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Journal of Differential Equations 373 (2023) 359–388

Journal of Differential Equations

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Radially symmetric spiral flows of the compressible Euler-Poisson system for semiconductors

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Received 17 December 2022; accepted 10 June 2023
Available online 26 July 2023

Abstract

In this paper, we study the steady flows to the compressible Euler-Poisson system for semiconductors with the nonzero angular velocity in a radially symmetric way in an annulus. The main purpose here is to elucidate the effect of the angular velocity in the structure of the steady flows. We show the well-posedness of all kinds of types of radially symmetric spiral flows including radial subsonic/supersonic/transonic flows, and further give a specific classification of the flow patterns under the assumption of various boundary conditions at the inner and the outer circle. Additionally, different from the purely radial case, the uniqueness of radial subsonic flow can not be obtained due to the nonlocal effect caused by the angular velocity, consequently we prove the uniqueness of the radial subsonic solution in the case without the semiconductor effect or with a small current assumption. Moreover, some new patterns of spiral flows with or without shock are observed, such as a smooth transonic flow and a supersonic-supersonic shock flow for a large relaxation time parameter.

MSC: 35L60; 35Q35; 35B40; 35C06

Keywords: Euler-Poisson system; Nonzero angular velocity; Radially symmetric; The nonlocal effect; Smooth transonic flow; Supersonic-supersonic shock flow

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https://doi.org/10.1016/j.jde.2023.06.017
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1. Introduction

In this paper, we consider the steady compressible Euler-Poisson system with the semiconductor effect as follows:

$$\begin{cases} \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + pI_n) = \rho \nabla \Phi - \dfrac{\rho \boldsymbol{u}}{\tau}, \\ \Delta_{\boldsymbol{x}} \Phi = \rho - b(\boldsymbol{x}), \end{cases} \tag{1.1}$$

which is usually described for the local behaviors of the electron density ρ and the electron average velocity $\boldsymbol{u}$ in semiconductor devices. The equations (1.1) express the conservation of electrons, the conservation of momentum and the local change of the electrostatic potential Φ, respectively. In the second equation of (1.1), $\rho\nabla\Phi$ represents the Coulomb force of electron particles and $\frac{\rho \boldsymbol{u}}{\tau}$ stands for the effect of the semiconductor damping where the constant parameter $\tau > 0$ is the momentum relaxation time. The given function $b(\boldsymbol{x}) > 0$ is the impurity density, also called the doping profile for semiconductors. Physically, p is the pressure as a function of ρ, typically stated as

$$\textit{the isothermal case}: \quad p(\rho) = T\rho \quad \text{and} \quad \textit{the isentropic case}: \quad p(\rho) = K\rho^{\gamma},$$

where T is the constant temperature, and $K > 0$, $\gamma > 1$ are constants.

The system (1.1) is a hyperbolic-elliptic coupled system, which is elliptic in the subsonic states ($|\boldsymbol{u}| < c(\rho)$) and hyperbolic in the supersonic states ($|\boldsymbol{u}| > c(\rho)$) for the local sound speed $c(\rho) = \sqrt{p'(\rho)}$. For this system, our aim of this paper is to investigate the structure of radially symmetric subsonic/supersonic/transonic steady-states with nonzero angular velocity in a two-dimensional annulus with different boundary data on the inner and outer circle, especially involving the sonic degenerate boundary.

Actually, the system (1.1) is simplified originally from the hydrodynamic model of semiconductors. The full model since its first introduction by Bløtejær [5] has been paid much attention for its ability of simulating hot electron effects, which is not considered in the classical drift-diffusion model [24]. Here we refer to [23,27] for more derivation in physics and mathematics. Importantly, the study of the stationary flows is known to be the cornerstone of the whole research subject for the hydrodynamic model of semiconductors. Especially for the unipolar steady model represented by Euler-Poisson equations, Degond and Markowich [10,11] first investigated the existence and uniqueness of subsonic steady-state with a strong subsonic condition in one dimension and for potential flow, in three dimension, respectively. Thereafter more contributions related to the steady subsonic flows were made in [2,3,17,18,26], see also the references therein. On the other hand, Peng and Violet [28] proved the existence of a unique supersonic steady-state with a strong supersonic background in one dimension, and Bae et al. [4] studied the case of two-dimensional supersonic flow without the semiconductor effect, i.e., $\tau = \infty$. Regarding the steady transonic flows, Ascher et al. [1] and Rosini [29] first observed the transonic shocks via phase plane analysis, where the boundary was subsonic but the doping profile was supersonic. Then, Gamba [14] and Gamba and Morawetz [15] technically constructed the transonic shocks via viscosity vanishing method, but the artificial boundary layers were presented. After then, Luo and Xin [22] and Luo et al. [21] studied the well-posedness of the one-dimensional transonic steady-states with non-sonic boundary conditions without the semiconductor effect. To sum up, all the results mentioned here were related to the non-degenerate states only.

When the boundary is assumed to be sonic, a degenerate case of the boundary, Li-Mei-Zhang-Zhang [19,20] first showed in great depth the structure of all types of the one-dimensional steady-states to the system (1.1) when the doping profile is restricted in subsonic/supersonic states. Latterly for one-dimensional case, there are some remarkable contributions in this topic, such as the transonic doping profile case [6], the case of structural stability [12,13], the case of C^∞-smoothness [30] of transonic steady-states, and even the bipolar case [25]. For the multi-dimensional case, Chen, et al. [7,8] only concerned purely radial flows in an annulus where the angular velocity is not considered, and they further proved the well-posedness of all types of radial steady-states to the system (1.1) with sonic boundary.

Subsequently to the previous work [7,8], it is quite natural for us to study spiral flows [9, Section 104] of the system (1.1), which are the superpositions of symmetric flows with the angular velocity and radial velocity being both nonzero. The so-called spiral flows substantially possess richer and more interesting properties on the structure of these steady-states due to the nonzero angular velocity. Particularly for transonic spiral flows with/or without shocks, we expect to discover more new flow patterns compared with the purely radial case. Therefore, combining the ideas of the obtained results for the steady Euler system [31] and Euler-Poisson system [7,8], we will study the well-posedness of radially symmetric spiral steady-state solutions of the system (1.1), and give a fairly complete classification to these solutions with suitable boundary conditions on the inner and outer circle of a two-dimensional annulus

$$\mathcal{A} := \{\boldsymbol{x} = (x_1, x_2) : r_0 < r = \sqrt{x_1^2 + x_2^2} < r_1\}, \quad 0 < r_0 < r_1 < +\infty.$$

Here the most significant point is that we consider the degenerate boundary in this paper.

For this purpose, we introduce the polar coordinate (r, θ) as

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta,$$

and denote the radially symmetric spiral solutions to (1.1) of the form

$$\boldsymbol{u} = u_1(r)\mathbf{e}_r + u_2(r)\mathbf{e}_\theta, \quad \rho = \rho(r), \quad \boldsymbol{E} = \nabla\Phi = E(r)\mathbf{e}_r$$

and the corresponding boundary conditions

$$(\rho, \rho\boldsymbol{u})|_{|\boldsymbol{x}|=r_0} = (\rho_0, \rho_0\boldsymbol{u}_0), \quad \rho|_{|\boldsymbol{x}|=r_1} = \rho_1, \tag{1.2}$$

with

$$\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{u}_0 = u_{10}\mathbf{e}_r + u_{20}\mathbf{e}_\theta.$$

Based on the assumption of the radial condition $b = b(r) \in L^\infty(r_0, r_1)$, the system (1.1) can be written as

$$\begin{cases} (r\rho u_1)_r = 0, & r_0 < r < r_1, \\ (r\rho u_1^2)_r - \rho u_2^2 + rp_r = r\rho\left(E - \dfrac{u_1}{\tau}\right), & r_0 < r < r_1, \\ (r\rho u_1 u_2)_r + \rho u_1 u_2 = -\dfrac{r\rho u_2}{\tau}, & r_0 < r < r_1, \\ (rE)_r = r(\rho - b), & r_0 < r < r_1, \end{cases} \tag{1.3}$$

and the boundary conditions (1.2) become

$$\rho(r_0) = \rho_0, \tag{1.4}$$

$$\rho(r_1) = \rho_1, \tag{1.5}$$

$$(u_1(r_0), u_2(r_0)) = (u_{10}, u_{20}), \tag{1.6}$$

for positive constants (ρ_0, ρ_1, u_{10}) and a nonzero parameter u_{20}. Consequently, the goal of our work below is to look for the steady-state solutions of system (1.3) with the conditions (1.4)-(1.6).

To solve the problem (1.3)-(1.6), one can set a steady current in the radial direction by

$$J_1 = \rho u_1,$$

and gets from the first equation of (1.3) and (1.6) that

$$rJ_1(r) = r_0 J_1(r_0) = r_0\rho_0 u_{10} =: r_0 J_{10} \quad \text{for} \quad r \in [r_0, r_1]. \tag{1.7}$$

Thus, one has

$$u_1 = \frac{J_1}{\rho} = \frac{r_0\rho_0 u_{10}}{r\rho} = \frac{r_0 J_{10}}{r\rho}. \tag{1.8}$$

For convenience, let us denote a new variable

$$n(r) := \frac{r\rho}{r_0 J_{10}} = \frac{\rho}{J_1} = \frac{1}{u_1}, \tag{1.9}$$

and consider the isothermal case, i.e., $p(\rho) = T\rho$. Here let $T = 1$ without loss of generality such that the local sound speed $c^2(\rho) = p'(\rho) = 1$ and the Mach number M can be defined by

$$M^2 := \frac{|\boldsymbol{u}|^2}{c^2(\rho)} = u_i^2, i = 1, 2,$$

then the flow is referred to be in the *subsonic/sonic/supersonic* state if

$$|\boldsymbol{u}|^2 = u_1^2 + u_2^2 \lesseqqgtr 1 : \text{the local sound speed.}$$

Now from (1.7), (1.8) and (1.9), the system (1.3) is reduced to

$$\begin{cases} \left(1 - \dfrac{1}{n^2}\right) n_r = n\left(E + \dfrac{1}{r} + \dfrac{u_2^2}{r} - \dfrac{1}{\tau n}\right), & r_0 < r < r_1, \\ (u_2)_r = -\left(\dfrac{1}{r} + \dfrac{n}{\tau}\right) u_2, & r_0 < r < r_1, \\ (rE)_r = n - rb, & r_0 < r < r_1, \end{cases} \tag{1.10}$$

with the corresponding boundary conditions

$$n(r_0) = n_0, \tag{1.11}$$

$$n(r_1) = n_1, \tag{1.12}$$

$$u_2(r_0) = u_{20}, \tag{1.13}$$

for the constants $n_0 = \frac{\rho_0}{J_{10}} > 0$, $n_1 = \frac{r_1 \rho_1}{r_0 J_{10}} > 0$ and $u_{20} \neq 0$.

What's more, from the second equation of (1.10) and the condition (1.13), it's easy to get

$$u_2(r) = \frac{u_{20} r_0}{r} \exp\left\{ -\frac{1}{\tau} \int_{r_0}^{r} n(s) ds \right\}, \quad r \in [r_0, r_1]. \tag{1.14}$$

Defining a function $\hat{E}$ by

$$\hat{E}(r) := E(r) + \frac{1}{r} + \frac{u_2^2(r)}{r}, \quad r \in (r_0, r_1), \tag{1.15}$$

we can simplify the system (1.10)-(1.13) to the following problem

$$\begin{cases} \left(1 - \frac{1}{n^2}\right) n_r = n\hat{E} - \frac{1}{\tau}, & r \in (r_0, r_1), \\ (r\hat{E})_r = n - rb - K(r; n), \\ n(r_0) = n_0, \quad n(r_1) = n_1, \end{cases} \tag{1.16}$$

where the function $K(r; n)$ is denoted by

$$K(r; n) := 2u_{20}^2 r_0^2 \left(\frac{1}{r^3} + \frac{n}{\tau r^2} \right) \exp\left\{ -\frac{2}{\tau} \int_{r_0}^{r} n(s) ds \right\}. \tag{1.17}$$

Similarly as formula (7) in [19], the system (1.16) can be changed as a nonlinear elliptic system

$$\begin{cases} \left[r\left(\frac{1}{n} - \frac{1}{n^3} \right) n_r + \frac{r}{\tau n} \right]_r = n - rb - K(r; n), & r \in (r_0, r_1), \\ n(r_0) = n_0, \quad n(r_1) = n_1. \end{cases} \tag{1.18}$$

It is clear that equation (1.18) is uniformly elliptic when $n(r) > 1$ or $0 < n(r) < 1$ for $r \in [r_0, r_1]$. Meanwhile, equation (1.18) will be degenerate at the boundary points when $n_0 = 1$ or $n_1 = 1$. We realize that, given different boundary data n_0 and n_1, the system (1.18) may present entirely different properties. Therefore, our goal of the paper is to investigate the well-posedness of the system (1.18) when the constants n_0 and n_1 are in a variety of sizes, and make a fairly complete classification of solutions as far as possible.

