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PLANAR TRAVELING WAVES FOR NONLOCAL DISPERSION EQUATION WITH MONOSTABLE NONLINEARITY

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Abstract. In this paper, we study a class of nonlocal dispersion equation with monostable nonlinearity in n-dimensional space

$$\begin{cases} u_t - J * u + u + d(u(t,x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t-\tau, x-y))dy, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}^n, \end{cases}$$

where the nonlinear functions $d(u)$ and $b(u)$ possess the monostable characters like Fisher-KPP type, $f_\beta(x)$ is the heat kernel, and the kernel $J(x)$ satisfies $\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^\alpha + o(|\xi|^\alpha)$ for $0 < \alpha \le 2$ and $\mathcal{K} > 0$. After establishing the existence for both the planar traveling waves $\phi(x \cdot \mathbf{e} + ct)$ for $c \ge c_*$ (c_* is the critical wave speed) and the solution $u(t,x)$ for the Cauchy problem, as well as the comparison principles, we prove that, all noncritical planar wavefronts $\phi(x \cdot \mathbf{e} + ct)$ are globally stable with the exponential convergence rate $t^{-n/\alpha} e^{-\mu_\tau t}$ for $\mu_\tau > 0$, and the critical wavefronts $\phi(x \cdot \mathbf{e} + c_* t)$ are globally stable in the algebraic form $t^{-n/\alpha}$, and these rates are optimal. As application,we also automatically obtain the stability of traveling wavefronts to the classical Fisher-KPP dispersion equations. The adopted approach is Fourier transform and the weighted energy method with a suitably selected weight function.

1. **Introduction.**

In this paper, we consider the Cauchy problem for the time-delayed nonlocal dispersion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - J * u + u + d(u(t,x)) = \displaystyle\int_{\mathbb{R}^n} f_\beta(y) b(u(t-\tau, x-y))dy, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $J(x)$ is a non-negative and radial kernel with unit integral, and

$$(J * u)(t,x) = \int_{\mathbb{R}^n} J(x-y)u(t,y)dy, \tag{2}$$

and $f_\beta(y)$, with $\beta > 0$, is the heat kernel in the form of

$$f_\beta(y) = \frac{1}{(4\pi\beta)^{\frac{n}{2}}} e^{\frac{-|y|^2}{4\beta}} \quad \text{with} \quad \int_{\mathbb{R}^n} f_\beta(y)dy = 1. \tag{3}$$

Equation (1) represents the dynamical population model of single species in ecology [11], where $u(t,x)$ is the density of population at location x and time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, and $J * u = \int_{\mathbb{R}^n} J(x-y)u(t,y)dy$ is the rate at which individuals are arriving to position x from all other places, while $-u(x,t) = -\int_{\mathbb{R}^n} J(x-y)u(t,x)dy$ stands the rate at which they are leaving the location x to travel to all other places.

When $\tau = 0$ (no time-delay), then the above equation is reduced to

$$\begin{cases} \dfrac{\partial u}{\partial t} - J * u + u + d(u) = \displaystyle\int_{\mathbb{R}^n} f_\beta(y) b(u(t, x-y))dy, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4}$$

Furthermore, noting the property of heat kernel

$$\lim_{\beta \to 0^+} \int_{\mathbb{R}^n} f_\beta(y) b(u(t,x-y))dy = b(u(t,x)),$$

and by taking the death rate $d(u) = u^2$ and the birth rate $b(u) = u$, we can then derive from the equation (4) to the classical Fisher-KPP equation with nonlocal dispersion

$$u_t = J * u - u + u(1-u). \tag{5}$$

Throughout this paper, we assume that the death rate $d(u)$ and birth rate $b(u)$ capture the following monostable characters:

(H_1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;
(H_2) $b'(0) > d'(0) \geq 0$ and $0 \leq b'(u_+) < d'(u_+)$;
(H_3) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$.

These characters are summarized from the classical Fisher-KPP equation, see also the monostable reaction-diffusion equations in ecology, for example, the Nicholson's blowflies equation [27, 28, 31, 37, 39] with

$$d(u) = \delta u \quad \text{and} \quad b(u) = pue^{-au}, \; p > 0, \delta > 0, a > 0$$

and $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta} > 0$ under the consideration of $1 < \frac{p}{\delta} \leq e$; and the age-structured population model [15, 16, 25, 31, 33, 35] with

$$d(u) = \delta u^2 \quad \text{and} \quad b(u) = pe^{-\gamma\tau} u, \quad \delta > 0, \; p > 0, \; \gamma > 0,$$

and $u_- = 0$ and $u_+ = \frac{p}{\delta} e^{-\gamma\tau}$.

Clearly, under the hypothesis (H_1)-(H_3), both $u_- = 0$ and $u_+ > 0$ are constant equilibria of the equation (1), where $u_- = 0$ is unstable and u_+ is stable for the spatially homogeneous equation associated with (1).

On the other hand, we also assume the kernel $J(x)$ satisfying:

(J_1) $J(x) = \prod_{i=1}^{n} J_i(x_i)$, where $J_i(x_i)$ is smooth, and $J_i(x_i) = J_i(|x_i|) \geq 0$ and $\int_{\mathbb{R}} J_i(x_i)dx_i = 1$ for $i = 1, 2\cdots, n$, and $\int_{\mathbb{R}} |y_1|J_1(y_1)e^{-\lambda_* y_1}dy_1 < \infty$ for $\lambda_* > 0$ defined in (16) and (17);

(J_2) Fourier transform of $J(x)$ satisfies $\hat{J}(\xi) = 1 - \mathcal{K}|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ with $\alpha \in (0, 2]$ and $\mathcal{K} > 0$.

A *planar traveling wavefront* to the equation (1) for $\tau \geq 0$ is a special solution in the form of $u(t, x) = \phi(x \cdot \mathbf{e} + ct)$ with $\phi(\pm\infty) = u_\pm$, where c is the wave speed, $\mathbf{e}$ is a unit vector of the basis of $\mathbb{R}^n$. Without loss of generality, we can always assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \cdots, 0)$ by rotating the coordinates. Thus, the planar traveling wavefront $\phi(x \cdot \mathbf{e}_1 + ct) = \phi(x_1 + ct)$ satisfies, for $\tau \geq 0$,

$$\begin{cases} c\phi' - J * \phi + \phi + d(\phi) = \displaystyle\int_{\mathbb{R}^n} f_\beta(y)b(\phi(\xi_1 - y_1 - c\tau))dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{6}$$

where $' = \frac{d}{d\xi_1}$ and $\xi_1 = x_1 + ct$. Let

$$f_{i\beta}(y_i) := \frac{1}{(4\pi\beta)^{1/2}} e^{-\frac{y_i^2}{4\beta}}. \tag{7}$$

Then

$$f_\beta(y) := \prod_{i=1}^{n} f_{i\beta}(y_i), \quad \text{and} \quad \int_{\mathbb{R}} f_{i\beta}(y_i)dy_i = 1, \quad i = 1, 2, \cdots, n, \tag{8}$$

and (6) is reduced to, for $\tau \geq 0$,

$$\begin{cases} c\phi' - J_1 * \phi + \phi + d(\phi) = \displaystyle\int_{\mathbb{R}} f_{1\beta}(y_1)b(\phi(\xi_1 - y_1 - c\tau))dy_1, \\ \phi(\pm\infty) = u_\pm. \end{cases} \tag{9}$$

The main purpose of this paper is to study the global asymptotic stability of planar traveling wavefronts of the equations (1) and (4) with or without time-delay, respectively, in particular, in the case of the *critical wave* $\phi(x_1 + c_*t)$. Here the number c_* is called the *critical speed* (or the *minimum speed*) in the sense that a traveling wave $\phi(x_1 + ct)$ exists if $c \geq c_*$, while no traveling wave $\phi(x_1 + ct)$ exists if $c < c_*$.

The nonlocal dispersion equation (1) has been extensively studied recently. For the local dispersion equation

$$u_t = J * u - u + F(u), \tag{10}$$

Chasseigne *et al* [3] and Cortazar *et al* [6] showed that the linear dispersion equation (10) (with $F(u) = 0$) is almost equivalent to the linear diffusion equation, and the asymptotic behavior of the solutions to the linear equation of nonlocal dispersion is exactly the same to the corresponding linear diffusion equation. Ignat and Rossi [19, 20] further obtained the asymptotic behavior of the solutions to the nonlinear equation (10). García-Melián and Quirós [14] investigated the blow up phenomenon of the solution to the equation (10) with $F(u) = u^p$, and gave the Fujita critical exponent. Regarding the structure of special solutions to (10) like traveling wave solutions, for (10) with monostable nonlinearity, recently Coville and his collaborators [7, 8, 9, 10] studied the existence and uniqueness (up to a shift) of traveling waves. See also the existence/nonexistence of traveling waves by Yagisita [40] and the existence of almost periodic traveling waves by Chen [4].

The stability of traveling waves for Fisher-KPP equations has been one of hot research spots and extensively investigated. The first framework on the stability of traveling waves for the regular Fisher-KPP equation was given by Sattinger [36] in 1976, where he proved that the non-critical traveling waves are exponentially stable by the spectral analysis method. Then, the local stability for the traveling waves, particularly for the critical waves, was obtained by Uchiyama [38] by the maximum principle method, where, no convergence rate was derived to the critical waves. Almost at the same time, by the Green function method, Moet [34] proved that the non-critical traveling waves are exponential stable and the critical waves are algebraic stable with the convergence rate $O(t^{-1/2})$. A similar algebraic convergence rate $O(t^{-1/4})$ to the critical traveling waves was also later derived by Kirchgassner in [22] by the spectral analysis method, which was further improved to be $O(t^{-3/2})$ by Gallay [13] by means of the renormalization group method under some stiff condition on the initial perturbation. In [2], by the probabilistic argument, Bramson gave some necessary and sufficient conditions on the initial data for the stability of both non-critical and critical traveling waves, respectively, which was then re-derived by Lau [24] in the analytic argument based on the maximum principle. For the multi-dimensional case, the stability of planar faster traveling waves with $c > c_*$ was obtained by Mallordy and Roquejoffre in [26], see also [18] for the stability on the manifolds but without convergence rates. Recently, Hamel and Roques [17] obtained the stability of pulsating fronts for the periodic spatial-temporal Fisher-KPP equations. On the other hand, when the diffusion equations involve the time-delays, which represent the dynamic models of population in ecology, the first result on the exponential stability for the fast traveling waves was obtained by Mei *et al* [29] by the technical weighted energy method, and the stability for the slower waves was then proved in [27, 28, 30]. Recently, by using the L^1-weighted energy method together with the Green function method, Mei, Ou and Zhao [31] further proved that, all non-critical waves are globally stable with an exponential convergence rate, and the critical waves are globally stable with the algebraic rate $O(t^{-1/2})$, which was then extended to the high-dimensional case in [32]. Instead of the regular spatial diffusion, math-biologically, the nonlocal dispersion equations (1) is regarded as an ideal model to describe the population distribution [11]. When the nonlinearity is bistable, the stability of traveling waves for (10) was obtained by Bates *et al* [1] and Chen [5], respectively. However, when the nonlinearity is monostable, the stability of traveling waves for the Fisher-KPP equations with nonlocal dispersion (1) is almost not related, except a special case for the fast waves with large wave speed to the 1-D age-structured population model by Pan *et al* [35]. As we know, such a problem is also very significant but challenging, because the equations of Fisher-KPP type possess an unstable node, different from the bistable case studied in [1, 5], this unstable node usually causes a serious difficulty in the stability proof, particularly, for the critical traveling waves. The main interest in this paper is to investigate the stability of traveling waves to (1) with $\tau > 0$ and (4) with $\tau = 0$.

