



Radial solutions of hydrodynamic model of semiconductors with sonic boundary



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ABSTRACT

The purpose of this paper is to study the multi-dimensional steady hydrodynamic model of semiconductors represented by Euler-Poisson equations with sonic boundary. We prove that, the steady Euler-Poisson equations with sonic boundary possess a unique subsonic solution and at least one supersonic solution in the radial form. The adopted approach for proof is the energy method combining the compactness analysis. For the n -D radial supersonic solutions, since it is more challenging to get the crucial energy estimates due to the effect by the multiple dimensions and the restriction by the sonic boundary, we propose a brand new two-steps iteration scheme to build up the key energy estimates. This is the first attempt to study the n -D steady-states with the sonic boundary, and the results obtained essentially improve and develop the previous studies in the one-dimensional case.

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1. Introduction

The hydrodynamic model of semiconductors, first introduced by Bløtekjær [7], usually characterizes the motion of the charged fluid particles such as electrons and holes in semiconductor devices [24]. This paper is a continuity of our series of study [8,19,20] on the Euler-Poisson equations of semiconductor models subjected to sonic boundary. Different from [8,19,20] on 1-D equations with sonic boundary, here we are mainly interested in the multiple-dimensional Euler-Poisson system [4,10] as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P I_n) = \rho \nabla \Phi - \frac{\rho \mathbf{u}}{\tau}, & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+, n = 2, 3, \\ \Delta \Phi = \rho - b(x). \end{cases} \quad (1.1)$$

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Here ρ , \mathbf{u} and Φ denote the electron density, the velocity and the electrostatic potential, respectively. I_n is the $n \times n$ identity matrix, the constant $\tau > 0$ is the momentum relaxation time and the function $b(x) > 0$ is the doping profile standing for the density of positively charged background ions. $P(\rho)$ is known as the pressure-density relation. As usual, for isentropic flows, $P(\rho) = \kappa\rho^\gamma$, $\kappa > 0$ with the adiabatic exponent $\gamma > 1$; for isothermal flows, $P = T\rho$ with the constant temperature $T > 0$. In present paper, we consider the isothermal case, and set $T = 1$ without loss of generality, i.e.

$$P(\rho) = \rho.$$

Throughout this paper, we consider the steady-state solutions of (1.1) in an annulus domain

$$\mathcal{A} := \{x \in \mathbb{R}^n | r_0 < |x| < r_1\}, \quad 0 < r_0 < r_1,$$

with the inner boundary

$$\Gamma_0 := \{x \in \mathbb{R}^n : |x| = r_0\},$$

and the outer boundary

$$\Gamma_1 := \{x \in \mathbb{R}^n : |x| = r_1\}.$$

Its closure is denoted by

$$\bar{\mathcal{A}} := \Gamma_0 \cup \mathcal{A} \cup \Gamma_1.$$

Note that

$$\operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) = \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \operatorname{div}(\rho\mathbf{u}) \cdot \mathbf{u},$$

and set $E := \nabla\Phi$ (the electric field), then the corresponding stationary equations of (1.1) can be written as

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\nabla\rho}{\rho} = E - \frac{\mathbf{u}}{\tau}, \quad x \in \mathcal{A}, \\ \operatorname{div} E = \rho - b(x). \end{cases} \tag{1.2}$$

The aim of our work is to investigate the structure of the steady-state solutions to (1.2), particularly, the radial subsonic/supersonic solutions of (1.2) in two and three dimensional annulus domains with sonic boundary, and to study various analytical features including the requirement of the doping profile and the adopted methods in the proofs by comparing with the one-dimensional case [19].

Additionally, we call $M := \frac{|\mathbf{u}|}{c(\rho)}$ the *Mach number* for $c(\rho) := \sqrt{P'(\rho)} = 1$. Here, $c(\rho)$ is called the *local sound speed*. Depending on the size of M , the analytic features of (1.2) vary: if $M > 1$, the stationary flow is called *supersonic*; if $M < 1$, the corresponding flow is called *subsonic*; otherwise, $M = 1$ is the sonic state.

In what follows, we assume that \tilde{b} is in $L^\infty(r_0, r_1)$ such that $b(x) := \tilde{b}(r)$ in $\bar{\mathcal{A}}$, and we denote

$$(\rho, \mathbf{u}, E)(x) := (\tilde{\rho}(r), \tilde{u}(r)\vec{\mathbf{e}}, \tilde{E}(r)\vec{\mathbf{e}}), \tag{1.3}$$

where $r = |x|$, and $\vec{\mathbf{e}} := \frac{x}{r}$ is a unit vector, and we prescribe the boundary conditions as follows:

$$(\rho|_{\Gamma_0}, \rho|_{\Gamma_1}, \rho\mathbf{u}|_{\Gamma_0}) = (\tilde{\rho}(r_0), \tilde{\rho}(r_1), \tilde{\rho}(r_0)\tilde{u}(r_0)\vec{\mathbf{e}}) = (\rho_0, \rho_1, j_0\vec{\mathbf{e}}) \tag{1.4}$$

for positive constants (ρ_0, ρ_1, j_0) . Therefore, (1.2) and (1.4) are reduced to

$$\begin{cases} (r^{n-1}\tilde{\rho}\tilde{u})_r = 0, \\ (r^{n-1}\tilde{\rho}\tilde{u}^2)_r + r^{n-1}\tilde{\rho}_r = r^{n-1}\tilde{\rho}(\tilde{E} - \frac{\tilde{u}}{\tau}), \\ (r^{n-1}\tilde{E})_r = r^{n-1}(\tilde{\rho} - \tilde{b}(r)), \\ (\tilde{\rho}(r_0), \tilde{\rho}(r_1), \tilde{u}(r_0)) = (\rho_0, \rho_1, j_0/\rho_0), \end{cases} \quad \text{for } r_0 < r < r_1, \tag{1.5}$$

so that the sonic state is redefined by $|\tilde{u}| = M = 1$. Clearly, each pair of the solution $(\tilde{\rho}, \tilde{u}, \tilde{E})$ to system (1.5) always corresponds to a solution (ρ, \mathbf{u}, E) to (1.2) and (1.4).

Definition 1.1 (*Radial subsonic/supersonic solution*). We call (ρ, \mathbf{u}, E) with $M < 1$ ($M > 1$) in \mathcal{A} radial subsonic (correspondingly, supersonic) to system (1.2) and (1.4) if the corresponding solution $(\tilde{\rho}, \tilde{u}, \tilde{E})$ of (1.5) satisfies $|\tilde{u}| < 1$ ($|\tilde{u}| > 1$) over (r_0, r_1) .

We now focus on (1.5). Let $\tilde{J} := \tilde{\rho}\tilde{u}$. Without loss of generality, let us also take $\tilde{J} > 0$. From the first equation of (1.5) we have

$$\tilde{J}(r) = j_0 \cdot \frac{r_0^{n-1}}{r^{n-1}}, \quad r \in [r_0, r_1]. \tag{1.6}$$

By (1.5) and (1.6), we can impose the sonic boundary conditions to (1.5) by

$$\rho_0 = j_0 \quad \text{and} \quad \rho_1 = j_0 \cdot \frac{r_0^{n-1}}{r_1^{n-1}}. \tag{1.7}$$

By dividing the second equation of (1.5) by $\tilde{\rho}$ and differentiating the resulting equation with respect to r , and using the third equation of (1.5), we obtain

$$\begin{cases} \left[r^{n-1} \left(\left(\frac{1}{\tilde{\rho}} - \frac{\tilde{J}^2}{\tilde{\rho}^3} \right) \tilde{\rho}_r + \frac{n-1}{r} \frac{\tilde{J}^2}{\tilde{\rho}^2} + \frac{\tilde{J}}{\tau\tilde{\rho}} \right) \right]_r = r^{n-1}(\tilde{\rho} - \tilde{b}), & r \in (r_0, r_1), \\ \rho_0 = j_0, \quad \rho_1 = j_0 \cdot \frac{r_0^{n-1}}{r_1^{n-1}}. \end{cases} \tag{1.8}$$

In order to classify the radial solutions, it is convenient to introduce a new variable

$$m(r) := r^{n-1}\tilde{\rho}(r), \quad r \in [r_0, r_1],$$

with a parameter $\mathcal{J} := j_0 r_0^{n-1} > 0$. Thus, by (1.6), it implies that

$$\tilde{J} = \frac{\mathcal{J}}{r^{n-1}} \quad \text{and} \quad \tilde{\rho} = \frac{m}{r^{n-1}}, \tag{1.9}$$

then (1.8) is reduced to

$$\begin{cases} \left[r^{n-1} \left(\frac{1}{m} - \frac{\mathcal{J}^2}{m^3} \right) m_r + \frac{r^{n-1}\mathcal{J}}{\tau m} \right]_r = m - B(r) + r^{n-3}(n-1)(n-2), & r \in (r_0, r_1), \\ m(r_0) = m(r_1) = \mathcal{J}, \end{cases} \tag{1.10}$$

where the function B is defined by

$$B(r) := r^{n-1}\tilde{b}(r)$$

on $[r_0, r_1]$. Obviously, $m > \mathcal{J}$ means that the flow is subsonic; correspondingly, $0 < m < \mathcal{J}$ stands for the supersonic flow. Moreover, equation (1.10) is elliptic but degenerate at the boundary, that causes us essential difficulties.

Now we define an interior subsonic/supersonic solution of (1.10) in the weak sense, which is first introduced by [19].