To this end, we first assume a doping function $\mathfrak{b}(r) := rb(r) \in L^\infty(r_0, r_1)$, and

$$\underline{\mathfrak{b}} := \operatorname*{essinf}_{r \in [r_0, r_1]} rb \quad \text{and} \quad \overline{\mathfrak{b}} := \operatorname*{esssup}_{r \in [r_0, r_1]} rb.$$

Then some different types of the solutions to (1.16)/(1.18) are introduced as follows.

Definition 1.1. Suppose that a function n satisfies

$$\int_{r_0}^{r_1}\left[r\left(\frac{1}{n}-\frac{1}{n^3}\right)n_r+\frac{r}{\tau n}\right]\varphi_r dr+\int_{r_0}^{r_1}(n-\mathfrak{b}-K(r;n))\varphi dr=0,$$

equivalently,

$$\frac{1}{2}\int_{r_0}^{r_1}\left[\frac{r(n+1)}{n^3}\left((n-1)^2\right)_r+\frac{r}{\tau n}\right]\varphi_r dr+\int_{r_0}^{r_1}(n-\mathfrak{b}-K(r;n))\varphi dr=0, \tag{1.19}$$

for any test function $\varphi\in H_0^1(r_0,r_1)$, and $(n-1)^2\in H^1(r_0,r_1)$. Now we define that

(i) If $0<n(r)<1$ for $r\in(r_0,r_1)$ with $0<n_0\le 1$ and $0<n_1\le 1$ such that $u_1(r)\ge 1$ and $|\boldsymbol{u}|(r)>1$ over $[r_0,r_1]$ with $u_{20}\neq 0$, then
- we call $n(r)$ a radial supersonic/or supersonic-sonic and totally supersonic solution to (1.18).

(ii) If $n(r)>1$ for $r\in(r_0,r_1)$ with $n_0\ge 1$ and $n_1\ge 1$ (correspondingly, $0<u_1(r)\le 1$ over $[r_0,r_1]$), then the solution $n(r)$ is called to be
- if $|\boldsymbol{u}|(r)<1$ on $[r_0,r_1]$, a totally subsonic solution to (1.18);
- if $|\boldsymbol{u}|(r)>1$ on $[r_0,r_1]$, a radial subsonic/or subsonic-sonic but totally supersonic solution to (1.18);
- if $|\boldsymbol{u}|>1$ and $|\boldsymbol{u}|<1$ both exist on $[r_0,r_1]$, accordingly a radial subsonic/or subsonic-sonic but totally transonic solution to (1.18).

Particularly, when $n_0\neq 1$ and $n_1\neq 1$, it follows that $n\in H^1(r_0,r_1)$; when $n_0=n_1=1$, one gets $(n-1)^2\in H_0^1(r_0,r_1)$ for the weak solution n.

Definition 1.2. Let $(n,\hat{E})(r)$ be a pair of the solution to the system (1.16).

(iii) If $(n,\hat{E})\in C^1(r_0,r_1)\times W^{1,\infty}(r_0,r_1)$ with $0<n_0\le 1$ and $n_1\ge 1$, and there exists a point $z_0\in(r_0,r_1)$ such that

$$(n,\hat{E})(r)=\begin{cases}(n_{sup},\hat{E}_{sup})(r) & \text{for} \quad r\in[r_0,z_0],\\ (n_{sub},\hat{E}_{sub})(r) & \text{for} \quad r\in[z_0,r_1],\end{cases}$$

where $0<n_{sup}(r)<1$ on (r_0,z_0), $n_{sub}(r)>1$ on (z_0,r_1), and

$$n_{sup}(z_0)=n_{sub}(z_0)=1,\quad n'_{sup}(z_0)=n'_{sub}(z_0)\quad \text{and}\quad \hat{E}_{sup}(z_0)=\hat{E}_{sub}(z_0), \tag{1.20}$$

then $n(r)$ is called to be a C^1-smooth transonic solution in the radial direction, corresponding to
- a totally supersonic solution when $|\boldsymbol{u}|(r)>1$ on $[r_0,r_1]$;
- a totally transonic solution when $|\boldsymbol{u}|(r)<1$ on some intervals of $[r_0,r_1]$.

(iv) If $0 < n_0 \leq 1$ and $n_1 \geq 1$, and the function $n(r)$ is split by a point z_0 in the form of

$$n(r) = \begin{cases} n_{sup}(r) < 1 & \text{for} \quad r \in (r_0, z_0), \\ n_{sub}(r) > 1 & \text{for} \quad r \in (z_0, r_1), \end{cases}$$

satisfying the entropy condition at z_0,

$$0 < n_{sup}(z_0) < 1 < n_{sub}(z_0), \tag{1.21}$$

and the Rankine-Hugoniot condition,

$$n_{sup}(z_0^-) + \frac{1}{n_{sup}(z_0^-)} = n_{sub}(z_0^+) + \frac{1}{n_{sub}(z_0^+)}, \quad \hat{E}_{sup}(z_0^-) = \hat{E}_{sub}(z_0^+), \tag{1.22}$$

then $n(r)$ is called to be a transonic shock solution in the radial direction, corresponding to
- a totally supersonic shock solution for $|\boldsymbol{u}|(r) > 1$ on $[r_0, r_1]$;
- a totally transonic shock solution for $|\boldsymbol{u}|(r) < 1$ on a subset of $[r_0, r_1]$.

For the smooth solutions in Definition 1.1 and 1.2, once we solve the boundary value problem (1.18) for the solution n, in virtue of the first equation of (1.16), then the solution $(n, \hat{E})$ of the problem (1.16) can be derived where

$$\hat{E} = \left(\frac{1}{n} - \frac{1}{n^3}\right) n_r + \frac{1}{\tau n} = \frac{n+1}{2n^3}\left[(n-1)^2\right]_r + \frac{1}{\tau n},$$

which in combination with (1.14) and (1.15) gets a solution (n, u_2, E) of the system (1.10)-(1.13), and further coincides with a corresponding solution (ρ, u_1, u_2, E) to the system (1.3)-(1.6).

Now by Definition 1.2, we are able to study the shock solutions to the system (1.3) with different boundary values. Recall that, a piecewise smooth function $(u_1^\pm, u_2^\pm, \rho^\pm, E^\pm) \in [C^1(\mathcal{A}^\pm)]^3 \times W^{1,\infty}(\mathcal{A}^\pm)$ with a jump on a circle $r = z_0$ is a radially symmetric spiral shock solution to the system (1.3) in $\mathcal{A}$, if $(u_1^\pm, u_2^\pm, \rho^\pm, E^\pm)$ satisfy the system (1.3) in $\mathcal{A}^\pm$, respectively, and satisfy the entropy condition $[p] > 0$ and the Rankine-Hugoniot conditions

$$\begin{cases} [\rho u_1] = [\rho u_1^2 + p] = 0, \\ [\rho u_1 u_2] = [E] = 0, \end{cases}$$

where $[X] = X^+(z_0) - X^-(z_0)$, which exactly corresponds to the conditions (1.21) and (1.22).

From now on, according to Definition 1.1 and 1.2, we divide the well-posedness problems (1.16)/(1.18) into the following four problems. That is,

(i) find the smooth solutions of (1.18) satisfying $0 < n \leq 1$, if $0 < n_0 \leq 1$, $0 < n_1 \leq 1$;
(ii) find the smooth solutions of (1.18) satisfying $n \geq 1$, if $n_0 \geq 1$, $n_1 \geq 1$;
(iii) find radial transonic solutions without shock to (1.16), if $0 < n_0 \leq 1$, $n_1 \geq 1$;
(iv) find radial transonic solutions with shock to (1.16), if $0 < n_0 \leq 1$, $n_1 \geq 1$.

For the above four problems, in the case of zero angular velocity, it has been proved in [7,8] that there exists at least one interior supersonic solutions $0 < n(r) \leq 1$, a unique interior subsonic solution $n(r) \geq 1$ over $[r_0, r_1]$, infinitely many transonic shock solutions for $\tau \gg 1$, and infinitely many C^1-smooth transonic solutions for $\tau \ll 1$ when the boundary is radially sonic and the doping profile is greater than the sonic curve, i.e., $\underline{b} > 1$.

In this paper, the presence of the nonzero angular velocity brings us more new phenomena on the structure of solutions. For this case, we observe the following features of problems (i)-(iv): (1) when the boundary is radially sonic, the difficulty caused by the degeneracy of the equation is still a key point we need to solve such like that of [7,8], which will affect the well-posedness and regularity of solutions. Precisely, for the degenerate equation, the standard methods applied for the uniformly elliptic equations in [10,28] can't be effective and the solutions only belong to $C^{\frac{1}{2}}$ Hölder continuous at the degenerate points; (2) the system (1.16) is non-autonomous, however, different from a truly one-dimensional autonomous system, we can not work on the overall phase plane by the phase plane analysis, thus the local analysis method will be the only way to solve the problems; (3) with the joining of the nonzero angular velocity, it could be a barrier to obtain the well-posedness of the solutions and draw a clear classification of the flow patterns.

In order to overcome these difficulties, we still adopt the approaches combined by the technical compactness method, the phase plane analysis and the local singularity analysis in the critical position (see [7,8]), to deal with a non-autonomous system with the degenerate boundary. More importantly, our purpose here is to explain the effect of the nonzero angular velocity in proving the well-posedness and classifying the flow patterns of all solutions. Based on the rigorous proof, we have two findings for the role of the angular velocity played in this paper. The first finding is that the nonzero angular velocity will bring a nonlocal term in the equations, which mainly prevents the proof of the uniqueness of the radial subsonic/subsonic-sonic flow. So we are obliged to confirm the uniqueness results for the radial subsonic flow without the semiconductor effect or with a small current condition. The second finding is that the system possesses more new types of the flows, and particularly there exist a smooth transonic flow of being subsonic in the radial direction and a supersonic-supersonic shock flow of being transonic in the radial direction when the relaxation time is large enough, that is quite different from the case of zero angular velocity. The representations are given in Theorem 2.1, Theorem 3.1, Theorem 4.1 and 4.3.

The paper is organized as follows. In section 2, we prove the existence of the supersonic solutions of problem (i) with the flow being supersonic in the radial direction. In section 3, we show the existence of the steady-states with $|u_{20}| \leq C\sqrt{\tau}$ and the uniqueness of solution with $\tau = +\infty$ or a small current constant $J_{10} = r_0\rho_0$ to problem (ii) when the radial velocity is subsonic/or subsonic-sonic, and classify the flow patterns by the size of $|u_{20}|$. In section 4, our aim is to investigate the steady flows with being transonic in the radial direction. We show that there exist infinitely many smooth radial transonic solutions of problem (iii) if $\tau \ll 1$, and there exist infinitely many transonic shock solutions if $\tau \gg 1$ and $|u_{20}| \ll 1$, and supersonic shock solutions if $\tau \gg 1$ and $|u_{20}| \gg 1$ for problem (iv).

Remark 1.3. Note that all there conclusions in this paper are obtained in the isothermal case, which can be similarly extended to the isentropic case. Besides, all radial spiral flows we study above start from the inner circle to the outer one. For the fluid flows from the outer circle to the inner direction, we believe there exist still some corresponding results.