In this paper, we will first investigate the linearized equation of (1), and derive the optimal decay rates of the solution to the linearized equation by means of Fourier transform. This is a crucial step for getting the optimal convergence for the nonlocal stability of traveling waves. Then, we will technically establish the global existence and comparison principles of the solution to the n-D nonlinear equation with nonlocal dispersion (1). Inspired by [34] for the classical Fisher-KPP equations and the further developments by [31], by ingeniously selecting a weight function which is dependent on the critical wave speed c_*, and using the weighted energy method and the Green function method with the comparison principles together, we will further prove that, all noncritical planar traveling waves $\phi(x\cdot\mathbf{e}+ct)$ are exponentially stable in the form of $t^{-\frac{n}{\alpha}}e^{-\mu_\tau}$ for some constant $\mu_\tau=\mu(\tau)$ such that $0<\mu_\tau\le\mu_0$ for $\tau\ge 0$; and all critical planar traveling waves $\phi(x\cdot\mathbf{e}+c_*t)$ are algebraically stable in the form of $t^{-\frac{n}{\alpha}}$. These convergence rates are optimal and the stability results significantly develop the existing studies on the nonlocal dispersion equations. We will also show that the time-delay τ will slow down the convergence of the the solution $u(t,x)$ to the noncritical planar traveling waves $\phi(x\cdot\mathbf{e}+ct)$ with $c>c_*$, and cause the higher requirement for the initial perturbation around the wavefronts.

The paper is organized as follows. In section 2, we will state the existence of the traveling waves, and their stability. In section 3, we will give the solution formulas to the linearized dispersion equations of (1) and (4), and derive the optimal decay rates by Fourier transform with energy method together. In section 4, we will prove the global existence of the solution to (1) and establish the comparison principle. In section 5, based on the results obtained in sections 3 and 4, by using the weighted energy method, we will further prove the stability of planar traveling waves including the critical and noncritical waves. Finally, in section 6, we will give some particular applications of our stability theory to the classical Fisher-KPP equation with nonlocal dispersion and the Nicholson's blowflies model, and make a concluding remark to a more general case.

Notation. Before ending this section, we make some notations. Throughout this paper, $C>0$ denotes a generic constant, while $C_i>0$ and $c_i>0$ $(i=0,1,2,\cdots)$ represent specific constants. $j=(j_1,j_2,\cdots,j_n)$ denotes a multi-index with non-negative integers $j_i\ge 0$ $(i=1,\cdots,n)$, and $|j|=j_1+j_2+\cdots+j_n$. The derivatives for multi-dimensional function are denoted as

$$\partial_x^j f(x):=\partial_{x_1}^{j_1}\cdots\partial_{x_n}^{j_n}f(x).$$

For a n-D function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\eta)=\hat{f}(\eta):=\int_{\mathbb{R}^n}e^{-\mathrm{i}x\cdot\eta}f(x)dx,\qquad \mathrm{i}:=\sqrt{-1},$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x):=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}x\cdot\eta}\hat{f}(\eta)d\eta.$$

Let I be an interval, typically $I=\mathbb{R}^n$. $L^p(I)$ $(p\ge 1)$ is the Lebesque space of the integrable functions defined on I, $W^{k,p}(I)$ $(k\ge 0,p\ge 1)$ is the Sobolev space of the L^p-functions $f(x)$ defined on the interval I whose derivatives $\partial_x^j f$ with $|j|=k$ also belong to $L^p(I)$, and in particular, we denote $W^{k,2}(I)$ as $H^k(I)$. Further, $L^p_w(I)$ denotes the weighted L^p-space for a weight function $w(x)>0$ with the norm defined as

$$\|f\|_{L^p_w}=\Big(\int_I w(x)\,|f(x)|^p\,dx\Big)^{1/p},$$

$W_w^{k,p}(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{W_w^{k,p}} = \Big(\sum_{|j|=0}^{k} \int_I w(x) \left|\partial_x^j f(x)\right|^p dx \Big)^{1/p},$$

and $H_w^k(I)$ is defined with the norm

$$\|f\|_{H_w^k} = \Big(\sum_{|j|=0}^{k} \int_I w(x) \left|\partial_x^j f(x)\right|^2 dx \Big)^{1/2}.$$

Let $T > 0$ be a number and $\mathbf{B}$ be a Banach space. We denote by $C^0([0,T],\mathbf{B})$ the space of the $\mathbf{B}$-valued continuous functions on $[0,T]$, $L^2([0,T],\mathbf{B})$ as the space of the $\mathbf{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of the $\mathbf{B}$-valued functions on $[0,\infty)$ are defined similarly.

2. **Traveling waves and their stabilities.** As we mentioned before, when $\tau = 0$ and $\beta \to 0^+$, the existence and uniqueness (up to a shift) of traveling waves for the equation (10) in the case of bistable or mono-stable $F(u)$ were proved in [7, 8, 9, 10], particular, in a recent work by Yagisita [40] for the existence and nonexistence of traveling waves, when the nonlinearity $F(u)$ is mono-stable. When $\beta \to 0^+$ but $\tau > 0$, the existence of traveling waves with a specially mono-stable $F(u)$ was presented in [35] by the upper-lower solutions method. Here we are going to state the existence of traveling waves to the time-delayed equation (1) with nonlocality for the birth rate function in a general case of mono-stability.

For the regular 1-D Fisher-KPP equation

$$u_t - u_{x_1x_1} = F(u) \tag{11}$$

with the mono-stable $F(u)$ satisfying

$$F(0) = F(u_+) = 0, \quad F'(0) > 0, \quad F'(u_+) < 0 \quad \text{and} \quad F'(0)u > F(u) \quad \text{for} \quad u \in [0, u_+],$$

it is well-known that the traveling wavefronts $\phi(x + ct)$ connecting with $\phi(-\infty) = 0$ and $\phi(+\infty) = u_+$ exist for all $c \geq c_*$, where $c_* = 2\sqrt{F'(0)}$ is the critical wave speed. To find the critical wave speed c_*, a heuristic but easy method is that, we first linearize (11) around $u = 0$

$$u_t - u_{x_1x_1} = F'(0)u,$$

then substitute $u = e^{\lambda(x_1+ct)}$ to the above equation to yield

$$\lambda c - \lambda^2 = F'(0),$$

namely,

$$\lambda = \frac{c \pm \sqrt{c^2 - 4F'(0)}}{2},$$

which implies the minimum speed such that $c_*^2 = 4F'(0)$, that is,

$$c_* = 2\sqrt{F'(0)}.$$

Similarly, for our nonlocal Fisher-KPP equation (9), we can formally derive its critical wave speed as follows. Let us linearize (9) around $\phi = 0$, we have

$$c\phi' - J_1 * \phi + \phi + d'(0)\phi = b'(0) \int_{\mathbb{R}} f_{1\beta}(y_1)\phi(\xi_1 - y_1 - c\tau))dy_1. \tag{12}$$

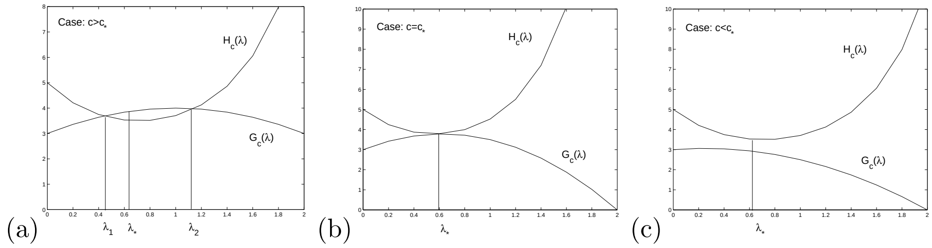

FIGURE 1. (a): the case of $c > c_*$; (b): the case of $c = c_*$; and (c): the case of $c < c_*$.

Setting $\phi(\xi_1) = e^{\lambda \xi_1}$ for some positive constant λ, we then have

$$c\lambda - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 + 1 + d'(0) = b'(0) e^{\beta\lambda^2 - \lambda c\tau}. \tag{13}$$

Denote

$$G_c(\lambda) := c\lambda - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 + 1 + d'(0), \tag{14}$$

$$H_c(\lambda) := b'(0) e^{\beta\lambda^2 - \lambda c\tau}. \tag{15}$$

Since

$$G_c''(\lambda) = -\int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} y_1^2 dy_1 < 0,$$

$$H_c''(\lambda) = b'(0) e^{\beta\lambda^2 - \lambda c\tau} [(2\beta\lambda - c\tau)^2 + 2\beta] > 0,$$

then $G_c(\lambda)$ is concave downward and $H_c(\lambda)$ is concave upward. Notice also

$$G_c(0) = d'(0) > b'(0) = H_c(0),$$

the graphs of $G_c(\lambda)$ and $H_c(\lambda)$ can be observed as in Figure 1. Clearly, when $c = c_*$, there exists a unique tangent point (c_*, λ_*) for these two curves $G_c(\lambda)$ and $H_c(\lambda)$, namely,

$$G_{c_*}(\lambda_*) = H_{c_*}(\lambda_*) \quad \text{and} \quad G'_{c_*}(\lambda_*) = H'_{c_*}(\lambda_*),$$

which determines the minimum speed c_* as follows

$$b'(0) e^{\beta\lambda_*^2 - \lambda_* c_* \tau} = c_*\lambda_* - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1 + 1 + d'(0), \tag{16}$$

$$b'(0)(2\beta\lambda_* - c_*\tau) e^{\beta\lambda_*^2 - \lambda_* c_* \tau} = c_* + \int_{\mathbb{R}} y_1 J_1(y_1) e^{-\lambda_* y_1} dy_1. \tag{17}$$

It is also noted that, when $c > c_*$, the equation $G_c(\lambda) = H_c(\lambda)$ has two roots

$$0 < \lambda_1 := \lambda_1(c) < \lambda_2 := \lambda_2(c), \tag{18}$$

and

$$G_c(\lambda) > H_c(\lambda) \quad \text{for } \lambda_1 < \lambda < \lambda_2;$$

while, when $c < c_*$, there is no solution for $G_c(\lambda) = H_c(\lambda)$.

By such an observation, we set the upper and lower solutions to the nonlinear equation (9) as

$$\bar{\phi}(\xi_1) := \min\{K,\ K e^{\lambda_1 \xi_1}\}, \quad \underline{\phi}(\xi_1) := \max\{0,\ K(1 - M e^{\epsilon \xi_1}) e^{\lambda_1 \xi_1}\},$$

for some suitably chosen constants $K > 0$, $M > 1$ and $\epsilon > 0$, where λ_1 is defined in (18), and $\lambda_1 = \lambda_*$ when $c = c_*$. Using the upper-lower solutions method as shown in [37, 35], then we can similarly prove the existence of traveling waves for (9). The proof is long but the procedure is straightforward to [37, 35], so we omit its detail.

Theorem 2.1 (Existence of traveling waves)**.** *Under the conditions (H_1)-(H_3) and (J_1)-(J_2), and $\int_{\mathbb{R}^1} J_1(y_1)e^{-\lambda y_1}dy_1 < +\infty$ for all $\lambda > 0$, then, for the time-delay $\tau \geq 0$, there exist a unique pair of numbers (c_*, λ_*) determined by*

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \qquad H'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*), \tag{19}$$

where

$$H_c(\lambda) = b'(0)e^{\beta\lambda^2-\lambda c\tau},\ G_c(\lambda) = c\lambda - E_c(\lambda) + d'(0),\ E_c(\lambda) = \int_{\mathbb{R}} J_1(y_1)e^{-\lambda y_1}dy_1 - 1, \tag{20}$$

namely, (c_*, λ_*) is the tangent point of $H_c(\lambda)$ and $G_c(\lambda)$, such that, when $c \geq c_*$, there exits a monotone traveling wavefront $\phi(x_1 + ct)$ of (6) connecting $u_\pm$ exists.

Furthermore, it can be verified:

- *In the case of $c > c_*$, there exist two numbers depending on c: $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $H_c(\lambda_i) = G_c(\lambda_i)$, i.e.,*

$$b'(0)e^{\beta\lambda_i^2-\lambda_i c\tau} = c\lambda_i - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda_i y_1}dy_1 + 1 + d'(0), \qquad i = 1, 2, \tag{21}$$

such that

$$H_c(\lambda) < G_c(\lambda) \quad \text{for} \ \ \lambda_1 < \lambda < \lambda_2, \tag{22}$$

and particularly,

$$H_c(\lambda_*) < G_c(\lambda_*) \ \ \text{with} \ \ \lambda_1 < \lambda_* < \lambda_2. \tag{23}$$

- *In the case of $c = c_*$, it holds*

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \ \ \text{with} \ \ \lambda_1 = \lambda_* = \lambda_2. \tag{24}$$

- *When $\xi_1 = x_1 + ct \to -\infty$, for all $c > c_*$, the traveling wavefronts $\phi(x_1 + ct)$ converge to $u_- = 0$ exponentially as follows*

$$|\phi(\xi_1)| = O(1)e^{-\lambda_1(c)|\xi_1|}. \tag{25}$$

Remark 1.

