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QUASI-NEUTRAL LIMIT TO STEADY-STATE HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH DEGENERATE BOUNDARY*

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Abstract. This paper is concerned with the quasi-neutral limit to a one-dimensional steady hydrodynamic model of semiconductors in the form of Euler–Poisson equations with degenerate boundary, a difficult case caused by the boundary layers and degeneracy. We establish a so-called convexity structure of the sequence of subsonic-sonic solutions near the boundary domains in this limit process, which efficiently overcomes the degenerate effect. We first show the strong convergence in the L^2 norm with the order $O(\lambda^{\frac{1}{2}})$ for the Debye length λ when the doping profile is continuous. Then we derive the uniform error estimates in the L^∞ norm with the order $O(\lambda)$ when the doping profile has higher regularity. The proof of L^∞ boundedness is based on a new bounded estimate method, which is used to replace the maximum principle utilized in the nondegenerate case. These newly proposed techniques in asymptotic limit analysis develop and improve the existing studies.

Key words. quasi-neutral limit, hydrodynamic model of semiconductors, degenerate boundary, boundary layers, subsonic-sonic solutions

MSC codes. 35L60, 35Q35, 35B40, 35C06

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1. Introduction.

Modeling equations. Hydrodynamic modeling of semiconductors, first introduced by Bløtejær [2], simulates the motion of a charged carrier in submicron semiconductor devices. This model contains several physical phenomena, such as hot electrons and velocity overshoots, which are missing in the classical drift-diffusion model. Related mathematical derivation of this model can be found in [22, 28, 29]. In the present paper, our aim is to investigate the quasi-neutral limit with degenerate sonic boundary for this model in the case of a one-dimensional steady-state system.

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Let the unknowns ρ, u, and Φ be the electron density, the electron average velocity, and the electrostatic potential, respectively; the one-dimensional transient model here is virtually governed by the unipolar Euler–Poisson equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = \rho \Phi_x - \dfrac{\rho u}{\tau}, \\ \lambda^2 \Phi_{xx} = \rho - b(x). \end{cases} \tag{1.1}$$

Here the function $p(\rho)$ is the pressure density relation, the parameter τ is the momentum relaxation time, and the fixed function $b(x)$ is the doping profile for semiconductors. The physical parameter $\lambda > 0$ represents the scaled Debye length, and its approximate value is always extremely small in actual semiconductors. Therefore, it is significant to study the zero-Debye-length (i.e., quasi-neutral) limit as $\lambda \to 0$ in (1.1).

The objective of this paper is concerned with the stationary equations of (1.1) in the bounded domain $[0,1]$. For the sake of clarity in studying this steady-state system, the following notation and assumptions are used throughout this paper:

(A1) $b \in L^\infty(0,1)$ and $0 < \underline{b} \le b(x) \le \overline{b}$ for $x \in (0,1)$, where

$$\underline{b} := \operatorname*{essinf}_{x\in(0,1)} b(x) \quad \text{and} \quad \overline{b} := \operatorname*{esssup}_{x\in(0,1)} b(x).$$

(A2) $\rho^2 p'(\rho)$ is strictly increasing with respect to ρ from $[0,+\infty)$ to $[0,+\infty)$. For convenience, we assume that

$$p(\rho) = \frac{\rho^\gamma}{\gamma} \quad \text{with the adiabatic exponent } \gamma \ge 1,$$

where $\gamma = 1$ represents the isothermal case and $\gamma > 1$ denotes the isentropic case.

(A3) It is assumed that the relaxation time $\tau \ge \tau_0 > 0$ is independent of λ for a constant τ_0.

Now we denote a prescribed constant current $j := \rho u$. From assumption (A2), the steady-state system of (1.1) then reduces to

$$\begin{cases} (F(\rho) - \Phi)_x = -\dfrac{j}{\tau\rho}, \\ \lambda^2 \Phi_{xx} = \rho - b(x), \end{cases} \qquad x \in (0,1), \tag{1.2}$$

where

$$F(\rho) := \begin{cases} \dfrac{j^2}{2\rho^2} + \dfrac{\rho^{\gamma-1}}{\gamma - 1} & \text{for} \quad \gamma > 1, \\ \dfrac{j^2}{2\rho^2} + \ln \rho & \text{for} \quad \gamma = 1. \end{cases} \tag{1.3}$$

For smooth solutions, after differentiating the first equation of (1.2) with respect to x and using the second equation of (1.2) to substitute Φ in the resultant equation, we get the density function ρ, satisfying

$$\lambda^2 F(\rho)_{xx} + \lambda^2 \left(\frac{j}{\tau\rho}\right)_x - (\rho - b(x)) = 0. \tag{1.4}$$

Note that, equation (1.4) is elliptic if $F'(\rho) > 0$, i.e., $\rho > J := j^{\frac{2}{\gamma+1}}$, which actually corresponds to subsonic flows (see [7]). In particular, since $\rho = J$ denotes the sonic state, we can impose a certain meaningful boundary condition, such as

$$\text{sonic boundary:} \quad \rho(0) = \rho(1) = J, \tag{1.5}$$

which causes the degeneracy at the boundary, the so-called degenerate boundary. In this situation, once we solve the BVP (1.4) and (1.5) for the solution ρ, then we may obtain the solution Φ from the second equation of (1.2) by using the supplementary condition

$$\Phi(0) = 0 \tag{1.6}$$

and the relation given by (1.2), (1.5), and (1.6),

$$\Phi(1) = \Phi(0) + \int_0^1 \left(F(\rho)_x + \frac{j}{\tau\rho(x)} \right) dx = \frac{j}{\tau} \int_0^1 \frac{1}{\rho(x)} dx.$$

As proved in [24], when $\underline{b} > J$, the problem (1.2), (1.5), (1.6) possesses a smooth subsonic-sonic solution in $C^{\frac{1}{2}}[0,1]$.

In order to consider the quasi-neutral limit, we rewrite the problem (1.2), (1.5), (1.6) in the form

$$\begin{cases} (F(\rho_\lambda) - \Phi_\lambda)_x = -\dfrac{j}{\tau\rho_\lambda}, \\ \lambda^2 (\Phi_\lambda)_{xx} = \rho_\lambda - b(x), \\ \rho_\lambda(0) = \rho_\lambda(1) = J, \quad \Phi_\lambda(0) = 0. \end{cases} \qquad x \in (0,1), \tag{1.7}$$

Following the methods and results of [24], here we give the existence and uniqueness of the subsonic-sonic solution to (1.7) with respect to λ.

PROPOSITION 1.1 ([24]). *Under assumptions* (A1)–(A3) *and the subsonic doping profile* $\underline{b} > J$, *for any* $\lambda > 0$, *system* (1.7) *admits a unique pair of subsonic-sonic solutions* $(\rho_\lambda, \Phi_\lambda) \in C^{1/2}[0,1] \times H^2(0,1)$ *satisfying* $(\rho_\lambda - J)^2 \in H^1(0,1)$ *and*

$$J + C\sin(\pi x) \le \rho_\lambda(x) \le \bar{b}, \quad x \in [0,1], \tag{1.8}$$

where $C = C(\tau_0, \underline{b})$ *is a positive constant independent of* λ.

In Proposition 1.1, the Hölder index $\frac{1}{2}$ is optimal for the global regularity of the solution ρ_λ. From (1.8), one can see that the solution ρ_λ is degenerate only at the boundary points $x = 0$ and $x = 1$. On this premise, we prepare to investigate the limit as $\lambda \to 0$ in (1.7).

Study background. We now draw a picture of the progress on the studies of well-posedness for the hydrodynamic model of semiconductors. For the stationary model over bounded domain, Degond and Markowich [7, 8] first proved the existence and uniqueness of subsonic solutions with the strong subsonic background in one dimension [7] and for potential flow in three dimensions [8], respectively. More results of stationary solutions in this model were studied in [1, 12, 38] and the references therein. Additionally for the one-dimensional case, the existence of the global weak solutions was shown in [30, 41], and the asymptotic stability of the stationary solution was investigated in [15, 17, 23, 31, 33]. Inspired by these results on the well-posedness, a series of studies were concerned with the asymptotic limits in the hydrodynamic model, such as the zero-relaxation-time limits [16, 19, 20, 37, 39], the zero-electron-mass limits [13, 14, 21], the quasi-neutral limits [6, 9, 10, 18, 34, 35, 36, 40], and even the Newtonian limits in the speed of light for the relativistic Euler–Poisson equations [26, 27, 32], for instance. These studies are significant and interesting but, to the best of our knowledge, do not involve the degenerate phenomenon and the relevant singularity.

For the system (1.7) with the degenerate sonic boundary and any fixed $\lambda > 0$, Li et al. [24] first presented the existence and uniqueness of the sonic-subsonic solution, the existence of sonic-supersonic solutions, and the existence of transonic solutions when the doping profile is subsonic, i.e., $\underline{b} > J$. For more results about the well-posedness problem of (1.7), we refer the reader to [3, 25] for the supersonic doping profile $(0 < b < J)$ and the transonic doping profile $(0 < \underline{b} < J < \bar{b})$. One-dimensional results have been extended to the high-dimensional model for the radial solutions in [4, 5]. However, the quasi-neutral limits problem related to the system with the degenerate boundary has not yet been touched upon and has remained open due to technical reasons.

Main difficulties and strategies. The main goal of this paper is to consider the quasi-neutral limit for the subsonic-sonic solution of (1.7) with the degenerate sonic boundary when $\underline{b} > J$, namely, the asymptotic analysis of the solution when the Debye length tends to zero as well as analysis of the relevant error estimates and convergence rates. Since the density function ρ_λ at the boundary is sonic, and the semiconductor doping function $b(x)$ is completely subsonic in the whole domain $[0, 1]$, the boundary layers will appear in the quasi-neutral limit problem. We are going to prove that when the doping profile $b(x)$ belongs to H^1, the solution sequence $(\rho_\lambda, \Phi_\lambda)$ will converge to the corresponding limit state in L^2 as the Debye length $\lambda \to 0^+$, and the corresponding convergence rate will be presented; further, when the doping profile $b(x)$ belongs to $W^{2,1}$, the solution sequence $(\rho_\lambda, \Phi_\lambda)$ will converge to the asymptotic state containing the boundary layers in L^∞, and the corresponding convergence rate will be shown, too, which clearly characterizes the quasi-neutral limit of the sonic-subsonic steady-state solution.