2. Supersonic/supersonic-sonic flows in the radial direction

In this section, we consider the steady-state solutions to the system (1.18) where the radial velocity is supersonic/or supersonic-sonic with the boundary condition $0 < n_0 \le 1$ and $0 < n_1 \le 1$. For the elliptic system

$$\begin{cases} \left[r\left(\frac{1}{n} - \frac{1}{n^3}\right) n_r + \frac{r}{\tau n} \right]_r = n - \mathfrak{b} - K(r; n), \quad r \in (r_0, r_1), \\ n(r_0) = n_0, \quad n(r_1) = n_1, \end{cases} \tag{2.1}$$

according to Definition 1.1, we summarize the following problem and results.

Problem I. Find the smooth function $0 < n \le 1$ in Type (i) of Definition 1.1, which solves the system (2.1) with the boundary conditions:

$$0 < n_0 \le 1 \quad \text{and} \quad 0 < n_1 \le 1,$$

when the doping function is in the subsonic state, i.e., $\underline{\mathfrak{b}} > 1$.

Theorem 2.1 *(the steady-states of Type (i)). Let the doping profile satisfy $\mathfrak{b} \in L^\infty(r_0, r_1)$ and $\underline{\mathfrak{b}} > 1$. When $0 < n_0 \le 1$ and $0 < n_1 \le 1$, there exists at least one smooth steady-state solution $n \in C^{\frac{1}{2}}[r_0, r_1]$ for the degenerate case, or $n \in C^{1+\alpha}[r_0, r_1]$ with $0 < \alpha < \frac{1}{2}$ for the non-degenerate case, to the system (2.1), satisfying*

$$0 < \underline{n} \le n(r) < 1 \quad \text{on} \quad (r_0, r_1),$$

where $\underline{n} = \underline{n}(n_0, n_1, \bar{\mathfrak{b}}, \tau, r_0, r_1, u_{20}^2)$ is a positive constant. Then in view of the fact that

$$u_1(r) = \frac{1}{n(r)} \ge 1 \quad \text{and} \quad |\boldsymbol{u}|(r) > 1 \quad \text{on} \quad [r_0, r_1],$$

the solution $n(r)$ corresponds to the totally supersonic flow of the equations (1.1).

Remark 2.2. In Theorem 2.1, the condition $\underline{\mathfrak{b}} > 1$ can be replaced by a weaker condition, i.e., $\underline{\mathfrak{b}} + \frac{1}{\tau} > 1$. Indeed, with a case of a strong supersonic background, just like $0 < n_0, n_1 \ll 1$, the above condition can be further weakened, inspired of [7,28]. Nevertheless, when the boundary data n_0, n_1 are less than the sonic value 1, there maybe exist no solution for the problem (see [20] for instance), which confirms the necessity of the condition $\underline{\mathfrak{b}} > 1$.

Remark 2.3. In Theorem 2.1, for the non-degenerate case $0 < n_0, n_1 < 1$, we further obtain a solution $(\rho, u_1, u_2, E) \in \left[C^{1+\alpha}[r_0, r_1]\right]^2 \times C^{2+\alpha}[r_0, r_1] \times W^{1,\infty}(r_0, r_1)$ to the system (1.3). However, the fact is that $(\rho, u_1, u_2, E) \in \left[C^{\frac{1}{2}}[r_0, r_1]\right]^2 \times C^{1+\frac{1}{2}}[r_0, r_1] \times W^{1,\infty}(r_0, r_1)$ for the degenerate case of $n_0 = 1$ or $n_1 = 1$.

Proof of Theorem 2.1. For this situation, we are interested to get a radial velocity $u_1(r) = 1/n(r) \ge 1$ over $[r_0, r_1]$. Thus it implies by (1.14) that the total velocity $|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2} > 1$, physically corresponding to the totally supersonic flow.

Note that (1.18) is a degenerate system whenever $n_0 = 1$ or $n_1 = 1$. To deal with the case of the degenerate boundary, we define an approximate system of (1.18) as follows (see [7]):

$$\begin{cases} \left[r\left(\frac{1}{n_k} - \frac{k^2}{(n_k)^3}\right)(n_k)_r + \frac{rk}{\tau n_k} \right]_r = n_k - \mathfrak{b} - K(r; n_k), \quad r \in (r_0, r_1), \\ n_k(r_0) = n_0, n_k(r_1) = n_1, \end{cases} \tag{2.2}$$

with a constant $k > 1$. Let the function $v_k = k/n_k$, then system (2.2) is changed as

$$\begin{cases} \left[r\left(v_k - \frac{1}{(v_k)^3}\right)(v_k)_r + \frac{rv_k}{\tau} \right]_r = \frac{k}{v_k} - \mathfrak{b} - K(r; k/v_k), \quad r \in (r_0, r_1), \\ v_k(r_0) = k/n_0 \geq k, v_k(r_1) = k/n_1 \geq k. \end{cases} \tag{2.3}$$

Clearly, (2.3) is a system of non-degenerate and nonlinear elliptic equation. So our task is to find a weak solution $v_k \in H^1(r_0, r_1)$ to (2.3) such that $k \leq v_k < +\infty$. Recalling [7, Theorem 3.2, Step 1], we first establish a mapping $\mathcal{F} : \eta \to v$ by solving the following quasi-linear elliptic equation

$$\begin{cases} \left[\frac{r(\eta+1)}{\eta}(v-1)v_r \right]_r + \frac{rv_r}{\tau} = \frac{k}{\eta} - \mathfrak{b} - \frac{\eta}{\tau} - K(r; k/\eta), \quad r \in (r_0, r_1), \\ v(r_0) = k/n_0 > 1, v(r_1) = k/n_1 > 1, \end{cases} \tag{2.4}$$

with $\eta \in \mathcal{D}$. Here the solution space $\mathcal{D} \in C^1[r_0, r_1]$ is denoted by

$$\begin{aligned} \mathcal{D} := \{\omega \in C^1[r_0, r_1] \big| k \leq \omega \leq \mathcal{M}, \omega(r_0) = k/n_0, \omega(r_1) = k/n_1, \\ \|\omega\|_{C^\alpha[r_0,r_1]} \leq \Lambda, \|\omega\|_{C^1[r_0,r_1]} \leq \Upsilon(\Lambda)\} \end{aligned}$$

for some undetermined constants $\mathcal{M}$, Λ and $\Upsilon(\Lambda)$. Next our intention is to look for a unique fixed point of the operator $\mathcal{F}$ in $\mathcal{D}$ by the Schauder fixed point theorem [16], that is exactly a solution of (2.3).

In order to obtain the fixed point, the first step is to assert that (2.4) has a unique solution $v \in C^{1+\alpha}[r_0, r_1]$ for $0 < \alpha < 1$. To prove this, an operator $\mathcal{F}_0 : \zeta \to \xi$ is defined for any fixed $\eta \in \mathcal{D}$ by solving the linear equations

$$\begin{cases} \left[\frac{r(\eta+1)}{\eta}(\zeta-1)\xi_r \right]_r + \frac{r\xi_r}{\tau} + G(r; \eta) = 0, \quad r \in (r_0, r_1), \\ \xi(r_0) = k/n_0 > 1, \xi(r_1) = k/n_1 > 1, \end{cases} \tag{2.5}$$

where

$$\zeta \in \mathcal{D}_0 := \left\{ \nu \in C^0[r_0, r_1] \big| k \leq \nu \leq \mathcal{K}, \nu(r_0) = k/n_0, \nu(r_1) = k/n_1 \right\}$$

for a constant $\mathcal{K}$ to be determined later and

$$G(r; \eta) := K(r; k/\eta) + \mathfrak{b} + \frac{\eta}{\tau} - \frac{k}{\eta}.$$

We prove that (2.5) has a solution $\xi \in H^1(r_0, r_1)$, and by the compact imbedding $H^1(r_0, r_1) \hookrightarrow C^0[r_0, r_1]$, the image set $\mathcal{F}_0(\mathcal{D}_0)$ is precompact. Also, it is seen that the operator $\mathcal{F}_0$ is continuous. Next we will prove $\mathcal{F}_0(\mathcal{D}_0) \subset \mathcal{D}_0$. As in [7, Theorem 3.2, Step 1], it is directly shown by $\mathfrak{b} + 1/\tau > 1$ and $K(r; k/\eta) > 0$ that

$$\xi(r) \geq k \quad \text{for} \quad r \in [r_0, r_1],$$

and

$$\|(\xi - k)_r\|_{L^2(r_0,r_1)} \leq \frac{r_1 - r_0}{r_0(k-1)} \|G(r;\eta)\|_{L^2(r_0,r_1)},$$

so it follows from Poincaré's inequality that

$$\xi \leq k + C(r_0, r_1, k)\|G(r;\eta)\|_{L^2(r_0,r_1)} \leq C(r_0, r_1, k, \tau, \overline{\mathfrak{b}}, u_{20}^2, \mathcal{M}) =: \mathcal{K}(\mathcal{M}),$$

which implies the result that $\xi \in \mathcal{D}_0$. Therefore, applying the Schauder fixed point theorem, one easily gets a fixed point $v \in \mathcal{D}_0$ for the operator $\mathcal{F}_0$ such that $\mathcal{F}_0(v) = v$, that is the solution of (2.4). Further, with the help of the regularity theory [16] and Sobolev imbedding theory, it holds that $v \in C^{1+\alpha}[r_0, r_1]$ with $0 < \alpha < \frac{1}{2}$ and $k \leq v \leq \mathcal{K}(\mathcal{M})$ satisfying

$$\|v\|_{C^\alpha[r_0,r_1]} \leq C_1(r_0, r_1, k, \tau, \overline{\mathfrak{b}}, u_{20}^2, \mathcal{M}),$$
$$\|v\|_{C^{1+\alpha}[r_0,r_1]} \leq C_2(C_1, r_0, r_1, k, \tau, \overline{\mathfrak{b}}, u_{20}^2, \mathcal{M}),$$

for constants C_1 and C_2. At the same time, the uniqueness of the solution v is directly obtained by using [7, Lemma 3.1]. Thus, the above assertion is proved.

Now the second step is to verify the condition $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$. As similar as [7, Theorem 3.2, Step 1], it is checked that

$$k \leq v \leq k + C(r_0, r_1)\sqrt{K_{\max} + \overline{\mathfrak{b}} + \frac{\mathcal{M}}{\tau}}, \tag{2.6}$$

where

$$K(r; k/\eta) \leq 2u_{20}^2 \left(\frac{1}{r_0} + \frac{1}{\tau}\right) =: K_{\max}(u_{20}^2, r_0, \tau). \tag{2.7}$$

One can see from (2.6) that $v(r) \leq \mathcal{M}$ over $[r_0, r_1]$ when $\mathcal{M}$ is large enough by taking

$$\mathcal{M} \geq k + \frac{C^2(r_0, r_1)}{2\tau} + \frac{1}{2}\sqrt{\left(2k + \frac{C^2(r_0, r_1)}{\tau}\right)^2 + 4C^2(r_0, r_1)(K_{\max} + \overline{\mathfrak{b}}) - 4k^2}.$$

Finally, we choose $\Lambda = C_1(r_0, r_1, k, \tau, \overline{\mathfrak{b}}, u_{20}^2)$ and $\Upsilon(\Lambda) = C_2(C_1, r_0, r_1, k, \tau, \overline{\mathfrak{b}}, u_{20}^2)$ such that $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$ is qualified. Meanwhile, we know that the continuous operator $\mathcal{F}$ is compact defined in the closed convex set $\mathcal{D} \subset C^1[r_0, r_1]$ due to the compact imbedding $C^{1+\alpha} \hookrightarrow C^1[r_0, r_1]$. Therefore, a unique fixed point v_k is a solution of (2.3) for any fixed k, which is obtained by the Schauder fixed point theorem. What's more, we find that $n_k = \frac{k}{v_k}$ is the solution of (2.2).

As in [7, Theorem 3.2, Step 2-3], we give a uniformly bounded estimate of the function v_k for any $k > 1$. Hence let $k \to 1^+$, then the limit of the sequence solution $\{n_k\}_{k>1}$ is a solution to the system (2.1) satisfying formula (1.19) and

$$0 < \underline{n} \leq n(r) < 1 \quad \text{for} \quad r \in (r_0, r_1),$$

where a constant $\underline{n}$ is a constant only depending on $(r_0, r_1, \tau, \overline{\mathfrak{b}}, u_{20}^2)$ with the degenerate case. But for the non-degenerate case of $0 < n_0, n_1 < 1$, it implies by [28] that the lower bound $\underline{n}$ is also dependent on n_0 and n_1. Afterwards we discuss the regularity of the solution n. Especially for the degenerate case of $n_0 = 1$ or $n_1 = 1$, the optimal regularity we only get is $n \in C^{\frac{1}{2}}[r_0, r_1] \cup C^1(r_0, r_1)$. But if $0 < n_0, n_1 < 1$, the regularity of n will be raised to $n \in C^{1+\alpha}[r_0, r_1]$ for any $0 < \alpha < \frac{1}{2}$ by the regularity theory and the Sobolev imbedding theorem. We can see [7,19] for details.