Definition 1.2. $m(r)$ is called an interior subsonic (correspondingly, interior supersonic) solution of system (1.10) if $m(r_0) = m(r_1) = \mathcal{J}$ and $m(r) > \mathcal{J}$ (correspondingly, $0 < m(r) < \mathcal{J}$) for $r \in (r_0, r_1)$, and $(m - \mathcal{J})^2 \in H_0^1(r_0, r_1)$, and it holds that

$$\int_{r_0}^{r_1} \left[r^{n-1} \left(\frac{1}{m} - \frac{\mathcal{J}^2}{m^3} \right) m_r + \frac{r^{n-1}\mathcal{J}}{\tau m} \right] \varphi_r dr + \int_{r_0}^{r_1} (m - B(r) + r^{n-3}(n-1)(n-2))\varphi dr = 0,$$

for any $\varphi \in H_0^1(r_0, r_1)$, which is equivalent to

$$\begin{aligned} & \int_{r_0}^{r_1} \left[r^{n-1} \frac{m + \mathcal{J}}{2m^3} ((m - \mathcal{J})^2)_r + \frac{r^{n-1}\mathcal{J}}{\tau m} \right] \varphi_r dr \\ & + \int_{r_0}^{r_1} (m - B(r) + r^{n-3}(n-1)(n-2))\varphi dr = 0. \end{aligned} \quad (1.11)$$

Once m is known from (1.10), in view of (1.3) and (1.9), ρ and \mathbf{u} can be determined. Then, by the second equation of (1.2), $\tilde{u} = \frac{\mathcal{J}}{m}$ and (1.9), $E(x)$ is computed by

$$E(x) = \tilde{E}(r)\tilde{\mathbf{e}} = \left(\tilde{u}\tilde{u}_r + \frac{\tilde{\rho}_r}{\tilde{\rho}} + \frac{\tilde{u}}{\tau} \right) \tilde{\mathbf{e}} = \left(\frac{(m + \mathcal{J})[(m - \mathcal{J})^2]_r}{2m^3} + \frac{\mathcal{J}}{\tau m} - \frac{n-1}{r} \right) \tilde{\mathbf{e}}.$$

Thus, finding the solution of (1.2), (1.4) and (1.7) amounts to solving (1.10).

Definition 1.3. (ρ, \mathbf{u}, E) is called a radial subsonic (correspondingly, supersonic) solution in \mathcal{A} to (1.2) and (1.4) with the sonic boundary conditions (1.7) if the corresponding solution of (1.10) is an interior subsonic (supersonic) solution.

The study on the hydrodynamic system of semiconductors has been one of hot research spots [17,23,25,30]. For the subsonic flows, Degong and Markowich [9,10] first proved the existence and uniqueness of smooth solutions with a fully subsonic background in one dimension, and for potential flow in three dimensions, respectively; see [2] for a non-isentropic case, and also [3,4,15,16,18,26,27] for more general subsonic case. For the supersonic flows, the existence and uniqueness of supersonic solutions were studied by Peng and Violet [28] when the flow is strongly supersonic in the one-dimensional case. The work was extended to the two-dimensional case by Bae [5]. Then, the transonic solutions have been a focus in the study of the stationary flows because of the forming of shock waves, we refer to [1,6,11–13,21,22,29].

For the one-dimensional case, if system (1.2) is with sonic boundary, the structures of all types of solutions for (1.2) have been intensively studied when the doping profile is subsonic [19], supersonic [20] or transonic [8]. In the case of the subsonic [19] and subsonic-dominated [8] doping profile, there exist a unique interior subsonic, at least one interior supersonic solution, infinitely many transonic shock solutions (the sufficiently large relaxation time, i.e. $\tau \gg 1$), and infinitely many C^1 -smooth transonic solutions (the sufficiently small

relaxation time, i.e. $\tau \ll 1$). The approach adopted consists of the technical compactness analysis, phase plane analysis and the energy method. Of course, interior subsonic/supersonic solutions may not exist with the subsonic-dominated [8] doping profile if the relaxation time is small enough. On the other hand, under the supersonic [20] and supersonic-dominated [8] doping profile, the non-existence of all types of the solutions can be obtained. However, the existence of supersonic and transonic shock solutions can be proved in an extreme case, where the doping profile is close to the sonic line and the semiconductor effect is small ($\tau \gg 1$).

Inspired by our previous studies mentioned above, we expect to establish the well-posedness of the solutions for high dimensional system with sonic boundary. Physically speaking, it is hard to put forward an acceptable critical boundary in a general domain, such as a flat nozzle. Therefore, we first pay attention to radial solutions of (1.2) in an annulus domain. In this paper, we show that there exist a unique radial subsonic solution and at least one radial supersonic solution to (1.2) and (1.4) with the sonic boundary conditions (1.7). Different from the one-dimensional case, the proof is more challenging. The first technical point is that we have to treat the case of the doping profile with the lower bound smaller than the sonic curve; the second technical point is that, in order to prove the existence of interior supersonic solutions for (1.10), the so-called the one-step iteration for the one-dimensional case is failed to the multi-dimensional case, and we propose a new two-steps iteration scheme for establishing the key energy estimates.

Throughout this paper we denote

$$\underline{B} = \operatorname{ess\,inf}_{r \in [r_0, r_1]} B(r) \quad \text{and} \quad \overline{B} := \operatorname{ess\,sup}_{r \in [r_0, r_1]} B(r),$$

and also define

$$\underline{\mathcal{B}} := \inf_{r \in [r_0, r_1]} \left\{ B(r) + \frac{2r}{\tau} - 2 \right\} \quad \text{and} \quad \overline{\mathcal{B}} := \sup_{r \in [r_0, r_1]} \left\{ B(r) + \frac{2r}{\tau} - 2 \right\},$$

which is necessary to prove the existence of the solutions in the three-dimensional case.

Now we state our main results about interior subsonic/supersonic solutions to (1.10) as follows.

Theorem 1.4 (*Interior subsonic solutions*).

1. *The case of $n = 2$: Let $B(r) \in L^\infty(r_0, r_1)$ and $\underline{B} \leq B(r) \leq \overline{B}$ satisfying $\overline{B} + \frac{1}{\tau} > \mathcal{J}$ and $\underline{B} + \frac{\mathcal{J}}{\tau(\overline{B} + 1/\tau)} > \mathcal{J}$, then system (1.10) admits a unique interior subsonic solution $m(r)$ over $[r_0, r_1]$. Further, $m \in C^{\frac{1}{2}}[r_0, r_1]$ satisfies a lower bound estimate*

$$m(r) \geq \mathcal{J} + \lambda \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where λ is a small and positive constant.

2. *The case of $n = 3$: Let $\overline{\mathcal{B}} > \mathcal{J}$ and $\min_{r \in [r_0, r_1]} \left(B(r) + \frac{2r\mathcal{J}}{\tau\overline{\mathcal{B}}} - 2 \right) > \mathcal{J}$, then equation (1.10) has a unique interior subsonic solution m satisfying $m \in C^{\frac{1}{2}}[r_0, r_1]$ and*

$$m(r) \geq \mathcal{J} + \bar{\lambda} \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where the constant $\bar{\lambda} > 0$ is also small.

Theorem 1.5 (*Interior supersonic solutions*).

1. The case of $n = 2$: Assume that $\underline{B} + \frac{1}{\tau} > \mathcal{J}$, then system (1.10) has at least one interior supersonic solution $m \in C^{1/2}[r_0, r_1]$ satisfying $\ell \leq m(r) < \mathcal{J}$ over (r_0, r_1) for a positive constant ℓ .
2. The case of $n = 3$: Suppose that $\underline{B} > \mathcal{J}$, then there exists an interior supersonic solution $m \in C^{1/2}[r_0, r_1]$ to system (1.10) satisfying $\bar{\ell} \leq m(r) < \mathcal{J}$ over (r_0, r_1) for a positive constant $\bar{\ell}$.

Remark 1.6.

1. If the hypotheses of Theorem 1.4 hold, we notice that subsonic solutions and supersonic solutions of (1.10) both exist. In addition, more restrictions on the doping profile are needed in 3-D case.
2. For any fixed $r_0 > 0$, there exist always an interior subsonic solution and an interior supersonic solution to (1.10) when the hypotheses of Theorem 1.4 and 1.5 are satisfied.
3. Affected by the multiple dimensions, (1.2) will be recast as a nonlinear non-autonomous ODE system, which is more complex than autonomous system in one dimensional case. Thus, the transonic solutions of (1.10) are not discussed in this paper. We will leave it in future.

Next the rest of this paper is organized as follows. The second section focuses on interior subsonic solutions of system (1.10). For clarity, we discuss this issue in the two-dimensional and three-dimensional cases, respectively. Under both two cases, there exists a unique interior subsonic solution to (1.10). In addition, the third section is devoted to interior supersonic solutions of (1.10) in two and three dimensions cases. The existence of interior supersonic solutions is proved by a two-steps iteration and the Schauder fixed point theorem.

2. Existence and uniqueness of interior subsonic solutions

In this section, we're going to prove that there exists a unique interior subsonic solution to (1.10) for both two-dimensional and three-dimensional cases. Here the main approach is the technical compactness method [19], which is inspired by the vanishing viscosity method.

2.1. The case of $n=2$

First we will prove the well-posedness of system (1.10) in the two-dimensional case. Actually, we consider the following equation,

$$\begin{cases} \left[r \left(\frac{1}{m} - \frac{\mathcal{J}^2}{m^3} \right) m_r + \frac{r\mathcal{J}}{\tau m} \right]_r = m - B(r), & r \in (r_0, r_1), \\ m(r_0) = m(r_1) = \mathcal{J}. \end{cases} \quad (2.1)$$

Our main theorem in this subsection is stated below.

Theorem 2.1. Assume that $B(r) \in L^\infty(r_0, r_1)$ and $\underline{B} \leq B(r) \leq \bar{B}$ satisfying $\bar{B} + \frac{1}{\tau} > \mathcal{J}$ and $\underline{B} + \frac{\mathcal{J}}{\tau(\bar{B} + 1/\tau)} > \mathcal{J}$, then we have a unique weak solution m to (2.1) satisfying $m \in C^{\frac{1}{2}}[r_0, r_1]$ and

$$m(r) \geq \mathcal{J} + \lambda \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where λ is a small and positive constant.

Since (2.1) is elliptic in (r_0, r_1) but degenerates at the boundary, we can't directly work on it. Therefore, we first consider the approximate equation of (2.1) as follows:

$$\begin{cases} \left[r \left(\frac{1}{m_j} - \frac{j^2}{(m_j)^3} \right) (m_j)_r + \frac{r\mathcal{J}}{\tau m_j} \right]_r = m_j - B(r), & r \in (r_0, r_1), \\ m_j(r_0) = m_j(r_1) = \mathcal{J}, \end{cases} \tag{2.2}$$

where the parameter j is a constant such that $0 < j < \mathcal{J}$. Obviously, one finds that (2.2) is uniformly elliptic in $[r_0, r_1]$ for the expected solution $m_j > \mathcal{J}$. The following comparison principle is the key ingredient to prove the uniqueness of interior subsonic solution to (1.10).