There are two essential difficulties in the study of quasi-neutral limit to the degenerate sonic boundary problem (1.7). The first difficulty is caused by the degeneracy of the sonic boundary. In the case when we consider the quasi-neutral limit problem, the degeneracy effect and the boundary layer effect will occur at the same time, and the singularity of the corresponding solution sequence will become very strong, so that the estimation near the boundary becomes particularly difficult and complicated. The second difficulty is that the maximum principle does not hold for the sonic degenerate boundary problem (1.7). When establishing the L^∞ boundedness of the error terms, the boundary degeneracy causes the coefficients and inhomogeneous terms of the error equations to no longer belong to L^∞ but only to L^1, so the existing procedure by the maximum principle for treating the case for the completely subsonic solutions no longer applies for the case of degenerate subsonic-sonic solutions.

In order to overcome these difficulties, we propose some new ideas for the proof. To treat the first obstacle mentioned above, we heuristically observe the special convexity structure of the solution sequence $v_\lambda = F(\rho_\lambda)$ near the boundary points such that the uniform bound estimate of order $O(\lambda^{\frac{1}{2}})$ can be technically derived. To overcome the second obstacle, we artfully develop a more useful energy method to get this convergence result, which is mainly based on the special properties of degeneracy equations.

Main results. In this paper, the main convergence results in $L^2(0,1)$ and $L^\infty(0,1)$ are shown as follows.

THEOREM 1.2 (L^2-estimate). *Let the doping profile be $b\in H^1(0,1)$. When the boundary layer profiles appear, there are a sequence of solutions $(\rho_\lambda,\Phi_\lambda)_{\lambda>0}$ to system* (1.7) *and a pair of corresponding limit solutions (ϱ,ϕ) such that the following L^2-estimates hold for an arbitrary small constant λ:*

$$\|\rho_\lambda-\varrho\|_{L^2(0,1)}\le C\lambda^{\frac12},\quad \|\Phi_\lambda-\phi\|_{L^2(0,1)}\le C\lambda^{\frac12},$$

where the limit solution ϱ satisfies $\varrho=b$, and C is a general constant independent of λ.

THEOREM 1.3 (L^∞-estimate). *Let the doping profile $b\in W^{2,1}(0,1)$ hold; then there exist the boundary layers profiles n_0 and n_1 satisfying the estimate*

$$\|\rho_\lambda-b-n_0(\cdot,\lambda)-n_1(\cdot,\lambda)\|_{L^\infty(0,1)}\le C\lambda,$$

with

$$|n_0(x,\lambda)|\le Ce^{-\frac{Cx}{\lambda}}\quad\text{and}\quad |n_1(x,\lambda)|\le Ce^{-\frac{C(1-x)}{\lambda}}\quad\text{for}\quad x\in[0,1].$$

Here C are some positive constants independent of λ, and ρ_λ is the solution to (1.7).

Remark 1.4. In Theorem 1.2, the continuity of the doping profile $b(x)$ is necessary. However, if the function $b(x)$ is discontinuous, the interior layers can appear in the neighborhood of the discontinuous points. In Theorem 1.3 for the $L^\infty(0,1)$ bounded result, the higher regularity of $b(x)$ is required.

Remark 1.5. In the proofs of Theorems 1.2 and 1.3, we realize that the thickness of the boundary layers is precisely equal to λ. A similar argument was obtained in [36].

The paper is organized as follows. Section 2 derives some crucial properties of the solution $v_\lambda=F(\rho_\lambda)$. Section 3 is devoted to establishing the $L^2(0,1)$-estimates of order $O(\lambda^{\frac12})$, with the boundary layer profiles under the assumption of $b\in H^1(0,1)$, so that Theorem 1.2 can be proved. Section 4 shows the convergence estimate in $L^\infty(0,1)$ with the convergence rate $O(\lambda)$ when the doping profile has better smoothness, which can guarantee Theorem 1.3. Finally, in section 5 we present some numerical simulations, which perfectly demonstrate and confirm our theoretical results.

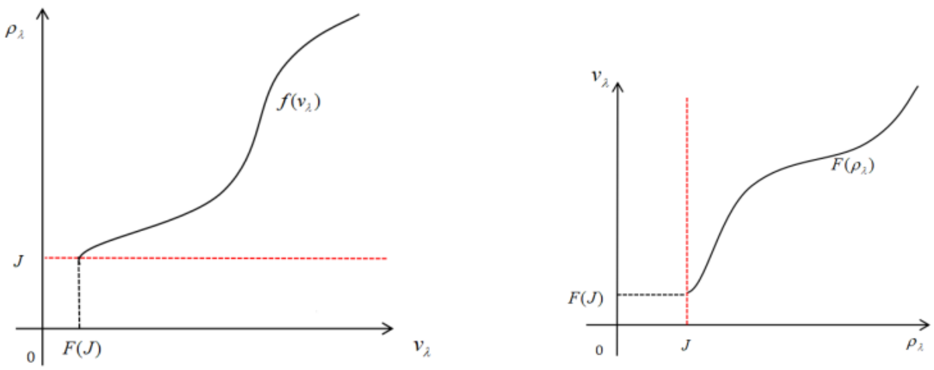
2. Preliminary observation.

In this section, to circumvent the effect of degenerate boundary, we are going to introduce some fundamental properties of the solution to equations (1.7) for sufficiently small $\lambda>0$.

It is well known that $(\rho_\lambda,\Phi_\lambda)_{\lambda>0}$ is a sequence of solutions to (1.7) and that from (1.3) F is strictly increasing on $\rho_\lambda>J$ in (1.7). Thus, we may denote $v_\lambda:=F(\rho_\lambda)$ and $f:=F^{-1}$ and rewrite (1.4) and (1.5) as follows:

$$\begin{cases}\lambda^2\dfrac{d^2v_\lambda}{dx^2}+\lambda^2\dfrac{j}{\tau}\dfrac{d}{dx}\left(\dfrac{1}{f(v_\lambda)}\right)-(f(v_\lambda)-b)=0,\quad x\in(0,1),\\ v_\lambda(0)=v_\lambda(1)=F(J).\end{cases}\tag{2.1}$$

Here f satisfies $\rho_\lambda(x)=f(v_\lambda(x))$ for $x\in[0,1]$ and is also increasing on the region $v_\lambda>F(J)$. See Figure 1 for details about the functions F and f with $\gamma>1$. Herewith $F'(J)=0$ and $f'(F(J))=\infty$ exactly correspond to some degenerate points.



(a) subsonic-sonic solution: $\rho_\lambda \geq J$ (b) subsonic-sonic solution: $v_\lambda \geq F(J)$

FIG. 1. *subsonic-sonic case.*

Before investigating the quasi-neutral limit to the problem (1.7), we need to illustrate some preliminaries about the functions v_λ and $f'(v_\lambda)$ in the following lemmas.

LEMMA 2.1. *There exist a small number $\delta > 0$ and a constant $B_1 > 0$ independent of λ such that for sufficiently small $\lambda < \delta$, it follows that*

$$v_\lambda(x) \geq F(J) + B_1 \frac{x}{\lambda}, \quad x \in [0, \lambda],$$
$$v_\lambda(x) \geq F(J) + B_1 x, \quad x \in [\lambda, \delta].$$

At the same time, it holds for a constant B_2 independent of λ that

$$v_\lambda(x) \geq F(J) + B_2 \frac{1-x}{\lambda}, \quad x \in [1-\lambda, 1],$$
$$v_\lambda(x) \geq F(J) + B_2(1-x), \quad x \in [1-\delta, 1-\lambda].$$

Proof. First, we notice that the function $v_\lambda(x)$ has some similar features with respect to λ near the endpoints $x = 0^+$ and $x = 1^-$. Therefore, let $\delta > 0$ be a small number independent of λ satisfying $\delta > \lambda > 0$; then it suffices to prove the inequalities on $[0, \delta]$,

$$v_\lambda(x) \geq F(J) + B_1 \frac{x}{\lambda}, \quad x \in [0, \lambda],$$

and

$$v_\lambda(x) \geq F(J) + B_1 x, \quad x \in [\lambda, \delta],$$

for sufficiently small λ. For simplicity, this proof is divided into four steps as follows.

Step 1. Denote $(\rho_\lambda, \Phi_\lambda, v_\lambda)$ by (n, Ψ, v) and reconsider (1.7) as the following equations:

$$\begin{cases} \dfrac{n^{\gamma+1} - J^{\gamma+1}}{n^3} n_x = \Psi_x - \dfrac{J^{\frac{2}{\gamma+1}}}{\tau n}, \\ \Psi_{xx} = \dfrac{n-b}{\lambda^2}. \end{cases} \tag{2.2}$$

Now we set the electric field $E := \Psi_x$ and let

$$\tilde{E} := E - \frac{J^{\frac{2}{\gamma+1}}}{\tau n}, \qquad c_\gamma := \frac{n^3(n-J)}{n^{\gamma+1} - J^{\gamma+1}};$$

then we reduce the system (2.2) to

$$\begin{cases} n_x = c_\gamma \cdot \dfrac{\tilde{E}}{n-J}, \\ \tilde{E}_x = \dfrac{n-b}{\lambda^2} + \dfrac{n_x}{\tau n^2}. \end{cases} \tag{2.3}$$

Moreover, the boundary conditions are subject to $n(0) = n(1) = J$. Of course, it is easy to see that the solution $(n, \tilde{E})$ of (2.3) satisfies

$$\frac{d\tilde{E}}{dn} = \frac{n-b}{\lambda^2 c_\gamma} \Big/ \frac{\tilde{E}}{n-J} + \frac{1}{\tau n^2}. \tag{2.4}$$

From the trajectories $(n, \tilde{E})$ of (2.4), we are ready to acquire some local features of the function $v = F(n)$ near $x = 0^+$.

