phi_x+\phi_t)(\phi_x+\phi_t),
\end{aligned}$$

taking integration by parts, using the Cauchy inequality and the time-exponential decay of $(\hat{n}, \hat{J})$ as shown in Lemma 3.1, we obtain

$$\int_{\mathbb{R}} F_2(\mathcal{H}\phi)_x\,dx \leqslant \int_{\mathbb{R}} \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2\,dx + O(1)\eta e^{-\nu_0 t} + O(1)\big[\eta+M(T)\big]\big\|(\phi_x,\phi_t)(t)\big\|^2, \tag{4.8}$$

and

$$\begin{aligned}
-2\int_{\mathbb{R}} F_2(\mathcal{H}\phi_t)_x\,dx &= 2\int_{\mathbb{R}} F_{2x}\mathcal{H}\phi_t\,dx \\
&\leqslant \frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2\,dx + O(1)\eta e^{-\nu_0 t} + O(1)\big[\eta+M(T)\big]\big\|(\phi_x,\phi_t)(t)\big\|^2.
\end{aligned} \tag{4.9}$$

From Lemma 3.1, we note that

$$\|\mathcal{N}-\breve{n}\|_{L^1} \leqslant C\eta, \qquad \big\|\hat{E}(t)\big\|_{L^\infty} \leqslant C\eta e^{-\nu_0 t}, \qquad \big\|\hat{n}(t)\big\|_{L^2} \leqslant C\eta e^{-\nu_0 t},$$

then we further estimate the nonlinear term involving F_3 as

$$\int_{\mathbb{R}} F_3\mathcal{H}(\phi+2\phi_t)\,dx \leqslant O(1)\eta M(T)e^{-\nu_0 t} + O(1)M(T)\big\|(\phi_x,\phi_t)(t)\big\|^2. \tag{4.10}$$

Summarizing (4.4), (4.5), (4.7), (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned}
\int_{\mathbb{R}} I_3(x,t)\,dx &\leqslant \frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2\,dx + \frac{d}{dt}\int_{\mathbb{R}} R_1(x,t)\,dx + \int_{\mathbb{R}} \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2\,dx \\
&\quad + O(1)\big[\eta+M(T)\big]\big\|(\phi_x,\phi_t)(t)\big\|^2 + O(1)\big[\eta+M(T)\big]e^{-\nu_0 t},
\end{aligned} \tag{4.11}$$

where

$$R_1(x,t) := -2\mathcal{H}\Big(p(\mathcal{N})\hat{n}_t + \frac{1}{2}p'(\mathcal{N})\hat{n}(\hat{n}+2\phi_x) + O(1)(\hat{n}+\phi_x)^3\Big)$$

satisfying

$$\int_{\mathbb{R}} R_1(x,t)\,dx \leqslant O(1)\eta e^{-\nu_0 t} + O(1)\big[\eta + M(T)\big]\big\|\phi_x(t)\big\|^2. \tag{4.12}$$

Similarly, we can also estimate I_4 as follows

$$\begin{aligned}\int_{\mathbb{R}} I_4(x,t)\,dx \leqslant{}& \frac{d}{dt}\int_{\mathbb{R}} \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2\,dx + \frac{d}{dt}\int_{\mathbb{R}} R_2(x,t)\,dx + \int_{\mathbb{R}} \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2\,dx \\ &+ O(1)\big[\eta + M(T)\big]\big\|(\psi_x,\psi_t)(t)\big\|^2 + O(1)\big[\eta + M(T)\big]e^{-\nu_0 t},\end{aligned} \tag{4.13}$$

where

$$R_2(x,t) := -2\mathcal{N}\left(q(\mathcal{H})\hat{k}_t + \frac{1}{2}q'(\mathcal{H})\hat{h}(\hat{h} + 2\psi_x) + O(1)(\hat{h} + \psi_x)^3\right)$$

satisfying

$$\int_{\mathbb{R}} R_2(x,t)\,dx \leqslant O(1)\eta e^{-\nu_0 t} + O(1)\big[\eta + M(T)\big]\big\|\psi_x(t)\big\|^2. \tag{4.14}$$