1. The results shown in Theorem 2.1 can be regarded as an extension of Lemma 2.1 in [31] for the existence of traveling waves of the regular diffusion equation with time-delay and mono-stable nonlinearity.

2. The existence of traveling waves in the mono-stable case studied in [8, 40] is a special example of ours, but seems not to be specific as ours. In fact, by taking $\tau = 0$ and $\beta \to 0^+$, as we mentioned in (5), the equation (1) reduces to the following regular equation [8, 40]

$$u_t - J * u + u = b(u) - d(u) =: F(u).$$

Notice that, our conditions (H_3) and (J_1) imply $F(u) \leq F'(0)u$ and $J(-x) = J(x)$. However, these restrictions $F(u) \leq F'(0)u$ and $J(-x) = J(x)$ are not assumed in [8, 40]. They proved that, there exists a critical wave speed c_*, such that a traveling wave $\phi(x+ct)$ exists for $c \geq c_*$ and no traveling wave $\phi(x+ct)$ exists for $c < c_*$. Such a result for existence/non-existence of traveling waves is better than ours presented in Theorem 2.1.

Next, we are going to state our stability results. First of all, let us technically choose a weight function:

$$w(x_1) = \begin{cases} e^{-\lambda_*(x_1 - x_*)}, & \text{for } x_1 \leq x_*, \\ 1, & \text{for } x_1 > x_*, \end{cases} \tag{26}$$

where $\lambda_* = \lambda_*(c_*) > 0$ is given in (16) and (17), and $x_* > 0$ is a sufficiently large number such that,

$$0 < d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_* - y_1 - c\tau))dy < d'(u_+) - b'(u_+). \tag{27}$$

The selection of x_* in (27) is valid, because of $d'(u_+) - b'(u_+) > 0$ (see(H_2)). In fact, we have

$$\begin{aligned} \lim_{\xi_1 \to \infty} d'(\phi(\xi_1)) &= d'(u_+) \\ &> b'(u_+) \\ &= \int_{\mathbb{R}^n} f_\beta(y) \Big[\lim_{\xi_1 \to \infty} b'(\phi(\xi_1 - y_1 - c\tau)) \Big] dy \\ &= \lim_{\xi_1 \to \infty} \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy, \end{aligned}$$

which implies that, by (H_3), there exists a unique $x_* \gg 1$ such that, for $\xi_1 \in [x_*, \infty)$

$$\begin{aligned} & d'(u_+) - b'(u_+) \\ & > d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \\ & \geq d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_* - y_1 - c\tau))dy \\ & > 0. \end{aligned} \tag{28}$$

Theorem 2.2 (Stability of planar traveling waves with time-delay)**.** *Under assumptions (H_1)-(H_3) and (J_1)-(J_2), for a given traveling wave $\phi(x_1 + ct)$ of the equation (1) with $c \geq c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data $u_0(s, x)$ is bounded in $[u_-, u_+]$ and $u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ with $m > \frac{n}{2}$, then the solution of (1) uniquely exists and satisfies:*

- *When $c > c_*$, the solution $u(t, x)$ converges to the noncritical planar traveling wave $\phi(x_1 + ct)$ exponentially*

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad t > 0, \tag{29}$$

where

$$0 < \mu_\tau < \min\{d'(u_+) - b'(u_+),\ \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}, \tag{30}$$

and $\varepsilon_1 = \varepsilon_1(\tau)$ such that $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $\varepsilon_1 = \varepsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$;

- *When $c = c_*$, the solution $u(t, x)$ converges to the critical planar traveling wave $\phi(x_1 + c_* t)$ algebraically*

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{n}{\alpha}}, \quad t > 0. \tag{31}$$

However, when the time-delay $\tau = 0$, then we have the following stronger stability for the traveling waves but with a weaker condition on initial perturbation.

Theorem 2.3 (Stability of planar traveling waves without time-delay)**.** *Under assumptions (H_1)-(H_3) and (J_1)-(J_2), for a given traveling wave $\phi(x_1+ct)$ of the equation (4) with $c \ge c_*$ and $\phi(\pm\infty) = u_\pm$, if the initial data $u_0(x)$ is bounded in $[u_-, u_+]$ and $u_0 - \phi \in H_w^m(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n)$ with $m > \frac{n}{2}$, then the solution of (4) uniquely exists and satisfies:*

- *When $c > c_*$, the solution $u(t,x)$ converges to the noncritical planar traveling wave $\phi(x_1+ct)$ exponentially*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1+ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad t > 0, \tag{32}$$

where

$$0 < \mu_0 < \min\{d'(u_+) - b'(u_+),\ G_c(\lambda_*) - H_c(\lambda_*)\}; \tag{33}$$

- *When $c = c_*$, the solution $u(t,x)$ converges to the critical planar traveling wave $\phi(x_1+c_*t)$ algebraically*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1+c_*t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \tag{34}$$

Remark 2.

1. Comparing Theorem 2.2 with time-delay and Theorem 2.3 without time-delay, we realize that, the sufficient condition on the initial perturbation around the wave in the case with time-delay is stronger than the case without time-delay, but the convergence rate to the noncritical waves $\phi(x_1+ct)$ for $c > c_*$ in the case with time-delay is weaker than the case without time-delay, see (30) for $\mu_\tau \le \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)] < G_c(\lambda_*) - H_c(\lambda_*)$, and (33) for $\mu_0 \le G_c(\lambda_*) - H_c(\lambda_*)$, and $\varepsilon_1 = \varepsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$. This means, the time-delay $\tau > 0$ affects the stability of traveling waves a lot, not only the higher requirement for the initial perturbation, but also the slower convergence rate for the solution to the noncritical traveling waves.

2. The convergence rates showed both in Theorem 2.2 and Theorem 2.3 are explicit and optimal in the sense of L^1-initial perturbations, particularly, the algebraic decay rates for the solution converging to the critical waves. Actually, all of them are derived from the linearized equations.

3. Notice that,

$$\lim_{c\to c_*} [G_c(\lambda_*) - H_c(\lambda_*)] = 0, \quad i.e., \quad \lim_{c\to c_*} \mu_\tau = 0 \quad \text{for all } \tau \ge 0,$$

From (29) and (30), or correspondingly, (32) and (33), we easily see that,

$$\lim_{c\to c_*} t^{-\frac{n}{\alpha}} e^{-\mu_\tau t} = t^{-\frac{n}{\alpha}}, \quad \tau \ge 0.$$

This implies that the exponential decay in the noncritical case will continuously degenerate to the algebraic decay in the critical case.

3. Linearized nonlocal dispersion equations.

In this section, we will derive the solution formulas for the linearized nonlocal dispersion equations with or without time-delay, as well as their optimal decay rates, which will play a key role in the stability proof in section 5.

Now let us introduce the solution formula for linear delayed ODEs [21] and the asymptotic behaviors of the solutions [32].

Lemma 3.1 ([21])**.** *Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $\tau > 0$*

$$\begin{cases} \dfrac{d}{dt}z(t) + k_1 z(t) = k_2 z(t-\tau) \\ z(s) = z_0(s), \quad s \in [-\tau, 0]. \end{cases} \tag{35}$$

Then

$$z(t) = e^{-k_1(t+\tau)} e_\tau^{\bar{k}_2 t} z_0(-\tau) + \int_{-\tau}^{0} e^{-k_1(t-s)} e_\tau^{\bar{k}_2(t-\tau-s)} [z_0'(s) + k_1 z_0(s)] ds, \tag{36}$$

where

$$\bar{k}_2 := k_2 e^{k_1 \tau}, \tag{37}$$

and $e_\tau^{\bar{k}_2 t}$ is the so-called `delayed exponential function` *in the form*

$$e_\tau^{\bar{k}_2 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \le t < 0, \\ 1 + \frac{\bar{k}_2 t}{1!}, & 0 \le t < \tau, \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!}, & \tau \le t < 2\tau, \\ \vdots & \vdots \\ 1 + \frac{\bar{k}_2 t}{1!} + \frac{\bar{k}_2^2 (t-\tau)^2}{2!} + \cdots + \frac{\bar{k}_2^m [t-(m-1)\tau]^m}{m!}, & (m-1)\tau \le t < m\tau, \\ \vdots & \vdots \end{cases} \tag{38}$$

and $e_\tau^{\bar{k}_2 t}$ is the fundamental solution to

$$\begin{cases} \dfrac{d}{dt}z(t) = \bar{k}_2 z(t-\tau) \\ z(s) \equiv 1, \quad s \in [-\tau, 0]. \end{cases} \tag{39}$$

Lemma 3.2 ([32])**.** *Let $k_1 \ge 0$ and $k_2 \ge 0$. Then the solution $z(t)$ to (35) (or equivalently (36)) satisfies*

$$|z(t)| \le C_0 e^{-k_1 t} e_\tau^{\bar{k}_2 t}, \tag{40}$$

where

$$C_0 := e^{-k_1 \tau} |z_0(-\tau)| + \int_{-\tau}^{0} e^{k_1 s} |z_0'(s) + k_1 z_0(s)| ds, \tag{41}$$

and the fundamental solution $e_\tau^{\bar{k}_2 t}$ with $\bar{k}_2 > 0$ to (39) satisfies

$$e_\tau^{\bar{k}_2 t} \le C(1+t)^{-\gamma} e^{\bar{k}_2 t}, \quad t > 0, \tag{42}$$

for arbitrary number $\gamma > 0$.

Furthermore, when $k_1 \ge k_2 \ge 0$, there exists a constant $\varepsilon_1 = \varepsilon_1(\tau)$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $\varepsilon_1 = 1$ for $\tau = 0$, and $\varepsilon_1 = \varepsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$, such that

$$e^{-k_1 t} e_\tau^{\bar{k}_2 t} \le C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0, \tag{43}$$

and the solution $z(t)$ to (35) satisfies

$$|z(t)| \le C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0. \tag{44}$$

Now, we consider the following linearized nonlocal time-delayed dispersion equation (which will be derived in section 5 for the proof of stability of traveling wavefronts)

$$\begin{cases} \dfrac{\partial v}{\partial t} - \displaystyle\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1} v(t, x-y)dy + c_1 v \\ \qquad\qquad = c_2 \displaystyle\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} v(t-\tau, x-y)dy, \\ v(s,x) = v_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}^n \end{cases} \tag{45}$$

for some given constant coefficients c, c_1 and c_2, where $c \geq c_*$ is the wave speed.