Step 2. In this step, we will show the existence of a constant $B_1 > 0$ and a small parameter $\varepsilon > 0$ independent of λ such that for sufficiently small λ,

$$\tilde{E}(n) \geq \frac{B_1}{\lambda}, \quad n \in [J, J+\varepsilon].$$

Suppose that there is a point $\check{n} \in [J, J+\varepsilon]$ such that $0 < \tilde{E}(\check{n}) < \frac{B_1}{\lambda}$. In this situation, choose $B_1 = \eta < 2\eta < \varepsilon$ with a constant η independent of λ. Next, as shown in Figure 2, we notice that the point $(\check{n}, \tilde{E}(\check{n}))$ must be in one of the areas (A) and (B). Here the triangular area (A) is bounded by the lines $n = J$, $\tilde{E} = \frac{\eta}{\lambda}$, and $\tilde{E} = \frac{c_1(n-J)}{\lambda}$, and the trapezoidal area (B) is bounded by the lines $\tilde{E} = \frac{c_1(n-J)}{\lambda}$, $n = J+\varepsilon$, $\tilde{E} = 0$, and $\tilde{E} = \frac{\eta}{\lambda}$, where the constant c_1 is independent of λ, ε. Note that

$$\frac{d\tilde{E}}{dn} \leq \frac{1}{\tau n^2} \leq \frac{1}{\tau_0 J^2}$$

in the area (A) and let λ be small enough. Hence if the point is in the area (A), it is easy to see that the trajectory $(n, \tilde{E})$ of (2.4) must intersect the line $\tilde{E} = \frac{c_1(n-J)}{\lambda}$ on the regions $n \in [J, J+\varepsilon]$ and $\tilde{E} \in [0, \frac{2\eta}{\lambda}]$. Additionally, if the trajectory passes the area (B), then we derive that $\frac{\tilde{E}}{n-J} \leq \frac{c_1}{\lambda}$ and

$$\frac{d\tilde{E}}{dn} \leq -\frac{c_2}{\lambda} \quad \text{for a constant } c_2 > 0 \text{ independent of } \lambda \text{ and } \varepsilon.$$

From this, the trajectory also intersects with the line $\tilde{E} = \frac{c_1(n-J)}{\lambda}$ in the above rectangular region $[J, J+\varepsilon] \times [0.\frac{2\eta}{\lambda}]$ for sufficiently small λ.

Next, we take a number x_1 such that the point $(n(x_1), \tilde{E}(x_1))$ is the intersection of the trajectory $(n, \tilde{E})$ and the line $\tilde{E} = \frac{c_1(n-J)}{\lambda}$. From the first equation of (2.3), it holds that

$$n(x_1) = n(0) + \int_0^{x_1} c_\gamma \cdot \frac{\tilde{E}}{n-J} dx \geq J + \frac{cc_1 x_1}{\lambda},$$

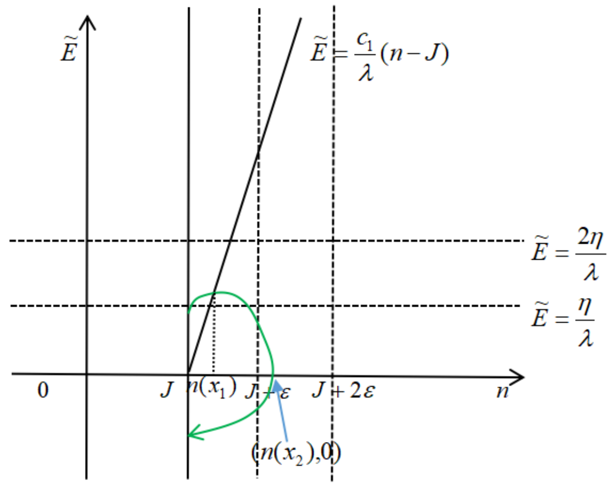


FIG. 2. *Counterexample 1: $0<\tilde{E}(n)<\frac{B_1}{\lambda}$ for some points n.*

and then

$$x_1 \le \frac{\varepsilon\lambda}{\underline{c}c_1},$$

where $\bar{c}$ and $\underline{c}$ are the upper and lower bounds of c_γ independent of λ. Starting from the point x_1, it is noted that the green trajectory in Figure 2 intersects with the line $\tilde{E}=0$ at the point $(n(x_2),0)$. In this process, we can easily get from

$$0<\tilde{E}\le\frac{c_1(n-J)}{\lambda} \quad \text{and} \quad \frac{d\tilde{E}}{dn}\le -\frac{C}{\lambda}$$

that $n\le J+2\varepsilon$ when λ is sufficiently small. Therefore, one can see from (2.3) that

$$\tilde{E}(x_2)=\tilde{E}(x_1)+\int_{x_1}^{x_2}\left(\frac{n-b}{\lambda^2}+\frac{n_x}{\tau n^2}\right)dx \le \frac{2\eta}{\lambda}+\int_{x_1}^{x_2}\left(\frac{J+2\varepsilon-\underline{b}}{\lambda^2}+\frac{\bar{c}c_1}{\tau J^2\lambda}\right)dx.$$

Choosing $\varepsilon\le\frac{\underline{b}-J}{4}$ and $\lambda\le\frac{\tau(\underline{b}-J)J^2}{\bar{c}c_1}$, we have

$$\tilde{E}(x_2)-\frac{\varepsilon}{\lambda}\le\left(\frac{J-\underline{b}}{2\lambda^2}+\frac{\bar{c}c_1}{\tau J^2\lambda}\right)(x_2-x_1)\le\frac{J-\underline{b}}{4\lambda^2}(x_2-x_1),$$

which leads to

$$x_2\le(x_2-x_1)+x_1\le\frac{4\lambda^2}{\underline{b}-J}\left(\frac{\varepsilon}{\lambda}-\tilde{E}(x_2)\right)+\frac{\varepsilon\lambda}{\underline{c}c_1}\le\frac{4\lambda\varepsilon}{\underline{b}-J}+\frac{\varepsilon\lambda}{\underline{c}c_1}=:C\lambda\ll 1.$$

In the proof of this step, we only discuss the shape of the trajectory in the region $\tilde{E}\ge 0$; the case of $\tilde{E}<0$ can be proved in a similar way. In conclusion, if λ is sufficiently small, we have shown that the length of the green trajectory of Figure 2 is less than 1. This is a contradiction. Thus, we have $\tilde{E}(n)\ge\frac{\eta}{\lambda}=\frac{B_1}{\lambda}$ for $J\le n\le J+\varepsilon$.

Step 3. In this step, we are going to prove that

$$v(x)\ge F(J)+B_1x \quad \text{on} \quad [0,\delta]$$

for a constant δ.

From the first equation of (2.2) and the result of Step 2, it follows that

$$v_x = F(n)_x = \tilde{E} \geq \frac{B_1}{\lambda} \quad \text{for} \quad n \in [J, J+\varepsilon];$$

then there is a number $\delta_0(\varepsilon) > 0$ such that v first reaches the line $v|_{x=\delta_0} = F(J+\varepsilon)$. Thus, for all $x \in [0, \delta_0]$, it holds that

$$v(x) \geq F(J) + \frac{B_1 x}{\lambda}.$$

Now we denote a small constant $\delta > \delta_0$, and we claim that $v(x) > F(J+\varepsilon)$ on $(\delta_0, \delta]$. In fact, if this is not true, then the function v must go back to the line $v = F(J+\varepsilon)$ at a constant $\delta_1 \in (\delta_0, \delta]$. Without loss of generality, we set $\delta_1 = \delta$. In this situation, we derive from (2.2) that near $x = \delta^+$,

$$v_{xx}(x) = \frac{n(x)-b}{\lambda^2} + \frac{n_x(x)}{\tau n^2} \leq \frac{n(x)-b}{\lambda^2} \leq \frac{J+\varepsilon-b}{\lambda^2} \leq \frac{J-b}{2\lambda^2} < 0 \quad \text{if} \quad 0 < \varepsilon \leq \frac{b-J}{2}.$$

Hence, by Taylor expansion

$$v(x) = v(\delta) + v_x(\delta)(x-\delta) + \int_\delta^x (x-s) v_{ss}(s) ds \quad \text{for} \quad x > \delta,$$

we have

$$v(x) - v(\delta) \leq (x-\delta)^2 \max_{s \in [\delta, x]} v_{ss}(s) \leq \frac{J-b}{2\lambda^2}(x-\delta)^2. \tag{2.5}$$

If $v(\delta_2) = J$ at $x = \delta_2 > \delta$, then it holds from (2.5) that

$$\delta_2 = (\delta_2 - \delta) + \delta \leq C\lambda + \delta < 2\delta \ll 1,$$

which contradicts the known condition $\delta_2 = 1$. See the blue trajectory in Figure 3 as a counterexample for details. Since, we get from

$$v(x) \geq F(J) + \frac{B_1 x}{\lambda} \quad \text{on} \quad [0, \delta_0]$$

and

$$v(x) > F(J+\varepsilon) \quad \text{on} \quad (\delta_0, \delta]$$

that

$$v(x) \geq F(J) + B_1 x \quad \text{on} \quad [0, \delta].$$

Step 4. In this step, we prove that if λ is small enough, then

$$v(x) \geq F(J) + \frac{B_1 x}{\lambda} \quad \text{on} \quad [0, \lambda].$$

From Step 3, we get

$$v(x) \geq F(J) + \frac{B_1 x}{\lambda} \quad \text{on} \quad [0, \delta_0].$$

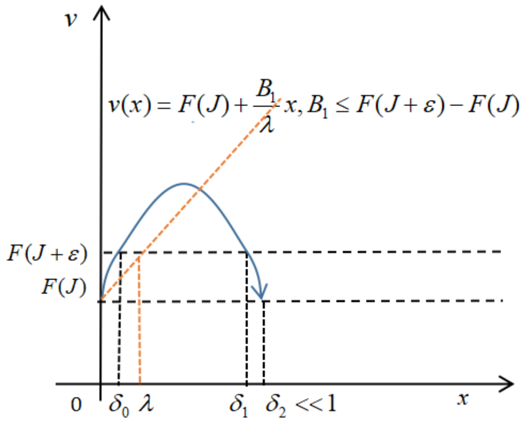


FIG. 3. *Counterexample* 2.

Now if $\lambda \le \delta_0$, then the result of this step directly holds. When $\delta_0 < \lambda < \delta$, we get from $B_1 = \eta < \varepsilon$ that

$$F(J) + \frac{B_1}{\lambda}x \le F(J) + \eta \quad \text{for} \quad x \in [0, \lambda];$$

see the red part of Figure 3. So, we redefine the constant $B_1 := \min\{\eta, F(J+\varepsilon) - F(J)\}$, and then this step can be finished.

Thus, this lemma is proved. $\square$

LEMMA 2.2. *$f'(v_\lambda)$ is uniformly bounded in $L^1(0,1)$ with respect to λ satisfying*

$$\int_0^1 f'(v_\lambda(x))dx \le B_3, \tag{2.6}$$

where $B_3 > 0$ is a constant independent of λ.