Substituting (4.11) and (4.13) into (4.3), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left[I_1(x,t) - \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2 - \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2\right]dx + \int_{\mathbb{R}}\left[I_2(x,t) - \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2 - \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2\right]dx \\ &\leqslant C\big[\eta + M(T)\big]e^{-\nu_0 t} + C\big[\eta + M(T)\big]\big\|(\phi_x,\phi_t,\psi_x,\psi_t)(t)\big\|^2 \\ &\quad + \frac{d}{dt}\int_{\mathbb{R}}\big[R_1(x,t) + R_2(x,t)\big]\,dx.\end{aligned} \tag{4.15}$$

From the uniform ellipticity (2.9), we obtain for some constant $C_4 > 0$ that

$$\begin{aligned}I_1(x,t) - \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2 - \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2 ={}& \mathcal{H}\left(\phi\phi_t + \frac{1}{2}\phi^2 + \phi_t^2\right) + \mathcal{H}\left(p'(\mathcal{N}) - \frac{\mathcal{J}^2}{\mathcal{N}^2}\right)\phi_x^2 \\ &+ \mathcal{N}\left(\psi\psi_t + \frac{1}{2}\psi^2 + \psi_t^2\right) + \mathcal{N}\left(q'(\mathcal{H}) - \frac{\mathcal{K}^2}{\mathcal{H}^2}\right)\psi_x^2 \\ &+ \mathcal{N}\mathcal{H}(\phi - \psi)^2 \\ \geqslant{}& C_4\big(\phi^2 + \phi_x^2 + \phi_t^2 + \psi^2 + \psi_x^2 + \psi_t^2 + \chi^2\big).\end{aligned} \tag{4.16}$$

By using $|\mathcal{H}_x|, |\mathcal{N}_x| \leqslant C\eta$ we get for some constants $C_5, C_6 > 0$ that

$$
\begin{aligned}
I_2(x,t) - \mathcal{H}\frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2 - \mathcal{N}\frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2 = {} & \mathcal{H}\phi_t^2 + \mathcal{H}\left(p'(\mathcal{N}) - \frac{\mathcal{J}^2}{\mathcal{N}^2}\right)\phi_x^2 + 2\mathcal{H}_x p'(\mathcal{N})\phi_x\phi_t \\
& + \mathcal{N}\psi_t^2 + \mathcal{N}\left(q'(\mathcal{H}) - \frac{\mathcal{K}^2}{\mathcal{H}^2}\right)\psi_x^2 + 2\mathcal{N}_x q'(\mathcal{H})\psi_x\psi_t \\
& + \mathcal{N}\mathcal{H}(\phi-\psi)^2 \\
\geqslant {} & C_5(1-\eta)\left(\phi_x^2+\phi_t^2+\psi_x^2+\psi_t^2\right) + C_5\chi^2. \qquad (4.17)
\end{aligned}
$$

Integrating (4.15) over $[0,t]$ and applying (4.12), (4.14), (4.16) and (4.17), we further obtain

$$
\begin{aligned}
& \left[C_4 - O(1)\left(\eta + M(T)\right)\right]\left\|(\phi,\phi_x,\phi_t,\psi,\psi_x,\psi_t)(t)\right\|^2 \\
& \quad + \left[C_4(1-\eta) - O(1)\left(\eta+M(T)\right)\right]\int_0^t \left\|(\phi_x,\phi_t,\psi_x,\psi_t)(s)\right\|^2 ds + C_6\int_0^t \left\|\chi(s)\right\|^2 ds \\
& \leqslant C\left[\eta + \left\|(\phi_0,\psi_0)\right\|_1^2 + \left\|(\theta_0,\vartheta_0)\right\|^2\right].
\end{aligned}
$$

Let $\eta + M(T) \ll 1$, then

$$
\left\|(\phi,\phi_x,\phi_t,\psi,\psi_x,\psi_t,\chi)(t)\right\|^2 + \int_0^t \left\|(\phi_x,\phi_t,\psi_x,\psi_t,\chi)(s)\right\|^2 ds \leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_1^2 + \left\|(\theta_0,\vartheta_0)\right\|^2 + \eta\right].
$$

The proof is complete. □

Lemma 4.2 *(Higher order energy estimates). It holds*

$$
\begin{aligned}
& \left\|(\phi,\psi)(t)\right\|_3^2 + \left\|(\phi_t,\psi_t)(t)\right\|_1^2 + \left\|\chi(t)\right\|_2^2 + \int_0^t \left[\left\|(\phi_t,\psi_t)(s)\right\|_1^2 + \left\|\chi(s)\right\|_2^2\right] ds \\
& \quad \leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_3^2 + \left\|(\theta_0,\vartheta_0)\right\|_2^2 + \eta\right] \qquad (4.18)
\end{aligned}
$$

provided with $M(T) + \eta \ll 1$.