Thus we complete the whole proof. □

3. Subsonic/subsonic-sonic flows in the radial direction

Our purpose of this section is to investigate the essential effect of the nonzero angular velocity on well-posedness and the flow pattern of the system (1.18) when the radial velocity is subsonic/or subsonic-sonic. Main problem and results are stated below.

Problem II. Find the smooth function $n \geq 1$ in Type (ii) of Definition 1.1, which solves the system (2.1) with the boundary conditions:

$$n_0 \geq 1 \quad \text{and} \quad n_1 \geq 1,$$

when $\underline{\mathfrak{b}} > 1$.

Theorem 3.1 *(the steady-states of Type (ii)).*

P1. the existence result: *Assume that $\mathfrak{b} \in L^\infty(r_0, r_1)$ and $\underline{\mathfrak{b}} > 1$ hold. For $n_0 \geq 1$ and $n_1 \geq 1$, there exists a constant $\sigma_0 = \sigma_0(\sqrt{\tau})$ such that if*

$$0 < |u_{20}| \leq \sigma_0,$$

then the system (2.1) admits a radial subsonic-sonic smooth solution $n \in C^{\frac{1}{2}}[r_0, r_1]$ for the degenerate case, or a radial subsonic smooth solution $n \in C^{1+\alpha}[r_0, r_1]$ with $0 < \alpha < \frac{1}{2}$ for the non-degenerate case. In addition, we have

$$1 + \lambda \sin\left(\pi \cdot \frac{r - r_0}{r_1 - r_0}\right) \leq n(r) \leq C \quad \textit{for} \quad r \in [r_0, r_1], \tag{3.1}$$

where $\lambda = \lambda(r_0, \tau, \underline{\mathfrak{b}}, n_0, n_1)$ and $C = C(r_0, \tau, \overline{\mathfrak{b}}, |u_{20}|, n_0, n_1)$ are the positive constants.

P2. the uniqueness result: (u1) *if $|u_{20}| > 0$ with $\tau = +\infty$, the above solution is unique;*

(u2) *if $0 < |u_{20}| \ll 1$ with $0 < \tau < +\infty$, further assume the constant $J_{10} = \rho_0 u_{10}$ is small enough, then there exists at most one smooth solution $n \in C^{1+\alpha}[r_0, r_1]$ to the system (2.1).*

P3. the flow patterns: *Assume that the relaxation time τ is large enough, i.e., $\tau \gg 1$.*

(1) *For the degenerate case of $n_0 = 1$ or $n_1 = 1$ or $n_0 = n_1 = 1$, it is proved that*

Subcase 1.1. *if $|u_{20}| \gg 1$, then $n(r)$ is called a totally supersonic solution with the fact that $|\boldsymbol{u}|(r) > 1$ over $[r_0, r_1]$;*

Subcase 1.2. *If $|u_{20}| \ll 1$, then $n(r)$ is called a totally transonic solution where the cases of $|\boldsymbol{u}|(r) < 1$ and $|\boldsymbol{u}|(r) > 1$ both exist for $r \in [r_0, r_1]$.*

In particular, there exists a positive constant $\sigma = \sigma_1$ such that if $|u_{20}| = \sigma_1$, then $n(r)$ is called a totally supersonic-sonic solution, corresponding to $|\boldsymbol{u}|(r) \geq 1$ over $[r_0, r_1]$.

(2) *For the non-degenerate case of $n_0 > 1$ and $n_1 > 1$, we have*

Subcase 2.1. *if $|u_{20}| \gg 1$, then $n(r)$ is a totally supersonic solution. That is, $|\boldsymbol{u}|(r) > 1$ on $[r_0, r_1]$;*

Subcase 2.2. *if $|u_{20}| \ll 1$, then $n(r)$ is a totally subsonic solution with the fact that $|\boldsymbol{u}|(r) < 1$ over $[r_0, r_1]$;*

Subcase 2.3. *if $|u_{20}|$ belongs to a domain $[\sigma_2, \sigma_3]$ with the constants $0 < \sigma_2 < \sigma_3$, then there exist both $|\boldsymbol{u}|(r) < 1$ and $|\boldsymbol{u}|(r) > 1$, so $n(r)$ is known to be a totally transonic solution on $[r_0, r_1]$.*

In addition, there must exist two constants σ_2' and σ_3' such that $0 < \sigma_2' \leq \sigma_2 < \sigma_3 \leq \sigma_3'$, then for $|u_{20}| = \sigma_2'$, $n(r)$ is a totally subsonic-sonic solution, i.e., $|\boldsymbol{u}|(r) \leq 1$ over $[r_0, r_1]$, and for $|u_{20}| = \sigma_3'$, $n(r)$ is a totally supersonic-sonic solution, i.e., $|\boldsymbol{u}|(r) \geq 1$ on $[r_0, r_1]$.

Assume that τ and $|u_{20}|$ is small enough, i.e., $\tau \ll 1$ and $|u_{20}| \ll 1$, there are still corresponding classifications of solutions as similar as the case of $\tau \gg 1$.

Remark 3.2. In P1 of Theorem 3.1, we emphasize the fact that the restriction $|u_{20}| \leq C\sqrt{\tau}$ is needed in the case of $0 < \tau < +\infty$. Obviously, the above condition is not necessary for the Euler-Poisson system without the effect of semiconductor, i.e., $\tau = +\infty$. Additionally, the Hölder index $\frac{1}{2}$ is optimal only in the degenerate case, which is also found in [7,19].

Remark 3.3. The uniqueness of the purely radial case, $0 < \tau < +\infty$ and $u_{20} = 0$, has been proved in [7]. However, in P2 of Theorem 3.1, the uniqueness results are obtained only in the two special case when $u_{20} \neq 0$. In fact, we have no way to prove the uniqueness of solutions in the general case since the comparison principle applied in [7,19] can't work owing to the nonlocal effect caused by the nonzero angular velocity.

Remark 3.4. In P3 of Theorem 3.1, the solution $n \geq 1$ only represents a subsonic-sonic flow in the radial direction, and physically the final categories of the solutions of system (2.1) are determined by the size of $|u_{20}|$. Therefore, it is essentially different from the purely radial result [7, Theorem 2.1].

Proof of Theorem 3.1. Part 1. The proof of the existence result. The part of the proof is split into three steps.

Step 1. We first deal with the degenerate case with $n_0 = 1$ or $n_1 = 1$ or $n_0 = n_1 = 1$, and focus on the solution $n(r) \geq 1$, i.e., $u_1(r) \leq 1$ to (2.1) over $[r_0, r_1]$. Without loss of generality, let's directly consider the case: $n_0 = n_1 = 1$. To prove this, one constructs an approximate system of (2.1) as follows:

$$\begin{cases} \left[r \left(\frac{1}{n_j} - \frac{j^2}{(n_j)^3} \right) (n_j)_r + \frac{r}{\tau n_j} \right]_r = n_j - \mathfrak{b} - K(r; n_j), \quad r \in (r_0, r_1), \\ n_j(r_0) = n_j(r_1) = 1, \end{cases} \tag{3.2}$$

with a constant $0 < j < 1$. Here the function K is given by (1.17).

Step 2. the existence of the approximate solution $n_j \in C^1[r_0, r_1]$ *to* (3.2). Concretely, we define a mapping $\mathcal{T} : \mathcal{S} \to C^0[r_0, r_1]$, $\mathcal{T}(m) = \bar{n}_j$ by solving the linearized system of (3.2)

$$\begin{cases} \left[r \left(\frac{1}{m} - \frac{j^2}{m^3} \right) (\bar{n}_j)_r \right]_r - \frac{r}{\tau m^2} (\bar{n}_j)_r = \bar{n}_j - \mathfrak{b} - \frac{1}{\tau m} - K(r; m), \quad r \in (r_0, r_1), \\ \bar{n}_j(r_0) = \bar{n}_j(r_1) = 1, \end{cases} \tag{3.3}$$

where we have denoted a closed subset of $C^0[r_0, r_1]$ by

$$\mathcal{S} := \{\omega \in C^0[r_0, r_1] | 1 \le \omega \le \mathcal{C}, \omega(r_0) = \omega(r_1) = 1\} \quad \text{for an undermined constant} \quad \mathcal{C},$$

and $m \in \mathcal{S}$. Therefore we have a solution $\bar{n}_j \in H^1(r_0, r_1)$ to (3.3) by the elliptic theory, and it is easy to see that $\mathcal{T}$ is continuous and compact by the continuity argument and the Sobolev imbedding theorem respectively. Now it suffices to prove the condition $\mathcal{T}(\mathcal{S}) \subset \mathcal{S}$ in applying the Schauder fixed point theorem. Similarly to the proof of [7, Lemma 2.3], and by the weak maximum principle [16, Theorem 8.1], one has

$$1 \le \bar{n}_j \le \mathcal{C},$$

provided that

$$1 \le \mathfrak{b} + \frac{1}{\tau m} + K(r; m) \le \mathcal{C} \quad \text{for any } m \in \mathcal{S}. \tag{3.4}$$

To get (3.4), noting that $\underline{\mathfrak{b}} > 1$ and

$$\mathfrak{b} + \frac{1}{\tau m} + K(r; m) \le \overline{\mathfrak{b}} + \frac{1}{\tau} + 2u_{20}^2 \left(\frac{1}{r_0} + \frac{m}{\tau} \right) \exp\left\{ -\frac{2}{\tau} \int_{r_0}^{r} m(s) ds \right\},$$

we choose

$$|u_{20}| \le \frac{\sqrt{\tau}}{2}$$

and

$$\mathcal{C} = 2 \left(\overline{\mathfrak{b}} + \frac{1}{\tau} + \frac{2u_{20}^2}{r_0} \right), \tag{3.5}$$

such that

$$\overline{\mathfrak{b}} + \frac{1}{\tau} + 2u_{20}^2\left(\frac{1}{r_0} + \frac{m}{\tau}\right) \le \mathcal{C}.$$

So the condition (3.4) holds. Therefore the system (3.2) admits a solution $n_j \in C^0[r_0, r_1]$ for any $0 < j < 1$ by using the Schauder fixed point theorem. Also, $n_j \in C^1[r_0.r_1]$ directly follows from the regularity theory and the Sobolev imbedding theorem.

Step 3. the existence and regularity of a weak solution $n(r) \ge 1$ of the system (2.1). For any $0 < j < 1$, the uniformly bounded estimate of $(n_j - 1)^2$ in $H_0^1(r_0, r_1)$ is derived by the same methods in [7, Theorem 2.1]. Let $j \to 1^-$, then the system (2.1) has a limit solution $n(r) \ge 1$ on $[r_0, r_1]$ satisfying (1.19) and $(n-1)^2 \in H_0^1(r_0, r_1)$.

From the proof of [7, Theorem 2.1] and [19, Theorem 2.1], we can see that the optimal regularity is $n \in C^{\frac{1}{2}}[r_0, r_1]$ for the degenerate case: $n_0 = n_1 = 1$, so does the case of $n_0 = 1$ or $n_1 = 1$. In addition, we define the solution of (3.3) by m_j when $K(r; m) \equiv 0$, and it follows from [7, Lemma 2.3] that

$$m_j(r) \ge 1 + \lambda \sin\left(\pi \cdot \frac{r - r_0}{r_1 - r_0}\right), \quad r \in [r_0, r_1],$$

where $\lambda = \lambda(r_0, \tau, \underline{\mathfrak{b}})$ is a small constant independent of j. Imitating the idea of the comparison principle in [7, Lemma 2.2], and due to $K > 0$, it is easy to obtain that

$$n_j(r) \ge m_j(r) \quad \text{for} \quad r \in [r_0, r_1].$$

Furthermore, we consider the non-degenerate boundary case: $n_0, n_1 > 1$. Actually, as in [10, Theorem 1], there exists a solution $n \in H^2(r_0, r_1) \hookrightarrow C^{1+\alpha}[r_0, r_1]$ with $0 < \alpha < \frac{1}{2}$, and

$$1 < \hat{C}_1(\min\{n_0, n_1\}, \underline{\mathfrak{b}}) \le n(r) \le \hat{C}_2(\max\{n_0, n_1\}, \overline{\mathfrak{b}}, |u_{20}|), \tag{3.6}$$

for the bounded constants $\hat{C}_1$ and $\hat{C}_2$. Therefore, the inequality (3.1) is true.