Proof. First, from $f = F^{-1}$ and $J \le \rho_\lambda \le \bar{b}$, we note that

$$f'(v_\lambda) = \frac{1}{F'(\rho_\lambda)} = \frac{\rho_\lambda^3}{\rho_\lambda^{\gamma+1} - J^{\gamma+1}} = O\left(\frac{1}{\rho_\lambda - J}\right), \tag{2.7}$$

where we have used

$$\rho_\lambda^{\gamma+1} = J^{\gamma+1} + (\gamma+1)\xi_1^\gamma(\rho - J) \quad \text{with some} \quad \xi_1 \in [J, \rho_\lambda].$$

From $F'(J) = 0$ and the bound of F'' in $[J, \bar{b}]$, we obtain, for some $\xi_2 \in [J, \rho_\lambda]$,

$$v_\lambda - F(J) = F(\rho_\lambda) - F(J) = \frac{F''(\xi_2)(\rho_\lambda - J)^2}{2} = O((\rho_\lambda - J)^2),$$

which yields from (2.7) that

$$f'(v_\lambda) = O\left(\frac{1}{\sqrt{v_\lambda - F(J)}}\right) \tag{2.8}$$

if $v_\lambda \in [J, J+\delta]$ with a small number $\delta > 0$. It follows from Lemma 2.1 and (2.8) that

$$\int_0^\delta f'(v_\lambda(x))dx \le C\int_0^\delta x^{-\frac{1}{2}}dx \le C\delta, \quad \int_{1-\delta}^1 f'(v_\lambda(x))dx \le C\int_{1-\delta}^1 (1-x)^{-\frac{1}{2}}dx \le C\delta.$$

Additionally, we derive from (1.8) and (2.7) that

$$\int_{\delta}^{1-\delta} f'(v_\lambda(x))dx \le C\int_{\delta}^{1-\delta}\frac{dx}{\sin(\pi x)} \le \frac{C}{\delta},$$

where $C>0$ is a constant independent of λ.

Hence, let $\delta \ge \delta_0 > 0$ for a constant δ_0; then (2.6) follows immediately. The proof is complete. ∎

Remark 2.3. Lemmas 2.1 and 2.2 explain some fundamental properties of the sequence of solutions $(v_\lambda)_{\lambda>0}$ to equations (2.1) in the neighborhood of the boundary points, which will be a key point in removing the effects of the degenerate boundary in the proof of the following convergence estimates.

3. L^2-estimate results. For the case when the boundary is degenerate, the purpose of this section is to show an L^2-estimate of order $O(\lambda^{\frac{1}{2}})$ to the error term $\rho_\lambda - b$, just as the results in [34]. In view of Sobolev imbedding $H^1(0,1)\hookrightarrow C^0[0,1]$, we state the following assumption to make sense of $b_0 := b(0)$ and $b_1 := b(1)$:

(A4) $b \in H^1(0,1)$.

Let $(\rho_\lambda, \Phi_\lambda)_{\lambda>0}$ be a sequence of solutions to equations (1.7), and let (ϱ,ϕ) be its limit as $\lambda \to 0$. Formally, (ϱ,ϕ) satisfies

$$\frac{d\left(F(\varrho)-\phi\right)}{dx} = -\frac{j}{\tau\varrho}, \quad \varrho = b(x). \tag{3.1}$$

Since $b(0) > J$ and $b(1) > J$, the boundary layers will occur near $x=0$ and $x=1$, respectively.

In order to solve the limit (ϱ,ϕ), some boundary conditions are needed. For this purpose, we define

$$G_\lambda := F(\rho_\lambda) - \Phi_\lambda \quad \text{and} \quad G := F(\varrho) - \phi, \tag{3.2}$$

which yield from (1.7) that

$$\frac{dG_\lambda}{dx} = -\frac{j}{\tau\rho_\lambda}. \tag{3.3}$$

From (3.3), $J \le \rho_\lambda \le \bar{b}$, and Poincaré's inequality, it is concluded that G_λ is bounded in $H^1(0,1)$ independent of λ. Then from (3.2), $G_\lambda \in L^\infty(0,1)$, and $\rho_\lambda \in L^\infty(0,1)$, it follows that Φ_λ is bounded in $L^\infty(0,1)$ independent of λ. Therefore, it is clear that G is bounded in $H^1(0,1)$, which, together with $\varrho = b \in H^1(0,1)$, leads to $\phi \in H^1(0,1)$. Due to the compact imbedding $H^1(0,1)\hookrightarrow C^0[0,1]$, $(G_\lambda)_{\lambda>0}$ is uniformly convergent to the limit G. Thus, we give the boundary condition $\phi(0)$ as

$$\phi(0) = F(\varrho(0)) - F(\rho_\lambda(0)) + \Phi_\lambda(0) = F(\varrho(0)) - F(J), \quad \varrho(0) = b_0. \tag{3.4}$$

It is obvious that system (3.1), (3.4) admits a unique solution (ϱ,ϕ). Also, we obtain the following relation:

$$\phi(1) = \phi(0) + \int_0^1 (F(\varrho)-G)_x dx = F(b_1) - F(J) + \int_0^1 \frac{j}{\tau b(x)}dx. \tag{3.5}$$

After that, to show some key information about the boundary layers, the solution $(\rho_\lambda, \Phi_\lambda)$ in a neighborhood of $x=0$ may be approximated by

$$(\rho_\lambda, \Phi_\lambda) := (\varrho(0) + \varrho_0(y), \phi(0) + \varphi(y))$$

for a fast variable $y = \frac{x}{\lambda}$. Similarly, we have also, in a neighborhood of $x = 1$, that

$$(\rho_\lambda, \Phi_\lambda) := (\varrho(1) + \varrho_1(z), \phi(1) + \psi(z)),$$

with the fast variable $z = \frac{1-x}{\lambda}$. Next, plugging the approximate solutions $(\varrho(0) + \varrho_0(y), \phi(0) + \varphi(y))$ and $(\varrho(1) + \varrho_1(z), \phi(1) + \psi(z))$ into equations (1.7) and neglecting the error term $O(\lambda)$, we obtain the following boundary layer equations:

$$\begin{aligned} &F(\varrho_0 + b_0)_y = \varphi_y, \quad \varphi_{yy} = \varrho_0, \quad y \in [0, +\infty), \\ &F(\varrho_1 + b_1)_z = \psi_z, \quad \psi_{zz} = \varrho_1, \quad z \in [0, +\infty). \end{aligned} \tag{3.6}$$

In general, the $L^2(0,1)$-estimate of $\rho_\lambda - \varrho$ depends exactly on λ, so that from the formulas (3.4) and (3.5), the boundary conditions of (3.6) are denoted by

$$\begin{aligned} &\varrho_0(0) = J - b_0, \quad \varphi(0) = F(J) - F(b_0), \quad \lim_{y \to +\infty} \varrho_0(y) = 0, \quad \lim_{y \to +\infty} \varphi(y) = 0, \\ &\varrho_1(0) = J - b_1, \quad \psi(0) = F(J) - F(b_1), \quad \lim_{z \to +\infty} \varrho_1(z) = 0, \quad \lim_{z \to +\infty} \psi(z) = 0. \end{aligned} \tag{3.7}$$

Recalling the results in [11, 36], we can prove the existence of two pairs of solutions (ϱ_0, φ) and (ϱ_1, ψ) to equations (3.6) and (3.7) and show the results below.

Lemma 3.1. *Let* (A1)–(A4) *hold; then the equations* (3.6) *and* (3.7) *admit the solutions* (ϱ_0, φ) *and* (ϱ_1, ψ)*, respectively, satisfying*

$$\begin{aligned} &|\varrho_0|, |\varrho_0'(y)|, |\varphi(y)|, |\varphi'(y)| \le C_1 e^{-C_2 y}, \\ &|\varrho_1|, |\varrho_1'(z)|, |\psi(z)|, |\psi'(z)| \le C_3 e^{-C_4 z} \end{aligned} \tag{3.8}$$

for any $y, z \in (0, +\infty)$*. Here* C_i $(i = 1, 2, 3, 4)$ *are positive constants.*

Proof. Let $m_i(y) = \varrho(0) + \varrho_0(y)$ and $m_e(z) = \varrho(1) + \varrho_1(z)$. We can directly check from (3.6) and (3.7) that $F(m_i) = \varphi + F(b_0)$, $F(m_e) = \psi + F(b_1)$, and $F'(J) = 0$. Therefore, we get a smooth, strictly increasing function f, given by $f = F^{-1}$, such that

$$\begin{aligned} \varphi_{yy} &= f(\varphi + F(b_0)) - b_0 =: f_1(\varphi), \\ \psi_{zz} &= f(\psi + F(b_1)) - b_1 =: f_2(\psi). \end{aligned}$$

Here the function f_1 (resp., f_2) is of class C^0 in φ (resp., ψ), strictly increasing for $\varphi \in [F(J) - F(b_0), 0]$ (resp., $\psi \in [F(J) - F(b_1), 0]$) and $f_1(0) = 0$ (resp., $f_2(0) = 0$).

Let us replace φ by $\bar\varphi = -\varphi$, satisfying

$$\begin{aligned} &\bar\varphi_{yy} = b_0 - f(F(b_0) - \bar\varphi) =: \bar f_1(\bar\varphi), \\ &\bar\varphi(0) = F(b_0) - F(J) > 0, \quad \bar\varphi(\infty) = 0. \end{aligned} \tag{3.9}$$

Indeed, we derive, for $\bar\varphi \in [0, F(b_0) - F(J)]$, that

$$(\bar f_1)_{\bar\varphi}(\bar\varphi) = f'(F(b_0) - \bar\varphi) > 0,$$

and there is a constant $c_0 > 0$ depending on $F(b_0)$ such that $(\bar f_1)_{\bar\varphi}(\bar\varphi) \ge c_0$. However, we note that

$$(\bar f_1)_{\bar\varphi}(F(b_0) - F(J)) = f'(F(J)) = +\infty, \tag{3.10}$$

which exactly corresponds to the degenerate boundary. Referring to Lemma 2.1 in [11], we see that problem (3.9) has a unique solution $\bar{\varphi}$ satisfying

$$y = \int_{\bar{\varphi}}^{F(b_0)-F(J)} \frac{ds}{\sqrt{2\mathcal{F}(s)}},$$

where

$$\mathcal{F}(s) = \int_0^s \bar{f}_1(t)dt. \tag{3.11}$$

So, it is easy to see that $\bar{\varphi}$ is deceasing in y, corresponding to the relation

$$0 \le \bar{\varphi}(y) \le F(b_0) - F(J) \quad \text{for any} \quad y \in [0, +\infty).$$

In fact, based on $\mathcal{F}(0) = 0$ and $\mathcal{F}'(0) = \bar{f}_1(0) = 0$, we get, for any $s \in [0, F(b_0) - F(J)]$, that

$$\mathcal{F}(s) = (\bar{f}_1)_{\bar{\varphi}}(\xi) \cdot \frac{s^2}{2} \quad \text{with} \quad \xi \in [0, s]. \tag{3.12}$$

Thus, it holds that

$$0 \le y \le \int_{\bar{\varphi}}^{F(b_0)-F(J)} \frac{ds}{\sqrt{c_0}s} \le \frac{1}{\sqrt{c_0}} \ln\left(\frac{F(b_0) - F(J)}{\bar{\varphi}}\right),$$

and further,

$$0 \le \bar{\varphi} \le [F(b_0) - F(J)]e^{-\sqrt{c_0}y}.$$

In addition, from (3.11), (3.12), and the relation $\bar{\varphi}_y = -\sqrt{\mathcal{F}(\bar{\varphi})}$, we have for some $\xi \in [0, \bar{\varphi}]$ that

$$\bar{\varphi}_y = -\sqrt{\frac{(\bar{f}_1)_{\bar{\varphi}}(\xi)}{2}} \cdot \bar{\varphi} = -\sqrt{\int_0^{\bar{\varphi}} \bar{f}_1(s)ds} \quad \forall \bar{\varphi} \in [0, F(b_0) - F(J)],$$

which implies that $(\bar{f}_1)_{\bar{\varphi}}(\xi)$ is bounded for any $\bar{\varphi} \in [0, F(b_0) - F(J)]$ although (3.10) holds. As a result, it follows that

$$|\bar{\varphi}_y| \le C|\bar{\varphi}| \le C_1 e^{-C_2 y}.$$

Thus, the first inequality of (3.8) follows immediately.