Proof. By calculating

$$
\int_0^t \int_{\mathbb{R}} \left[\partial_x(4.1)_1 \cdot \mathcal{H}(\phi_x + 2\phi_{xt}) + \partial_x(4.1)_2 \cdot \mathcal{N}(\psi_x + 2\psi_{xt})\right] dx\, ds,
$$

and applying (4.2), we can similarly prove

$$
\begin{aligned}
& \left\|(\phi_x,\phi_{xx},\phi_{xt},\psi_x,\psi_{xx},\psi_{xt},\chi_x)(t)\right\|^2 + \int_0^t \left\|(\phi_{xx},\phi_{xt},\psi_{xx},\psi_{xt},\chi_x)(s)\right\|^2 ds \\
& \quad \leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_2^2 + \left\|(\theta_0,\vartheta_0)\right\|_1^2 + \eta\right] \qquad (4.19)
\end{aligned}
$$

provided $\eta + M(T) \ll 1$.

Furthermore, by taking

$$\int_0^t \int_{\mathbb{R}} \left[\partial_x^2 (4.1)_1 \cdot \mathcal{H}(\phi_{xx} + 2\phi_{xxt}) + \partial_x^2 (4.1)_2 \cdot \mathcal{N}(\psi_{xx} + 2\psi_{xxt})\right] dx\, ds,$$

and applying (4.2) and (4.19), we can prove

$$\begin{aligned} &\left\|(\phi_{xx}, \phi_{xxx}, \phi_{xxt}, \psi_{xx}, \psi_{xxx}, \psi_{xxt}, \chi_{xx})(t)\right\|^2 + \int_0^t \left\|(\phi_{xxx}, \phi_{xxt}, \psi_{xxx}, \psi_{xxt}, \chi_{xx})(s)\right\|^2 ds \\ &\quad \leqslant C\left[\left\|(\phi_0, \psi_0)\right\|_3^2 + \left\|(\theta_0, \vartheta_0)\right\|_1^2 + \eta\right] \end{aligned} \tag{4.20}$$

provided $\eta + M(T) \ll 1$.

Combining (4.2), (4.19) and (4.20), we prove (4.18). □

Now we are going to derive the decay rates for the derivatives of $(\phi, \psi, \bar{\chi})$.

Lemma 4.3. *It holds*

$$\begin{aligned} &(1+t)\left\|(\phi_x, \psi_x, \phi_t, \psi_t, \chi)(t)\right\|^2 + \int_0^t (1+s)\left\|(\phi_t, \psi_t)(s)\right\|^2 ds \\ &\quad \leqslant C\left[\left\|(\phi_0, \psi_0)\right\|_1^2 + \left\|(\theta_0, \vartheta_0)\right\|^2 + \eta\right] \\ &\qquad + C\eta \int_0^t (1+s)\left\|(\phi_{xx}, \psi_{xx})(s)\right\|^2 ds + C\eta \int_0^t (1+s)^2\left\|(\phi_{xt}, \psi_{xt})(s)\right\|^2 ds \end{aligned} \tag{4.21}$$

provided that $M(T) + \eta \ll 1$.

Proof. Multiplying $(4.1)_1$ by $(1+t)\mathcal{H}\phi_t$ and $(4.1)_2$ by $(1+t)\mathcal{N}\psi_t$, adding them together and then integrating the resultant equation with respect to x over $(-\infty, \infty)$, we have