Part 2. the proof of the uniqueness result.

Case 1. $|u_{20}| > 0$ with $\tau = +\infty$ for the degenerate case. When $\tau = +\infty$, it is checked that $K(r; n) = \frac{2u_{20}^2 r_0^2}{r^3}$. In this case, we also have

$$n_j(r) \ge 1 + \lambda \sin\left(\pi \cdot \frac{r - r_0}{r_1 - r_0}\right), \quad r \in [r_0, r_1],$$

where $\lambda = \lambda(r_0, \underline{\mathfrak{b}})$ is a small constant independent of j. Afterwards the uniqueness of the limit solution n in Part 1 is obtained by coping the idea of the proof of [7, Theorem 2.1].

Case 2. $0 < |u_{20}| \ll 1$ with $0 < \tau < +\infty$ and $0 < J_{10} \ll 1$ for the non-degenerate case. In this case, we have the equations for the solution $\hat{n} := \frac{r\rho}{r_0} = \frac{J_{10}}{u_1} = J_{10} n$ as follows:

$$\begin{cases} \left[r\left(\frac{1}{\hat{n}} - \frac{J_{10}^2}{\hat{n}^3}\right)\hat{n}_r + \frac{rJ_{10}}{\tau \hat{n}}\right]_r = r_0\hat{n} - \mathfrak{b} - K(r; \hat{n}/J_{10}), \quad r \in (r_0, r_1), \\ \hat{n}(r_0) = \rho_0 > 1, \quad \hat{n}(r_1) = \frac{r_1\rho_1}{r_0} > 1. \end{cases} \tag{3.7}$$

Here the solution $\hat{n} \in C^1[r_0, r_1]$ has been derived by Part 1 satisfying $\hat{n} > 1$. Now referring to [10, Theorem 2], it suffices to show the uniqueness of the solution $\hat{n}$. Let $\hat{n}_1$ and $\hat{n}_2$ be two solutions of the system (3.7), then subtracting two equations one gets

$$[r(a_1(r)e)_r]_r - J_0(ra_2(r)e)_r = r_0 e - a_3(r) \tag{3.8}$$

where $e(r) = \hat{n}_2 - \hat{n}_1$ and

$$a_1(r) = \int_0^1 \mathcal{F}'(\hat{n}_1 + \eta(\hat{n}_2 - \hat{n}_1))d\eta > 0, \quad \mathcal{F}'(\hat{n}) = \frac{1}{\hat{n}} - \frac{J_{10}^2}{\hat{n}^3},$$

$$a_2(r) = \frac{1}{\tau \hat{n}_1 \hat{n}_2} > 0, \quad a_3(r) = K(\hat{n}_2) - K(\hat{n}_1).$$

Multiplying (3.8) by $a_1 e$ and integrating it over $[r_0, r_1]$ we have

$$\int_{r_0}^{r_1} r|(a_1 e)_r|^2 dr - J_{10} \int_{r_0}^{r_1} ra_2 e(a_1 e)_r dr + \int_{r_0}^{r_1} r_0 a_1 e^2 - a_1 a_3 e dr = 0. \tag{3.9}$$

We estimate, using Young's inequality:

$$\left| \int_{r_0}^{r_1} ra_2 e(a_1 e)_r dr \right| \le \frac{1}{2} \int_{r_0}^{r_1} r\sqrt{a_1} a_2 e^2 dr + \frac{1}{2} \int_{r_0}^{r_1} r \frac{a_2}{\sqrt{a_1}} |(a_1 e)_r|^2 dr, \tag{3.10}$$

and calculate a_3:

$$a_3(r) = u_{20}^2 [g_1(r; \hat{n}_2) e(r) + g_2(r; \hat{n}_1, \hat{n}_2)], \quad r \in [r_0, r_1],$$

where

$$g_1(r; \hat{n}_2) = \frac{2r_0^2}{J_{10} \tau r^2} \exp\left\{ -\frac{2}{J_{10}\tau} \int_{r_0}^{r} \hat{n}_2(s) ds \right\} \le \frac{2}{J_{10}\tau}$$

and

$$g_2(r; \hat{n}_1, \hat{n}_2) = 2r_0^2 \left(\frac{1}{r^3} + \frac{\hat{n}_1}{J_{10}\tau r^2} \right) \left(\exp\left\{ -\frac{2}{J_{10}\tau} \int_{r_0}^{r} \hat{n}_2(s) ds \right\} - \exp\left\{ -\frac{2}{J_{10}\tau} \int_{r_0}^{r} \hat{n}_1(s) ds \right\} \right)$$

$$\le \frac{C(r_0, \tau, \overline{\mathfrak{b}})}{J_{10}^2} \left| \int_{r_0}^{r} (\hat{n}_2 - \hat{n}_1)(s) ds \right| \le \frac{C(r_0, r_1, \tau, \overline{\mathfrak{b}})}{J_{10}^2} \|\hat{n}_2 - \hat{n}_1\|_{L^2(r_0, r_1)},$$

such that

$$
\begin{aligned}
\int_{r_0}^{r_1} a_1 a_3 e dr &= \int_{r_0}^{r_1} a_1 e(g_1 e + g_2) dr \\
&\le \frac{2u_{20}^2}{J_{10}\tau}\int_{r_0}^{r_1} a_1 e^2 dr + \frac{C(r_0,r_1,\tau,\overline{\mathfrak{b}})u_{20}^2}{J_{10}^2}\|\hat{n}_2-\hat{n}_1\|_{L^2(r_0,r_1)}\int_{r_0}^{r_1} a_1 e dr \\
&\le \frac{2u_{20}^2}{J_{10}\tau}\int_{r_0}^{r_1} a_1 e^2 dr + \frac{C(r_0,r_1,\tau,\overline{\mathfrak{b}})u_{20}^2}{J_{10}^2}\|a_1\|_{L^2(r_0,r_1)}\int_{r_0}^{r_1} e^2 dr,
\end{aligned}
\tag{3.11}
$$

where we have used Hölder's inequality. Therefore it follows from (3.9), (3.10) and (3.11) that

$$
\begin{aligned}
&\int_{r_0}^{r_1} r\left(1-\frac{J_{10}a_2}{2\sqrt{a_1}}\right)|(a_1 e)_r|^2 dr \\
&+\int_{r_0}^{r_1}\left[r_0 a_1 - \frac{J_{10}}{2} r\sqrt{a_1}a_2 - \frac{2u_{20}^2}{J_{10}\tau}a_1 - \frac{C\cdot u_{20}^2}{J_{10}^2}\|a_1\|_{L^2(r_0,r_1)}\right] e^2 dr \le 0.
\end{aligned}
\tag{3.12}
$$

Accordingly, there exists a small constant $\hat{J}_{10}=\hat{J}_{10}(r_0,r_1,\tau,\overline{\mathfrak{b}})$ such that if $0<J_{10}<\hat{J}_{10}$ and $0<|u_{20}|<\hat{U}_2$ for a small constant $\hat{U}_2=\hat{U}_2(r_0,r_1,\tau,\overline{\mathfrak{b}},J_{10})$, then both terms on the left hand side of (3.12) are positive, which get a contradiction to the assumption that $\hat{n}_1\neq\hat{n}_2$. The proof of uniqueness result is complete.

Part 3. the classification of flow patterns. We note from (1.14) that

$$
|\boldsymbol{u}|^2(r) = u_1^2+u_2^2 = \frac{1}{n^2} + u_{20}^2\frac{r_0^2}{r^2}\exp\left\{-\frac{2}{\tau}\int_{r_0}^{r} n(s)ds\right\},
$$

and the function u_2^2 is strictly decreasing in r. Here let's define

$$
u_M := \max_{r\in[r_0,r_1]} |\boldsymbol{u}|^2(r) \quad \text{and} \quad u_m := \min_{r\in[r_0,r_1]} |\boldsymbol{u}|^2(r).
$$

For the degenerate case, it is seen that

$$
u_M \ge \max\left\{|\boldsymbol{u}|^2(r_0), |\boldsymbol{u}|^2(r_1)\right\} > 1,
$$

and one has

$$
\begin{aligned}
u_m &= \min\left\{|\boldsymbol{u}|^2(r_0), |\boldsymbol{u}|^2(r_1), |\boldsymbol{u}|^2(r_M)\right\} \\
&\ge \frac{1}{n(r_M)^2} + u_{20}^2\frac{r_0^2}{r_1^2}\exp\left\{-\frac{2(r_1-r_0)n(r_M)}{\tau}\right\}
\end{aligned}
$$

where r_M is the maximum point of the function $n(r)$ in (r_0, r_1). By (3.5), we get $1 < n(r_M) \leq 4(\overline{\mathfrak{b}} + \frac{2u_{20}^2}{r_0})$ if $\tau \geq \frac{r_0}{r_0\overline{\mathfrak{b}}+2u_{20}^2}$. Thus, it's checked that

$$u_m \geq u_{20}^2 \frac{r_0^2}{r_1^2} \exp\left\{ -\frac{8(r_1 - r_0)(\overline{\mathfrak{b}} + \frac{2u_{20}^2}{r_0})}{\tau} \right\}.$$

Now if $\tau \gg 1$ such that

$$u_{20}^2 \leq \frac{\ln 2 \cdot r_0 \tau}{8(r_1 - r_0)} - r_0 \overline{\mathfrak{b}} \leq \frac{\ln 2 \cdot r_0 \tau}{16(r_1 - r_0)}, \tag{3.13}$$

we have $8(r_1 - r_0)(\overline{\mathfrak{b}} + \frac{2u_{20}^2}{r_0}) \leq \ln 2 \cdot \tau$. Thus,

$$u_m \geq \frac{r_0^2 u_{20}^2}{2r_1^2}.$$

So it follows from (3.13) that $u_m > 1$ for sufficiently large τ and $\frac{r_1}{r_0\sqrt{2}} < |u_{20}| \leq \frac{\ln 2}{4}\sqrt{\frac{r_0 \tau}{(r_1 - r_0)}}$, which means that the flow is totally supersonic. As we know by [7], the flow is subsonic-sonic when $|u_{20}| = 0$. When $0 < |u_{20}| \ll 1$, it is easy to see from (3.1) that $u_M > 1 > u_m$. This is a totally transonic flow. Of course, due to the continuous dependence on the parameter $|u_{20}| > 0$ for the solution $n(r)$, there must exist a constant $\sigma_1 \in [0, \frac{r_1}{r_0\sqrt{2}}]$ such that $|u_{20}| = \sigma_1$, satisfying $u_M > u_m = 1$, which corresponds to a totally supersonic-sonic flow.

For the non-degenerate case with $n_0, n_1 > 1$, we show that the flow is also totally supersonic if $\tau \gg 1$ and $|u_{20}| \gg 1$ similarly to the foregoing case. In fact, it is checked from (3.6) that

$$n(r_m) := \min_{r \in [r_0, r_1]} n(r) \geq \hat{C}_1(\min\{n_0, n_1\}, \underline{\mathfrak{b}}) > 1.$$

Hence if $\tau \gg 1$ and $|u_{20}| \ll 1$, it is easy to see that $0 < u_m < u_M < 1$. This is certainly a totally subsonic flow. What's more, because of the continuous dependence between the function $|\boldsymbol{u}|$ and the parameter $|u_{20}|$, there exist two critical points σ_2' and σ_3' such that

$$|u_{20}| = \sigma_2' \Rightarrow u_m < u_M = 1 \quad \text{(the subsonic-sonic flow)},$$

and

$$|u_{20}| = \sigma_3' \Rightarrow u_m = 1 < u_M \quad \text{(the supersonic-sonic flow)}.$$

When $|u_{20}|$ belongs to a certain subarea of $[\sigma_2, \sigma_3] \subseteq [\sigma_2', \sigma_3']$, we can show that $u_M > 1 > u_m$, which points directly at a totally transonic flow.