We are going to derive its solution formula as well as the asymptotic behavior of the solution. By taking Fourier transform to (45), and noting that,

$$\begin{aligned} &\mathcal{F}\Big[\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1} v(t, x-y)dy\Big](t,\eta) \\ &= \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\eta}\Big(\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1} v(t,x-y)dy\Big)dx \\ &= \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}\Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\eta} v(t,x-y)dx\Big)dy \\ &= \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}\Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}(x+y)\cdot\eta} v(t,x)dx\Big)dy \\ &= \Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}y\cdot\eta} J(y)e^{-\lambda_* y_1} dy\Big)\hat{v}(t,\eta), \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\mathcal{F}\Big[c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} v(t-\tau, x-y)dy\Big](t-\tau,\eta) \\ &= c_2 \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\eta}\Big(\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} v(t-\tau,x-y)dy\Big)dx \\ &= c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)}\Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\eta} v(t-\tau,x-y)dx\Big)dy \\ &= c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)}\Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}(x+y)\cdot\eta} v(t-\tau,x)dx\Big)dy \\ &= c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}y\cdot\eta}\Big(\int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\eta} v(t-\tau,x)dx\Big)dy \\ &= \Big(c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}y\cdot\eta} dy\Big)\hat{v}(t-\tau,\eta), \end{aligned} \tag{47}$$

we have

$$\frac{d\hat{v}}{dt} + A(\eta)\hat{v} = B(\eta)\hat{v}(t-\tau,\eta), \quad \text{with } \hat{v}(s,\eta) = \hat{v}_0(s,\eta), \ s \in [-\tau, 0], \tag{48}$$

where

$$A(\eta) := c_1 - \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1} e^{-\mathrm{i}y\cdot\eta} dy \tag{49}$$

and

$$B(\eta) := c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)} e^{-\mathrm{i}y\cdot\eta} dy. \tag{50}$$

By using the formula of the delayed ODE (36) in Lemma 3.1, we then solve (48) as follows

$$\begin{aligned}\hat{v}(t,\eta) &= e^{-A(\eta)(t+\tau)}e_{\tau}^{\mathcal{B}(\eta)t}\hat{v}_0(-\tau,\eta)\\ &\quad +\int_{-\tau}^{0} e^{-A(\eta)(t-s)}e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)}\Big[\partial_s\hat{v}_0(s,\eta)+A(\eta)\hat{v}_0(s,\eta)\Big]ds, \end{aligned}\tag{51}$$

where

$$\mathcal{B}(\eta) := B(\eta)e^{A(\eta)\tau}. \tag{52}$$

Then, by taking the inverse Fourier transform to (51), we get

$$\begin{aligned} v(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\eta}e^{-A(\eta)(t+\tau)}e_{\tau}^{\mathcal{B}(\eta)t}\hat{v}_0(-\tau,\eta)d\eta\\ &\quad +\int_{-\tau}^{0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\eta}e^{-A(\eta)(t-s)}e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)}\\ &\quad \times\Big[\partial_s\hat{v}_0(s,\eta)+A(\eta)\hat{v}_0(s,\eta)\Big]d\eta ds, \end{aligned}\tag{53}$$

and its derivatives

$$\begin{aligned} \partial_{x_j}^k v(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\eta}(\mathrm{i}\eta_j)^k e^{-A(\eta)(t+\tau)}e_{\tau}^{\mathcal{B}(\eta)t}\hat{v}_0(-\tau,\eta)d\eta\\ &\quad +\int_{-\tau}^{0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\cdot\eta}(\mathrm{i}\eta_j)^k e^{-A(\eta)(t-s)}e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)}\\ &\quad \times\Big[\partial_s\hat{v}_0(s,\eta)+A(\eta)\hat{v}_0(s,\eta)\Big]d\eta ds \end{aligned}\tag{54}$$

for $k = 0, 1, \cdots$ and $j = 1, \cdots, n$.

Now we are going to derive the asymptotic behavior of $v(t,x)$.

Proposition 1 (Optimal decay rates for $\tau > 0$)**.** *Suppose that $v_0 \in C([-\tau,0]; H^{m+1}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$ and $\partial_s v_0 \in L^1([-\tau,0]; H^m(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$ for $m \geq 0$, and let*

$$\begin{cases} \tilde{c}_1 := c_1 - \displaystyle\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}dy,\\ c_3 := c_2\displaystyle\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)}dy > 0. \end{cases}\tag{55}$$

If $\tilde{c}_1 \geq c_3$, then there exists a constant $\varepsilon_1 = \varepsilon_1(\tau)$ as showed in (43) satisfying $0 < \varepsilon_1 < 1$ for $\tau > 0$, such that the solution of the linearized equation (45) satisfies

$$\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)} \leq C\mathcal{E}_{v_0}^k t^{-\frac{n+2k}{2\alpha}}e^{-\varepsilon_1(\tilde{c}_1-c_3)t},\ t>0, \tag{56}$$

for $k = 0, 1, \cdots, [m]$ and $j = 1, \cdots, n$, where

$$\begin{aligned}\mathcal{E}_{v_0}^k : &= \|v_0(-\tau)\|_{L^1(\mathbb{R}^n)} + \|v_0(-\tau)\|_{H^k(\mathbb{R}^n)}\\ &\quad +\int_{-\tau}^{0}[\|(v_{0s}', v_0)(s)\|_{L^1(\mathbb{R}^n)} + \|(v_{0s}', v_0)(s)\|_{H^k(\mathbb{R}^n)}]ds. \end{aligned}\tag{57}$$

Furthermore, if $m > \frac{n}{2}$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\mathcal{E}_{v_0}^m t^{-\frac{n}{\alpha}}e^{-\varepsilon_1(\tilde{c}_1-c_3)t},\quad t>0. \tag{58}$$

Particularly, when $\tilde{c}_1 = c_3$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\mathcal{E}_{v_0}^m t^{-\frac{n}{\alpha}},\quad t>0. \tag{59}$$

Proof. Let

$$I_1(t,\eta): = (\mathrm{i}\eta_j)^k e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t}\hat{v}_0(-\tau,\eta), \tag{60}$$

$$I_2(t-s,\eta): = (\mathrm{i}\eta_j)^k e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}\Big[\partial_s\hat{v}_0(s,\eta)+A(\eta)\hat{v}_0(s,\eta)\Big]. \tag{61}$$

Then, (54) is reduced to

$$\partial_{x_j}^k v(t,x) = \mathcal{F}^{-1}[I_1](t,x) + \int_{-\tau}^0 \mathcal{F}^{-1}[I_2](t-s,x)ds. \tag{62}$$

So, by using Parseval's equality, we have

$$\begin{aligned}\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)} &\le \|\mathcal{F}^{-1}[I_1](t)\|_{L^2(\mathbb{R}^n)} + \int_{-\tau}^0 \|\mathcal{F}^{-1}[I_2](t-s)\|_{L^2(\mathbb{R}^n)}ds \\ &= \|I_1(t)\|_{L^2(\mathbb{R}^n)} + \int_{-\tau}^0 \|I_2(t-s)\|_{L^2(\mathbb{R}^n)}ds. \end{aligned}\tag{63}$$

$$\begin{aligned}|e^{-A(\eta)t}| &= e^{-c_1 t}\Big|\exp\Big(t\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}e^{-\mathrm{i}y\cdot\eta}dy\Big)\Big| \\ &= e^{-c_1 t}\exp\Big(t\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}\cos(y\cdot\eta)dy\Big) \\ &= e^{-\tilde{c}_1 t}\exp\Big(-t\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}(1-\cos(y\cdot\eta))dy\Big) \\ &=: e^{-k_1 t}, \qquad \text{with } k_1 := \tilde{c}_1 + \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}(1-\cos(y\cdot\eta))dy, \end{aligned}\tag{64}$$

Note that, using (49), (50), and the facts $\frac{e^x+e^{-x}}{2}\ge 1$ for all $x\in\mathbb{R}$, and $\int_{\mathbb{R}^n} J(y)\sin(y\cdot\eta)dy = 0$ because $J(y)$ is even and $\sin(y\cdot\eta)$ is odd, and $\int_{\mathbb{R}^n} J(y)dy = 1$, we have

$$\begin{aligned}&\exp\Big(-t\int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}(1-\cos(y\cdot\eta))dy\Big) \\ &= \exp\Big(-t\int_{\mathbb{R}^n} J(y)\frac{e^{-\lambda_* y_1}+e^{\lambda_* y_1}}{2}(1-\cos(y\cdot\eta))dy\Big) \\ &\le \exp\Big(-t\int_{\mathbb{R}^n} J(y)(1-\cos(y\cdot\eta))dy\Big) \\ &= \exp\Big(-t\int_{\mathbb{R}^n} J(y)[1-[\cos(y\cdot\eta)+\mathrm{i}\sin(y\cdot\eta)]]dy\Big) \\ &= e^{(\hat{J}(\eta)-1)t} \end{aligned}\tag{65}$$

and

$$|B(\eta)| \le c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)}dy = c_3 =: k_2, \tag{66}$$

and

$$|\mathcal{B}(\eta)| = |B(\eta)e^{A(\eta)\tau}| \le c_3 e^{k_1\tau} = k_2 e^{k_1\tau} =: \bar{k}_2, \tag{67}$$

and further

$$|e_\tau^{\mathcal{B}(\eta)t}| \le e_\tau^{\bar{k}_2 t}. \tag{68}$$

If $\tilde{c}_1 \ge c_3$, from (J$_2$), namely, $1-\hat{J}(\eta) = \mathcal{K}|\eta|^\alpha - o(|\eta|^\alpha) > 0$ as $\eta\to 0$, then $k_1 = \tilde{c}_1 + 1 - \hat{J}(\eta) \ge c_3 = k_2$. Using (64), (65), (68) and (43) in Lemma 3.2, we obtain

$$\begin{aligned}\|I_1(t)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t}\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\le C\int_{\mathbb{R}^n}(e^{-k_1(t+\tau)}e_\tau^{\bar{k}_2 t})^2|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\le C\int_{\mathbb{R}^n}(e^{-\varepsilon_1(k_1-k_2)t})^2|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &= Ce^{-2\varepsilon_1(\tilde{c}_1-c_3)t}\int_{\mathbb{R}^n} e^{-2\varepsilon_1(1-\hat{J}(\eta))t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta. \quad (69)\end{aligned}$$

Again from (J_2), there exist some numbers $0<\mathcal{K}_1<\mathcal{K}$, $0<\delta<1$ and $\tilde{a}>0$, such that

$$\begin{cases}\mathcal{K}_1|\eta|^\alpha \le 1-\hat{J}(\eta)\le \mathcal{K}|\eta|^\alpha, & \text{as } |\eta|\le\tilde{a},\\ \delta:=\mathcal{K}_1\tilde{a}^\alpha\le 1-\hat{J}(\eta)\le\mathcal{K}|\eta|^\alpha, & \text{as } |\eta|\ge\tilde{a}.\end{cases} \tag{70}$$

Therefore, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{-2\varepsilon_1(1-\hat{J}(\eta))t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &= \int_{|\eta|\le\tilde{a}} e^{-2\varepsilon_1(1-\hat{J}(\eta))t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\quad + \int_{|\eta|\ge\tilde{a}} e^{-2\varepsilon_1(1-\hat{J}(\eta))t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\le \int_{|\eta|\le\tilde{a}} e^{-2\varepsilon_1\mathcal{K}_1|\eta|^\alpha t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta + \int_{|\eta|\ge\tilde{a}} e^{-2\varepsilon_1\delta t}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\le \|\hat{v}_0(-\tau)\|^2_{L^\infty(\mathbb{R}^n)} t^{-\frac{n+2k}{\alpha}}\int_{|\eta|\le\tilde{a}} e^{-2\varepsilon_1\mathcal{K}_1|\eta t^{\frac{1}{\alpha}}|^\alpha}|\eta_j t^{\frac{1}{\alpha}}|^{2k}d(\eta t^{\frac{1}{\alpha}}) \\ &\quad + e^{-2\varepsilon_1\delta t}\int_{|\eta|\ge\tilde{a}}|\hat{v}_0(-\tau,\eta)|^2|\eta_j|^{2k}d\eta \\ &\le C(\|v_0(-\tau)\|^2_{L^1(\mathbb{R}^n)}+\|v_0(-\tau)\|^2_{H^k(\mathbb{R}^n)})t^{-\frac{n+2k}{\alpha}}. \end{aligned} \tag{71}$$

Substitute (71) into (69), we obtain

$$\|I_1(t)\|_{L^2(\mathbb{R}^n)} \le C(\|v_0(-\tau)\|_{L^1(\mathbb{R}^n)}+\|v_0(-\tau)\|_{H^k(\mathbb{R}^n)})t^{-\frac{n+2k}{2\alpha}}e^{-\varepsilon_1(\tilde{c}_1-c_3)t} \tag{72}$$

Thus, in a similar way, we can also prove

$$\begin{aligned}&\|I_2(t-s)\|_{L^2(\mathbb{R}^n)} \\ &= \Big(\int_{\mathbb{R}^n}|e^{-A(\eta)(t-s)}e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}|^2\Big|\partial_s\hat{v}_0(s,\eta)+A(\eta)\hat{v}_0(s,\eta)\Big|^2\cdot|\eta_j|^{2k}d\eta\Big)^{\frac{1}{2}} \\ &\le Ce^{-\varepsilon_1(\tilde{c}_1-c_3)t}\Big(\int_{\mathbb{R}^n} e^{-2\varepsilon_1(1-\hat{J}(\eta))t}\Big(|\eta|^{2k}|\partial_s\hat{v}_0(s,\eta)|+|\eta|^{2k}|\hat{v}_0(s,\eta)|^2\Big)d\eta\Big)^{\frac{1}{2}} \\ &\le Ct^{-\frac{n+2k}{2\alpha}}e^{-\varepsilon_1(\tilde{c}_1-c_3)t}\Big(\|(\partial_s v_0,v_0)(s)\|_{L^1(\mathbb{R}^n)}+\|(\partial_s v_0,v_0)(s)\|_{H^k(\mathbb{R}^n)}\Big). \end{aligned} \tag{73}$$

Substituting (72) and (73) to (63), we immediately obtain (56).