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}(1+t)\int_{\mathbb{R}} \left(\mathcal{H}\left[\phi_t^2 + p'(\mathcal{N})\phi_x^2\right] + \mathcal{N}\left[\psi_t^2 + q'(\mathcal{H})\psi_x^2\right] + \mathcal{N}\mathcal{H}(\phi - \psi)^2\right) dx\right\} \\ &\quad + (1+t)\int_{\mathbb{R}} \left[\mathcal{H}\phi_t^2 + \mathcal{N}\psi_t^2\right] dx + (1+t)\int_{\mathbb{R}} p'(\mathcal{N})\mathcal{H}_x\phi_x\phi_t\, dx + (1+t)\int_{\mathbb{R}} q'(\mathcal{H})\mathcal{N}_x\psi_x\psi_t\, dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} \left(\mathcal{H}\phi_t^2 + \mathcal{H}p'(\mathcal{N})\phi_t^2 + \mathcal{N}\psi_t^2 + \mathcal{N}q'(\mathcal{H})\psi_t^2 + \mathcal{N}\mathcal{H}(\phi - \psi)^2\right) dx \\ &= (1+t)\int_{\mathbb{R}} \mathcal{H}(F_{1x} + F_{2x} - F_3)\phi_t\, dx + (1+t)\int_{\mathbb{R}} \mathcal{N}(G_{1x} + G_{2x} - G_3)\psi_t\, dx. \end{aligned} \tag{4.22}$$

As shown before, we can estimate

$$(1+t)\int_{\mathbb{R}} \mathcal{H}(F_{1x}+F_{2x}-F_3)\phi_t\,dx \leqslant \frac{\mathcal{H}}{2}\frac{d}{dt}\left\{(1+t)\int_{\mathbb{R}} \frac{\mathcal{J}^2}{\mathcal{N}^2}\phi_x^2\,dx\right\} + O(1)\left\|\phi_x(t)\right\|^2 + O(1)(1+t)\left[\eta+M(T)\right]\left\|\phi_t(t)\right\|^2 + O(1)(1+t)e^{-\nu_0 t}, \tag{4.23}$$

and

$$(1+t)\int_{\mathbb{R}} \mathcal{N}(G_{1x}+G_{2x}-G_3)\psi_t\,dx \leqslant \frac{\mathcal{N}}{2}\frac{d}{dt}\left\{(1+t)\int_{\mathbb{R}} \frac{\mathcal{K}^2}{\mathcal{H}^2}\psi_x^2\,dx\right\} + O(1)\left\|\psi_x(t)\right\|^2 + O(1)(1+t)\left[\eta+M(T)\right]\left\|\psi_t(t)\right\|^2 + O(1)(1+t)e^{-\nu_0 t}. \tag{4.24}$$

By using (4.23) and (4.24) in (4.22) and integrating the resultant equation with respect to t over $[0,t]$, we have

$$\begin{aligned}
&(1+t)\int_{\mathbb{R}}\left\{\mathcal{H}\left[\phi_t^2+\left(p'(\mathcal{N})-\frac{\mathcal{J}^2}{\mathcal{N}^2}\right)\phi_x^2\right]+\mathcal{N}\left[\psi_t^2+\left(q'(\mathcal{H})-\frac{\mathcal{K}^2}{\mathcal{H}^2}\right)\psi_x^2\right]+\mathcal{N}\mathcal{H}\chi^2\right\}dx\\
&\quad+2\int_0^t(1+s)\int_{\mathbb{R}}\left[\mathcal{H}\phi_t^2+\mathcal{N}\psi_t^2\right]dx\,ds+2\int_0^t(1+s)\int_{\mathbb{R}}p'(\mathcal{N})\mathcal{H}_x\phi_x\phi_t\,dx\,ds\\
&\quad+2\int_0^t(1+s)\int_{\mathbb{R}}q'(\mathcal{H})\mathcal{N}_x\psi_x\psi_t\,dx\,ds\\
&\leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_1^2+\left\|(\theta_0,\vartheta_0)\right\|^2+\eta\right]+O(1)\int_0^t\left\|(\phi_x,\psi_x)(s)\right\|^2ds\\
&\quad+O(1)\left[\eta+M(T)\right]\int_0^t(1+s)\left\|(\phi_t,\psi_t)(s)\right\|^2ds.
\end{aligned}\tag{4.25}$$