Lemma 2.2. *Let $p \in C^1[r_0, r_1]$ be a weak solution of (2.2) satisfying $p \geq \mathcal{J}$ on $[r_0, r_1]$, and*

$$\int_{r_0}^{r_1} \left[r \left(\frac{1}{p} - \frac{j^2}{p^3} \right) p_r + \frac{r\mathcal{J}}{\tau p} \right] \varphi_r dr + \int_{r_0}^{r_1} (p - B(r)) \varphi dr = 0$$

for any $\varphi \in H_0^1(r_0, r_1)$ where $0 < j < \mathcal{J}$. Further, let $q \in C^1[0, 1]$ be such that $q(x) > 0$ on $[r_0, r_1]$, $q(r_0) \leq \mathcal{J}$, $q(r_1) \leq \mathcal{J}$, and for any $\varphi \geq 0$, $\varphi \in H_0^1(r_0, r_1)$,

$$\int_{r_0}^{r_1} \left[r \left(\frac{1}{q} - \frac{j^2}{q^3} \right) q_r + \frac{r\mathcal{J}}{\tau q} \right] \varphi_r dr + \int_{r_0}^{r_1} (q - B(r)) \varphi dr \leq 0.$$

Then $p(r) \geq q(r)$ over $[r_0, r_1]$.

Proof. This proof is same as that of Lemma 2.2 [19] and we omit it here. \square

Now let's prove the well-posedness of (2.2) first.

Lemma 2.3. *Assume that $B(r) \in L^\infty(r_0, r_1)$ and $\underline{B} \leq B(r) \leq \overline{B}$ satisfying $\overline{B} + \frac{1}{\tau} > \mathcal{J}$ and $\underline{B} + \frac{\mathcal{J}}{\tau(\overline{B} + 1/\tau)} > \mathcal{J}$, then there exists a unique weak solution m_j to (2.2) satisfying $m_j - \mathcal{J} \in H_0^1(r_0, r_1)$ and*

$$m_j(r) \geq \mathcal{J} + \lambda \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where λ is a small and positive constant, independent of j .

Proof. First denote a closed subset of $C^0[r_0, r_1]$ by

$$\mathcal{C} := \{ \omega \in C^0[r_0, r_1] \mid \mathcal{J} \leq \omega(r) \leq N, \omega(r_0) = \omega(r_1) = \mathcal{J} \}$$

for an undetermined constant $N > \mathcal{J}$. Then we define an operator $\mathfrak{P} : \mathcal{C} \rightarrow C^0[r_0, r_1]$, $\mathfrak{P}(\bar{m}) = m_j$, by solving the linearized system of (2.2),

$$\begin{cases} \left[r \left(\frac{1}{\bar{m}} - \frac{j^2}{\bar{m}^3} \right) (m_j)_r \right]_r - \frac{r\mathcal{J}}{\tau \bar{m}^2} (m_j)_r = m_j - B(r) - \frac{\mathcal{J}}{\tau \bar{m}}, & r \in (r_0, r_1), \\ m_j(r_0) = m_j(r_1) = \mathcal{J}, \end{cases} \tag{2.3}$$

with $\bar{m} \in \mathcal{C}$. Due to the L^2 theory of elliptic equations, we have $m_j \in H^1(r_0, r_1)$ for system (2.3). By the compact imbedding $H^1(r_0, r_1) \hookrightarrow C^0[r_0, r_1]$, one can see that $\mathfrak{P}(\bar{m})$ is precompact. Further, \mathfrak{P} is continuous by a standard continuity argument. In order to use the Schauder fixed point theorem [14], it remains to prove $\mathfrak{P}(\mathcal{C}) \subset \mathcal{C}$.

Here we only need to show $\mathcal{J} \leq m_j(r) \leq N$ over $[r_0, r_1]$ by selecting a suitable N . In fact, if $B(r) + \frac{\mathcal{J}}{\tau\bar{m}} \geq \mathcal{J}$ over $[r_0, r_1]$, we obtain

$$\begin{cases} \left[r \left(\frac{1}{\bar{m}} - \frac{j^2}{\bar{m}^3} \right) (m_j)_r \right]_r - \frac{r\mathcal{J}}{\tau\bar{m}^2} (m_j)_r - (m_j - \mathcal{J}) \leq 0, & r \in (r_0, r_1), \\ m_j(r_0) = m_j(r_1) = \mathcal{J}. \end{cases}$$

Thus, by the weak maximum principle (Theorem 8.1 [14]), it is easy to see that $m_j - \mathcal{J} \geq 0$. Similarly, suppose that $B(r) + \frac{\mathcal{J}}{\tau\bar{m}} \leq N$ over $[r_0, r_1]$, then it follows that

$$\begin{cases} \left[r \left(\frac{1}{\bar{m}} - \frac{j^2}{\bar{m}^3} \right) (m_j)_r \right]_r - \frac{r\mathcal{J}}{\tau\bar{m}^2} (m_j)_r - (m_j - N) \geq 0, & r \in (r_0, r_1), \\ m_j(r_0) = m_j(r_1) = \mathcal{J}, \end{cases}$$

which yields that $m_j - N \leq 0$. In brief, we can derive that $\mathcal{J} \leq m_j(r) \leq N$ over $[r_0, r_1]$ while

$$\mathcal{J} \leq B(r) + \frac{\mathcal{J}}{\tau\bar{m}} \leq N, \quad r \in [r_0, r_1], \quad (2.4)$$

for arbitrary $\mathcal{J} \leq \bar{m} \leq N$. Now we choose $N = \bar{B} + \frac{1}{\tau} > \mathcal{J}$ so that the right-side inequality of (2.4) directly holds. Moreover, a simple computation using the condition

$$\underline{B} + \frac{\mathcal{J}}{\tau(\bar{B} + 1/\tau)} > \mathcal{J}$$

yields that the left-side inequality of (2.4) also holds. Therefore, $\mathfrak{P}(\mathcal{C}) \subset \mathcal{C}$, and one can see that there exists a fixed point m_j of \mathfrak{P} such that $\mathfrak{P}(m_j) = m_j$. Recalled Theorem 1 of [9], (2.2) has a weak solution $m_j \in H^2(r_0, r_1)$. Thanks to the compact imbedding $H^2(r_0, r_1) \hookrightarrow C^1[r_0, r_1]$, we have $m_j \in C^1[r_0, r_1]$.

Then we need to prove the uniqueness of the solution of (2.2) and build a lower bound estimate. Suppose that there exist two solutions m_j^1 and m_j^2 satisfying $m_j^1, m_j^2 \geq \mathcal{J}$ and $m_j^1, m_j^2 \in C^1[r_0, r_1]$. Thus, Lemma 2.2 implies that $m_j^1(r) = m_j^2(r)$ over $[r_0, r_1]$. Furthermore, define

$$\mathfrak{m}(r) := \mathcal{J} + \lambda \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where λ is a positive constant. Note that $\underline{B} + \frac{\mathcal{J}}{\tau(\bar{B} + 1/\tau)} > \mathcal{J}$, then a direct calculation shows that

$$\begin{aligned} - \left[r \left(\frac{1}{\mathfrak{m}} - \frac{j^2}{\mathfrak{m}^3} \right) \mathfrak{m}_r + \frac{r\mathcal{J}}{\tau\mathfrak{m}} \right]_r + \mathfrak{m} - B(r) &\leq C(\lambda^2 + \lambda) + \left(\mathcal{J} - B(r) - \frac{\mathcal{J}}{\tau(\mathcal{J} + \lambda)} \right) \\ &< C(\lambda^2 + \lambda) + \left(\mathcal{J} - B(r) - \frac{\mathcal{J}}{\tau(\bar{B} + 1/\tau)} \right) \\ &< 0, \end{aligned}$$

by choosing λ sufficiently small to satisfy $\lambda < \bar{B} + \frac{1}{\tau} - \mathcal{J}$ and $C(\lambda^2 + \lambda) < \underline{B} + \frac{\mathcal{J}}{\tau(\bar{B} + 1/\tau)} - \mathcal{J}$. Here $C = C(\tau, r_0)$ is a positive constant independent of j . Hence, by Lemma 2.2, we get that

$$m_j(r) \geq \mathbf{m}(r) \quad \text{over} \quad [r_0, r_1], \tag{2.5}$$

and the constant λ is positive and small, independent of j . The proof is complete. \square

Next we return to prove Theorem 2.1.

Proof of Theorem 2.1. Multiplying (2.2) by $(m_j - \mathcal{J})$, we get

$$\begin{aligned} & (\mathcal{J}^2 - j^2) \int_{r_0}^{r_1} r \frac{|(m_j)_r|^2}{(m_j)^3} dr + \frac{4}{9} \int_{r_0}^{r_1} r \frac{m_j + \mathcal{J}}{(m_j)^3} |[(m_j - \mathcal{J})^{\frac{3}{2}}]_r|^2 dr \\ & + \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \frac{r(m_j)_r}{m_j} dr + \int_{r_0}^{r_1} (m_j - B)(m_j - \mathcal{J}) dr = 0. \end{aligned} \tag{2.6}$$

Combining $\underline{B} + \frac{1}{\tau} \geq \mathcal{J}$ with integration by parts and Cauchy inequality, we obtain

$$\frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \frac{r(m_j)_r}{m_j} dr = \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} r d(\ln m_j) = \frac{\mathcal{J}}{\tau} [(r_1 - r_0) \cdot \ln \mathcal{J}] - \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \ln m_j dr,$$

and

$$\begin{aligned} \int_{r_0}^{r_1} (m_j - B)(m_j - \mathcal{J}) dr & \geq \int_{r_0}^{r_1} (m_j - \mathcal{J})^2 dr - \int_{r_0}^{r_1} \left(B + \frac{1}{\tau} - \mathcal{J} \right) (m_j - \mathcal{J}) dr \\ & \quad + \frac{1}{\tau} \int_{r_0}^{r_1} (m_j - \mathcal{J}) dr \\ & \geq \frac{1}{2} \int_{r_0}^{r_1} (m_j - \mathcal{J})^2 dr - \frac{1}{2} \int_{r_0}^{r_1} \left(B + \frac{1}{\tau} - \mathcal{J} \right)^2 dr \\ & \quad + \frac{1}{\tau} \int_{r_0}^{r_1} (m_j - \mathcal{J}) dr. \end{aligned}$$