In the same way, one can see that the second result of (3.8) also holds. Hence, the proof is complete. □

Now we are going to investigate the quasi-neutral limit of (1.7) as $\lambda \to 0$ and to obtain the convergence rate of the solution $(\rho_\lambda, \Phi_\lambda)$.

THEOREM 3.2. *Assume that* (A1)–(A4) *hold, let* $(\rho_\lambda, \Phi_\lambda)_{\lambda>0}$ *be a sequence of solutions to system* (1.7), *and let* (ϱ, Φ) *be the unique solution to equations* (3.1) *and* (3.4). *Then, as* $\lambda \to 0$, *it holds that*

$$\|\rho_\lambda - \varrho\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}}, \quad \|\Phi_\lambda - \Phi\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}},$$

with some positive constants C *independent of* λ.

Proof. In the proof of this lemma, all the constants C denote general constants independent of λ. Let $R_\lambda(x) := \rho_\lambda(x) - b(x) - m_0(x) - m_1(x)$ so that

$$R_\lambda(0) = -m_1(0) \quad \text{and} \quad R_\lambda(1) = -m_0(1),$$

where $m_0(x) := \varrho_0\left(\frac{x}{\lambda}\right)$ and $m_1(x) := \varrho_1\left(\frac{1-x}{\lambda}\right)$. In view of the Poisson equation $(1.7)_2$, we get

$$\begin{aligned}
&\|R_\lambda\|^2_{L^2(0,1)} \\
&\quad = \int_0^1 (\rho_\lambda(x) - b(x) - m_0(x) - m_1(x)) R_\lambda(x) dx \\
&\quad = \int_0^1 (\rho_\lambda(x) - b(x)) R_\lambda(x) dx - \int_0^1 (m_0(x) + m_1(x)) R_\lambda(x) dx \\
&\quad = \lambda^2 \int_0^1 (\Phi_\lambda)_{xx} R_\lambda(x) dx - \int_0^1 (m_0(x) + m_1(x)) R_\lambda(x) dx \\
&\quad = -\lambda^2 \int_0^1 (\Phi_\lambda)_x (\rho_\lambda(x) - b(x) - m_0(x) - m_1(x))_x dx - \int_0^1 (m_0(x) m_1(x)) R_\lambda(x) dx \\
&\qquad - \lambda^2 [(\Phi_\lambda)_x(0) m_1(0) + (\Phi_\lambda)_x(1) m_0(1)] \\
&\quad = -\lambda^2 \int_0^1 (\Phi_\lambda)_x [f(v_\lambda)]_x dx + \lambda^2 \int_0^1 (\Phi_\lambda)_x (b(x) + m_0(x) + m_1(x))_x dx \\
&\qquad - \int_0^1 (m_0(x) + m_1(x)) R_\lambda(x) dx - \lambda^2 [(\Phi_\lambda)_x(0) m_1(0) + (\Phi_\lambda)_x(1) m_0(1)] \\
&\quad = -\lambda^2 \int_0^1 f'(v_\lambda) |(\Phi_\lambda)_x|^2 dx - \lambda^2 \int_0^1 f'(v_\lambda) (\Phi_\lambda)_x (G_\lambda)_x dx \\
&\qquad + \lambda^2 \int_0^1 (\Phi_\lambda)_x (b(x) + m_0(x) + m_1(x))_x dx - \int_0^1 (m_0(x) + m_1(x)) R_\lambda(x) dx \\
&\qquad - \lambda^2 [(\Phi_\lambda)_x(0) m_1(0) + (\Phi_\lambda)_x(1) m_0(1)] \\
&\quad =: -\lambda^2 \int_0^1 f'(v_\lambda) |(\Phi_\lambda)_x|^2 dx + I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where $v_\lambda = F(\rho_\lambda) = G_\lambda + \Phi_\lambda$ and $f(v_\lambda) = f(G_\lambda + \Phi_\lambda)$. Since $\{G_\lambda\}_{\lambda>0}$ is bounded in $L^\infty(0,1)$, Hölder's inequality and (2.6) imply

$$\begin{aligned}
I_1 &= -\lambda^2 \int_0^1 f'(v_\lambda)(\Phi_\lambda)_x (G_\lambda)_x dx \le \frac{\lambda^2}{4} \int_0^1 f'(v_\lambda) |(\Phi_\lambda)_x|^2 dx + \lambda^2 \int_0^1 f'(v_\lambda) |G_\lambda)_x|^2 dx \\
&\le \frac{\lambda^2}{4} \int_0^1 f'(v_\lambda) |(\Phi_\lambda)_x|^2 dx + \lambda^2 ||G_\lambda)_x||^2_{L^\infty(0,1)} \int_0^1 f'(v_\lambda) dx \\
&\le \frac{\lambda^2}{4} \int_0^1 f'(v_\lambda) |(\Phi_\lambda)_x|^2 dx + C\lambda^2.
\end{aligned}$$

Note from Lemma 3.1 that $\{\lambda (m_0)_x\}_{\lambda>0}$ and $\{\lambda (m_1)_x\}_{\lambda>0}$ are bounded in $L^\infty(0,1)$. Additionally, the elliptic condition $\rho_\lambda \ge J$, together with (2.7), yields that $f'(v_\lambda) \ge \underline{C}$ for some constant $\underline{C} > 0$. Therefore, the above conditions combined with $b \in H^1(0,1)$ give

$$
\begin{aligned}
I_2 &= \lambda^2 \int_0^1 (\Phi_\lambda)_x (b(x)+m_0(x)+m_1(x))_x dx \\
&\le \frac{\underline{C}\lambda^2}{4} \int_0^1 |(\Phi_\lambda)_x|^2 dx + \frac{\lambda^2}{\underline{C}} \int_0^1 |(b(x)+m_0(x)+m_1(x))_x|^2 dx \\
&\le \frac{\underline{C}\lambda^2}{4} \int_0^1 |(\Phi_\lambda)_x|^2 dx + C\lambda.
\end{aligned}
$$

On the other hand, again from Lemma 3.1, it is concluded that

$$
\|m_0\|_{L^1(0,1)}, \|m_1\|_{L^1(0,1)} \le C\lambda \quad \text{and} \quad |m_0(1)|, |m_1(0)| \le C\lambda,
$$

which, in combination with the $L^\infty(0{,}1)$ boundedness of the sequences $\{R_\lambda\}_{\lambda>0}$ and $\{\lambda^2(\Phi_\lambda)_x\}_{\lambda>0}$, implies that

$$
I_3 + I_4 = -\int_0^1 (m_0(x)+m_1(x)) R_\lambda(x) dx - \lambda^2 \left[(\Phi_\lambda)_x(0) m_1(0) + (\Phi_\lambda)_x(1) m_0(1)\right] \le C\lambda.
$$

Consequently, it is easy to verify that

$$
\|R_\lambda\|^2_{L^2(0,1)} + \frac{\underline{C}\lambda^2}{2} \|(\Phi_\lambda)_x\|^2_{L^2(0,1)} dx \le C\lambda.
$$

Noticing by (3.8) that $\|m_0\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}}$ and $\|m_1\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}}$, we easily get

$$
\|\rho_\lambda - b\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}}.
$$

Afterwards, it follows from (3.1), (3.3), $\varrho = b$, and Poincaré's inequality that

$$
\|G_\lambda - G\|_{H^1(0,1)} \le C\|\rho_\lambda - \varrho\|_{L^2(0,1)},
$$

and then, from the definition of G_λ and G, it holds that

$$
\begin{aligned}
\|\Phi_\lambda - \Phi\|_{L^2(0,1)} &\le C\left(\|\rho_\lambda - \varrho\|_{L^2(0,1)} + \|G_\lambda - G\|_{L^2(0,1)}\right) \\
&\le C\|\rho_\lambda - \varrho\|_{L^2(0,1)} \le C\lambda^{\frac{1}{2}}.
\end{aligned}
$$

Thus, we end the proof of this theorem. ∎

Finally, Theorem 1.2 can be obtained directly by Theorem 3.2.

4. L^∞-estimate results. In section 3, we mainly studied the quasi-neutral limit as $\lambda \to 0$ for the solution $(\rho_\lambda, \Phi_\lambda)$ of system (1.7) and showed an L^2-estimate of order $O(\lambda^{\frac{1}{2}})$ for $\rho_\lambda - b$ when the boundary is degenerate and the doping profile $b(x)$ satisfies $b \in H^1(0,1)$. In [36], a stronger estimate in $L^\infty(0,1)$ has been obtained with the assumption of higher regularity of the function $b(x)$ for the case of nondegenerate boundary. Therefore, when the boundary is degenerate, we also want to study an $L^\infty(0,1)$-estimate of $\rho_\lambda - b - n_0 - n_1$, where n_0 and n_1 are the boundary layers profiles.

It is obvious that finding the solution v_λ of (2.1) is essentially consistent with solving the subsonic-sonic solution ρ_λ of (1.7). Thus, owing to the existence of a unique solution $J \le \rho_\lambda \le \bar{b}$, the limit equation of v_λ can be denoted by $v = F(\varrho) = F(b)$. In addition, since $b(0) \ne J$ and $b(1) \ne J$, there exist the boundary layer profiles $v_0(y)$ and $v_1(z)$ satisfying

$$
\begin{cases}
\dfrac{d^2 v_0}{dy^2} = f(F(b_0) + v_0) - b_0, y \in (0, +\infty), \\
v_0(0) = F(J) - F(b_0), \quad \lim\limits_{y \to +\infty} v_0(y) = 0,
\end{cases}
\tag{4.1}
$$

$$
\begin{cases} \dfrac{d^2 v_1}{dz^2} = f(F(b_1)+v_1) - b_1, z \in (0,+\infty), \\ v_1(0) = F(J) - F(b_1), \quad \lim\limits_{z\to+\infty} v_1(z) = 0. \end{cases} \tag{4.2}
$$

As showed in Lemma 3.1, the unique solution v_0 (resp., v_1) decays exponentially as $y \to +\infty$ (resp., $z \to +\infty$), that is,

$$
|v_0(y)|, \quad |v_0'(y)| \le Ce^{-\mu_1 y} \quad \big(|v_1(z)|, \quad |v_1'(z)| \le Ce^{-\mu_2 z}\big), \tag{4.3}
$$

where C, μ_1, and μ_2 are the positive constants independent of λ.