All the above are the results of $\tau \gg 1$. Then when $\tau \ll 1$, it is easy to get that the flow is totally transonic only if $|u_{20}| \ll 1$ for the degenerate case, and the flow is totally subsonic if $|u_{20}| \ll 1$ for the non-degenerate case.

The proof is complete. □

4. Transonic flows in the radial direction

In this section, we concern with the steady flows being transonic in the radial direction. Here the system (1.16) is written as

$$\begin{cases} \left(1-\frac{1}{n^2}\right)n_r = n\hat{E}-\frac{1}{\tau}, \quad r\in(r_0,r_1), \\ (r\hat{E})_r = n-\mathfrak{b}-K(r;n), \\ n(r_0)=n_0, \quad n(r_1)=n_1. \end{cases} \tag{4.1}$$

4.1. Radial transonic flows without shock

We now study the structure of the smooth solutions with a transonic flow in the radial direction.

Problem III. Find the smooth function n in Type (iii) of Definition 1.2, which solves the system (4.1) with the boundary conditions:

$$0<n_0\le 1 \quad \text{and} \quad n_1\ge 1,$$

when $\underline{\mathfrak{b}}>1$.

Theorem 4.1 *(the steady-states of Type (iii)). Let $\mathfrak{b}\in L^\infty(r_0,r_1)$ and $\underline{\mathfrak{b}}>1$. For $0<n_0\le 1$ and $n_1\ge 1$,*

(1) *if τ is large enough, then there is no smooth steady-state solution of Definition 1.2 to the system (4.1);*
(2) *if τ is small enough with $n_0=n_1=1$, and assume that $|u_{20}|\le C\sqrt{\tau}$ and $\mathfrak{b}\in C^0[r_0,r_1]$, then there exist infinitely many C^1-smooth radial transonic solutions to (4.1) satisfying*

$$0<n(r)<1 \quad on \quad (r_0,z_0), \quad n(r)>1 \quad on \quad (z_0,r_1),$$

with a point $z_0\in(r_0,r_1)$, and the smoothness condition (1.20) follows.

Remark 4.2. From Theorem 4.1(2), we can recognize the C^1-smooth radial transonic solutions in the situation of $n_0=n_1=1$, and the conclusions are true even when the boundary value n_0,n_1 be extremely close to the sonic state, i.e., $1-n_0\le C\tau$ and $n_1-1\le C\tau$. But for more general cases with any $n_0<1$ or $n_1>1$, it is found that whether the continuous solution exists remains unanswered.

Proof of Theorem 4.1. (1) The non-existence of the smooth solutions with $\tau\gg 1$. For convenience, we denote

$$F=r\left(\hat{E}-\frac{1}{\tau n}\right), \quad \varrho=n-1,$$

then (4.1) can be converted into

$$\begin{cases} \varrho_r = \dfrac{(\varrho+1)^3}{r(\varrho+2)} \cdot \dfrac{F}{\varrho}, \\ F_r = \varrho + 1 - \mathfrak{b} + \dfrac{\varrho+1}{\tau(\varrho+2)} \cdot \dfrac{F}{\varrho} - \dfrac{1}{\tau(\varrho+1)} - K(r;\varrho+1), \end{cases} \tag{4.2}$$

and

$$\varrho(r_0) = n_0 - 1 \le 0, \quad \varrho(r_1) = n_1 - 1 \ge 0. \tag{4.3}$$

Obviously, each solution of (4.2)-(4.3) always corresponds to a solution of (4.1). Here and in what follows, we extend the continuous function $\mathfrak{b}(r)$ periodically to $[r_0, +\infty)$. Therefore, all trajectories of the non-autonomous system (4.2) satisfy

$$\frac{dF}{d\varrho} = \frac{r(\varrho+1-\mathfrak{b})(\varrho+2)}{(\varrho+1)^3} \cdot \frac{\varrho}{F} + \frac{r}{\tau(\varrho+1)^2} - \frac{r(\varrho+2)}{\tau(\varrho+1)^4} \cdot \frac{\varrho}{F} - \frac{r(\varrho+2)K(r;\varrho+1)}{(\varrho+1)^3} \cdot \frac{\varrho}{F}. \tag{4.4}$$

Now suppose that $(\varrho, F)(r)$ is a smooth solution to the system (4.2)-(4.3) over $[r_0, r_1]$ such that

$$(\varrho, F)(r) = \begin{cases} (\varrho_{sup}, F_{sup})(r), & r \in [r_0, z_0], \\ (\varrho_{sub}, F_{sub})(r), & r \in [z_0, r_1], \end{cases}$$

where

$$-1 < \varrho_{sup}(r) < 0 \quad \text{on} \quad (r_0, z_0), \qquad \varrho_{sub}(r) > 0 \quad \text{on} \quad (z_0, r_1),$$
$$\varrho_{sup}(z_0) = \varrho_{sub}(z_0) = 1, \quad F_{sup}(z_0) = F_{sub}(z_0) \quad \text{with} \quad z_0 \in (r_0, r_1).$$

Firstly, let's assert that $F_{sup}(z_0) < 0$ for a sufficiently large τ. Aiming to get a contradiction, we assume that $F_{sup}(z_0) \ge 0$. For this situation, we argue by two cases. That is, $F_{sup}(r) > 0$ and $F_{sup}(r) < 0$ as r tends to z_0^-. As a result, if $\varrho_{sup}(r) < 0$ and $F_{sup}(r) > 0$ near z_0^-, we get from $(4.2)_1$ and (4.4) that $\varrho_r(r) < 0$ and $\frac{dF}{d\varrho} > 0$ in a small neighborhood of z_0^-, which implies that the trajectory of the smooth solution can't be close to the sonic line $\varrho = 0$ in the region of $F > 0$. This is a contradiction to $\varrho_{sup}(z_0) = 1$. If $\varrho_{sup}(r) < 0$ and $F_{sup}(r) < 0$ near z_0^-, we should prove $F_{sup}(z_0^-) = 0$. Now it implies by $(4.2)_1$ that $\varrho_r > 0$ near z_0^-. From (4.4), it follows that

$$\frac{dF}{d\varrho} \le \frac{1}{(\varrho+1)^2} \left[-C(r_0, \max_{r \in [r_0, r_1]} \varrho_{sup}(r), \underline{\mathfrak{b}}, |u_{20}|) \frac{\varrho}{F} + \frac{r_1}{\tau} \right] < 0,$$

if $\frac{\varrho}{F} \le \frac{C}{\tau}$ near z_0^- and $\tau \gg 1$. Hence, the trajectory of the solution can't go to the line $F = 0$ at the point $r = z_0$, which leads to a contradiction; so we have to consider the case of $\frac{dF}{d\varrho} \ge 0$, however, it is check that $\frac{dF}{d\varrho} \le \frac{C}{\tau}$ near z_0^-. Clearly, if $\tau \gg 1$, then the trajectory of the solution also can't reach to the point (0,0) in the (ϱ, F) plane at $r = z_0$. Therefore, the assertion of $F_{sup}(z_0) < 0$ is true.

Secondly, since $F_{sub}(z_0) = F_{sup}(z_0) < 0$, then $\varrho_r < 0$ near z_0^+. Due to $\varrho_{sub}(z_0) = 1$, it is impossible to derive a radial subsonic trajectory of $\varrho > 0$ on (z_0, r_1). Thus the smooth solution does not exist if $\tau \gg 1$.

(2) The existence of infinitely many C^1-smooth transonic solutions with $n_0 = n_1 = 1$ and $\tau \ll 1$. In this case, our goal is to find the C^1-smooth steady-state $(\varrho, F)(r)$ over (r_0, r_1) to equations (4.2) with the boundary conditions

$$\varrho(r_0) = \varrho(r_1) = 0. \tag{4.5}$$

satisfying

$$\varrho(r) < 0 \quad \text{for} \quad r \in (r_0, z_0),$$
$$\varrho(r) > 0 \quad \text{for} \quad r \in (z_0, r_1),$$

and

$$\varrho(z_0^-) = \varrho(z_0^+) = 0, \quad \varrho'(z_0^-) = \varrho'(z_0^+), \quad F(z_0^-) = F(z_0^+)$$

for a number $z_0 \in (r_0, r_1)$.

In other words, by analyzing the characters of the trajectories of (4.4), we will construct a C^1-smooth radial transonic trajectory to equations (4.2) with the sonic boundary (4.5) in the (ϱ, F) plane. The proof is divided into three steps.

Step 1. The radial supersonic part of the trajectory. In this step, we prove that there exists a number $z_0 \in (r_0, r_1)$, then the system (4.2) with (4.5) has a trajectory in radial supersonic region $-1 < \varrho < 0$ over (r_0, z_0), such that this trajectory ends at the point $(0, 0)$ in the (ϱ, F) plane.

To do this, for $\varrho \leq 0$, we denote an autonomy system

$$\begin{cases} \varrho_r = \dfrac{(\varrho+1)^3}{z_0(\varrho+2)} \cdot \dfrac{F_1}{\varrho}, \\ (F_1)_r = \varrho + 1 - \overline{\mathfrak{b}} + \dfrac{\varrho+1}{\tau(\varrho+2)} \cdot \dfrac{F_1}{\varrho} - \dfrac{1}{\tau(\varrho+1)} - K_{\max}, \end{cases} \quad \text{for} \quad r_0 \leq r \leq z_0, \tag{4.6}$$

where $K_{\max}$ is defined by (2.7). Thus,

$$\begin{aligned} \frac{dF_1}{d\varrho} &= z_0 \left(\frac{(\varrho+1-\overline{\mathfrak{b}})(\varrho+2)}{(\varrho+1)^3} \cdot \frac{\varrho}{F_1} + \frac{1}{\tau(\varrho+1)^2} - \frac{(\varrho+2)}{\tau(\varrho+1)^4} \cdot \frac{\varrho}{F_1} - \frac{(\varrho+2)K_{\max}}{(\varrho+1)^3} \cdot \frac{\varrho}{F_1} \right) \\ &=: z_0 \cdot H(\varrho, F_1; \overline{\mathfrak{b}}, K_{\max}). \end{aligned} \tag{4.7}$$

With regard to the equation $H(\varrho, F_1; \overline{\mathfrak{b}}, K_{\max}) = 0$, the critical curve is shown as follows:

$$\begin{aligned} \Xi(\varrho) &= \frac{(\varrho+2)\varrho}{(\varrho+1)^2} + \frac{\tau(\varrho+2)\varrho K_{\max}}{\varrho+1} - \frac{\tau(\varrho+1-\overline{\mathfrak{b}})(\varrho+2)\varrho}{\varrho+1} \\ &= \frac{[2u_{20}^2(\varrho+1)+1](\varrho+2)\varrho}{(\varrho+1)^2} - \frac{\tau(\varrho+1-\frac{2u_{20}^2}{r_0}-\overline{\mathfrak{b}})(\varrho+2)\varrho}{\varrho+1} \\ &=: \frac{[2u_{20}^2(\varrho+1)+1](\varrho+2)\varrho}{(\varrho+1)^2} + \Psi(\varrho). \end{aligned} \tag{4.8}$$

Thanks to the fact that

$$\Psi'(\varrho) = -\tau\left(2\varrho + 2 - \frac{2u_{20}^2}{r_0} - \overline{\mathfrak{b}} - \frac{\frac{2u_{20}^2}{r_0} + \overline{\mathfrak{b}}}{(\varrho+1)^2}\right),$$

$$\Psi''(\varrho) = -2\tau\left(1 + \frac{\frac{2u_{20}^2}{r_0} + \overline{\mathfrak{b}}}{(\varrho+1)^3}\right),$$

we check from $\overline{\mathfrak{b}} > 1$ that

$$\begin{aligned} &\Xi(\varrho) < \Xi(0) = 0,\\ &\Xi'(\varrho) = 2u_{20}^2 + \frac{2[u_{20}^2(\varrho+1)+1]}{(\varrho+1)^3} + \Psi'(\varrho) > 2\tau(\overline{\mathfrak{b}} - 1) > 0,\\ &\Xi''(\varrho) < 0, \quad \lim_{\varrho\to -1} \Xi(\varrho) = -\infty, \end{aligned} \tag{4.9}$$

on the domain $\varrho \in (-1, 0)$.