In order to estimate

$$\int_0^t(1+s)\int_{\mathbb{R}}p'(\mathcal{N})\mathcal{H}_x\phi_x\phi_t\,dx\,ds+\int_0^t(1+s)\int_{\mathbb{R}}q'(\mathcal{H})\mathcal{N}_x\psi_x\psi_t\,dx\,ds,$$

since we don't have some positive terms like

$$\int_0^t(1+s)\int_{\mathbb{R}}\phi_x^2\,dx\,ds+\int_0^t(1+s)\int_{\mathbb{R}}\psi_x^2\,dx\,ds,$$

to control it, we need a careful but technical treatment as follows

$$\begin{aligned}
&\left|\int_0^t (1+s)\int_{\mathbb{R}} p'(\mathcal{N})\mathcal{H}_x\phi_x\phi_t\,dx\,ds\right| \\
&\quad\leqslant \int_0^t (1+s)\big\|\phi_x(s)\big\|_{L^\infty}\big\|\phi_t(s)\big\|_{L^\infty}\,ds\left|\int_{\mathbb{R}} p'(\mathcal{N})\mathcal{H}_x\,dx\right| \\
&\quad\leqslant C\eta\int_0^t (1+s)\big\|\phi_x(s)\big\|_{L^\infty}\big\|\phi_t(s)\big\|_{L^\infty}\,ds \quad \big[\text{by Sobolev's inequality: } \|f\|_{L^\infty}\leqslant\sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}\big] \\
&\quad\leqslant C\eta\int_0^t (1+s)\big\|\phi_x(s)\big\|^{\frac12}\big\|\phi_{xx}(s)\big\|^{\frac12}\big\|\phi_t(s)\big\|^{\frac12}\big\|\phi_{xt}(s)\big\|^{\frac12}\,ds \\
&\quad= C\eta\int_0^t \big\|\phi_x(s)\big\|^{\frac12}(1+s)^{\frac14}\big\|\phi_{xx}(s)\big\|^{\frac12}(1+s)^{\frac14}\big\|\phi_t(s)\big\|^{\frac12}(1+s)^{\frac12}\big\|\phi_{xt}(s)\big\|^{\frac12}\,ds \\
&\qquad [\text{by Hölder's inequality}] \\
&\quad\leqslant C\eta\left(\int_0^t \big\|\phi_x(s)\big\|^2\,ds\right)^{\frac14}\left(\int_0^t (1+s)\big\|\phi_{xx}(s)\big\|^2\,ds\right)^{\frac14} \\
&\qquad\times\left(\int_0^t (1+s)\big\|\phi_t(s)\big\|^2\,ds\right)^{\frac14}\left(\int_0^t (1+s)^2\big\|\phi_{x}(s)\big\|^2\,ds\right)^{\frac14} \quad [\text{by Cauchy–Schwart inequality}] \\
&\quad\leqslant C\eta\int_0^t \big\|\phi_x(s)\big\|^2\,ds + C\eta\int_0^t (1+s)\big\|\phi_{xx}(s)\big\|^2\,ds \\
&\qquad + C\eta\int_0^t (1+s)\big\|\phi_t(s)\big\|^2\,ds + C\eta\int_0^t (1+s)^2\big\|\phi_{xt}(s)\big\|^2\,ds,
\end{aligned} \tag{4.26}$$

and similarly,

$$\begin{aligned}
\left|\int_0^t (1+s)\int_{\mathbb{R}} q'(\mathcal{H})\mathcal{N}_x\psi_x\psi_t\,dx\,ds\right| &\leqslant C\eta\int_0^t \big\|\psi_x(s)\big\|^2\,ds + C\eta\int_0^t (1+s)\big\|\psi_{xx}(s)\big\|^2\,ds \\
&\quad + C\eta\int_0^t (1+s)\big\|\psi_t(s)\big\|^2\,ds + C\eta\int_0^t (1+s)^2\big\|\psi_{xt}(s)\big\|^2\,ds.
\end{aligned} \tag{4.27}$$

Thus we can submit (4.26) and (4.27) to (4.25), and apply the uniform ellipticity condition (or say, the subsonic condition) (2.9) and Lemma 4.2, to have

$$\big[C_4 - O(1)\big(\eta + M(T)\big)\big](1+t)\big\|(\phi_x, \phi_t, \psi_x, \psi_t, \chi)(t)\big\|^2 + \big[C_4 - O(1)\big(\eta + M(T)\big)\big]\int_0^t (1+s)\big\|(\phi_t, \psi_t)(s)\big\|^2\,ds$$

$$\leqslant C\big[\big\|(\phi_0, \psi_0)\big\|_1^2 + \big\|(\theta_0, \vartheta_0)\big\|^2 + \eta\big] + C\eta\int_0^t (1+s)\big\|(\phi_{xx}, \psi_{xx})(s)\big\|^2\,ds + C\eta\int_0^t (1+s)^2\big\|(\phi_{xt}, \psi_{xt})(s)\big\|^2 ds.$$