Now define $v_\lambda(x) = F(b(x)) + v_0\left(\frac{x}{\lambda}\right) + v_1\left(\frac{1-x}{\lambda}\right) + \lambda r_\lambda(x)$. Due to the uniqueness of the solutions v_0 and v_1, we find that the existence of a unique solution r_λ is actually equivalent to the existence of a unique solution v_λ of (2.1). Therefore, substituting the solution v_λ into equation (2.1), we have a unique solution r_λ to the equation

$$
\begin{aligned}
&-\lambda\frac{d^2 r_\lambda}{dx^2} - \lambda g(F(b)+v_0+v_1+\lambda r_\lambda)\frac{dr_\lambda}{dx} \\
&\quad + \frac{1}{\lambda^2}\big[f(F(b)+v_0+v_1+\lambda r_\lambda) - f(F(b)+\chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1)\big] \\
&= \frac{d^2F(b)}{dx^2} + g(F(b)+v_0+v_1+\lambda r_\lambda)\frac{d(F(b)+v_0+v_1)}{dx} \\
&\quad + \frac{d^2(v_0+v_1)}{dx^2} - \frac{1}{\lambda^2}\big[f(F(b)+\chi_{[0,1-\delta]}v_0+\chi_{[\delta,1]}v_1) - f(F(b))\big] \\
&=: h(x;\lambda, r_\lambda, v_0, v_1),
\end{aligned} \tag{4.4}
$$

with the boundary conditions

$$
r_\lambda(0) = -\frac{1}{\lambda}v_1\left(\frac{1}{\lambda}\right) \quad \text{and} \quad r_\lambda(1) = -\frac{1}{\lambda}v_0\left(\frac{1}{\lambda}\right). \tag{4.5}
$$

Here we have defined $g(F(b)+v_0+v_1+\lambda r_\lambda) = \dfrac{j}{\tau}\dfrac{f'}{f^2}(v_\lambda)$ and

$$
f(F(b)+\chi_{[0,1-\delta]}v_0+\chi_{[\delta,1]}v_1) = \begin{cases} f(F(b)+v_0), & x\in[0,\delta], \\ f(F(b)+v_0+v_1), & x \in (\delta, 1-\delta), \\ f(F(b)+v_1), & x\in[1-\delta,1], \end{cases}
$$

for characteristic functions $\chi_{[0,1-\delta]}(x)$ and $\chi_{[\delta,1]}(x)$. By the theory of elliptic equations, one can easily see that $\|r_\lambda\|_{H^1(0,1)} \le \frac{C}{\lambda^2}$ for all fixed $\lambda > 0$. Moreover, if λ is sufficiently small, then $F(b)+\chi_{[0,1-\delta]}v_0+\chi_{[\delta,1]}v_1$ always stays in the domain of definition of f for all $x\in[0,1]$ and for a constant δ independent of λ.

Based on the above fact, the equality $v_\lambda - F(b) - v_0 - v_1 = O(\lambda)$ follows if r_λ is uniformly bounded in $L^\infty(0,1)$ with respect to λ. Therefore, we need to show an $L^\infty(0,1)$ uniform bounded estimate of r_λ in the following lemma.

LEMMA 4.1. *Assume that the doping profile satisfies $b \in W^{2,1}(0,1)$; then there exists a constant M_0 independent of λ such that for sufficiently small $\lambda > 0$,*

$$
\|r_\lambda\|_{L^\infty(0,1)} \le M_0. \tag{4.6}
$$

In order to prove Lemma 4.1, we first state two key propositions as follows.

PROPOSITION 4.2. *Denote*

$$\begin{aligned} H(x) &:= H(v_0, F(b), F(b_0))(x) \\ &= \int_0^1 \int_0^1 f''\big((1-\eta)[\theta(F(b)+v_0)+(1-\theta)F(b)] + \eta[\theta(F(b_0)+v_0) \\ &\quad +(1-\theta)F(b_0)]\big) d\eta d\theta. \end{aligned}$$

Let $b \in W^{2,1}(0,1)$; then, for sufficiently small $\lambda > 0$, it follows that

$$\int_0^\delta \left| \frac{[F(b)-F(b_0)]v_0}{\lambda^2} \cdot H(x) \right| dx \le C, \tag{4.7}$$

where the constants C and δ are independent of λ.

Proof. Clearly, it is checked from (4.1) that $(F(b)+v_0)|_{x=0} = (F(b_0)+v_0)|_{x=0} = F(J)$; also note from (2.8) that

$$f''(s) = O\left(\frac{1}{(s-F(J))^{\frac{3}{2}}} \right) \quad \text{near} \quad s = F(J)^+.$$

Hence, we get from $b > J$ that

$$|H(v_0, F(b), F(b_0))| \le \max\{f''(F(b)+v_0), f''(F(b_0)+v_0)\}.$$

If $b(x) \le b_0$ near $x = 0^+$, then $f''(F(b)+v_0) > f''(F(b_0)+v_0)$, so that $|H(v_0, F(b), F(b_0))| \le f''(F(b)+v_0)$. From (4.3), $b \in C^0[0,1]$, and $F(b)+v_0 \ge F(J)$, it follows that near $x = 0^+$,

$$\begin{aligned} |H(v_0, F(b), F(b_0))| &\le \frac{C}{[(v_0 + F(b_0) - F(J)) + F(b) - F(b_0)]^{\frac{3}{2}}} \\ &\le \frac{C}{[v_0 - (F(J) - F(b_0)) - Cx]^{\frac{3}{2}}} \\ &\le \frac{C}{[(1 - e^{\frac{-Cx}{\lambda}}) - Cx]^{\frac{3}{2}}}, \end{aligned}$$

where we can see that $1 - e^{\frac{-Cx}{\lambda}} \ge Cx$ for sufficiently small λ and near $x = 0^+$. Now for $x \in [0, \lambda]$, a direct calculation indicates that there exists a constant $\hat{c}$ independent of λ such that $(1 - e^{-\frac{Cx}{\lambda}}) - Cx \ge \frac{\hat{c}x}{\lambda}$ when λ is small enough. Therefore, we have

$$\int_0^\lambda \left| \frac{[F(b)-F(b_0)]v_0}{\lambda^2} \cdot H(x) \right| dx \le C \int_0^\lambda \frac{xe^{-\frac{Cx}{\lambda}}}{\lambda^2} \frac{1}{\hat{c}(\frac{x}{\lambda})^{\frac{3}{2}}} dx \le C \int_0^1 \frac{e^{-Cy}}{\sqrt{y}} dy \le C. \tag{4.8}$$

In addition, if δ is small enough, then

$$(1 - e^{\frac{-Cx}{\lambda}}) - Cx \ge (1 - e^{-C}) - C\delta \ge \frac{1 - e^{-C}}{2}$$

for $x \in [\lambda, \delta]$. Hence, it follows that

$$\begin{aligned} \int_\lambda^\delta \left| \frac{[F(b)-F(b_0)]v_0}{\lambda^2} \cdot H(x) \right| dx &\le C \int_\lambda^\delta \frac{xv_0}{\lambda^2} dx \le C \int_\lambda^\delta \frac{xe^{-\frac{Cx}{\lambda}}}{\lambda^2} dx \\ &\le C \int_1^\infty ye^{-Cy} dy \le C, \end{aligned}$$

which, together with (4.8), leads to (4.7).

When $b(x) \ge b_0$ near $x = 0^+$, we have the same result by a similar way. Thus, the proof is complete. ∎

PROPOSITION 4.3. *Let $b \in W^{2,1}(0,1)$; then for sufficient small λ,*

$$\|h\|_{L^1(0,1)} \le C,$$

where the constant C is independent of λ, and the function h is defined as in (4.4).

Proof. Now set $h(x) := h_1(x) + h_2(x)$, where

$$h_1(x) = \frac{d^2 F(b)}{dx^2} + g(F(b) + v_0 + v_1 + \lambda r_\lambda)\frac{d(F(b) + v_0 + v_1)}{dx}$$

and

$$h_2(x) = \frac{d^2(v_0 + v_1)}{dx^2} - \frac{1}{\lambda^2}\left[f(F(b) + \chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1) - f(F(b))\right].$$

To derive $h \in L^1(0,1)$, it suffices to show $h_1 \in L^1(0,1)$ and $h_2 \in L^1(0,1)$ in the rest of proof.

First, by the definition of $g(v_\lambda)$, we can get

$$\begin{aligned}\int_0^1 |g(v_\lambda)(v_0+v_1)_x|dx \le{}& C\int_\delta^{1-\delta} |(v_0+v_1)_x|dx \\ &+ \frac{C}{\lambda}\left[\int_0^\delta f'(v_\lambda)e^{-\frac{Cx}{\lambda}}dx + \int_{1-\delta}^1 f'(v_\lambda)e^{-\frac{C(1-x)}{\lambda}}dx\right].\end{aligned}$$

Hereunto, we have

$$\int_\delta^{1-\delta} |(v_0 + v_1)_x|dx \le \int_\delta^{1-\delta}\left[e^{-\frac{Cx}{\lambda}} + e^{-\frac{C(1-x)}{\lambda}}\right]dx \le C_\delta,$$

$$\begin{aligned}\frac{1}{\lambda}\int_0^\delta f'(v_\lambda)e^{-\frac{Cx}{\lambda}}dx &\le \int_0^\lambda f'(v_\lambda)\frac{e^{-\frac{Cx}{\lambda}}}{\lambda}dx + \int_\lambda^\delta f'(v_\lambda)\frac{e^{-\frac{Cx}{\lambda}}}{\lambda}dx \\ &\le C\int_0^\lambda \frac{e^{-\frac{Cx}{\lambda}}}{(\frac{x}{\lambda})^{\frac{1}{2}}\lambda}dx + C\int_\lambda^\delta \frac{e^{-\frac{Cx}{\lambda}}}{\lambda}dx \\ &\le \int_0^1 e^{-C\xi}\xi^{-\frac{1}{2}}d\xi + C\int_1^{\frac{\delta}{\lambda}} e^{-C\xi}d\xi \\ &\le C,\end{aligned}$$

where we have applied (2.8), (4.3), and Lemma 2.1. Similarly, it is easy to see that

$$\frac{1}{\lambda}\int_{1-\delta}^1 f'(v_\lambda)e^{-\frac{C(1-x)}{\lambda}}dx \le C.$$

Thus, we obtain

$$\int_0^1 |g(v_\lambda)(v_0 + v_1)_x|dx \le C.$$

Obviously, by the regularity of $b(x)$, it holds that $\frac{dF(b)}{dx} \in L^\infty(0,1)$ and $\frac{d^2F(b)}{dx^2} \in L^1(0,1)$ with respect to λ. From (2.6), $g(v_\lambda) \in L^1(0,1)$ for any $\lambda > 0$. As a result, it is directly checked that $h_1 \in L^1(0,1)$ for any λ.