After that, because of $\mathcal{J} \leq m_j \leq \bar{B} + \frac{1}{\tau}$, we derive from (2.6) that

$$\begin{aligned} & \frac{(\mathcal{J}^2 - j^2)r_0}{(\bar{B} + \frac{1}{\tau})^3} \int_{r_0}^{r_1} |(m_j)_r|^2 dr + \frac{8r_0\mathcal{J}}{9(\bar{B} + \frac{1}{\tau})^3} \int_{r_0}^{r_1} |[(m_j - \mathcal{J})^{\frac{3}{2}}]_r|^2 dr + \frac{1}{2} \int_{r_0}^{r_1} (m_j - \mathcal{J})^2 dr \\ & \leq \frac{1}{2} \int_{r_0}^{r_1} \left(B + \frac{1}{\tau} - \mathcal{J} \right)^2 dr + \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \ln m_j dr - \frac{\mathcal{J}}{\tau} [(r_1 - r_0) \cdot \ln \mathcal{J}] \end{aligned}$$

$$\leq \frac{1}{2} \int_{r_0}^{r_1} \left(\bar{B} + \frac{1}{\tau} - \mathcal{J} \right)^2 dr + \frac{\mathcal{J}(r_1 - r_0)}{\tau} \left[\ln \left(\bar{B} + \frac{1}{\tau} \right) - \ln \mathcal{J} \right],$$

which gives

$$\| (m_j - \mathcal{J})^{\frac{3}{2}} \|_{H^1(r_0, r_1)} \leq C_1(\bar{B}, \tau, r_0), \quad \| (\mathcal{J}^2 - j^2)(m_j)_r \|_{L^2(r_0, r_1)} \leq C_2(\bar{B}, \tau, r_0)(\mathcal{J}^2 - j^2)^{\frac{1}{2}}.$$

Here C_1 and C_2 are positive constants independent of j . Thus, by the compact imbedding $H^1(r_0, r_1) \hookrightarrow C^\alpha[r_0, r_1]$, $0 < \alpha < \frac{1}{2}$, there exists a function m , as $j \rightarrow \mathcal{J}^-$, such that up to a subsequence,

$$(m_j - \mathcal{J})^{\frac{3}{2}} \rightharpoonup (m - \mathcal{J})^{\frac{3}{2}} \text{ weakly in } H^1(r_0, r_1), \tag{2.7}$$

$$(m_j - \mathcal{J})^{\frac{3}{2}} \rightarrow (m - \mathcal{J})^{\frac{3}{2}} \text{ strongly in } C^\alpha[r_0, r_1], \tag{2.8}$$

$$(\mathcal{J}^2 - j^2)(m_j)_r \rightarrow 0 \text{ strongly in } L^2(r_0, r_1). \tag{2.9}$$

Noticing that $[(m_j - \mathcal{J})^2]_r = \frac{4}{3}(m_j - \mathcal{J})^{\frac{1}{2}}[(m_j - \mathcal{J})^{\frac{3}{2}}]_r$, we get

$$\| (m_j - \mathcal{J})^2 \|_{H^1(r_0, r_1)} \leq C \| (m_j - \mathcal{J})^{\frac{3}{2}} \|_{H^1(r_0, r_1)} \leq C(r_0, \bar{B}, \tau),$$

which leads to

$$(m_j - \mathcal{J})^2 \rightharpoonup (m - \mathcal{J})^2 \text{ weakly in } H^1(r_0, r_1) \text{ as } j \rightarrow \mathcal{J}^-.$$

Thus, multiplying (2.2) by $\varphi \in H_0^1(r_0, r_1)$, we have

$$\begin{aligned} & \int_{r_0}^{r_1} r \frac{m_j + \mathcal{J}}{2m_j^3} [(m_j - \mathcal{J})^2]_r \varphi_r dr + \int_{r_0}^{r_1} \frac{r}{m_j^3} (\mathcal{J}^2 - j^2)(m_j)_r \varphi_r dr \\ & + \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \frac{r}{m_j} \varphi_r dr + \int_{r_0}^{r_1} (m - B(r)) \varphi dr = 0. \end{aligned}$$

As $j \rightarrow \mathcal{J}^-$, by (2.7)-(2.9), (1.11) holds in the case of $n = 2$. The lower bound estimate is directly obtained from (2.5) and (2.8).

To prove the uniqueness of the interior subsonic solution, we first define $w(r) := (m(r) - \mathcal{J})^2$ and it is easy to see that $w \in H_0^1(r_0, r_1)$ satisfies the equality

$$\left(\frac{r(\sqrt{w} + 2\mathcal{J})w_r}{2(\sqrt{w} + \mathcal{J})^3} + \frac{r\mathcal{J}}{\tau(\sqrt{w} + \mathcal{J})} \right)_r = \sqrt{w} + \mathcal{J} - B(r), \quad r \in (r_0, r_1). \tag{2.10}$$

Then, recalled from the proof of Theorem 2.1 [19], it implies by (2.10) that $w \in C^{1+\frac{1}{4}}[r_0, r_1]$. Letting

$$G_w(r) := \frac{r(\sqrt{w} + 2\mathcal{J})w_r}{2(\sqrt{w} + \mathcal{J})^3} + \frac{r\mathcal{J}}{\tau(\sqrt{w} + \mathcal{J})},$$

we have

$$\begin{cases} \frac{r(\sqrt{w} + 2\mathcal{J})w_r}{2(\sqrt{w} + \mathcal{J})^3} = G_w - \frac{r\mathcal{J}}{\tau(\sqrt{w} + \mathcal{J})}, \\ G_w(r) = G_w(r_0) + \int_{r_0}^r (\sqrt{w(s)} + \mathcal{J} - B(r)) ds. \end{cases} \tag{2.11}$$

First, suppose that (2.1) has two different interior subsonic solutions $m_1(r)$ and $m_2(r)$ over $[r_0, r_1]$. Next there exists a nonempty domain $[\bar{r}_0, \bar{r}_1] \subset [r_0, r_1]$ such that (2.10) has two corresponding solutions $w_1(r)$ and $w_2(r)$ satisfying

$$w_1(\bar{r}_0) = w_2(\bar{r}_0), \quad w_1(\bar{r}_1) = w_2(\bar{r}_1) \quad \text{and} \quad w_1(r) > w_2(r) \quad \text{for} \quad r \in (\bar{r}_0, \bar{r}_1).$$

Because of the C^1 -continuity of w_1 and w_2 , it holds that

$$(w_1)_r(\bar{r}_0) \geq (w_2)_r(\bar{r}_0) \quad \text{and} \quad (w_1)_r(\bar{r}_1) \leq (w_2)_r(\bar{r}_1). \tag{2.12}$$

Hence, it follows from the first equation of (2.11) that $G_{w_1}(\bar{r}_1) \leq G_{w_2}(\bar{r}_1)$. Then by the second equation of (2.11), we derive

$$G_{w_1}(\bar{r}_0) + \int_{\bar{r}_0}^{\bar{r}_1} (\sqrt{w_1(s)} + \mathcal{J} - B(r)) ds \leq G_{w_2}(\bar{r}_0) + \int_{\bar{r}_0}^{\bar{r}_1} (\sqrt{w_2(s)} + \mathcal{J} - B(r)) ds.$$

Since $w_1(r) > w_2(r)$ for $r \in (\bar{r}_0, \bar{r}_1)$, we get

$$G_{w_1}(\bar{r}_0) < G_{w_2}(\bar{r}_0),$$

which gives $(w_1)_r(\bar{r}_0) < (w_2)_r(\bar{r}_0)$. This is a contradiction to (2.12). Therefore, the interior subsonic solution of (2.1) is unique.

In the end, we show that $m \in C^{\frac{1}{2}}[r_0, r_1]$. Since $m(r) \geq \mathcal{J}$ over $[r_0, r_1]$, then

$$|m(a) - \mathcal{J} + m(c) - \mathcal{J}| = |m(a) - \mathcal{J}| + |m(c) - \mathcal{J}| \geq |(m(a) - \mathcal{J}) - (m(c) - \mathcal{J})| = |m(a) - m(c)|.$$

Thus, by $(m - \mathcal{J})^2 \in C^1[r_0, r_1]$, it is easy to see that

$$\begin{aligned} \frac{|m(a) - m(c)|^2}{|a - c|} &= \frac{|m(a) - m(c)| |(m(a) - \mathcal{J})^2 - (m(c) - \mathcal{J})^2|}{|a - c| |m(a) - \mathcal{J} + m(c) - \mathcal{J}|} \\ &\leq \frac{|(m(a) - \mathcal{J})^2 - (m(c) - \mathcal{J})^2|}{|a - c|} \\ &\leq C, \end{aligned}$$

for any $a, c \in [r_0, r_1]$, which implies $m \in C^{\frac{1}{2}}[r_0, r_1]$. This finishes the proof. \square

2.2. The case of $n=3$

In the subsection, we prove the existence and uniqueness of interior subsonic solutions of (1.10) in the three-dimensional case. Here (1.10) is rewritten as

$$\begin{cases} \left[r^2 \left(\frac{1}{m} - \frac{\mathcal{J}^2}{m^3} \right) m_r + \frac{r^2 \mathcal{J}}{\tau m} \right]_r = m - B(r) + 2, & r \in (r_0, r_1), \\ m(r_0) = m(r_1) = \mathcal{J}. \end{cases} \tag{2.13}$$

Now we list some results for interior subsonic solution of (2.13).

Theorem 2.4. Suppose that $\bar{\mathcal{B}} > \mathcal{J}$ and $\inf_{r \in [r_0, r_1]} \left\{ B(r) + \frac{2r\mathcal{J}}{\tau\bar{\mathcal{B}}} - 2 \right\} > \mathcal{J}$, then (2.13) admits a unique interior subsonic solution $m(r)$ over $[r_0, r_1]$ satisfying $m \in C^{\frac{1}{2}}[r_0, r_1]$ and

$$m(r) \geq \mathcal{J} + \bar{\lambda} \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

where $\bar{\lambda}$ is a small and positive constant.

Proof. First we divide the process into three steps.