Similarly, we can prove (58). We omit the details. Thus, we complete the proof of Proposition 1. □

For $\tau=0$, the equation (45) is reduced to

$$\begin{cases}\dfrac{\partial v}{\partial t}+c\dfrac{\partial v}{\partial x_1}-\displaystyle\int_{\mathbb{R}^n}J(y)e^{-\lambda_* y_1}v(t,x-y)dy+c_1 v\\ \qquad\qquad =c_2\displaystyle\int_{\mathbb{R}^n}f_\beta(y)e^{-\lambda_*(y_1+c\tau)}v(t,x-y-c\tau\mathbf{e}_1)dy,\\ v(s,x)=v_0(x),\quad x\in\mathbb{R}^n.\end{cases}\tag{74}$$

Taking Fourier transform to (74), as showed in (48), we have

$$\frac{d\hat{v}}{dt}=[B(\eta)-A(\eta)]\hat{v},\quad \text{with } \hat{v}(0,\eta)=\hat{v}_0(\eta),\tag{75}$$

where $A(\eta)$ and $B(\eta)$ are given in (49) and (50) with $\tau=0$, respectively. Integrating (75) yields

$$\hat{v}(t,\eta)=e^{-[A(\eta)-B(\eta)]t}\hat{v}_0(\eta).$$

Taking the inverse Fourier transform, we get the solution formula

$$v(t,x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{\mathrm{i}x\cdot\eta}e^{-[A(\eta)-B(\eta)]t}\hat{v}_0(\eta)d\eta.$$

Then, a similar analysis as showed before can derive the optimal decay of the solution in the case without time-delay as follows. The detail of proof is omitted.

Proposition 2 (Optimal decay rates for $\tau=0$)**.** *Suppose that $v_0\in H^m(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$ for $m\ge 0$, then the solution of the linearized equation (74) satisfies*

$$\|\partial_{x_j}^k v(t)\|_{L^2(\mathbb{R}^n)}\le C(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)})t^{-\frac{n+2k}{2\alpha}}e^{-(\tilde{c}_1-c_3)t},\ t>0,\tag{76}$$

for $k=0,1,\cdots,[m]$ and $j=1,\cdots,n$, where the positive constants $\tilde{c}_1$ and c_3 are defined in (55) for $\tau=0$.

Furthermore, if $m>\frac{n}{2}$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)}\le C(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)})t^{-\frac{n}{\alpha}}e^{-(\tilde{c}_1-c_3)t},\quad t>0.\tag{77}$$

Particularly, when $\tilde{c}_1=c_3$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)}\le C(\|v_0\|_{L^1(\mathbb{R}^n)}+\|v_0\|_{H^k(\mathbb{R}^n)})t^{-\frac{n}{\alpha}},\quad t>0.\tag{78}$$

4. **Global existence and comparison principle.**

In this section, we prove the global existence and uniqueness of the solution for the Cauchy problem to the nonlinear equation with nonlocal dispersion (1), and then establish the comparison principle in n-D case by a different proof approach to the previous work [4, 10].

Proposition 3 (Existence and Uniqueness)**.** *Let $u_0(s,x)\in C([-\tau,0];C(\mathbb{R}^n))$ with $0=u_-\le u_0(s,x)\le u_+$ for $(s,x)\in[-\tau,0]\times\mathbb{R}^n$, then the solution to (1) uniquely and globally exists, and satisfies that $u\in C^1([0,\infty);C(\mathbb{R}^n))$, and $u_-\le u(t,x)\le u_+$ for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$).*

Proof. Multiplying (1) by $e^{\eta_0 t}$ and integrating it over $[0,t]$ with respect to t, where $\eta_0>0$ will be technically selected in (82) below, we then express (1) in the integral form

$$\begin{aligned}u(t,x)\quad=\quad & e^{-\eta_0 t}u(0,x)+\int_0^t e^{-\eta_0(t-s)}\Big[\int_{\mathbb{R}^n}J(x-y)u(s,y)dy+(\eta_0-1)u(s,x)\\ &-d(u(s,x))+\int_{\mathbb{R}^n}f_\beta(y)b(u(s-\tau,x-y))dy\Big]ds.\end{aligned}\tag{79}$$

Let us define the solution space as, for any $T \in [0, \infty]$,

$$\mathfrak{B} = \Big\{ u(t,x) | u(t,x) \in C([0,T] \times \mathbb{R}^n) \text{ with } u_- \le u \le u_+, \\ u(s,x) = u_0(s,x), (s,x) \in [-\tau, 0] \times \mathbb{R}^n \Big\}, \tag{80}$$

with the norm

$$\|u\|_{\mathfrak{B}} = \sup_{t \in [0,T]} e^{-\eta_0 t} \|u(t)\|_{L^\infty(\mathbb{R}^n)}, \tag{81}$$

where

$$\eta_0 := 1 + \eta_1 + \eta_2, \quad \eta_1 := \max_{u \in [u_-, u_+]} |d'(u)|, \quad \eta_2 := \max_{u \in [u_-, u_+]} |b'(u)|. \tag{82}$$

Clearly, $\mathfrak{B}$ is a Banach space.

Define an operator $\mathcal{P}$ on $\mathfrak{B}$ by

$$\begin{aligned} &\mathcal{P}(u)(t,x) \\ &= e^{-\eta_0 t} u_0(0,x) + \int_0^t e^{-\eta_0 (t-s)} \bigg[\int_{\mathbb{R}^n} J(x-y) u(s,y) dy + (\eta_0 - 1) u(s,x) \\ &\quad - d(u(s,x)) + \int_{\mathbb{R}^n} f_\beta(y) b(u(s-\tau, x-y)) dy \bigg] ds, \quad \text{for} \quad 0 \le t \le T, \end{aligned} \tag{83}$$

and

$$\mathcal{P}(u)(s,x) := u_0(s,x), \quad \text{for} \quad s \in [-\tau, 0]. \tag{84}$$

Now we are going to prove that $\mathcal{P}$ is a contracting operator from $\mathfrak{B}$ to $\mathfrak{B}$.

Firstly, we prove that $\mathcal{P} : \mathfrak{B} \to \mathfrak{B}$. In fact, if $u \in \mathfrak{B}$, from (H$_1$)-(H$_3$), namely, $0 = d(0) \le d(u) \le d(u_+)$, $0 = b(0) \le b(u) \le b(u_+)$, and $d(u_+) = b(u_+)$, and using the facts $\int_{\mathbb{R}^n} J(x-y) dy = 1$, $\int_{\mathbb{R}^n} f_\beta(y) dy = 1$, and

$$g(u) := (\eta_0 - 1) u - d(u) \text{ is increasing}, \tag{85}$$

which implies $g(u_+) \ge g(u) \ge g(0) = 0$ for $u \in [u_-, u_+]$, then we have

$$\begin{aligned} 0 = u_- \le \mathcal{P}(u) \quad &\le \quad e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0 (t-s)} \bigg[\int_{\mathbb{R}^n} J(x-y) u_+ dy \\ &\qquad + (\eta_0 - 1) u_+ - d(u_+) + \int_{\mathbb{R}^n} f_\beta(y) b(u_+) dy \bigg] ds \\ &= \quad e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0 (t-s)} [\eta_0 u_+ - d(u_+) + b(u_+)] ds \\ &= \quad u_+. \end{aligned} \tag{86}$$

This plus the continuity of $\mathcal{P}(u)$ based on the continuity of u proves $\mathcal{P}(u) \in \mathfrak{B}$, namely, $\mathcal{P}$ maps from $\mathfrak{B}$ to $\mathfrak{B}$.

Secondly, we prove that $\mathcal{P}$ is contracting. In fact, let $u_1,\ u_2 \in \mathfrak{B}$, and $v = u_1 - u_2$, then we have

$$\mathcal{P}(u_1) - \mathcal{P}(u_2) \tag{87}$$

$$\begin{aligned} &= \int_0^t e^{-\eta_0 (t-s)} \bigg[\int_{\mathbb{R}^n} J(x-y) v(s,y) dy + (\eta_0 - 1) v(s,x) \\ &\quad - [d(u_1(s,x)) - d(u_2(s,x))] \\ &\quad + \int_{\mathbb{R}^n} f_\beta(y) [b(u_1(s-\tau, x-y)) - b(u_2(s-\tau, x-y))] dy \bigg] ds. \end{aligned} \tag{88}$$

So, we have

$$
\begin{aligned}
&|\mathcal{P}(u_1)-\mathcal{P}(u_2)|e^{-\eta_0 t}\\
&\le \int_0^t e^{-2\eta_0(t-s)}\Big(\eta_0+\max_{u\in[u_-,u_+]}|d'(u)|\Big)\|v\|_{\mathfrak{B}}ds\\
&\quad+\max_{u\in[u_-,u_+]}|b'(u)|\begin{cases}\displaystyle\int_0^{t-\tau} e^{-2\eta_0(t-s)}\|v\|_{\mathfrak{B}}ds, & \text{for } t\ge\tau\\ 0, & \text{for } 0\le t\le\tau\end{cases}\\
&\le \frac{1}{2\eta_0}\Big((\eta_0+\eta_1)(1-e^{-2\eta_0 t})+\eta_2(e^{-2\eta_0\tau}-e^{-2\eta_0 t})\Big)\|v\|_{\mathfrak{B}}\\
&\le \frac{\eta_0+\eta_1+\eta_2}{2\eta_0}\|v\|_{\mathfrak{B}}\\
&=\frac{2\eta_0-1}{2\eta_0}\|v\|_{\mathfrak{B}}\\
&=:\rho\|v\|_{\mathfrak{B}}
\end{aligned}\tag{89}
$$

for $0<\rho:=\frac{2\eta_0-1}{2\eta_0}<1$, namely, we prove that the mapping $\mathcal{P}$ is contracting:

$$\|\mathcal{P}(u_1)-\mathcal{P}(u_2)\|_{\mathfrak{B}}\le\rho\|u_1-u_2\|_{\mathfrak{B}}<\|u_1-u_2\|_{\mathfrak{B}}. \tag{90}$$

Hence, by the Banach fixed-point theorem, $\mathcal{P}$ has a unique fixed point u in $\mathfrak{B}$, i.e, the integral equation (79) has a unique classical solution on $[0,T]$ for any given $T>0$. Differentiating (79) with respect to t, we get back to the original equation (1), i.e.,

$$u_t=J*u-u+d(u(t,x))+\int_{\mathbb{R}^n}f_\beta(y)b(u(t-\tau,x-y))dy, \tag{91}$$

then we can easily confirm from the right-hand-side of (91) that $u_t\in C([0,T]\times\mathbb{R}^n)$. This completes our proof. □

Remark 3. From the proof of Proposition 3, we realize that, when $u_0(s,x)\in C^k([-\tau,0]\times\mathbb{R}^n)$, then the solution of the time-delayed equation (1) holds $u(t,x)\in C^{k+1}([0,\infty);C(\mathbb{R}^n))$; while for the non-delayed equation (4) (i.e., $\tau=0$), if $u_0(x)\in C(\mathbb{R}^n)$, then the solution of the non-delayed equation (4) holds $u(t,x)\in C^\infty([0,\infty);C(\mathbb{R}^n))$. This means that the solution to the nonlocal dispersion equation (1) possesses a really good regularity in time. However, the solutions for (1) lack the regularity in space.