In addition, from (4.1) and (4.2), one can prove that

$$
\begin{aligned}
&h_2(x, v_0, v_1, F(b)) \\
&\quad = \frac{1}{\lambda^2}\Big\{\big[f(F(b_0)+v_0) - f(F(b_0))\big] + \big[f(F(b_1)+v_1) - f(F(b_1))\big] \\
&\qquad - \big[f(F(b) + \chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1) - f(F(b))\big]\Big\} \\
&\quad = \frac{1}{\lambda^2}\Big[v_0\int_0^1 f'(\theta(F(b_0)+v_0) + (1-\theta)F(b_0))d\theta + v_1\int_0^1 f'(\theta(F(b_1)+v_1) \\
&\qquad + (1-\theta)F(b_1))d\theta - \big(\chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1\big)\int_0^1 f'(\theta(F(b) + \chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1) \\
&\qquad + (1-\theta)F(b))d\theta\Big].
\end{aligned}
$$

To show $h_2 \in L^1(0,1)$, we next prove the three parts $h_2 \in L^1(0,\delta)$, $h_2 \in L^1(\delta, 1-\delta)$, and $h_2 \in L^1(1-\delta, 1)$, respectively.

Part 1. *Proof of* $h_2 \in L^1(\delta, 1-\delta)$. In the case of $x \in [\delta, 1-\delta]$, one can see that

$$
\begin{aligned}
h_2(x) = \frac{1}{\lambda^2}\Big[&v_0\int_0^1 f'(\theta(F(b_0)+v_0) + (1-\theta)F(b_0))d\theta \\
&+ v_1\int_0^1 f'(\theta(F(b_1)+v_1) + (1-\theta)F(b_1))d\theta \\
&- \big(v_0+v_1\big)\int_0^1 f'(\theta(F(b)+v_0+v_1) + (1-\theta)F(b))d\theta\Big].
\end{aligned}
$$

Here $f'(\theta(F(b_0)+v_0)+(1-\theta)F(b_0))$, $f'(\theta(F(b_1)+v_1)+(1-\theta)F(b_1))$, and $f'(\theta(F(b)+v_0+v_1)+(1-\theta)F(b))$ are bounded for all $\theta \in [0,1]$ and $x \in [\delta, 1-\delta]$ independently of λ. Hence, for sufficiently small λ, we have

$$
\int_\delta^{1-\delta} |h_2(x)|dx \le C_\delta \int_\delta^{1-\delta} \frac{v_0+v_1}{\lambda^2}dx \le \frac{C_\delta}{\delta^2} \le C_{\delta_0},
$$

where $\delta > 0$ is a constant satisfying $\delta \in [\delta_0, 1/2]$, and the constant C_{δ_0} depends only upon $\delta_0 > 0$ independently of λ.

Part 2. *Proof of* $h_2 \in L^1(0,\delta)$. That is, over $[0,\delta]$,

$$
\begin{aligned}
h_2(x) &= \frac{v_0}{\lambda^2}\int_0^1 \Big[f'(\theta(F(b_0)+v_0) + (1-\theta)F(b_0)) - f'(\theta(F(b)+v_0) + (1-\theta)F(b))\Big]d\theta \\
&\quad + \frac{v_1}{\lambda^2}\int_0^1 f'(\theta(F(b_1)+v_1) + (1-\theta)F(b_1))d\theta \\
&= \frac{[F(b_0)-F(b)]v_0}{\lambda^2}\int_0^1\int_0^1 f''\big((1-\eta)[\theta(F(b)+v_0) + (1-\theta)F(b)] \\
&\quad + \eta[\theta(F(b_0)+v_0) + (1-\theta)F(b_0)]\big)d\eta d\theta \\
&\quad + \frac{v_1}{\lambda^2}\int_0^1 f'(\theta(F(b_1)+v_1) + (1-\theta)F(b_1))d\theta \\
&= \frac{[F(b)-F(b_0)]v_0}{\lambda^2}\cdot H(v_0, F(b), F(b_0)) + \frac{v_1}{\lambda^2}\int_0^1 f'(\theta, F(b_1), v_1)d\theta,
\end{aligned}
$$

where the function H is as denoted in Proposition 4.2. From (4.7), we have $\frac{[F(b)-F(b_0)]v_0}{\lambda^2}$.

$H \in L^1(0,\delta)$. Then a direct calculation indicates that by (2.6) and (4.3),

$$\int_0^\delta \left| \frac{v_1}{\lambda^2} \int_0^1 f'(\theta, F(b_1), v_1) d\theta \right| dx \le C \int_0^\delta \frac{f'(F(b_1)+v_1)v_1}{\lambda^2} dx \le \frac{C}{\lambda^2} \int_0^\delta e^{-\frac{1-x}{\lambda}} dx$$
$$\le C \frac{e^{-\frac{1-\delta}{\lambda}}}{\lambda^2} \le C,$$

where we have applied $e^{-\frac{C}{\lambda}}/\lambda^2 \le C$ for all $\lambda > 0$ with a constant C independent of λ. Therefore, we realize that, for some constant C independent of λ,

$$\int_0^\delta |h_2(x)| dx \le C.$$

Part 3. *Proof of* $h_2 \in L^1(1-\delta, 1)$. The proof is similar to Part 2. In a word, we get the result that $h \in L^1(0,1)$ for any λ. The proof is finished. ∎

Proof of Lemma 4.1. By rearranging equation (4.4), we have

$$\begin{aligned}
&\frac{1}{\lambda^2}\left[f(F(b)+v_0+v_1+\lambda r_\lambda) - f(F(b)+\chi_{[0,1-\delta]}v_0 + \chi_{[\delta,1]}v_1)\right] \\
&= \frac{1}{\lambda} \cdot \begin{cases} \dfrac{v_1+\lambda r_\lambda}{\lambda} \int_0^1 f'(F(b)+v_0+\vartheta(v_1+\lambda r_\lambda)) d\vartheta \quad \text{for} \quad x \in [0,\delta], \\[2ex] \int_0^1 f'(F(b)+v_0+v_1+\vartheta\lambda r_\lambda) d\vartheta \cdot r_\lambda \quad \text{for} \quad x \in [\delta, 1-\delta], \\[2ex] \dfrac{v_0+\lambda r_\lambda}{\lambda} \int_0^1 f'(F(b)+v_1+\vartheta(v_0+\lambda r_\lambda)) d\vartheta \quad \text{for} \quad x \in [1-\delta, 1], \end{cases} \\
&= \frac{r_\lambda}{\lambda} \cdot \Big[\chi_{[0,\delta]} \int_0^1 f'(F(b)+v_0+\vartheta(v_1+\lambda r_\lambda)) d\vartheta + \chi_{[\delta,1-\delta]} \int_0^1 f'(F(b)+v_0+v_1+\vartheta\lambda r_\lambda) d\vartheta \\
&\quad + \chi_{[1-\delta,1]} \int_0^1 f'(F(b)+v_1+\vartheta(v_0+\lambda r_\lambda)) d\vartheta\Big] + \frac{\chi_{[0,\delta]}v_1}{\lambda^2} \int_0^1 f'(F(b)+v_0+\vartheta(v_1+\lambda r_\lambda)) d\vartheta \\
&\quad + \frac{\chi_{[1-\delta,1]}v_0}{\lambda^2} \int_0^1 f'(F(b)+v_1+\vartheta(v_0+\lambda r_\lambda)) d\vartheta \\
&=: \frac{r_\lambda}{\lambda} \cdot l_1 + l_2,
\end{aligned}$$

where we note that $(F(b)+v_0+\vartheta(v_1+\lambda r_\lambda))_{x\in[0,\delta]}$, $(F(b)+v_0+v_1+\vartheta\lambda r_\lambda)_{x\in[\delta,1-\delta]}$, and $(F(b)+v_1+\vartheta(v_0+\lambda r_\lambda))_{x\in[1-\delta,1]}$ are certainly in the domain of definition of f' for all $\theta \in [0,1]$ and sufficiently small λ. In what follows, equations (4.4) and (4.5) become

$$\begin{cases} -\lambda(r_\lambda)_{xx} - \lambda g(v_\lambda)(x)(r_\lambda)_x + \dfrac{l_1(x)}{\lambda} r_\lambda = (h-l_2)(x), \quad x \in (0,1), \\ r_\lambda(0) = -\dfrac{1}{\lambda} v_1\left(\dfrac{1}{\lambda}\right), \quad r_\lambda(1) = -\dfrac{1}{\lambda} v_0\left(\dfrac{1}{\lambda}\right). \end{cases} \tag{4.9}$$

Here we check by Lemma 2.2 and Proposition 4.3 that $g(v_\lambda) \in L^1(0,1)$ and $h \in L^1(0,1)$ for any λ. Furthermore, it is easy to verify that $l_1(x) \ge \kappa > 0$ for all $x \in [0,1]$ and a constant κ independent of λ. From the properties of the function f', we get $l_2 \in L^1(0,1)$.