Step 1. In this step, we concern the following approximate equation of (2.13)

$$\begin{cases} \left[r^2 \left(\frac{1}{m_j} - \frac{j^2}{(m_j)^3} \right) (m_j)_r + \frac{r^2 \mathcal{J}}{\tau m_j} \right]_r = m_j - B(r) + 2, & r \in (r_0, r_1), \\ m(r_0) = m(r_1) = \mathcal{J}, \end{cases} \quad (2.14)$$

and prove the existence and uniqueness of the solution to (2.14). In order to apply the Schauder fixed point theorem, we define an operator $\mathcal{P} : \bar{m} \rightarrow m_j$, by solving the linear equation

$$\begin{cases} \left[r^2 \left(\frac{1}{\bar{m}} - \frac{j^2}{\bar{m}^3} \right) (m_j)_r \right]_r - \frac{r^2 \mathcal{J}}{\tau \bar{m}^2} (m_j)_r = m_j - B(r) + 2 - \frac{2r\mathcal{J}}{\tau \bar{m}}, & r \in (r_0, r_1), \\ m_j(r_0) = m_j(r_1) = \mathcal{J}. \end{cases}$$

Now it is easy to verify that the operator \mathcal{P} is precompact and continuous. What's important is to prove $\mathcal{P}(\mathcal{C}) \subset \mathcal{C}$. As similar as that of Lemma 2.3, and by applying the weak maximum principle, we get the result

$$\mathcal{J} \leq m_j(r) \leq \bar{\mathcal{B}}$$

provided that

$$\bar{\mathcal{B}} > \mathcal{J} \quad \text{and} \quad \inf_{r \in [r_0, r_1]} \left\{ B(r) + \frac{2r\mathcal{J}}{\tau\bar{\mathcal{B}}} - 2 \right\} > \mathcal{J}.$$

Hereafter there exists a fixed point m_j of \mathcal{P} such that $\mathcal{P}(m_j) = m_j$, which is also a weak solution to (2.14) satisfying $m_j \in H^2(r_0, r_1)$.

The uniqueness of the solution of (2.14) can be obtained by a comparison principle, just like Lemma 2.2. Of course, we calculate that the comparison principle must be derived in the three-dimensional case. Hence, define

$$\bar{m}(r) := \mathcal{J} + \bar{\lambda} \sin \left(\pi \cdot \frac{r - r_0}{r_1 - r_0} \right), \quad r \in [r_0, r_1],$$

and note that $\inf_{r \in [r_0, r_1]} \left\{ B(r) + \frac{2r\mathcal{J}}{\tau\bar{\mathcal{B}}} - 2 \right\} > \mathcal{J}$. Then if $\bar{\lambda} > 0$ is sufficiently small, we also obtain

$$\begin{aligned} - \left[r^2 \left(\frac{1}{\bar{m}} - \frac{j^2}{\bar{m}^3} \right) \bar{m}_r + \frac{r^2 \mathcal{J}}{\tau \bar{m}} \right]_r + \bar{m} - B(r) + 2 &\leq C(\bar{\lambda}^2 + \bar{\lambda}) + \left(\mathcal{J} - B(r) - \frac{2r\mathcal{J}}{\tau(\mathcal{J} + \bar{\lambda})} + 2 \right) \\ &< C(\bar{\lambda}^2 + \bar{\lambda}) + \left(\mathcal{J} - B(r) - \frac{2r\mathcal{J}}{\tau\bar{\mathcal{B}}} + 2 \right) \\ &< 0. \end{aligned}$$

Here C is a positive constant independent of j . By the comparison principle, we also get

$$m_j(r) \geq \bar{m}(r) \quad \text{over} \quad [r_0, r_1].$$

Step 2. The second step is to give a uniform bound estimate of the approximate solution $m_j(r)$ for all $0 < j < \mathcal{J}$. As in (2.6), we have

$$\begin{aligned} & (\mathcal{J}^2 - j^2) \int_{r_0}^{r_1} r^2 \frac{|(m_j)_r|^2}{(m_j)^3} dr + \frac{4}{9} \int_{r_0}^{r_1} r^2 \frac{m_j + \mathcal{J}}{(m_j)^3} \cdot |[(m_j - \mathcal{J})^{\frac{3}{2}}]_r|^2 dr \\ & + \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \frac{r^2 (m_j)_r}{m_j} dr + \int_{r_0}^{r_1} (m_j - B + 2)(m_j - \mathcal{J}) dr = 0. \end{aligned} \tag{2.15}$$

Then because of $\underline{\mathcal{B}} > \mathcal{J}$, it holds that

$$\frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} \frac{r^2 (m_j)_r}{m_j} dr = \frac{\mathcal{J}}{\tau} [(r_1^2 - r_0^2) \cdot \ln \mathcal{J}] - \frac{\mathcal{J}}{\tau} \int_{r_0}^{r_1} 2r \ln m_j dr,$$

and

$$\begin{aligned} \int_{r_0}^{r_1} (m_j - B + 2)(m_j - \mathcal{J}) dr & \geq \frac{1}{2} \int_{r_0}^{r_1} (m_j - \mathcal{J})^2 dr - \frac{1}{2} \int_{r_0}^{r_1} \left(B + \frac{2r}{\tau} - 2 - \mathcal{J} \right)^2 dr \\ & \quad + \frac{1}{\tau} \int_{r_0}^{r_1} 2r(m_j - \mathcal{J}) dr, \end{aligned}$$

where we used Young’s inequality and integration by parts. Therefore, it follows from (2.15) and $\mathcal{J} \leq m_j \leq \bar{\mathcal{B}}$ that

$$\begin{aligned} & \frac{(\mathcal{J}^2 - j^2)r_0^2}{\bar{\mathcal{B}}^3} \int_{r_0}^{r_1} |(m_j)_r|^2 dr + \frac{8r_0^2 \mathcal{J}}{9\bar{\mathcal{B}}^3} \int_{r_0}^{r_1} |[(m_j - \mathcal{J})^{\frac{3}{2}}]_r|^2 dr + \frac{1}{2} \int_{r_0}^{r_1} (m_j - \mathcal{J}_0)^2 dr \\ & \leq \frac{1}{2} \int_{r_0}^{r_1} (\bar{\mathcal{B}} - \mathcal{J})^2 dr + \frac{\mathcal{J}(r_1^2 - r_0^2)(\ln \bar{\mathcal{B}} - \ln \mathcal{J})}{\tau}, \end{aligned}$$

which also gives

$$\|(m_j - \mathcal{J})^{\frac{3}{2}}\|_{H^1} \leq C \quad \text{and} \quad \|(\mathcal{J}^2 - j^2)(m_j)_r\|_{L^2} \leq C(\mathcal{J}^2 - j^2)^{\frac{1}{2}},$$

for some constant C depending on $(\tau, \bar{\mathcal{B}}, r_0, r_1)$, but independent of j . Hence, by the above estimates, there exists a subsequence $\{m_j\}_{0 < j < \mathcal{J}}$, converging weakly to a limit m as $j \rightarrow \mathcal{J}^-$. In fact, the limit function m is certainly a weak solution of (2.13) such that $(m - \mathcal{J})^2 \in H_0^1(r_0, r_1)$ and (1.11) holds.

Step 3. The last step is to prove the uniqueness of this interior subsonic solution $m(r)$ and to show $m \in C^{\frac{1}{2}}[r_0, r_1]$. This part of the proof is referring to that of Theorem 2.1 directly, and we don’t repeat it here. The proof is finished. \square

3. Existence of interior supersonic solutions

In this section, we are going to prove the existence of interior supersonic solutions of (1.10) in the two and three dimensional cases, respectively.

3.1. The case of $n = 2$

As similar as Lemma 2.2, we introduce a comparison principle first.

Lemma 3.1. *Let $V \in C^1[r_0, r_1]$ satisfying $V(r) \geq k_0 > 1$ over $[r_0, r_1]$ be a weak solution of the following equation*

$$\begin{cases} \left[rd_1(r) \cdot (V - 1)V_r + \frac{rV}{\tau} \right]_r - \left(\frac{V}{\tau} - d_2(r) \right) = 0, & r \in (r_0, r_1), \\ V(r_0) = V(r_1) = k_0, \end{cases}$$

where $d_1, d_2 \in L^\infty(r_0, r_1)$ and $d_1(r) > 0$ on $[r_0, r_1]$. Thus, for any $\varphi \in H_0^1(r_0, r_1)$, it holds that

$$\int_{r_0}^{r_1} \left[rd_1(r) \cdot (V - 1)V_r + \frac{rV}{\tau} \right] \varphi_r dr + \int_{r_0}^{r_1} \left(\frac{V}{\tau} - d_2(r) \right) \varphi dr = 0.$$

In addition, let $U \in C^1[r_0, r_1]$ be such that $U(r) > 0$ over $[r_0, r_1]$, $U(r_0) \leq k_0$, $U(r_1) \leq k_0$, and for any $\varphi \geq 0$, $\varphi \in H_0^1(r_0, r_1)$,

$$\int_{r_0}^{r_1} \left[rd_1(r) \cdot (U - 1)U_r + \frac{rU}{\tau} \right] \varphi_r dr + \int_{r_0}^{r_1} \left(\frac{U}{\tau} - d_2(r) \right) \varphi dr \leq 0.$$

Then $V(r) \geq U(r)$ over $[r_0, r_1]$.