Now we establish two comparison principle for (1). Although the comparison principle in 1D case were proved in [4, 10]. Here we give a comparison principle in n-D case with much weaker restriction on the initial data. The proof is also new and easy to follow. Different from the previous works [4, 10], instead of the differential equation (1), we will work on the integral equation (79), and sufficiently use the property of contracting operator $\mathcal{P}$.

Let $\bar{u}(t,x)$ be an upper solution to (1), namely

$$\begin{cases}\dfrac{\partial\bar{u}}{\partial t}-J*\bar{u}+\bar{u}+d(\bar{u}(t,x))\ge\displaystyle\int_{\mathbb{R}^n}f_\beta(y)b(\bar{u}(t-\tau,x-y))dy,\\ \bar{u}(s,x)\ge u_0(s,x),\quad s\in[-\tau,0],\ x\in\mathbb{R}^n,\end{cases}\tag{92}$$

where its integral form can be written as

$$\bar{u}(t,x) \geq e^{-\eta_0 t}\bar{u}(0,x) + \int_0^t e^{-\eta_0(t-s)}\Big[\int_{\mathbb{R}^n} J(x-y)\bar{u}(s,y)dy + (\eta_0-1)\bar{u}(x,s) \\ -d(\bar{u}(s,x)) + \int_{\mathbb{R}^n} f_\beta(y)b(\bar{u}(s-\tau,x-y))dy\Big]ds, \tag{93}$$

and let $\underline{u}(t,x)$ be an lower solution to (1) satisfying (92) or (93) conversely. Then we have the following comparison result.

Proposition 4 (Comparison Principle)**.** *Let $\underline{u}(t,x)$ and $\bar{u}(t,x)$ be the classical lower and upper solutions to (1), with $u_- \leq \underline{u}(t,x)$, $\bar{u}(t,x) \leq u_+$, respectively, and satisfy $0 \leq \underline{u}(t,x) \leq u_+$ and $0 \leq \bar{u}(t,x) \leq u_+$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then $\underline{u}(t,x) \leq \bar{u}(t,x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^n$.*

Proof. . We need to prove $\bar{u}(t,x) - \underline{u}(t,x) \geq 0$ for $(t,x) \in [0,\infty)\times\mathbb{R}^n$, namely, $r(t) := \inf_{x\in\mathbb{R}^n} v(t,x) \geq 0$, where $v(t,x) := \bar{u}(t,x) - \underline{u}(t,x)$.

If this is not true, then there exist some constants $\varepsilon > 0$ and $T > 0$ such that $r(t) > -\varepsilon e^{3\eta_0 t}$ for $t \in [0,T)$ and $r(T) = -\varepsilon e^{3\eta_0 T}$, where η_0 given in (82).

Since $\underline{u}(t,x)$ and $\bar{u}(t,x)$ are the lower and upper solutions to (1) and $\bar{u}(s,x) - \underline{u}(s,x) \geq 0$, for $s \in [-\tau, 0]$, and using (82) and (85), and noting $\bar{u}(t,x) - \underline{u}(t,x) \geq -\varepsilon e^{3\eta_0 T}$ for $(t,x) \in [0,T]\times\mathbb{R}^n$, then we have, for $0 \leq t \leq T$,

$$\begin{aligned}
&\bar{u}(t,x) - \underline{u}(t,x) \\
&\geq e^{-\eta_0 t}[\bar{u}(0,x) - \underline{u}(0,x)] \\
&\quad + \int_0^t e^{-\eta_0(t-s)}\Big(\int_{\mathbb{R}^n} J(x-y)[\bar{u}(s,y) - \underline{u}(s,y)]dy \\
&\quad + g(\bar{u}(s,x)) - g(\underline{u}(s,x)) \\
&\quad + \int_{\mathbb{R}^n} f_\beta(y)[b(\bar{u}(s-\tau,x-y)) - b(\underline{u}(s-\tau,x-y))]dy\Big)ds \\
&\geq \int_0^t e^{-\eta_0(t-s)}\Big(-\varepsilon e^{3\eta_0 s} - \max_{\zeta\in[u_-,u_+]} g'(\zeta)\varepsilon e^{3\eta_0 s}\Big)ds \\
&\quad - \max_{u\in[u_-,u_+]}|b'(u)| \begin{cases} \int_\tau^t e^{-\eta_0(t-s)}\varepsilon e^{3\eta_0(s-\tau)}ds, & \text{for } t \geq \tau \\ 0, & \text{for } 0 \leq t \leq \tau \end{cases} \\
&\geq \begin{cases} -(\eta_0+1)\varepsilon e^{-\eta_0 t}\int_0^t e^{4\eta_0 s}ds - \eta_0\varepsilon e^{-3\eta_0\tau}e^{-\eta_0 t}\int_\tau^t e^{4\eta_0 s}ds, & \text{for } t \geq \tau \\ -(\eta_0+1)\varepsilon e^{-\eta_0 t}\int_0^t e^{4\eta_0 s}ds, & \text{for } 0 \leq t \leq \tau \end{cases} \\
&\geq -\frac{2\eta_0+1}{4\eta_0}\varepsilon e^{3\eta_0 t}.
\end{aligned} \tag{94}$$

Thus, from the assumption we know

$$-\varepsilon e^{3\eta_0 T} = \inf_{x\in\mathbb{R}^n}(\bar{u}(T,x) - \underline{u}(T,x)) \geq -\frac{2\eta_0+1}{4\eta_0}\varepsilon e^{3\eta_0 T}, \tag{95}$$

which is a contradiction for $\eta_0 > \frac{1}{2}$. Here, our η_0 defined in (82) satisfies $\eta_0 > 1$. Thus the proof is complete. □

5. **Global stability of planar traveling waves.** The main purpose in this section is to prove Theorems 2.2 for all traveling waves including the critical traveling waves.

For given traveling wave $\phi(x_1+ct)$ with the speed $c\ge c_*$ and the given initial data $u_-\le u_0(s,x)\le u_+$ for $(s,x)\in[-\tau,0]\times\mathbb{R}^n$, let us define $U_0^+(s,x)$ and $U_0^-(s,x)$ as

$$\begin{aligned} U_0^-(s,x): &= \min\{\phi(x_1+cs),u_0(s,x)\}\\ U_0^+(s,x): &= \max\{\phi(x_1+cs),u_0(s,x)\} \end{aligned}\tag{96}$$

for $(s,x)\in[-\tau,0]\times\mathbb{R}^n$. So,

$$u_0-\phi=(U_0^+-\phi)+(U_0^--\phi).$$

Since $u_0-\phi\in C([-\tau,0];H_w^{m+1}(\mathbb{R}^n)\cap L_w^1(\mathbb{R}^n))$ with $m>\frac{n}{2}$ and $w(x)\ge 1$ (see (26)), and noting Sobolev's embedding theorem $H^m(\mathbb{R}^n)\hookrightarrow C(\mathbb{R}^n)$, we have $u_0-\phi\in C([-\tau,0];C(\mathbb{R}^n))$. On the other hand, the traveling wave $\phi(x_1+cs)$ is smooth, then we can guarantee $U_0^\pm(s,x)\in C([-\tau,0];C(\mathbb{R}^n))$. Thus, applying Proposition 3, we know that the solutions of (1) with the initial data $U_0^+(s,x)$ and $U_0^-(s,x)$ globally exist, and denote them by $U^+(t,x)$ and $U^-(t,x)$, respectively, that is,

$$\begin{cases}\dfrac{\partial U^\pm}{\partial t}-J*U^\pm+U^\pm+d(U^\pm)=\displaystyle\int_{\mathbb{R}^n}f_\beta(y)b(U^\pm(t-\tau,x-y))dy,\\ U^\pm(s,x)=U_0^\pm(s,x),\quad x\in\mathbb{R}^n,\ s\in[-\tau,0].\end{cases}\tag{97}$$

Then the comparison principle (Proposition 4) further implies

$$\begin{cases}u_-\le U^-(t,x)\le u(t,x)\le U^+(t,x)\le u_+\\ u_-\le U^-(t,x)\le \phi(x_1+ct)\le U^+(t,x)\le u_+\end{cases}\quad \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}^n.\tag{98}$$

In what follows, we are going to complete the proof for the stability in three steps.

Step 1. The convergence of $U^+(t,x)$ to $\phi(x_1+ct)$

Let

$$V(t,x):=U^+(t,x)-\phi(x_1+ct),\qquad V_0(s,x):=U_0^+(s,x)-\phi(x_1+cs).\tag{99}$$

It follows from (98) that

$$V(t,x)\ge 0,\qquad V_0(s,x)\ge 0.\tag{100}$$

We see from (1) that $V(t,x)$ satisfies (by linearizing it around 0)

$$\begin{aligned}&\frac{\partial V}{\partial t}-\int_{\mathbb{R}^n}J(y)V(t,x-y)dy+V+d'(0)V\\ &\quad -b'(0)\int_{\mathbb{R}^n}f_\beta(y)V(t-\tau,x-y)dy\\ &=-Q_1(t,x)+\int_{\mathbb{R}^n}f_\beta(y)Q_2(t-\tau,x-y)dy+[d'(0)-d'(\phi(x_1+ct))]V\\ &\quad+\int_{\mathbb{R}^n}f_\beta(y)[b'(\phi(x_1-y_1+c(t-\tau))-b'(0)]V(t-\tau,x-y)dy\\ &=:I_1(t,x)+I_2(t-\tau,x)+I_3(t,x)+I_4(t-\tau,x),\end{aligned}\tag{101}$$

with the initial data

$$V(s,x)=V_0(s,x),\ s\in[-\tau,0],\tag{102}$$

where

$$Q_1(t,x) = d(\phi + V) - d(\phi) - d'(\phi)V \tag{103}$$

with $\phi = \phi(x_1 + ct)$ and $V = V(t,x)$, and

$$Q_2(t-\tau, x-y) = b(\phi + V) - b(\phi) - b'(\phi)V \tag{104}$$

with $\phi = \phi(x_1 - y_1 + c(t-\tau))$ and $V = V(t-\tau, x-y)$. Here I_i, $i = 1,2,3,4$, denotes the i-th term in the right-side of line above (101).

From (H_3), i.e., $d''(u) \geq 0$ and $b''(u) \leq 0$, applying Taylor formula to (103) and (104), we immediately have

$$Q_1(t,x) \geq 0 \quad \text{and} \quad Q_2(t-\tau, x-y) \leq 0,$$

which implies

$$I_1(t,x) \leq 0 \quad \text{and} \quad I_2(t-\tau, x) \leq 0. \tag{105}$$

From (H_3) again, since $d'(\phi)$ is increasing and $b'(\phi)$ is decreasing, then $d'(0) - d'(\phi(x_1+ct)) \leq 0$ and $b'(\phi(x_1 - y_1 + c(t-\tau))) - b'(0) \leq 0$, which imply, with $V \geq 0$,

$$I_3(t,x) \leq 0 \quad \text{and} \quad I_4(t-\tau, x) \leq 0. \tag{106}$$

Thus, applying (105) and (106) to (101), we obtain

$$\frac{\partial V}{\partial t} - J * V + V + d'(0)V - b'(0)\int_{\mathbb{R}^n} f_\beta(y)V(t-\tau, x-y)dy \leq 0. \tag{107}$$

Let $\bar{V}(t,x)$ be the solution of the following equation with the same initial data $V_0(s,x)$:

$$\begin{cases} \dfrac{\partial \bar{V}}{\partial t} - J * \bar{V} + \bar{V} + d'(0)\bar{V} - b'(0)\displaystyle\int_{\mathbb{R}^n} f_\beta(y)\bar{V}(t-\tau, x-y)dy = 0, \\ \bar{V}(s,x) = V_0(s,x), \quad s \in [-\tau, 0], x \in \mathbb{R}^n. \end{cases} \tag{108}$$

From Proposition 3, we know that $\bar{V}(t,x)$ globally exists. Furthermore, (108) is actually a linear equation, and its solution is as smooth as its initial data. By the comparison principle (Proposition 4), we have

$$0 \leq V(t,x) \leq \bar{V}(t,x), \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{109}$$