This immediately gives (4.21) by letting $M(T) + \eta \ll 1$. The proof is complete. □

Lemma 4.4 *(Decay rates for $(\phi_x, \psi_x, \chi, \phi_{xx}, \psi_{xx}, \chi_x)$). It holds*

$$(1+t)\big\|(\phi_x, \psi_x, \phi_t, \chi)(t)\big\|^2 + \int_0^t (1+s)\big\|(\phi_t, \psi_t)(s)\big\|^2\,ds + (1+t)^2\big\|(\phi_{xx}, \psi_{xx}, \phi_{xt}, \psi_{xt}, \chi_x)(t)\big\|^2 + \int_0^t \big[(1+s)\big\|(\phi_{xx}, \psi_{xx})(s)\big\|^2 + (1+s)^2\big\|(\phi_{xt}, \psi_{xt})(s)\big\|^2\big]\,ds \leqslant C\big[\big\|(\phi_0, \psi_0)\big\|_2^2 + \big\|(\theta_0, \vartheta_0)\big\|_1^2 + \eta\big] \tag{4.28}$$

provided that $M(T) + \eta \ll 1$.

Proof. By carrying out the following calculation

$$\int_0^t (1+s)\int_{\mathbb{R}} \big[\partial_x(4.1)_1 \cdot \mathcal{H}\phi_{xt} + \partial_x(4.1)_2 \cdot \mathcal{N}\psi_{xt}\big]\,dx\,ds,$$

and applying the energy estimate proved in Lemma 4.2 and the decay estimate of Lemma 4.3, with the same method used in Lemma 4.3, we can prove

$$(1+t)\big\|(\phi_x, \phi_{xx}, \phi_{xt}, \psi_x, \psi_{xx}, \psi_{xt}, \chi_x)(t)\big\|^2 + \int_0^t (1+s)\big\|(\phi_{xx}, \phi_{xt}, \psi_{xx}, \psi_{xt})(s)\big\|^2\,ds \leqslant C\big[\eta + \big\|(\phi_0, \psi_0)\big\|_2^2 + \big\|(\theta_0, \vartheta_0)\big\|_1^2\big] + C\eta\int_0^t (1+s)^2\big\|(\phi_{xt}, \psi_{xt})(s)\big\|^2\,ds \tag{4.29}$$

as long as $M(T) + \eta \ll 1$.

Furthermore, by taking

$$\int_0^t (1+s)^2\int_{\mathbb{R}} \big[\partial_x\,(4.1)_1 \cdot \mathcal{H}\phi_{xt} + \partial_x(4.1)_2 \cdot \mathcal{N}\psi_{xt}\big]\,dx\,ds,$$

and applying (4.29), we then obtain

$$(1+t)^2\big\|(\phi_{xx},\phi_{xt},\psi_{xx},\psi_{xt},\chi_x)(t)\big\|^2+\int_0^t(1+s)^2\big\|(\phi_{xt},\psi_{xt})(s)\big\|^2\,ds$$
$$\leqslant C\big[\eta+\big\|(\phi_0,\psi_0)\big\|_2^2+\big\|(\theta_0,\vartheta_0)\big\|_1^2\big] \tag{4.30}$$

by taking $M(T)+\eta\ll 1$. Thus, combining (4.21), (4.29) and (4.30) gives (4.28). The proof is complete. □

Lemma 4.5 *(Decay rate for $(\phi_{xxx},\psi_{xxx},\chi_{xx})$). It holds*

$$(1+t)^3\big\|(\phi_{xxx},\psi_{xxx},\phi_{xxt},\psi_{xxt},\chi_{xx})(t)\big\|^2$$
$$+\int_0^t\big[(1+s)^2\big\|(\phi_{xxx},\psi_{xxx})(s)\big\|^2+(1+s)^3\big\|(\phi_{xxt},\psi_{xxt})(s)\big\|^2\big]\,ds$$
$$\leqslant C\big[\big\|(\phi_0,\psi_0)\big\|_3^2+\big\|(\theta_0,\vartheta_0)\big\|_2^2+\eta\big] \tag{4.31}$$

provided that $M(T)+\eta\ll 1$.