Proof. Referring to the textbook [14] (see Theorem 10.7) and Theorem 2.2 [19], we set

$$I(r, z_1, z_2) := rd_1(r)(z_1 - 1)z_2 + \frac{rz_1}{\tau}.$$

Then, for any $\varphi \in H_0^1(r_0, r_1)$, $\varphi \geq 0$, we obtain

$$\int_{r_0}^{r_1} [I(r, U, U_r) - I(r, V, V_r)] \varphi_r dr + \frac{1}{\tau} \int_{r_0}^{r_1} (U - V) \varphi dr \leq 0. \tag{3.1}$$

Denote $W =: U - V$ and $U_t := tU + (1 - t)V$. A simple computation indicates that

$$\begin{aligned} I(r, U, U_r) - I(r, V, V_r) &= I(r, U, U_r) - I(r, V, U_r) + I(r, V, U_r) - I(r, V, V_r) \\ &= \int_0^1 \frac{\partial I}{\partial z_1}(r, U_t, U_r) dt \cdot W(r) + \int_0^1 \frac{\partial I}{\partial z_2}(r, V, (U_t)_r) dt \cdot W_r(r). \end{aligned}$$

Let $\varphi(r) = \frac{W^+(r)}{W^+(r) + \epsilon}$ with $W^+(r) := \max\{0, W(r)\}$ and a positive constant ϵ , and note that

$$\left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r = \frac{W_r^+(r)}{W^+(r) + \epsilon}, \quad \varphi_r(r) = \frac{\epsilon}{W^+(r) + \epsilon} \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r.$$

Because $k_0 > 1$, $U \in C^1[r_0, r_1]$ and $d_m := \min_{r \in [r_0, r_1]} d_1(r) > 0$, this yields that

$$\int_0^1 \frac{\partial I}{\partial z_1}(r, U_t, U_r) dt = r d_1(r) U_r + \frac{r}{\tau} \leq C$$

and

$$\int_0^1 \frac{\partial I}{\partial z_2}(r, V, (U_t)_r) dt = r d_1(r) (V - 1) \geq r_0 (k_0 - 1) d_m.$$

Then it follows from (3.1) that

$$\begin{aligned} & \epsilon r_0 (k_0 - 1) d_m \int_{r_0}^{r_1} \left| \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r \right|^2 dr + \frac{1}{\tau} \int_{r_0}^{r_1} \frac{(W^+(r))^2}{W^+(r) + \epsilon} dr \\ & \leq C \epsilon \int_{r_0}^{r_1} \frac{W^+(r)}{W^+(r) + \epsilon} \left| \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r \right| dr \\ & \leq \frac{\epsilon r_0 (k_0 - 1) d_m}{2} \int_{r_0}^{r_1} \left| \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r \right|^2 dr + \frac{C^2 \epsilon (r_1 - r_0)}{2 r_0 (k_0 - 1) d_m}, \end{aligned}$$

where we used Young’s inequality in the second inequality. Thus, we get for any ϵ

$$\int_{r_0}^{r_1} \left| \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r \right|^2 dr \leq \frac{C^2 (r_1 - r_0)}{r_0^2 (k_0 - 1)^2 d_m^2},$$

which further by Poincaré’s inequality gives

$$\begin{aligned} \int_{r_0}^{r_1} \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]^2 dr & \leq (r_1 - r_0)^2 \int_{r_0}^{r_1} \left| \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]_r \right|^2 dr \\ & \leq \frac{C^2 (r_1 - r_0)^3}{r_0^2 (k_0 - 1)^2 d_m^2} < \infty. \end{aligned}$$

Now letting $\epsilon \rightarrow 0^+$, one can see that if $W^+(r) \neq 0$ for some $r \in [r_0, r_1]$,

$$\int_{r_0}^{r_1} \left[\ln \left(1 + \frac{W^+(r)}{\epsilon} \right) \right]^2 dr = \infty,$$

which gets a contradiction. Therefore, $U(r) \leq V(r)$ for all $r \in [r_0, r_1]$. \square

Then let's show the existence theorem as follows.

Theorem 3.2. *Assume that $\underline{B} + \frac{1}{\tau} > \mathcal{J}$, then system (2.1) admits at least one interior supersonic solution $m \in C^{1/2}[r_0, r_1]$ satisfying $\ell \leq m(r) \leq \mathcal{J}$ over $[r_0, r_1]$ for a positive constant ℓ . Moreover, the function m possesses the property that $m(r) < \mathcal{J}$ for any $r \in (r_0, r_1)$.*

Proof. This proof is divided into three steps for clarity.

Step 1. We first consider the following approximate equation of (2.1)

$$\begin{cases} \left[r \left(\frac{1}{m_k} - \frac{k^2}{m_k^3} \right) (m_k)_r + \frac{rk}{\tau m_k} \right]_r = m_k - B(r), \\ m_k(r_0) = m_k(r_1) = \mathcal{J}, \end{cases} \quad (3.2)$$

with the parameter $k > \mathcal{J}$. Let $v_k(r) := \frac{k}{m_k(r)}$, thus, (3.2) becomes

$$\begin{cases} \left[r \left(v_k - \frac{1}{v_k} \right) (v_k)_r + \frac{rv_k}{\tau} \right]_r - \left(\frac{k}{v_k} - B \right) = 0, r \in (r_0, r_1), \\ v_k(r_0) = v_k(r_1) = \frac{k}{\mathcal{J}} \triangleq k_0 > 1. \end{cases} \quad (3.3)$$

Next we only need to prove that there exists a weak solution $v_k(r)$ to (3.3) satisfying $v_k \geq k_0$. Here our adopted approach is the iterative method. Due to the effect of high dimensions space, we apply a so-called two-steps iteration to complete the following proof.

Let X be a solution space, denoted by

$$\begin{aligned} X := \{ \phi(r) : \phi \in C^1[r_0, r_1], k_0 \leq \phi(r) \leq \mathcal{M}, \phi(r_0) = \phi(r_1) = k_0, \\ \|\phi\|_{C^\alpha[r_0, r_1]} \leq \Lambda, \|\phi\|_{C^1[r_0, r_1]} \leq \Upsilon(\Lambda) \}. \end{aligned}$$

Here some positive constants \mathcal{M} , Λ and $\Upsilon(\Lambda)$ are determined later. Then we define an operator $\Psi : \eta \rightarrow v$ by solving the quasi-linear system

$$\begin{cases} \left[\frac{r(\eta+1)}{\eta} \cdot (v-1)v_r \right]_r + \frac{rv_r}{\tau} - \left(\frac{k}{\eta} - B - \frac{\eta}{\tau} \right) = 0, \quad r \in (r_0, r_1), \\ v(r_0) = v(r_1) = k_0, \end{cases} \quad (3.4)$$

where $\eta \in X$. To use the Schauder fixed point theorem, we first claim that system (3.4) has a unique solution $v \in C^{1+\alpha}[r_0, r_1]$ for $0 < \alpha < 1$ and arbitrary fixed $\eta \in X$.

To this end, set

$$S := \{ \omega \in C^0[r_0, r_1] | k_0 \leq \omega \leq \mathcal{K} \quad \text{and} \quad \omega(r_0) = \omega(r_1) = k_0 \}$$

for an undetermined constant \mathcal{K} , and let's define an operator $i : S \rightarrow C^0[r_0, r_1]$, $i(\xi) = \zeta$ by solving the linearized system of (3.4)

$$\begin{cases} [rg_1(\eta, \xi) \cdot \zeta_r]_r + \frac{r\zeta_r}{\tau} + g_2(\eta, \tau) = 0, \quad r \in (r_0, r_1), \\ \zeta(r_0) = \zeta(r_1) = k_0. \end{cases} \quad (3.5)$$

Here $\xi \in S$ and we have defined $g_1 := \frac{r(\eta + 1)(\xi - 1)}{\eta}$ and $g_2 := B + \frac{\eta}{\tau} - \frac{k}{\eta}$. Furthermore, note that g_1 and g_2 are C^1 -continuous with respect to η .

Actually, one finds that the fixed point of the operator i is a solution of (3.4), so we need to prove the existence of the fixed point of i . Since (3.5) has a solution $\zeta \in H^1(r_0, r_1)$, the operator i is precompact by the compact imbedding $H^1(r_0, r_1) \hookrightarrow C^0[r_0, r_1]$. The continuity of i is based on the standard argument, obviously. Next, we only need to prove $k_0 \leq \zeta(r) \leq \mathcal{K}$ over $[r_0, r_1]$. Multiplying (3.5) by $(\zeta - k_0)^-(r) := \min\{0, (\zeta - k_0)(r)\}$, we have

$$\int_{r_0}^{r_1} r g_1(r) |[(\zeta - k_0)^-]_r|^2 dr + \frac{1}{2\tau} \int_{r_0}^{r_1} [(\zeta - k_0)^-]^2 dr + \int_{r_0}^{r_1} (-g_2(r)) (\zeta - k_0)^- dr = 0 \tag{3.6}$$

where we have used

$$\frac{1}{\tau} \int_{r_0}^{r_1} r \zeta_r (\zeta - k_0)^- dr = -\frac{1}{2\tau} \int_{r_0}^{r_1} [(\zeta - k_0)^-]^2 dr.$$

Here each term of (3.6) is non-negative since $g_1 \geq r_0(k_0 - 1) > 0$, $\eta \geq k_0$, $B + \frac{k_0}{\tau} > \mathcal{J}$ and

$$g_2(\eta, \tau) = B + \frac{\eta}{\tau} - \frac{k}{\eta} = \left(B + \frac{k_0}{\tau} - \mathcal{J} \right) + \left(\mathcal{J} - \frac{k}{\eta} \right) + \frac{\eta - k_0}{\tau} > 0.$$

Then this implies by (3.6) that $\zeta(r) \geq k_0$ over $[r_0, r_1]$. Now multiplying (3.5) by $(\zeta - k_0)(r)$, we show that

$$r_0(k_0 - 1) \int_{r_0}^{r_1} |(\zeta - k_0)_r|^2 dr + \frac{1}{2\tau} \int_{r_0}^{r_1} (\zeta - k_0)^2 dr \leq \int_{r_0}^{r_1} g_2(\eta, \tau) (\zeta - k_0) dr.$$

Thus, by Young’s inequality and Poincaré’s inequality

$$\int_{r_0}^{r_1} (\zeta - k_0)^2 dr \leq (r_1 - r_0)^2 \int_{r_0}^{r_1} |(\zeta - k_0)_r|^2 dr,$$

we get

$$\begin{aligned} r_0(k_0 - 1) \int_{r_0}^{r_1} |(\zeta - k_0)_r|^2 dr &\leq \int_{r_0}^{r_1} g_2(\eta, \tau) (\zeta - k_0) dr \\ &\leq \frac{r_0(k_0 - 1)}{2(r_1 - r_0)^2} \int_{r_0}^{r_1} (\zeta - k_0)^2 dr + \frac{(r_1 - r_0)^2}{2r_0(k_0 - 1)} \int_{r_0}^{r_1} g_2^2(\eta, \tau) dr \\ &\leq \frac{r_0(k_0 - 1)}{2} \int_{r_0}^{r_1} |(\zeta - k_0)_r|^2 dr + \frac{(r_1 - r_0)^2}{2r_0(k_0 - 1)} \|g_2(\eta)\|_{L^2}^2, \end{aligned}$$

which indicates that

$$\|(\zeta - k_0)_r\|_{L^2} \leq \frac{r_1 - r_0}{r_0(k_0 - 1)} \|g_2(\eta)\|_{L^2}.$$

Further, we conclude that

$$\zeta \leq k_0 + C(r_0, r_1, k_0) \|g_2(\eta)\|_{L^2},$$

then choose $\mathcal{K}(\eta) := k_0 + C(r_0, r_1, k_0) \|g_2(\eta)\|_{L^2}$ such that $k_0 \leq \zeta \leq \mathcal{K}$. Applying the Schauder fixed point theorem, we have a fixed point $v \in S$ such that $i(v) = v$, which is also a weak solution to (3.4). Thanks to the regularity theory and Sobolev imbedding theory [14], it's proved that $v \in C^{1+\alpha_0}[r_0, r_1]$ such that $k_0 \leq v \leq \mathcal{K}(\eta)$,

$$\|v\|_{C^{\alpha_0}[r_0, r_1]} \leq C_0(k_0, \eta, \tau, \mathcal{K}(\eta)) \quad \text{and} \quad \|v\|_{C^{1+\alpha_0}[r_0, r_1]} \leq C(C_0, k_0, \eta, \tau, \mathcal{K}(\eta)) \quad (3.7)$$

for constants C_0 and $0 < \alpha_0 < \frac{1}{2}$. Moreover, we can prove that the solution of (3.4) is unique by Lemma 3.1. Thus the claim is verified.