Let

$$v(t,x) := e^{-\lambda_*(x_1 + ct - x_*)}\bar{V}(t,x). \tag{110}$$

From (108), $v(t,x)$ satisfies

$$\begin{aligned} \frac{\partial v}{\partial t} &- \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}v(t, x-y)dy + c_1 v \\ &= c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1 + c\tau)}v(t-\tau, x-y)dy, \end{aligned} \tag{111}$$

where

$$c_1 := c\lambda_* + 1 + d'(0) > 0, \quad \text{and} \quad c_2 := b'(0). \tag{112}$$

When $\tau = 0$, then (74) is reduced to

$$\frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y)e^{-\lambda_* y_1}v(t,x-y)dy + c_1 v = c_2 \int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_* y_1}v(t,x-y)dy. \tag{113}$$

Applying Proposition 1 to (111) for $\tau > 0$ and Proposition 2 to (113) for $\tau = 0$, we obtain the following decay rates:

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\alpha}}e^{-\varepsilon_1(\tilde{c}_1 - c_3)t}, \quad \text{for } \tau > 0, \tag{114}$$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{\alpha}}e^{-(\tilde{c}_1 - c_3)t}, \quad \text{for } \tau = 0, \tag{115}$$

where $0 < \varepsilon_1 = \varepsilon_1(\tau) < 1$, and c_3 is defined in (55), which can be directly calculated as, by using the property (8),

$$\begin{aligned} c_3 &= b'(0)\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_*(y_1+c\tau)}dy \\ &= b'(0)\int_{\mathbb{R}} f_{1\beta}(y_1)e^{-\lambda_*(y_1+c\tau)}dy_1 \\ &= b'(0)e^{\beta\lambda_*^2-\lambda_*c\tau} > 0. \end{aligned} \tag{116}$$

and

$$\tilde{c}_1 = c\lambda_* + 1 + d'(0) - \int_{\mathbb{R}} J(y_1)e^{-\lambda_* y_1}dy_1 = c\lambda_* + d'(0) - E_c(\lambda_*). \tag{117}$$

When $c > c_*$, namely, the wave $\phi(x_1+ct)$ is non-critical, from (23) in Theorem 2.1, we realize

$$\tilde{c}_1 := c\lambda_* + d'(0) - E_c(\lambda_*) = G_c(\lambda_*) > H_c(\lambda_*) = b'(0)e^{\beta\lambda_*^2-\lambda_*c\tau} =: c_3. \tag{118}$$

Thus, (114) and (115) immediately imply the following exponential decay for $c > c_*$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n}{\alpha}}e^{-\varepsilon_1\tilde{\mu}t}, \quad \text{for } \tau > 0, \tag{119}$$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n}{\alpha}}e^{-\tilde{\mu}t}, \quad \text{for } \tau = 0, \tag{120}$$

where

$$\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_*) - H_c(\lambda_*) > 0. \tag{121}$$

When $c = c_*$, namely, the wave $\phi(x_1+c_*t)$ is critical, from (24) in Proposition 2.1, we realize

$$\tilde{c}_1 := c\lambda_* + d'(0) - E_c(\lambda_*) = G_c(\lambda_*) = H_c(\lambda_*) = b'(0)e^{\beta\lambda_*^2-\lambda_*c\tau} := c_3. \tag{122}$$

Then, from (114) and (115), we immediately obtain the following algebraic decay for $c = c_*$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n}{\alpha}}, \quad \text{for all} \tau \ge 0. \tag{123}$$

Since $V(t,x) \le \bar{V}(t,x) = e^{\lambda_*(x_1+ct-x_*)}v(t,\xi)$, and $0 < e^{\lambda_*(x_1+ct-x_*)} \le 1$ for $x_1 \in (-\infty, x_*-ct]$, we immediately obtain the following decay for V.

Lemma 5.1. *Let $V = V(t,x)$. Then*
- *When $c > c_*$, then*

$$\|V(t)\|_{L^\infty((-\infty,\ x_*-ct]\times\mathbb{R}^{n-1})} \le C(1+t)^{-\frac{n}{\alpha}}e^{-\varepsilon_1\tilde{\mu}t}, \quad \text{for } \tau > 0, \tag{124}$$

$$\|V(t)\|_{L^\infty((-\infty,\ x_*-ct]\times\mathbb{R}^{n-1})} \le C(1+t)^{-\frac{n}{\alpha}}e^{-\tilde{\mu}t}, \quad \text{for } \tau = 0; \tag{125}$$

Here $\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_*) - H_c(\lambda_*) > 0$ for $c > c_*$.
- *When $c = c_*$, then*

$$\|V(t)\|_{L^\infty((-\infty,\ x_*-ct]\times\mathbb{R}^{n-1})} \le C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \ge 0. \tag{126}$$

Next we prove $V(t,x)$ exponentially decay for $x \in [x_*-ct,\infty)\times\mathbb{R}^{n-1}$.

Lemma 5.2. *For $\tau > 0$, it holds that*

$$\|V(t)\|_{L^\infty([x_*-ct,\infty)\times\mathbb{R}^{n-1})} \le Ct^{-\frac{n}{\alpha}}e^{-\mu_\tau t}, \quad \text{for } c > c_*, \tag{127}$$

$$\|V(t)\|_{L^\infty([x_*-ct,\infty)\times\mathbb{R}^{n-1})} \le Ct^{-\frac{n}{\alpha}}, \quad \text{for } c = c_*, \tag{128}$$

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1\tilde{\mu}\}$ for $c > c_*$.

Proof. From (97) and (6), as set in (99) $V(t,x) := U^+(t,x) - \phi(x_1+ct)$, we have

$$\frac{\partial V}{\partial t} - J * V + V + d(\phi + V) - d(\phi) = \int_{\mathbb{R}^n} f_\beta(y)[b(\phi+V) - b(\phi)]dy. \tag{129}$$

Applying Taylor expansion formula and noting (H_3) for $d''(u) \geq 0$ and $b''(u) \leq 0$, we have

$$d(\phi+V) - d(\phi) = d'(\phi)V + d''(\bar\phi_1)V^2 \geq d'(\phi)V, \tag{130}$$

$$b(\phi+V) - b(\phi) = b'(\phi)V + b''(\bar\phi_2)V^2 \leq b'(\phi)V, \tag{131}$$

where $\bar\phi_i$ $(i=1,2)$ are some functions between ϕ and $\phi+V$. Substituting (130) and (131) into (129), and noticing Lemma 5.1, we have

$$\begin{cases} \dfrac{\partial V}{\partial t} - J * V + V + d'(\phi)V \leq \displaystyle\int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_1 - y_1 + c(t-\tau)))V(t-\tau, x-y)dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t>0, x \in \mathbb{R}^n \\ V|_{x_1 \leq x_* - ct} \leq C_2(1+t)^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde\mu t}, \qquad \text{for } t > 0, (x_2, \cdots, x_n) \in \mathbb{R}^{n-1} \\ V|_{t=s} = V_0(s,x), \qquad\qquad\qquad\qquad \text{for } s \in [-\tau, 0], x \in \mathbb{R}^n \end{cases} \tag{132}$$

for some positive constant C_2.

Let

$$\tilde V(t) = C_3(1+\tau+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \tag{133}$$

for $C_3 \geq C_2 \geq \max_{(s,x)\in[-\tau,0]\times\mathbb{R}^n} |V_0(s,x)|$. As in (27), for given $0<\varepsilon_0<1$, we can select a sufficiently large number x_* such that, for $\xi_1 \geq x_* \gg 1$,

$$d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \geq \varepsilon_0[d'(u_+) - b'(u_+)] > 0. \tag{134}$$

Thus, we have

$$\begin{aligned}
&\frac{\partial \tilde V}{\partial t} - J * \tilde V + \tilde V + d'(\phi)\tilde V - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))\tilde V(t-\tau)dy \\
&= -\frac{n}{\alpha} C_3(1+t+\tau)^{-\frac{n}{\alpha}-1} e^{-\mu_\tau t} - \mu_\tau C_3 (1+t+\tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \\
&\quad + C_3(1+t+\tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} d'(\phi(\xi_1)) \\
&\quad - C_3(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau (t-\tau)} \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \\
&= C_3(1+t+\tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \Big\{ \Big[d'(\phi(\xi_1)) \\
&\quad - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \Big] - \mu_\tau - \frac{n}{\alpha}(1+t+\tau)^{-1} \\
&\quad - \left(e^{\mu_\tau \tau} \left(\frac{1+t}{1+t+\tau} \right)^{-\frac{n}{\alpha}} - 1 \right) \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \Big\} \\
&\geq C_3(1+t+\tau)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} \Big\{ \varepsilon_0[d'(u_+) - b'(u_+)] - \mu_\tau - \frac{n}{\alpha}(1+t+\tau)^{-1} \\
&\quad - \left(e^{\mu_\tau \tau} \left(\frac{1+t}{1+t+\tau} \right)^{-\frac{n}{\alpha}} - 1 \right) \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))dy \Big\} \\
&\geq 0
\end{aligned} \tag{135}$$

by selecting a sufficiently small number

$$0 < \mu_\tau < d'(u_+) - b'(u_+) \quad \text{for } c > c_*, \tag{136}$$

$$\mu_\tau = 0 \quad \text{for } c = c_*, \tag{137}$$

and taking $t \geq l_0\tau$ for a sufficiently large integer $l_0 \gg 1$. Hence, we proved that

$$\begin{cases} \dfrac{\partial \tilde{V}}{\partial t} - J * \tilde{V} + \tilde{V} + d'(\phi)\tilde{V} \geq \displaystyle\int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau))\tilde{V}(t-\tau)dy, \\ \qquad\qquad\qquad\qquad \text{for } t > l_0\tau, \xi \in [x_*, +\infty) \times \mathbb{R}^{n-1} \\ \tilde{V}|_{\xi_1 = x_*} = C_3(1+\tau+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} > C_2(1+t)^{-\frac{n}{\alpha}} e^{-\varepsilon_1 \tilde{\mu} t}, \\ \qquad\qquad\qquad\qquad \text{for } t > 0, (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1} \\ \tilde{V}(t) = C_3(1+\tau+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t} > V_0(t,\xi), \qquad \text{for } t \in [-\tau, l_0\tau], \xi \in \mathbb{R}^n. \end{cases} \tag{138}$$

Denote $\Omega := \{(x,t) | x_1 \geq x_* - ct,\ t \geq l_0\tau,\ (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}\}$. Noticing the construction of (132) and (138), then similar to the proof of Proposition 4 , we know that

$$\tilde{V}(t) - V(t,x) \geq 0, \text{ for } (x,t) \in \mathbb{R}^n \times [-\tau, \infty) \setminus \Omega. \tag{139}$$

Thus the proof is complete. □

For $\tau = 0$, it is easy to prove the corresponding results as follows.

Lemma 5.3. *For $\tau = 0$, it holds that*

$$\|V(t)\|_{L^\infty([x_* - ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \text{ for } c > c_*, \tag{140}$$

$$\|V(t)\|_{L^\infty([x_* - ct, \infty) \times \mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}}, \qquad \text{for } c = c_*, \tag{141}$$

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1\tilde{\mu}\}$ for $c > c_*$.

Combing Lemma 5.1-Lemma 5.3, we obtain the decay rates for $V(t,x)$ in $L^\infty(\mathbb{R}^n)$.

Lemma 5.4. *It holds that:*

- *When $c > c_*$, then*

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \tag{142}$$

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \tag{143}$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

- *When $c = c_*$,*

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\alpha}}, \quad \text{for all } \tau \geq 0. \tag{144}$$

Since $V(t,x) = U^+(t,x) - \phi(x_1 + ct)$, Lemma 5.4 give directly the following convergence for the solution in the cases with time-delay.