Proof. In the same fashion of Lemma 4.4, by computing

$$\int_0^t(1+s)^2\int_{\mathbb{R}}\big[\partial_x^2(4.1)_1\cdot\mathcal{H}\phi_{xxt}+\partial_x^2(4.1)_2\cdot\mathcal{N}\psi_{xxt}\big]\,dx\,ds,$$

we first obtain

$$(1+t)^2\big\|(\phi_{xx},\phi_{xxx},\phi_{xxt},\psi_{xx},\psi_{xxx},\psi_{xxt},\chi_{xx})(t)\big\|^2+\int_0^t(1+s)^2\big\|(\phi_{xxx},\phi_{xxt},\psi_{xxx},\psi_{xxt})(s)\big\|^2\,ds$$
$$\leqslant C\big[\eta+\big\|(\phi_0,\psi_0)\big\|_3^2+\big\|(\theta_0,\vartheta_0)\big\|_2^2\big] \tag{4.32}$$

provided that $M(T)+\eta\ll 1$, and by calculating

$$\int_0^t(1+s)^3\int_{\mathbb{R}}\big[\partial_x^2(4.1)_1\cdot\mathcal{H}\phi_{xxt}+\partial_x^2(4.1)_2\cdot\mathcal{N}\psi_{xxt}\big]\,dx\,ds,$$

we further have

$$(1+t)^3\big\|(\phi_{xxx},\phi_{xxt},\psi_{xxx},\psi_{xxt},\chi_{xx})(t)\big\|^2+\int_0^t(1+s)^3\big\|(\phi_{xxt},\psi_{xxt})(s)\big\|^2\,ds$$
$$\leqslant C\big[\eta+\big\|(\phi_0,\psi_0)\big\|_3^2+\big\|(\theta_0,\vartheta_0)\big\|_2^2\big] \tag{4.33}$$

as long as $M(T)+\eta\ll 1$. Thus, combining (4.32) and (4.33) gives (4.31). The proof is complete. □

Lemma 4.6 *(Decay rate for (ϕ_t, ψ_t)). It holds*

$$(1+t)^2\big\|(\phi_t, \psi_t, \phi_{xt}, \psi_{xt}, \phi_{tt}, \psi_{tt}, \chi_t)(t)\big\|^2 + \int_0^t (1+s)^2\big\|(\phi_{xt}, \psi_{xt}, \phi_{tt}, \psi_{tt})(s)\big\|^2\,ds$$
$$\leqslant C\big[\big\|(\phi_0, \psi_0)\big\|_2^2 + \big\|(\theta_0, \vartheta_0)\big\|_1^2 + \eta\big]$$

provided that $M(T) + \eta \ll 1$.

Proof. By calculating

$$\int_0^t \int_{\mathbb{R}} \big[\partial_t(4.1)_1 \cdot \mathcal{H}(\phi_t + 2\phi_{tt}) + \partial_t(4.1)_2 \cdot \mathcal{N}(\psi_t + \psi_{tt})\big]\,dx\,ds,$$

and using Lemma 4.2, we obtain the energy estimate for (ϕ_{tt}, ψ_{tt})

$$\big\|(\phi_t, \phi_{xt}, \phi_{tt}, \psi_t, \psi_{xt}, \psi_{tt}, \chi_t)(t)\big\|^2 + \int_0^t \big\|(\phi_{xt}, \phi_{tt}, \psi_{xt}, \psi_{tt}, \chi_t)(s)\big\|^2\,ds$$
$$\leqslant C\big[\eta + \big\|(\phi_0, \psi_0)\big\|_2^2 + \big\|(\theta_0, \vartheta_0)\big\|_1^2\big] \tag{4.34}$$

when $M(T) + \eta \ll 1$.

By taking

$$\int_0^t \int_{\mathbb{R}} (1+s)\big[\partial_t(4.1)_1 \cdot \mathcal{H}(\phi_t + 2\phi_{tt}) + \partial_t(4.1)_2 \cdot \mathcal{N}(\psi_t + 2\psi_{tt})\big]\,dx\,ds,$$

and using (4.34), we then obtain

$$(1+t)\big\|(\phi_t, \phi_{xt}, \phi_{tt}, \psi_t, \psi_{xt}, \psi_{tt}, \chi_t)(t)\big\|^2 + \int_0^t (1+s)\big\|(\phi_{xt}, \phi_{tt}, \psi_{xt}, \psi_{tt}, \chi_t)(s)\big\|^2\,ds$$
$$\leqslant C\big[\eta + \big\|(\phi_0, \psi_0)\big\|_2^2 + \big\|(\theta_0, \vartheta_0)\big\|_1^2\big] \tag{4.35}$$

if $M(T) + \eta \ll 1$.