Next, we go back to show that Ψ has a fixed point, so it is necessary to prove $\Psi(X) \subset X$. Since $v(r) \geq k_0$ over $[r_0, r_1]$, it remains to determine the upper bound of the solution $v(r)$ for (3.4). Multiplying (3.4) by $(v - k_0)^2$, we derive

$$\int_{r_0}^{r_1} \frac{r(\eta+1)}{2\eta} \cdot |(v - k_0)^2|_r|^2 dr + \frac{1}{3\tau} \int_{r_0}^{r_1} (v - k_0)^3 dr \leq \int_{r_0}^{r_1} \left(B + \frac{\eta}{\tau} - \frac{k}{\eta} \right) (v - k_0)^2 dr,$$

which leads to

$$\begin{aligned} \frac{r_0}{2} \int_{r_0}^{r_1} |(v - k_0)^2|_r|^2 dr &\leq \int_{r_0}^{r_1} \left(B + \frac{\eta}{\tau} \right) (v - k_0)^2 dr \\ &\leq \frac{r_0}{4(r_1 - r_0)^2} \int_{r_0}^{r_1} (v - k_0)^4 dr + \frac{(r_1 - r_0)^2}{r_0} \int_{r_0}^{r_1} \left(B + \frac{\eta}{\tau} \right)^2 dr \\ &\leq \frac{r_0}{4} \int_{r_0}^{r_1} |(v - k_0)^2|_r|^2 dr + \frac{(r_1 - r_0)^3}{r_0} \left(\bar{B} + \frac{\mathcal{M}}{\tau} \right)^2. \end{aligned}$$

Here we have used Poincaré's inequality

$$\int_{r_0}^{r_1} (v - k_0)^4 dr \leq (r_1 - r_0)^2 \int_{r_0}^{r_1} |(v - k_0)^2|_r|^2 dr.$$

It then follows that

$$\|[(v - k_0)^2]_r\|_{L^2}^2 \leq \frac{4(r_1 - r_0)^3}{r_0^2} \left(\bar{B} + \frac{\mathcal{M}}{\tau} \right)^2.$$

Moreover it holds that

$$0 < v(r) \leq k_0 + C \sqrt{\bar{B} + \frac{\mathcal{M}}{\tau}}$$

for a positive constant C depending on (r_0, r_1) . Thus by a simple calculation, we can choose

$$\mathcal{M} = \mathcal{M}(\overline{B}, \tau) \geq k_0 + \frac{C^2}{2\tau} + C\sqrt{\overline{B} + \frac{k_0}{\tau} + \frac{C^2}{4\tau^2}}$$

such that $k_0 + C\sqrt{\overline{B} + \frac{\mathcal{M}}{\tau}} \leq \mathcal{M}$. Then we can see that $v(r) \leq \mathcal{M}$ over $[r_0, r_1]$ for any $k_0 \leq \eta \leq \mathcal{M}$. Hereafter it implies by (3.7) that for a constant $0 < \alpha_0 < \frac{1}{2}$,

$$\|v\|_{C^{\alpha_0}[r_0, r_1]} \leq C_0(\mathcal{M}, \mathcal{K}(\mathcal{M}), \tau, k_0) \quad \text{and} \quad \|v\|_{C^{1+\alpha}[r_0, r_1]} \leq C(\mathcal{M}, \mathcal{K}(\mathcal{M}), \tau, k_0, C_0),$$

and we determine $\alpha = \alpha_0$, $\Lambda = C_0(\mathcal{M}, \mathcal{K}(\mathcal{M}), \tau, k_0)$ and $\Upsilon(\Lambda) = C(\mathcal{M}, \mathcal{K}(\mathcal{M}), \tau, k_0, \Lambda)$. Now it can be verified that $v \in X$ and X is a bounded and closed convex subset of $C^1[r_0, r_1]$. Also, the operator Ψ is a compact map of X into itself by the compact imbedding $C^{1+\alpha}[r_0, r_1] \hookrightarrow C^1[r_0, r_1]$. Using the continuity theory, one can see that the operator Ψ is continuous. Hence, a fixed point of the map Ψ can be obtained by the Schauder fixed point theorem. In the end, (3.3) has a weak solution $v_k \in C^1[r_0, r_1]$, and $m_k(r) = k/v_k(r)$ is an interior supersonic solution of (3.2) over $[r_0, r_1]$.

Step 2. This step is to prove the existence of the interior supersonic solutions of (2.1). Multiplying (3.3) by $(v_k - k_0)(r)$, and using Young’s inequality and Poincaré’s inequality, we have

$$\begin{aligned} & (k_0 - 1) \int_{r_0}^{r_1} \frac{r(v_k + 1)}{v_k} |(v_k)_r|^2 dr + \frac{4}{9} \int_{r_0}^{r_1} \frac{r(v_k + 1)}{v_k} |(v_k - k_0)^{\frac{3}{2}}|_r|^2 dr \\ &= \int_{r_0}^{r_1} \left(B + \frac{v_k + k_0}{2\tau} - \frac{k}{v_k} \right) (v_k - k_0) dr \\ &\leq \frac{2}{3} \int_{r_0}^{r_1} (v_k - k_0)^{\frac{3}{2}} dr + \frac{1}{3} \int_{r_0}^{r_1} \left(B + \frac{v_k + k_0}{2\tau} \right)^3 dr \\ &\leq \frac{r_0}{3(r_1 - r_0)^2} \int_{r_0}^{r_1} (v_k - k_0)^3 dr + \frac{(r_1 - r_0)^3}{r_0} + \frac{r_1 - r_0}{3} \left(\overline{B} + \frac{\mathcal{M} + k_0}{2\tau} \right)^3 \\ &\leq \frac{r_0}{3} \int_{r_0}^{r_1} |(v_k - k_0)^{\frac{3}{2}}|_r|^2 dr + C(\overline{B}, \tau, \mathcal{M}, k_0, r_0, r_1), \end{aligned}$$

where we used

$$\int_{r_0}^{r_1} \frac{rv_k}{\tau} (v_k - k_0)_r dr = -\frac{1}{2\tau} \int_{r_0}^{r_1} (v_k + k_0)(v_k - k_0) dr.$$

Thus, it follows that

$$\|(k_0 - 1)^{\frac{1}{2}}(v_k)_r\|_{L^2} + \|(v_k - k_0)^{\frac{3}{2}}\|_{H^1} \leq C,$$

for some constants C only depending on $(\overline{B}, \tau, \mathcal{M}, k_0, r_0, r_1)$. In fact, as $k_0 \rightarrow 1^+$, i.e. $k \rightarrow \mathcal{J}^+$, given that a suitable choice of \mathcal{M} , we can obtain

$$\|(v_k - k_0)^{\frac{3}{2}}\|_{L^\infty} \leq C(\overline{B}, \tau, r_0, r_1),$$

which gives

$$v_k \leq k_0 + C^{\frac{2}{3}}.$$

Then,

$$m_k(r) = \frac{k}{v_k(r)} \geq \frac{k}{k + C^{\frac{2}{3}}} \geq \frac{1}{1 + C^{\frac{2}{3}}} \triangleq \ell \quad \text{for } r \in [r_0, r_1]. \tag{3.8}$$

A direct computation yields that

$$(m_k)_r = -\frac{k(v_k)_r}{v_k^2} \quad \text{and} \quad ((\mathcal{J} - m_k)^2)_r = \frac{4\mathcal{J}k(v_k - k_0)^{\frac{1}{2}}((v_k - k_0)^{\frac{3}{2}})_r}{3v_k^3},$$

which together with (3.8) implies

$$\|(k_0 - 1)^{\frac{1}{2}}(m_k)_r\|_{L^2} + \|(\mathcal{J} - m_k)^2\|_{H^1} \leq C(\overline{B}, \tau, r_0, r_1).$$

Finally, one can see that there exists a function m such that, as $k \rightarrow \mathcal{J}^+$, up to a subsequence,

$$\begin{aligned} (\mathcal{J} - m_k)^2 &\rightharpoonup (\mathcal{J} - m)^2 \quad \text{weakly in } H^1(r_0, r_1), \\ (\mathcal{J} - m_k)^{\frac{3}{2}} &\rightharpoonup (\mathcal{J} - m)^{\frac{3}{2}} \quad \text{weakly in } H^1(r_0, r_1), \\ (\mathcal{J} - m_k)^{\frac{3}{2}} &\rightarrow (\mathcal{J} - m)^{\frac{3}{2}} \quad \text{strongly in } C^\alpha[r_0, r_1], \quad 0 < \alpha < \frac{1}{2}, \\ (k_0 - 1)(m_k)_r &\rightarrow 0 \quad \text{strongly in } L^2(r_0, r_1). \end{aligned} \tag{3.9}$$

Hence equation (2.1) has an interior supersonic solution $m(r)$ over $[r_0, r_1]$, and (1.11) holds. The lower bound of the solution m is obtained by (3.8) and (3.9), and $m \in C^{1/2}[r_0, r_1]$ is easily obtained as similar as that of Theorem 2.1.