Lemma 5.5. *It holds that:*

- *When $c > c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |U^+(t,x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } \tau > 0, \tag{145}$$

$$\sup_{x \in \mathbb{R}^n} |U^+(t,x) - \phi(x_1 + ct)| \leq C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \tag{146}$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

- *When $c = c_*$, then*

$$\sup_{x\in\mathbb{R}^n} |U^+(t,x) - \phi(x_1 + c_*t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad \textit{for all } \tau \ge 0. \tag{147}$$

Step 2. The convergence of $U^-(t,x)$ to $\phi(x_1+ct)$
For the traveling wave $\phi(x_1+ct)$ with $c \ge c_*$, let

$$V(t,x) = \phi(x_1+ct) - U^-(t,x), \quad V_0(s,x) = \phi(x_1+cs) - U_0^-(s,x). \tag{148}$$

As in Step 1, we can similarly prove that $U^-(t,x)$ converges to $\phi(x_1+ct)$ as follows.

Lemma 5.6. *It holds that:*

- *When $c > c_*$, then*

$$\sup_{x\in\mathbb{R}^n} |U^-(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \textit{for } \tau > 0, \tag{149}$$

$$\sup_{x\in\mathbb{R}^n} |U^-(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \textit{for } \tau = 0, \tag{150}$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

- *When $c = c_*$, then*

$$\sup_{x\in\mathbb{R}^n} |U^-(t,x) - \phi(x_1 + c_*t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad \textit{for all } \tau \ge 0. \tag{151}$$

Step 3. The convergence of $u(t,x)$ to $\phi(x_1+ct)$
Finally, we prove that $u(t,x)$ converges to $\phi(x_1+ct)$. Since the initial data satisfy $U_0^-(s,x) \le u_0(s,x) \le U_0^+(s,x)$ for $(s,x) \in [-\tau,0]\times\mathbb{R}^n$, then the comparison principle implies that

$$U^-(t,x) \le u(t,x) \le U^+(t,x), \quad (t,x) \in R_+ \times \mathbb{R}^n.$$

Thanks to Lemmas 5.5 and 5.6, by the squeeze argument, we have the following convergence results.

Lemma 5.7. *It holds that:*

- *When $c > c_*$, then*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \textit{for } \tau > 0, \tag{152}$$

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_0 t}, \quad \textit{for } \tau = 0, \tag{153}$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

- *When $c = c_*$, then*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1 + c_*t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad \textit{for all } \tau \ge 0. \tag{154}$$

6. **Applications and concluding remark.** In this section, we first give the direct applications of Theorem 2.1-2.2 to the Nicholson's blowflies type equation with nonlocal dispersion, and the classical Fisher-KPP equation with nonlocal dispersion. Then we point out that, the developed stability theory above can be also applied to the more general case.

6.1. **Nicholson's blowflies equation with nonlocal dispersion.** For the equation (1), by taking $d(u) = \delta u$ and $b(u) = pue^{-au}$ with $\delta > 0$, $p > 0$ and $a > 0$, we get the so-called Nicholson's blowflies equation with nonlocal dispersion

$$\begin{cases} \dfrac{\partial u}{\partial t} - J * u + u + \delta u(t,x)) = p \displaystyle\int_{\mathbb{R}^n} f_\beta(y) u(t-\tau, x-y) e^{-au(t-\tau,x-y)} dy, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \ x \in \mathbb{R}^n. \end{cases} \tag{155}$$

Clearly, there exist two constant equilibria $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta}$, and the selected $d(u)$ and $b(u)$ satisfy the hypothesis (H_1)-(H_3) automatically under the consideration of $1 < \frac{p}{\delta} \le e$. Let $J(x)$ satisfy the hypothesis (J_1) and (J_2), from Theorem 2.1 and Theorem 2.2, we have the following existence of monostable traveling waves and their stabilities.

Theorem 6.1 (Traveling waves)**.** *Let $J(x)$ satisfy (J_1) and (J_2). For* (155)*, there exists the minimal speed $c_* > 0$, such that when $c \ge c_*$, the planar traveling waves $\phi(x \cdot \mathbf{e}_1 + ct)$ exist uniquely (up to a shift). Here $c_* > 0$ and $\lambda_* > 0$ are determined by*

$$H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \textit{and} \quad H'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),$$

where

$$H_c(\lambda) = pe^{\beta\lambda^2 - \lambda c\tau} \quad \textit{and} \quad G_c(\lambda) = c\lambda - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 + 1 + \delta.$$

Particularly, when $c > c_*$, then $H_c(\lambda_*) < G_c(\lambda_*)$.

Theorem 6.2 (Stability of traveling waves)**.** *Let $J(x)$ satisfy (J_1) and (J_2), and the initial data be $u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H^{m+1}_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \le u_0 \le u_+$ for $(s,x) \in [-\tau, 0] \times \mathbb{R}^n$. Then the solution of* (155) *uniquely exists and satisfies:*

- *When $c > c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |u(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad t > 0, \tag{156}$$

for $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \ \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$, and $\varepsilon_1 = \varepsilon_1(\tau)$ such that $0 < \varepsilon_1 < 1$ for $\tau > 0$ and $\varepsilon_1 = 1$ for $\tau = 0$

- *When $c = c_*$, then*

$$\sup_{x \in \mathbb{R}^n} |u(t,x) - \phi(x_1 + c_* t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \tag{157}$$

6.2. **Fisher-KPP equation with nonlocal dispersion.** For the equation (1), let $d(u) = u^2$, $b(u) = u$ and the delay $\tau = 0$, and take the limit of (1) as $\beta \to 0^+$, we get the classical Fisher-KPP equation with nonlocal dispersion without time-delay

$$\begin{cases} \dfrac{\partial u}{\partial t} - J * u + u = u(1-u) \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{158}$$

Then we have the existence of the monostable traveling waves and their stabilities from Theorem 2.1 and Theorem 2.2.

Theorem 6.3 (Traveling waves)**.** *Let $J(x)$ satisfy (J_1) and (J_2). For* (158)*, there exists the minimal speed $c_* > 0$, such that when $c \ge c_*$, the planar traveling waves*

$\phi(x\cdot\mathbf{e}_1+ct)$ exist uniquely (up to a shift) connecting with $\phi(-\infty)=0$ and $\phi(+\infty)=1$. Here

$$c_* := \min_{\lambda>0}\frac{1}{\lambda}\int_{\mathbb{R}} J_1(y_1)e^{-\lambda y_1}dy_1 = \frac{1}{\lambda_*}\int_{\mathbb{R}} J_1(y_1)e^{-\lambda_* y_1}dy_1,$$

and $\lambda_*>0$ is determined by

$$\int_{\mathbb{R}}(1+\lambda_* y_1)J_1(y_1)e^{-\lambda_* y_1}dy_1 = 0.$$

Theorem 6.4 (Stability of traveling waves)**.** *Let $J(x)$ satisfy (J_1) and (J_2), and the initial data be $u_0-\phi\in H_w^m(\mathbb{R}^n)\cap L_w^1(\mathbb{R}^n)$ with $m>\frac{n}{2}$, and $u_-\le u_0\le u_+$ for $x\in\mathbb{R}^n$. Then the solution of* (158) *uniquely exists and satisfies:*

- *When $c>c_*$, then*

$$\sup_{x\in\mathbb{R}^n}|u(t,x)-\phi(x_1+ct)| \le C(1+t)^{-\frac{n}{\alpha}}e^{-\mu_0 t}, \quad t>0, \tag{159}$$

for $0<\mu_0<\min\{d'(u_+)-b'(u_+),\ G_c(\lambda_*)-H_c(\lambda_*)\}=\min\{1,\ (c-c_*)\lambda_*\}$;

- *When $c=c_*$, then*

$$\sup_{x\in\mathbb{R}^n}|u(t,x)-\phi(x_1+c_*t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad t>0. \tag{160}$$

6.3. **Concluding remark.** Here we give a remark on the wave stability to the generalized equations with nonlocal dispersion. Let us consider a more general monostable equation with nonlocal dispersion

$$\begin{cases} \dfrac{\partial u}{\partial t} - J*u+u+d(u(t,x)) = F\Big(\displaystyle\int_{\mathbb{R}^n}\kappa(y)b(u(t-\tau,x-y))dy\Big), \\ u(s,x)=u_0(s,x), \quad s\in[-\tau,0],\ x\in\mathbb{R}^n, \end{cases} \tag{161}$$

where $J(x)$ satisfies (J_1) and (J_2) as mentioned before, and $F(\cdot)$, $d(u)$, $b(u)$ and $\kappa(x)$ satisfy

($\mathcal{H}_1$) There exist $u_-=0$ and $u_+>0$ such that $d(0)=b(0)=F(0)=0$, $d(u_+)=F(b(u_+))$, $d\in C^2[0,u_+]$, $b\in C^2[0,u_+]$ and $F\in C^2[0,b(u_+)]$;
($\mathcal{H}_2$) $F'(0)b'(0)>d'(0)\ge 0$ and $0<F'(b(u_+))b'(u_+)<d'(u_+)$;
($\mathcal{H}_3$) $d'(u)\ge 0$, $b'(u)\ge 0$, $d''(u)\ge 0$ and $b''(u)\le 0$ for $u\in[0,u_+]$;
($\mathcal{H}_4$) $F'(u)\ge 0$ and $F''(u)\le 0$ for $u\in[0,b(u_+)]$;
($\mathcal{H}_5$) $\kappa(x)$ is a smooth, positive and radial kernel with $\int_{\mathbb{R}^n}\kappa(x)dx=1$ and $\int_{\mathbb{R}^n}\kappa(x)\cdot e^{-\lambda x_1}dx<+\infty$ for all $\lambda>0$.

Then, by a similar calculation, we can prove the existence of the traveling waves $\phi(x_1+ct)$ for $c\ge c_*$, where $c_*>0$ is a specified minimal wave speed, and that the noncritical traveling waves with $c>c_*$ are exponentially stable and the critical waves with $c=c_*$ are algebraically stable.

Theorem 6.5 (Traveling waves)**.** *Assume that (J_1)-(J_2) and ($\mathcal{H}_1$)-($\mathcal{H}_5$) hold. For* (161)*, there exist a pair of numbers $c_*>0$ and $\lambda_*>0$, such that when $c\ge c_*$, the planar traveling waves $\phi(x\cdot\mathbf{e}_1+ct)$ exist uniquely (up to a shift). Here $c_*>0$ and $\lambda_*=\lambda_*(c_*)>0$ are determined by*

$$\mathcal{H}_{c_*}(\lambda_*)=\mathcal{G}_{c_*}(\lambda_*) \quad \text{and} \quad \mathcal{H}'_{c_*}(\lambda_*)=\mathcal{G}'_{c_*}(\lambda_*),$$

where

$$\mathcal{H}_c(\lambda)=F'(0)b'(0)\int_{\mathbb{R}^n}e^{-\lambda y_1}\kappa(y)dy, \quad \mathcal{G}_c(\lambda)=c\lambda-\int_{\mathbb{R}}J_1(y_1)e^{-\lambda y_1}dy_1+1+d'(0).$$

When $c>c_*$, then

$$\mathcal{H}_c(\lambda_*)<\mathcal{G}_c(\lambda_*).$$

Theorem 6.6 (Stability of traveling waves). *Assume that* (J_1)-(J_2) *and* $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ *hold. Let the initial data be* $u_0 - \phi \in C([-\tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ *and* $\partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H_w^{m+1}(\mathbb{R}^n) \cap L_w^1(\mathbb{R}^n))$ *with* $m > \frac{n}{2}$, *and* $u_- \le u_0 \le u_+$ *for* $x \in \mathbb{R}^n$. *Then the solution of* (161) *uniquely exists and satisfies:*

- *When* $c > c_*$, *then*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1 + ct)| \le C(1+t)^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad t > 0, \tag{162}$$

for $0 < \mu_\tau < \min\{d'(u_+) - F'(b(u_+))b'(u_+),\ \varepsilon_1[\mathcal{G}_c(\lambda_*) - \mathcal{H}_c(\lambda_*)]\}$, *and* $0 < \varepsilon_1 < 1$ *for* $\tau > 0$ *and* $\varepsilon_1 = 1$ *for* $\tau = 0$;

- *When* $c = c_*$, *then*

$$\sup_{x\in\mathbb{R}^n} |u(t,x) - \phi(x_1 + c_* t)| \le C(1+t)^{-\frac{n}{\alpha}}, \quad t > 0. \tag{163}$$

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