One can see that $r_\lambda \in W^{1,\infty}(0,1)$, and then $\max\{|r_\lambda(0)|, |r_\lambda(1)|\} < \epsilon < \frac{1}{2}$ for sufficiently small $\lambda < \epsilon$. Now denote a test function $\zeta(x) = \mathcal{L}(x)w(x) \in H_0^1(0,1)$ by

$$\mathcal{L}(x) = e^{\int_0^x g(v_\lambda)(s) ds} \quad \text{and} \quad w(x) = \max\{0, r_\lambda - 2\epsilon\},$$

with any fixed λ. Then multiplying (4.9) by $\zeta(x)$, one gets

$$\lambda\int_0^1 \mathcal{L}(x)|w_x|^2dx+\frac{1}{\lambda}\int_0^1 \mathcal{L}(x)l_1(x)w^2dx=\int_0^1 \mathcal{L}(x)(h-l_2)(x)w(x)dx.$$

Since $l_1\geq\kappa>0$ and $0<\underline{\mathcal{L}}\leq\mathcal{L}\leq\overline{\mathcal{L}}$ for constants $\underline{\mathcal{L}}$ and $\overline{\mathcal{L}}$ independent of λ, we have

$$\lambda\underline{\mathcal{L}}\int_0^1 |w_x|^2dx+\frac{\underline{\mathcal{L}}\kappa}{\lambda}\int_0^1 w^2dx\leq\overline{\mathcal{L}}\int_0^1 |(h-l_2)(x)|w(x)dx. \tag{4.10}$$

Note that, for any $\omega\in H_0^1(0,1)$, it holds that $\|\omega\|^2_{L^\infty(0,1)}\leq C\|\omega_x\|_{L^2(0,1)}\|\omega\|_{L^2(0,1)}$, which, together with Young's inequality with parameter λ, implies that

$$\nu\|w\|^2_{L^\infty(0,1)}\leq\lambda\underline{\mathcal{L}}\int_0^1 |w_x|^2dx+\frac{\underline{\mathcal{L}}\kappa}{\lambda}\int_0^1 w^2dx\leq\overline{\mathcal{L}}\|w\|_{L^\infty(0,1)}\cdot\|(h-l_2)\|_{L^1(0,1)}, \tag{4.11}$$

where the constant $\nu>0$ is selected by $\nu=\max\{1,4\underline{\mathcal{L}}^2\kappa\}/C$. Thus there exists a constant

$$M_0:=\frac{\overline{\mathcal{L}}\|(h-l_2)\|_{L^1(0,1)}}{\nu}+1,$$

so that for any λ, we get

$$\sup_{x\in[0,1]} r_\lambda(x)\leq M_0.$$

Similarly, multiplying (4.9) by $\mathcal{L}(x)$ and the function $\tilde{w}=-\min\{0,r_\lambda-\min\{r_\lambda(0), r_\lambda(1)\}\}$, we also get

$$\lambda\int_0^1 \mathcal{L}(x)|\tilde{w}_x|^2dx+\frac{1}{\lambda}\int_0^1 \mathcal{L}(x)l_1(x)\tilde{w}^2dx=-\int_0^1 \mathcal{L}(x)(h-l_2)(x)\tilde{w}(x)dx.$$

As in (4.10) and (4.11), it follows that

$$-\inf_{x\in[0,1]} r_\lambda(x)\leq M_0.$$

Thus, the proof is complete. ∎

Now we return to problem (1.7). From Lemma 4.1, a uniform estimate of the error term in $L^\infty(0,1)$ is stated as follows.

THEOREM 4.4. *Under the assumptions of Lemma* 4.1 *and Proposition* 4.3, *there exists a unique pair of smooth solutions* $(\rho_\lambda,\Phi_\lambda)$ *to system* (1.7), *and then the boundary layer profiles* n_0 *and* n_1 *satisfy*

$$\begin{aligned} &n_0(0,\lambda)=J-b_0, \quad |n_0(x,\lambda)|\leq Ce^{-\mu_3x/\lambda},\\ &n_1(0,\lambda)=J-b_1, \quad |n_1(x,\lambda)|\leq Ce^{-\mu_4(1-x)/\lambda}, \end{aligned} \tag{4.12}$$

such that when $\lambda\to0$, *we have*

$$\|\rho_\lambda-b-n_0(\cdot,\lambda)-n_1(\cdot,\lambda)\|_{L^\infty(0,1)}=O(\lambda), \tag{4.13}$$

where C, μ_3, and μ_4 are positive constants independent of λ. Furthermore, note that

$$\phi(x) = F(b) - F(J) + \int_0^x \frac{j}{\tau b(y)} dy;$$

then it holds that

$$\|\Phi_\lambda - \phi - v_0 - v_1\|_{L^\infty(0,1)} = O(\lambda) \quad \text{when} \quad \lambda \to 0. \tag{4.14}$$

Here $v_0(x/\lambda)$ and $v_1((1-x)/\lambda)$ are given by (4.1) *and* (4.2).

Proof. Obviously, the existence of the unique solution $(\rho_\lambda, \Phi_\lambda)$ is determined by the unique solution v_λ of (2.1). Referring to the proof of Theorem 2 in [36], we choose

$$n_0(x,\lambda) = \left[\int_0^1 F'(b + y(\rho_\lambda - b))dy\right]^{-1} v_0\left(\frac{x}{\lambda}\right)$$

and

$$n_1(x,\lambda) = \left[\int_0^1 F'(b + y(\rho_\lambda - b))dy\right]^{-1} v_1\left(\frac{1-x}{\lambda}\right),$$

where there exist the positive upper and lower bounds to

$$\int_0^1 F'(b + y(\rho_\lambda - b))dy = \frac{F(\rho_\lambda) - F(b)}{\rho_\lambda - b}.$$

Thus, a direct calculation yields that (4.12) and (4.13) follow. Also, by the definitions of Φ_λ and ϕ, we get

$$\begin{aligned}
&\Phi_\lambda(x) - \phi(x) - v_0\left(\frac{x}{\lambda}\right) - v_1\left(\frac{1-x}{\lambda}\right) \\
&= v_\lambda(x) - F(b) - v_0\left(\frac{x}{\lambda}\right) - v_1\left(\frac{1-x}{\lambda}\right) - \frac{j}{\tau}\int_0^x \frac{\rho_\lambda(x) - b(x)}{\rho_\lambda(x) b(x)} dx,
\end{aligned}$$

which, in combination with (4.6), (4.12), and (4.13), implies the estimate (4.14). This proof is verified. ∎

Therefore, based on Theorem 4.4, we immediately prove Theorem 1.3.

5. Numerical simulations. In this section, we are going to carry out some numerical simulations. We numerically show that $b(x)$ is the background solution of $\rho_\lambda(x)$ for $x \in (0,1)$ as $\lambda \to 0^+$, and the boundary layers at $x = 0$ and $x = 1$ are clearly demonstrated.

Example 5.1. Consider the steady-state Euler–Poisson equations with the sonic boundary,

$$\begin{cases} (F(\rho_\lambda) - \Phi_\lambda)_x = -\frac{1}{\rho_\lambda}, \ x \in (0,1), \\ \lambda^2 (\Phi_\lambda)_{xx} = \rho_\lambda - b(x), \ x \in (0,1), \\ \rho_\lambda(0) = \rho_\lambda(1) = 1, \\ \Phi_\lambda(0) = 0, \end{cases} \tag{5.1}$$

where

$$b(x) = 3 + \sin(\pi x)$$

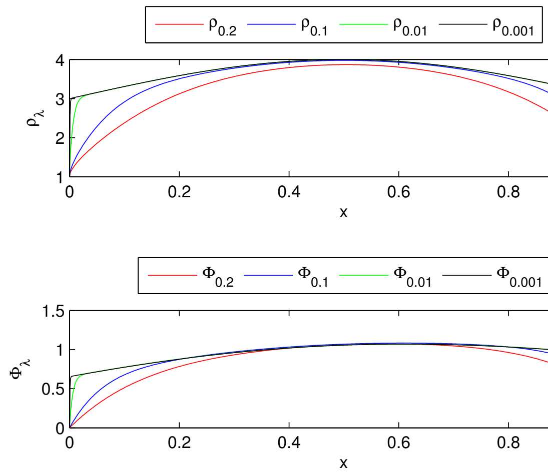


FIG. 4. *Numerical approximations with $\gamma = 1$ and different λ.*

and

$$F(\rho_\lambda) = \begin{cases} \frac{1}{2\rho_\lambda} + \frac{\rho_\lambda^{\gamma-1}}{\gamma-1} & \text{if} \quad \gamma > 1, \\ \frac{1}{2\rho_\lambda} + \ln \rho_\lambda & \text{if} \quad \gamma = 1. \end{cases}$$

The first equation of (5.1) implies that

$$(F(\rho_\lambda))_{xx} - (\Phi_\lambda)_{xx} = -\left(\frac{1}{\rho_\lambda}\right)_x. \tag{5.2}$$

Substituting (5.2) into the second equation of (5.1) gives

$$(F(\rho_\lambda))_{xx} - \frac{1}{\lambda^2}(\rho_\lambda - b(x)) = -\left(\frac{1}{\rho_\lambda}\right)_x.$$

Now, we are ready to numerically solve the boundary problem. Let $h = 1/1000$ be the spatial stepsize. Denote by ρ_i and Φ_i the numerical approximations to $\rho_\lambda(ih)$ and $\Phi_\lambda(ih)$, where $i = 0, 1, 2, \ldots, 1000$. Applying the second-order finite difference method to the above equation, we have

$$\frac{F(\rho_{i+1}) - 2F(\rho_i) + F(\rho_{i-1})}{h^2} - \frac{1}{\lambda^2}(\rho_i - b(ih)) = -\frac{2h}{\rho_{i+1} - \rho_{i-1}}.$$

Together with the boundary condition, we can obtain the numerical approximation ρ_i. Then, applying the finite difference method to the first equation of (5.1), we get the numerical solutions of Φ_i. Numerical approximations with $\gamma = 1$ for the isothermal case and $\gamma = 2$ for the isentropic case are shown in Figures 4 and 5, respectively. From the figures, one can see the quasi-neutral limit,

$$\rho_\lambda(x) \to b(x) \quad \text{for } x \in (0, 1), \text{ as } \lambda \to 0^+,$$

and, obviously, there are boundary layers at the boundaries $x = 0$ and $x = 1$, respectively. These numerical computations further confirm our theoretical results.

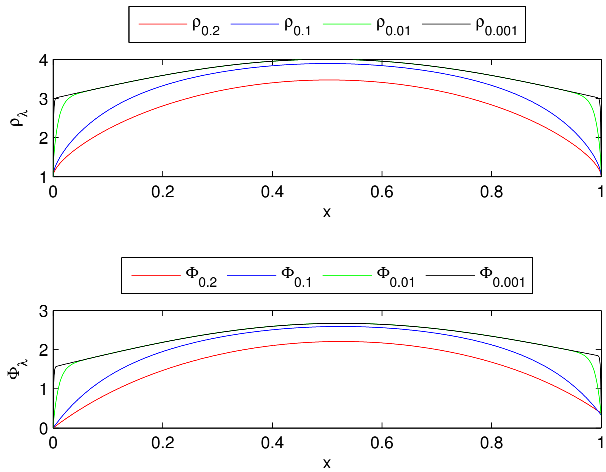


FIG. 5. *Numerical approximations with $\gamma = 2$ and different λ.*

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