Step 3. At the last step, we need to prove that $m(r) < \mathcal{J}$ over (r_0, r_1) . If a function m satisfies $m(r) \equiv \mathcal{J}$ on any interval $[s_1, s_2] \subset [r_0, r_1]$, then m is not a solution of (2.1) because $\underline{B} + \frac{1}{\tau} > \mathcal{J}$. Thus, there exist two points \hat{s}_1 and \hat{s}_2 satisfying $0 < \hat{s}_1 - r_0 \ll 1$ and $0 < r_1 - \hat{s}_2 \ll 1$. Then let $\varepsilon > 0$ be a small number such that $m(\hat{s}_1), m(\hat{s}_2) \leq \mathcal{J} - \varepsilon < \mathcal{J}$. Next, we are going to prove that $m(r) \leq \mathcal{J} - \varepsilon$ over $[\hat{s}_1, \hat{s}_2]$. After that, set $w = (\mathcal{J} - m)^2$, further we know $w \in H_0^1(r_0, r_1)$ and $w(\hat{s}_1), w(\hat{s}_2) \geq \varepsilon^2$. From (1.11), taking $\varphi(r) = (w - \varepsilon^2)^-(r)$, we have

$$\begin{aligned} &\int_{\hat{s}_1}^{\hat{s}_2} r \frac{2\mathcal{J} - \sqrt{w}}{2(\mathcal{J} - \sqrt{w})^3} |[(w - \varepsilon^2)^-]_r|^2 dr + \int_{\hat{s}_1}^{\hat{s}_2} \frac{r\mathcal{J}[(w - \varepsilon^2)^-]_r}{\tau(\mathcal{J} - \sqrt{w})} dr \\ &+ \int_{\hat{s}_1}^{\hat{s}_2} (\mathcal{J} - \sqrt{w} - B(r))(w - \varepsilon^2)^- dr = 0. \end{aligned} \tag{3.10}$$

Since $2\mathcal{J} - \sqrt{w} > \mathcal{J} - \sqrt{w} \geq 0$, one can see that the first term of (3.10) is non-negative. Then, by a direct computation, we change the last two of (3.10) as

$$\begin{aligned}
 & \int_{\hat{s}_1}^{\hat{s}_2} r \mathcal{J} \frac{[(w - \varepsilon^2)^-]_r}{\tau(\mathcal{J} - \sqrt{w})} dr + \int_{\hat{s}_1}^{\hat{s}_2} (\mathcal{J} - \sqrt{w} - B(r))(w - \varepsilon^2)^- dr \\
 &= \frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} \frac{r\sqrt{w} [(w - \varepsilon^2)^-]_r}{\mathcal{J} - \sqrt{w}} dr + \frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} r [(w - \varepsilon^2)^-]_r dr \\
 &+ \int_{\hat{s}_1}^{\hat{s}_2} (\mathcal{J} - \sqrt{w} - B(r))(w - \varepsilon^2)^- dr \\
 &= \frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} \frac{r\sqrt{w} [(w - \varepsilon^2)^-]_r}{\mathcal{J} - \sqrt{w}} dr + \int_{\hat{s}_1}^{\hat{s}_2} \left(\mathcal{J} - \sqrt{w} - B(r) - \frac{1}{\tau} \right) (w - \varepsilon^2)^- dr.
 \end{aligned} \tag{3.11}$$

Here we notice that the second term on the right-hand side of (3.11) is also non-negative because of $\underline{B} + \frac{1}{\tau} > \mathcal{J}$. It remains to show the non-negativity of the first term on the right-hand side of (3.11). Note that

$$-[h(\sqrt{w})]_r := -[2\mathcal{J}\sqrt{w} + w + 2\mathcal{J}^2 \ln(\mathcal{J} - \sqrt{w})]_r = \frac{\sqrt{w}w_r}{\mathcal{J} - \sqrt{w}},$$

then it follows that

$$\frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} \frac{r\sqrt{w} [(w - \varepsilon^2)^-]_r}{\mathcal{J} - \sqrt{w}} dr = -\frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} r[h(\sqrt{w^\varepsilon})]_r dr = \frac{1}{\tau} \int_{\hat{s}_1}^{\hat{s}_2} [h(\sqrt{w^\varepsilon}) - h(\varepsilon)] dr, \tag{3.12}$$

where $w^\varepsilon := (w - \varepsilon^2)^- + \varepsilon^2$. Hence, $0 \leq \sqrt{w^\varepsilon} \leq \varepsilon$. Then a simple computation yields that

$$h'(s) = 2\mathcal{J} + 2s - \frac{2\mathcal{J}^2}{\mathcal{J} - s} < 0 \quad \text{for } s \in (0, \varepsilon],$$

because $h''(s) < 0$ on $(0, \varepsilon)$ and $h'(0) = 0$. Thus, it holds that the right side of (3.12) is non-negative, which leads to $(w - \varepsilon^2)^- = 0$. We derive that $m(r) \leq \mathcal{J} - \varepsilon$ over $[\hat{s}_1, \hat{s}_2]$ for some small constants ε . The proof is finished. \square

3.2. The case of $n = 3$

In the subsection, we state the results of interior supersonic solutions to (1.10) in three dimensional case.

Theorem 3.3. *Assume that $\underline{\mathcal{B}} > \mathcal{J}$, then system (2.13) admits an interior supersonic solution $m \in C^{1/2}[r_0, r_1]$ satisfying $\underline{\ell} \leq m(r) \leq \mathcal{J}$ over $[r_0, r_1]$ for a positive constant $\underline{\ell}$, moreover, $0 < m(r) < \mathcal{J}$ over (r_0, r_1) .*

Proof. This proof is similar as that of Theorem 3.2, so we sketch it as follows. The approximate system of (2.13) is the following equation

$$\begin{cases} \left[r^2 \left(\frac{1}{m_k} - \frac{k^2}{m_k^3} \right) (m_k)_r + \frac{r^2 k}{\tau m_k} \right]_r = m_k - B(r) + 2, \\ m_k(r_0) = m_k(r_1) = \mathcal{J}, \end{cases} \tag{3.13}$$

with the parameter $k > \mathcal{J}$. Let $v_k(r) := \frac{k}{m_k(r)}$, thus (3.13) can be recast as

$$\begin{cases} \left[r^2 \left(v_k - \frac{1}{v_k} \right) (v_k)_r + \frac{r^2 v_k}{\tau} \right]_r - \left(\frac{k}{v_k} - B + 2 \right) = 0, r \in (r_0, r_1), \\ v_k(0) = v_k(1) = k_0. \end{cases} \quad (3.14)$$

Then we define an operator $\tilde{\Psi} : \eta \rightarrow v$ by solving the following system

$$\begin{cases} \left[\frac{r^2(\eta+1)}{\eta} \cdot (v-1)v_r \right]_r + \frac{r^2 v_r}{\tau} - \left(\frac{k}{\eta} - B + 2 - \frac{2r\eta}{\tau} \right) = 0, \quad r \in (r_0, r_1), \\ v(0) = v(1) = k_0, \end{cases} \quad (3.15)$$

where $\eta \in X$. As similar to that of Theorem 3.2, and by applying the Schauder fixed point theorem, we show that there exists a unique solution $v \in C^{1+\alpha}[r_0, r_1]$ to the quasi-linear system (3.15), and there exists a constant $\tilde{\mathcal{K}}$ depending on η such that $k_0 \leq v \leq \tilde{\mathcal{K}}$,

$$\|v\|_{C^\alpha[r_0, r_1]} \leq C_0(k_0, \eta, \tau, \tilde{\mathcal{K}}(\eta)) \quad \text{and} \quad \|v\|_{C^{1+\alpha}[r_0, r_1]} \leq C(C_0, k_0, \eta, \tau, \tilde{\mathcal{K}}(\eta)) \quad (3.16)$$

for constants $\tilde{\mathcal{K}}(\eta)$ and $0 < \alpha < \frac{1}{2}$. In the following, we only need to prove $k_0 \leq v_k \leq \mathcal{M}$ with a proper constant \mathcal{M} . Obviously, since $v \geq k_0$, then we only prove $v_k \leq \mathcal{M}$. Now multiplying (3.15) by $(v - k_0)^2$, we derive

$$\begin{aligned} & \int_{r_0}^{r_1} \frac{r^2(\eta+1)}{2\eta} \cdot |[v - k_0]_r|^2 dr + \frac{2}{3\tau} \int_{r_0}^{r_1} r(v - k_0)^3 dr \\ & \leq \int_{r_0}^{r_1} \left(B + \frac{2r\eta}{\tau} - 2 - \frac{k}{\eta} \right) (v - k_0)^2 dr, \end{aligned}$$

which follows from the proof in Theorem 3.2 that

$$\|[v - k_0]_r\|_{L^2} \leq C(r_0, r_1, k_0) (\bar{\mathcal{B}} + \mathcal{M}).$$

Thus, we get

$$0 < v(r) \leq k_0 + C(r_0, r_1, k_0) \sqrt{\bar{\mathcal{B}} + \mathcal{M}}.$$

Then take $\mathcal{M} = \mathcal{M}(\bar{\mathcal{B}}, k_0)$ sufficiently large such that

$$k_0 + C(r_0, r_1, k_0) \sqrt{\bar{\mathcal{B}} + \mathcal{M}} \leq \mathcal{M}.$$

As a result, it holds that $v(r) \leq \mathcal{M}$ over $[r_0, r_1]$. Next it follows from (3.16) that $v \in X$ and X is also a bounded and closed convex subset of $C^1[r_0, r_1]$. Hereafter the Sobolev imbedding theorem and the Schauder fixed point theorem yield that there exists a fixed point v_k of the operator $\tilde{\Psi}$ such that

$$\tilde{\Psi}(v_k) = v_k.$$

Hence equation (3.14) has a weak solution v_k , then a solution of (3.13) would be obtained, that is

$$m_k(r) = \frac{k}{v_k(r)}, \quad r \in [r_0, r_1].$$

Next, the bound estimate of m_k can be verified by

$$\|(k_0 - 1)^{\frac{1}{2}}(m_k)_r\|_{L^2} + \|(\mathcal{J} - m_k)^2\|_{H^1} \leq C(\bar{\mathcal{B}}, k_0),$$

whose proof is shown in Theorem 3.2. Moreover, as similar to that of Theorem 3.2, there exists a limit of convergence $m(r)$ by a subsequence $\{m_k\}_{\mathcal{J} < k < +\infty}$ as $k \rightarrow \mathcal{J}^+$, which is an interior supersonic solution of (2.13). For a constant $0 < \bar{\ell} < \mathcal{J}$, it is easy to check that $\bar{\ell} \leq m(r) < \mathcal{J}$ over (r_0, r_1) and $m \in C^{1/2}[r_0, r_1]$. The proof is complete. \square

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