Finally, by carrying out

$$\int_0^t \int_{\mathbb{R}} (1+s)^2\big[\partial_t(4.1)_1 \cdot \mathcal{H}(\phi_t + 2\phi_{tt}) + \partial_t(4.1)_2 \cdot \mathcal{N}(\psi_t + 2\psi_{tt})\big]\,dx\,ds,$$

and using (4.35), we then obtain

$$(1+t)^2\left\|(\phi_t,\phi_{xt},\phi_{tt},\psi_t,\psi_{xt},\psi_{tt},\chi_t)(t)\right\|^2+\int_0^t(1+s)^2\left\|(\phi_{xt},\phi_{tt},\psi_{xt},\psi_{tt},\chi_t)(s)\right\|^2ds$$
$$\leqslant C\left[\eta+\left\|(\phi_0,\psi_0)\right\|_2^2+\left\|(\theta_0,\vartheta_0)\right\|_1^2\right]$$

provided that $M(T)+\eta\ll 1$. This proves the lemma. □

Lemma 4.7 *(Decay rate for $(\phi_{xt},\psi_{xt},\chi_t)$). It holds*

$$(1+t)^3\left\|(\phi_{xt},\psi_{xt},\phi_{tt},\psi_{tt},\chi_t)(t)\right\|^2+\int_0^t(1+s)^3\left\|(\phi_{tt},\psi_{tt})(s)\right\|^2ds$$
$$\leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_3^2+\left\|(\theta_0,\vartheta_0)\right\|_2^2+\eta\right] \tag{4.36}$$

provided that $M(T)+\eta\ll 1$.

Proof. As shown before, one can take

$$\int_0^t\int_{\mathbb{R}}(1+s)^3\left[\partial_t(4.1)_1\cdot\mathcal{H}\phi_{tt}+\partial_t(4.1)_2\cdot\mathcal{N}\psi_{tt}\right]dx\,ds,$$

then we can similarly prove (4.36). The details are omitted. □

Lemma 4.8 *(Decay rate for $(\phi_{xxt},\psi_{xxt},\chi_{xt})$). It holds*

$$(1+t)^4\left\|(\phi_{xxt},\psi_{xxt},\phi_{xtt},\psi_{xtt},\chi_{xt})(t)\right\|^2$$
$$+\int_0^t\left[(1+s)^3\left\|(\phi_{xxt},\psi_{xxt})(s)\right\|^2+(1+s)^4\left\|(\phi_{xtt},\psi_{xtt})(s)\right\|^2\right]ds$$
$$\leqslant C\left[\left\|(\phi_0,\psi_0)\right\|_3^2+\left\|(\theta_0,\vartheta_0)\right\|_2^2+\eta\right] \tag{4.37}$$

provided that $M(T)+\eta\ll 1$.

Proof. By the same manner as before, let us take

$$\int_0^t\int_{\mathbb{R}}(1+s)^4\left[\partial_t(4.1)_1\cdot\mathcal{H}\phi_{xtt}+\partial_t(4.1)_2\cdot\mathcal{N}\psi_{xtt}\right]dx\,ds,$$

then we can similarly prove (4.37). □

Combining Lemmas 4.3–4.8, we immediately establish the following estimates.

Lemma 4.9 *(Decay rate for the derivatives of (ϕ,ψ,χ)). It holds*

$$\sum_{l=0}^{3}(1+t)^l \left\| \partial_x^l(\phi,\psi)(t) \right\|^2 + \sum_{l=0}^{2}(1+t)^{l+2} \left\| \partial^l(\phi_t,\psi_t)(t) \right\|^2$$
$$+ \sum_{l=0}^{2}(1+t)^{l+1} \left\| \partial_x^l \chi(t) \right\|^2 + \sum_{l=0}^{1}(1+t)^{l+3} \left\| \partial_x^l \chi_t(t) \right\|^2$$
$$\leqslant C\left[\left\| (\phi_0,\psi_0) \right\|_3^2 + \left\| (\theta_0,\vartheta_0) \right\|_2^2 + \eta \right]$$

provided that $M(T)+\eta \ll 1$.

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