Suppose that $F_1(0) = -l/2 < 0$, and because of $\frac{dF_1}{d\varrho}(0) = \frac{1}{\tau} > 0$, we will consider the region of $\varrho \le 0$ and $F \le 0$. As similar as [8, Lemma 3.4], it is seen from (4.7) and (4.8) that

$$\frac{dF_1}{d\varrho} = \frac{z_0}{\tau(\varrho+1)^2}\cdot\left(\frac{F_1 - \beta\Xi}{F_1} + \frac{(\beta-1)\Xi}{F_1}\right), \tag{4.10}$$

where $\beta > 1$ is a constant to be determined later. From (4.9) and (4.10), we show

$$\begin{aligned} (F_1^2 - \beta^2\Xi^2)' &= 2F_1F_1' - 2\beta^2\Xi\Xi'\\ &= \frac{2z_0(F_1 - \beta\Xi)}{\tau(\varrho+1)^2} + 2\Xi\left[\frac{z_0(\beta-1)}{\tau(\varrho+1)^2} - \beta^2\Xi'\right]\\ &= \frac{2z_0(F_1^2 - \beta^2\Xi^2)}{\tau(\varrho+1)^2(F_1+\beta\Xi)} + 2\Xi\cdot I(\varrho), \end{aligned} \tag{4.11}$$

where

$$I(\varrho) = \frac{z_0(\beta-1)}{\tau(\varrho+1)^2} - 2u_{20}^2\beta^2 - \frac{2u_{20}^2\beta^2}{(\varrho+1)^2} - \frac{2\beta^2}{(\varrho+1)^3} + \beta^2\tau\left(2\varrho + 2 - \frac{2u_{20}^2}{r_0} - \overline{\mathfrak{b}} - \frac{\frac{2u_{20}^2}{r_0} + \overline{\mathfrak{b}}}{(\varrho+1)^2}\right).$$

First observing that the function $I(\varrho)$ must change sign in $(-1, 0)$, but as in [8, Lemma 3.4], one can get

$$I(\varrho) > 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right], \quad 1 \le k_0 < +\infty,$$

with a sufficiently small constant $\tau = \tau_1(r_0, z_0, k_0, \overline{\mathfrak{b}}, u_{20}^2)$ and a constant $\beta = \beta_1\left(z_0, \frac{1}{\tau}\right) > 1$. Therefore it follows from (4.10), (4.11) and $F_1^2(0) - \beta^2\Xi^2(0) > 0$ that

$$F_1^2(\varrho) - \beta^2\Xi^2(\varrho) > 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right],$$

and further

$$F_1(\varrho) < \beta\Xi(\varrho) < 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right]. \tag{4.12}$$

From (4.10), we obtain

$$\frac{dF_1}{d\varrho}(\varrho) > 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right].$$

Now let $\hat{F} := F - F_1$ and suppose that $F(0) = -l < 0$, so $\hat{F}(0) = F(0) - F_1(0) = -l/2 < 0$. Note that

$$\frac{d\hat{F}}{d\varrho} = rH(\varrho, F; \overline{\mathfrak{b}}, K_{\max}) - z_0 H(\varrho, F_1; \overline{\mathfrak{b}}, K_{\max}) + \frac{rT(\varrho)\varrho}{F} \quad \text{for} \quad -1 + \frac{1}{2^{k_0}} \le \varrho \le 0, r_0 \le r \le z_0,$$

where we know

$$T(\varrho) = \frac{(\overline{\mathfrak{b}} - \mathfrak{b} + K_{\max} - K(r; \varrho + 1))(\varrho + 2)}{(\varrho + 1)^3} > 0 \quad \text{for} \quad -1 + \frac{1}{2^{k_0}} \le \varrho \le 0, r_0 \le r \le z_0.$$

By local continuation method similarly as [8, Lemma 3.3], it is easy to see that

$$\frac{dF}{d\varrho}(\varrho) > \frac{dF_1}{d\varrho}(\varrho) > 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right],$$

which leads to

$$F(\varrho) \le \beta\Xi(\varrho) < 0 \quad \text{for} \quad \varrho \in \left[-1 + \frac{1}{2^{k_0}}, 0\right].$$

Now it is certain that the trajectory ending at $(0, -l)$ with $l > 0$ can pass through the line $F = 0$ at a point $r = s_0$ for $\varrho \in \left[-1, -1 + \frac{1}{2^{k_0}}\right]$. As a result, by integrating the equation (2.1) over (s_0, z_0), we have

$$\left[\frac{r(\varrho + 2)\varrho}{(\varrho + 1)^3}\varrho_r + \frac{r}{\tau(\varrho + 1)}\right]\Bigg|_{r=s_0}^{r=z_0} = \int_{s_0}^{z_0} (\varrho + 1 - \overline{\mathfrak{b}} - K(r))dr.$$

Notice by the first equation of (4.2) that

$$F = \frac{r(\varrho+2)\varrho}{(\varrho+1)^3}\varrho_r \le 0 \quad \text{on} \quad (s_0, z_0),$$

then it implies

$$\left[\frac{r}{\tau(\varrho+1)}\right]\Big|_{r=s_0}^{r=z_0} = \frac{1}{\tau}\left(z_0 - \frac{s_0}{\varrho(s_0)+1}\right) \ge \int_{s_0}^{z_0} (\varrho + 1 - \overline{\mathfrak{b}} - K)(r)dr,$$

which indicates to

$$\frac{s_0}{\varrho(s_0)+1} - z_0 \le \tau(z_0 - s_0) \sup_{r\in(s_0,z_0)} (\overline{\mathfrak{b}} + K - 1 - \varrho)(r) \le C\tau(z_0 - s_0), \text{ if } |u_{20}| \le C\sqrt{\tau}.$$

Choose τ sufficiently small such that $\varrho(s_0) + 1 \le \frac{1}{2^{k_0}} \le \frac{r_0}{r_1+1}$, then we obtain

$$z_0 - s_0 \ge \frac{C}{\tau}\left(2^{k_0}s_0 - z_0\right) \ge \frac{C}{\tau}\left(\frac{r_1+1}{r_0}s_0 - z_0\right) \ge \frac{C}{\tau}, \text{ if } \tau \ll 1,$$

which is contradiction to $0 < r_0 < s_0 < z_0 < r_1 < +\infty$. Hence the trajectory of radial supersonic solutions ending at the point $(0, -l)$ can't satisfy the boundary condition $\varrho(r_0) = 0$. Obviously, the trajectory can't end at the point $(0, l)$. In conclusion, the solution trajectory of (4.4) with (4.5) ends at the point (0, 0) in the (ϱ, F) plane if $|u_{20}| \le C\sqrt{\tau}$.

Step 2. The radial subsonic part of the trajectory. In this step, we prove the existence of a radial subsonic solution $\varrho > 0$ to the following system

$$\begin{cases} \varrho_r = \dfrac{(\varrho+1)^3}{r(\varrho+2)} \cdot \dfrac{F}{\varrho}, \qquad r \in (z_0, r_1) \\ F_r = \varrho + 1 - \mathfrak{b} + \dfrac{\varrho+1}{\tau(\varrho+2)} \cdot \dfrac{F}{\varrho} - \dfrac{1}{\tau(\varrho+1)} - K_0(r; \varrho+1), \\ \varrho(z_0) = \varrho(r_1) = 0, \end{cases} \tag{4.13}$$

with

$$K_0(r; \varrho+1) = 2u_{20}^2 r_0^2 \left(\frac{1}{r^3} + \frac{\varrho+1}{\tau r^2}\right) \exp\left\{-\frac{2}{\tau}\left(\int_{r_0}^{z_0} (\hat{\varrho}(s)+1)ds + \int_{z_0}^{r} (\varrho(s)+1)ds\right)\right\}.$$

Here $-1 < \hat{\varrho} \le 0$ is the radial supersonic solution of (4.2) and (4.5) on $[r_0, z_0]$. The second result we want to show is that the solution trajectory of (4.13) starts from the original point (0, 0) in radial subsonic region $\varrho > 0$ of the plane (ϱ, F).

First of all, it is known from Theorem 3.1 that the solution $\varrho(r) > 0$ of (4.13) exists over $[z_0, r_1]$ when $|u_{20}| \le C\sqrt{\tau}$. Then let's consider the system (4.6) on $[z_0, r_1]$. That is,

$$\begin{cases} \varrho_r = \dfrac{(\varrho+1)^3}{z_0(\varrho+2)} \cdot \dfrac{F_2}{\varrho}, \\ (F_2)_r = \varrho + 1 - \overline{\mathfrak{b}} + \dfrac{\varrho+1}{\tau(\varrho+2)} \cdot \dfrac{F_2}{\varrho} - \dfrac{1}{\tau(\varrho+1)} - K_{\max}, \end{cases} \quad \text{for} \quad z_0 \le r \le r_1. \tag{4.14}$$

Assume that $\varrho(z_0) = 0$ and $F_2(z_0) = \frac{l}{2} > 0$. Similar as the proof of [8, Lemma 3.3], we can get the results that

$$F_2(\varrho) \geq \beta \Xi(\varrho) \geq 0 \quad \text{for} \quad 0 \leq \varrho \leq \overline{\mathfrak{b}} + \frac{2u_{20}^2}{r_0},$$

and

$$\frac{dF_2}{d\varrho} > 0 \quad \text{for} \quad \varrho \geq \overline{\mathfrak{b}} + \frac{2u_{20}^2}{r_0},$$

with a small constant $\tau = \tau_2(r_0, z_0, \overline{\mathfrak{b}}, u_{20}^2) \ll 1$ and a constant $\beta = \beta_2\left(r_0, \frac{1}{\tau}\right) > 1$. Therefore the trajectories of (4.14) starting from the point $(0, \frac{l}{2})$ go to infinity. Let $F(z_0) = l$, then we apply the local continuation method in [8, Lemma 3.3] to obtain

$$\frac{dF}{d\varrho} > 0 \quad \text{for} \quad \varrho \geq 0,$$

which indicates that the trajectories of (4.13) starting from the point $(0, l)$ can't go back to the line $\varrho = 0$. Furthermore, it is impossible that the trajectories of (4.13) start from the point $(0, -l)$. So the second result is proved if $\tau \ll 1$.

Step 3. C^1-smoothness of the solution. From Step 1-2, it has been shown that there exists a continuous solution to the system (4.2) and (4.5) with a critical point z_0. Thus it just needs to prove the C^1-smoothness of the continuous solution in the neighborhood of z_0.

By (4.4), the slope of $F(\varrho)$ at the point $\varrho = 0$ can be calculated in the form of

$$\theta_1 = \frac{1}{2}\left(\frac{z_0}{\tau} - \sqrt{\left(\frac{z_0}{\tau}\right)^2 - 8z_0\big[(\mathfrak{b}(z_0) - 1) + 1/\tau + \hat{c}\big]}\right)$$

or

$$\theta_2 = \frac{1}{2}\left(\frac{z_0}{\tau} + \sqrt{\left(\frac{z_0}{\tau}\right)^2 - 8z_0\big[(\mathfrak{b}(z_0) - 1) + 1/\tau + \hat{c}\big]}\right)$$

where

$$\hat{c} = 2u_{20}^2 r_0^2 \left(\frac{1}{z_0^3} + \frac{1}{\tau z_0}\right) \exp\left\{-\frac{2}{\tau}\int_{r_0}^{z_0} (\varrho_{sup}(s) + 1)ds\right\}.$$

Then copying the methods used in the proof of [8, Theorem 3.6], one can get

$$\lim_{\varrho \to 0^-} \frac{dF(\varrho)}{d\varrho} \text{ exists} = \lim_{\varrho \to 0^+} \frac{dF(\varrho)}{d\varrho} \text{ exists} = F'(0) = \theta_1.$$

Accordingly we have a C^1-smooth solution of (4.1) with $n_0 = n_1 = 1$. Since the choice of the smooth point $z_0 \in (r_0, r_1)$ is arbitrary, the C^1-smooth radial transonic solutions are infinitely many. The proof is finished. □

4.2. Radial transonic flows with shock

In the study of radial transonic flows with shock, we find that the radial velocity will jump from supersonic to subsonic and the angular velocity is continuous across the shock, but the total velocity may be supersonic, sonic or subsonic after the shock. Based on this, we prepare to list various types of the flows with shock with different conditions.

Problem IV. Find the discontinuous function n in Type (iv) of Definition 1.2, which solves the system (4.1) with the boundary conditions:

$$0 < n_0 \le 1 \quad \text{and} \quad n_1 \ge 1,$$

when $\underline{\mathfrak{b}} > 1$.

Theorem 4.3. *Let $\mathfrak{b} \in L^\infty(r_0, r_1)$ and $\underline{\mathfrak{b}} > 1$, and $0 < n_0 \le 1$ and $n_1 \ge 1$, then*

(a) *assume further that τ is large enough, i.e., $\tau \gg 1$; the system (4.1) admits a shock solution in the form of*

$$(n, \hat{E})(r) = \begin{cases} (n_{sup}, \hat{E}_{sup})(r), & r \in (r_0, z_0), \\ (n_{sub}, \hat{E}_{sub})(r), & r \in (z_0, r_1), \end{cases} \tag{4.15}$$

satisfying the entropy condition (1.21) and the Rankine-Hugoniot condition (1.22) at the jump point z_0, where z_0 can be uniquely determined when the value of $n_{sup}(z_0)$ is fixed, and since the choice of $n_{sup}(z_0)$ is arbitrary, the shock solutions will be infinitely many.
Furthermore, we have the following results:
1. *for $|u_{20}| \gg 1$, there exists a supersonic-supersonic shock at the point z_0, and the pair of the shock solution $(n, \hat{E})(r)$ is totally supersonic;*
2. *for $|u_{20}| \ll 1$, there exists a supersonic-subsonic shock at the point z_0, and the pair of the shock solution $(n, \hat{E})(r)$ is totally transonic. Here the solution jumps from supersonic to subsonic at the critical point z_0.*

(b) *assume further that τ is small enough, i.e., $\tau \ll 1$; there is no shock solution to the system (4.1) with $n_0 = n_1 = 1$.*

Remark 4.4. In Theorem 4.3(a), when $|u_{20}| = \sigma_4$ with a constant σ_4 and $n_1 > 1$, the solution (4.15) is totally supersonic-sonic. If $|u_{20}|$ belongs to a subset of $[0, \sigma_4)$, there may exist a totally transonic solution and a supersonic-supersonic shock, and the solution (4.15) changes smoothly from supersonic to subsonic in (z_0, r_1).

Proof of Theorem 4.3. (a) the existence of the infinitely many shock solutions with $\tau \gg 1$. Without loss of generality, let's consider the degenerate case of $n_0 = n_1 = 1$ directly, whose adopt method can be also used in the non-degenerate situation. The proof is divided into four steps.

Step 1. First we focus on the following system with $\tau \gg 1$,

$$\begin{cases} \left(1 - \dfrac{1}{n^2}\right) n_r = n\hat{E} - \dfrac{1}{\tau}, \quad r \in (r_0, r_0 + L), \\ (r\hat{E})_r = n - \mathfrak{b} - K(r; n), \\ n(r_0) = 1, \quad n(r_0 + L) = 1, \end{cases} \tag{4.16}$$

where $L \geq \frac{r_1 - r_0}{4}$ is a length of interval, and the doping function $\mathfrak{b}$ has been periodically extended from $[r_0, r_1]$ to $[r_0, +\infty]$ here and below. It follows from Theorem 2.1 that there exists a solution $(n_L, \hat{E}_L)(r)$ to (4.16) on $[r_0, r_0 + L]$.

For the solution $(\tilde{n}_L, \tilde{E}_L)$ of the system

$$\begin{cases} \left(1 - \dfrac{1}{\tilde{n}^2}\right) \tilde{n}_r = \tilde{n}\tilde{E}, \quad r \in (r_0, r_0 + L), \\ (r\tilde{E})_r = \tilde{n} - \mathfrak{b} - \frac{2u_{20}^2}{r^2}, \\ \tilde{n}(r_0) = 1, \quad \tilde{n}(r_0 + L) = 1, \end{cases} \tag{4.17}$$

by [7, Lemma 2.1] and a standard energy estimate, it is seen that

$$\tilde{E}_L(r_0 + L) \leq -\mu(L, |u_{20}|, \mathfrak{b}) < 0.$$

Then as in [19, Theorem 4.2, Step 3], and subtracting (4.16) by (4.17) we have

$$|n_L - \tilde{n}_L| + |\hat{E}_L - \tilde{E}_L| \leq \frac{C}{\tau}.$$

Therefore it holds that

$$\hat{E}_L(r_0 + L) \leq \tilde{E}_L(r_0 + L) + \frac{C}{\tau} \leq -\frac{\mu}{2} \quad \text{if} \quad \tau \gg 1. \tag{4.18}$$

Step 2. Let ϵ be a small number such that $0 < \epsilon \ll 1$. In light of (4.18), it is easy to observe that the solution $n_L(r)$ of (4.16) keeps decreasing in r near the end point $r = r_0 + L$. So as in the proof of [19, Theorem 4.2] and [8, Theorem 2.2], we show that there exists a number $r_0 < y_1 < r_0 + L$ at which the solution n last arrives the line $n = 1 - \epsilon$, and it follows that

$$|\hat{E}_L(y_1) - \hat{E}_L(r_0 + L)| \leq C\epsilon, \quad n_L(y_1) = 1 - \epsilon, \quad r_0 + L - y_1 \leq C\epsilon.$$

Hence, for the ODE system

$$\begin{cases} \left(1 - \dfrac{1}{n^2}\right) n_r = n\hat{E} - \dfrac{1}{\tau}, \quad r > r_0, \\ (r\hat{E})_r = n - \mathfrak{b} - K(r; n), \\ n(r_0) = n_0 \leq 1, \hat{E}(r_0) = \hat{E}_L(r_0), \end{cases} \tag{4.19}$$

we are able to construct a shock solution on an interval $[r_0, y_2]$ for a number y_2 in the form of

$$(n,\hat{E})(r)=\begin{cases}(n_{sup},\hat{E}_{sup})(r), & r\in[r_0,y_1],\\ (n_{sub},\hat{E}_{sub})(r), & r\in[y_1,y_2],\end{cases}$$

where we have $0<n_{sup}\le 1$, $n_{sub}\ge 1$, $(n_{sup},\hat{E}_{sup})(r)=(n_L,\hat{E}_L)(r)$ for $r\in[r_0,y_1]$, and

$$n_{sup}(y_1)=1-\epsilon<1,\quad n_{sub}(y_1)=\frac{1}{1-\epsilon}>1,\quad \hat{E}_{sup}(y_1)=\hat{E}_{sub}(y_1).$$

Indeed, it follows that

$$n_{sub}(y_2)=1,\quad |\hat{E}_{sub}(y_2)-\hat{E}_{sub}(y_1)|\le C\epsilon,\quad y_2-y_1\le C\epsilon,$$

whose proof is referred to [8, Theorem 2.2, Step 2] and necessarily needs the inequality (4.18) in Step 1 and the conditions $\underline{b}>1$ and $K>0$.

Step 3. We will apply the continuity argument to obtain a shock solution. Let $L=L_1=\frac{r_1-r_0}{2}$ and we denote the solution of the system (4.16) by $(n_{L_1},\hat{E}_{L_1})$. Thus there exist a jump point z_1 and a number y_3, such that the corresponding shock solution of (4.19), denoted by $(n^{(1)},\hat{E}^{(1)})$, satisfies

$$n^{(1)}(z_1^-)=1-\epsilon,\quad n^{(1)}(z_1^+)=\frac{1}{1-\epsilon}>1,\quad \hat{E}^{(1)}(z_1)<0,$$
$$n^{(1)}(y_3)=1,\quad \left|y_3-\frac{r_0+r_1}{2}\right|\le C\epsilon\quad\text{and}\quad \left|\hat{E}^{(1)}(y_3)-\hat{E}_{L_1}\left(\frac{r_0+r_1}{2}\right)\right|\le C\epsilon.$$

That is, we derive a shock solution by Step 1-2 as follows:

$$(n^{(1)},\hat{E}^{(1)})(r)=\begin{cases}(n^{(1)}_{sup},\hat{E}^{(1)}_{sup})(r), & r\in[r_0,z_1],\\ (n^{(1)}_{sub},\hat{E}^{(1)}_{sub})(r), & r\in[z_1,y_3],\end{cases}$$

where the Rankine-Hugoniot condition and the entropy condition are satisfied at the point z_1.

Similarly, let $L=L_2=2(r_1-r_0)$, so there exist a jump point z_2 and a number y_4, then it shows a shock solution $(n^{(2)},\hat{E}^{(2)})$ to the system (4.19) in the form of

$$(n^{(2)},\hat{E}^{(2)})(r)=\begin{cases}(n^{(2)}_{sup},\hat{E}^{(2)}_{sup})(r), & r\in[r_0,z_2],\\ (n^{(2)}_{sub},\hat{E}^{(2)}_{sub})(r), & r\in[z_2,y_4],\end{cases}$$

satisfying

$$|y_4-(2r_1-r_0)|\le C\epsilon,\quad n^{(2)}_{sub}(y_4)=1,\quad \left|\hat{E}^{(2)}_{sub}(y_4)-\hat{E}_{L_2}(2r_1-r_0)\right|\le C\epsilon,$$
$$n^{(2)}_{sup}(z_2)=1-\epsilon<1<\frac{1}{1-\epsilon}=n^{(2)}_{sub}(z_2),$$

which also meets the Rankine-Hugoniot condition and the entropy condition at the point z_2.

Hence we could take ϵ small such that $r_1 \in [y_3, y_4]$. It is seen that the solution of (4.19) continuously depends on the initial value $\hat{E}(r_0)$. Then by the continuity argument, there must exist a constant $\hat{E}_{r_1-r_0}(r_0) \in [\hat{E}_{L_1}(r_0), \hat{E}_{L_2}(r_0)]$ as a initial value of (4.19), corresponding to the length of interval $r_1 - r_0$. As a consequence, we obtain a shock solution to (4.19), written by

$$(n_{shock}, \hat{E}_{shock})(r) = \begin{cases} (n_{front}, \hat{E}_{front})(r), & r \in [r_0, z_0], \\ (n_{back}, \hat{E}_{back})(r), & r \in [z_0, r_1], \end{cases} \tag{4.20}$$

satisfying $0 < n_{front}(r) < 1$ on (r_0, z_0), $n_{back}(r) > 1$ on (z_0, r_1), and

$$n_{front}(z_0) = 1 - \epsilon \le 1 < \frac{1}{1-\epsilon} = n_{back}(z_0), \quad \hat{E}_{front}(z_0) = \hat{E}_{back}(z_0) < 0,$$
$$n_{front}(r_0) = 1, \quad n_{back}(r_1) = 1.$$

Clearly, the function (4.20) is truly a shock solution to (4.1) with $n_0 = n_1 = 1$. Additionally, there exists a constant ϵ_0 such that $0 < \epsilon < \epsilon_0$, and due to the arbitrary choice of ϵ, the shock solutions are infinitely many.

Step 4. As in the proof of Part 3 of Theorem 3.1, it is known that the shock solution produced in Step 1-3 is totally supersonic if $\tau \gg 1$ and $|u_{20}| \gg 1$, and naturally the solution jumps from supersonic to supersonic at the point z_0. Here we call the shock supersonic-supersonic. Additionally, the shock solution is totally transonic if $\tau \gg 1$ and $|u_{20}| \ll 1$, and the shock is supersonic-subsonic.

(b) the non-existence of the radial transonic shock solution with $\tau \ll 1$. If $\tau \ll 1$, we know that there exist infinitely many smooth steady-states shown in Theorem 4.1. Applying the proof by contradiction similar as [19, Theorem 5.13], we prove that there is no radial transonic shock solution to (4.1).

The proof is complete. □

Data availability

Data will be made available on request.

Acknowledgments

The research of L. Chen was supported by China Postdoctoral Science Foundation (2022M711309). The research of M. Mei was supported by the NSERC grant RGPIN 354724-2016, and the research of G.J. Zhang was supported by the National Natural Science Foundation of China grant 11871012.

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