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Global and blow-up solutions for compressible Euler equations with time-dependent damping

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Abstract

This paper deals with the Cauchy problem for the compressible Euler equations with time-dependent damping, where the time-vanishing damping in the form of $\frac{\mu}{(1+t)^{\lambda}}$ makes some fantastic variety of the dynamic system. For $0 < \lambda < 1$ and $\mu > 0$, or $\lambda = 1$ but $\mu > 2$, the solutions are proved to exist globally in time, when the derivatives of the initial data are small, but the initial data themselves can be arbitrarily large. This is the so-called challenging case of global solutions with large initial data; while, when the initial Riemann invariants are monotonic and their derivatives with absolute value are large at least at one point, then the solutions are still bounded, but their derivatives will blow up at finite time, somewhat like the singularity formed by shock waves. For $\lambda > 1$ and $\mu > 0$, or $\lambda = 1$ but $0 < \mu \le 1$, the derivatives of solutions will blow up even for all initial data, including the interesting case of blow-up solutions with small initial data. Here the initial Riemann invariants are monotonic. In fact, such a blow-up phenomenon is determined by the mechanism of the dynamic system itself. In order to prove the global existence of solutions with large initial data, we introduce a new energy functional related to the Riemann invariants, which crucially enables us to build up the maximum principle for the corresponding Riemann invariants, and the uniform boundedness for the local solutions. Finally, numerical simulations in different cases are carried out, which further confirm our theoretical results.

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1. Introduction and main results

We consider the following Cauchy problem for the 1D compressible Euler equations (the so-called p-system) with time-degenerate damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha(x,t)u, & x \in R,\ t \in R_+, \\ (v,u)|_{t=0} = (v_0(x), u_0(x)) \to (v_\pm, u_\pm), & \text{as } x \to \pm\infty. \end{cases} \tag{1.1}$$

Physically, (1.1) models a compressible flow through porous media, where, $v = v(x,t) > 0$ is the specific volume of the flow at time t and the location x, $u = u(x,t)$ is the fluid velocity, and $p(v) = v^{-\gamma}/\gamma$ with $\gamma > 1$ is the pressure of the flow. This means the flow is the polytropic gas. The term $-\alpha(x,t)u$ in the second equation of (1.1) is the so-called damping effect on the fluid, and $\alpha(x,t) \geq 0$ satisfies $\alpha \in C_b^1(R \times R_+)$, where $C_b^1(R \times R_+)$ is the set of bounded and continuous functions whose first partial derivatives are also bounded. The well-known example is $\alpha(x,t) = \mu/(1+t)^\lambda$, which represents the time-gradually-vanishing friction effect with $\lambda > 0$. $(v_0, u_0)(x)$ are the initial data, and $(v_\pm, u_\pm)$ with $v_\pm > 0$ are the state constants.

Let us depict a background picture of the relevant research. In order to describe the progress in different cases more explicitly, let us take, at this moment, $\alpha(x,t) = \frac{\mu}{(1+t)^\lambda}$ for example, although our study in this paper treats a more general form of the damping coefficient.

When $\mu = 0$, the system (1.1) is the standard Euler equations which has been extensively studied, and the solutions in general blow up due to the formation of shocks [2,3,6,8,20,25,39].

When $\mu > 0$ and $\lambda = 0$, the system (1.1) becomes the damped Euler system. For the case with "boundary layer", namely $v_+ \neq v_-$, Marcati and Milani [27], based on Darcy's Law, first studied the relaxation limit of the hyperbolic solutions to the parabolic solutions in the weak sense. Later then, Hsiao and Liu [15] investigated the convergence in the smooth sense, and observed that the damping effect makes the compressible Euler system behave like the corresponding nonlinear porous-media diffusion equation

$$\begin{cases} \bar{v}_t = \bar{u}_x, \\ p(\bar{v})_x = -\mu\bar{u}, \ \text{(Darcy's law)} \end{cases} \quad \text{equivalently,} \quad \begin{cases} \bar{v}_t + (p'(\bar{v})\bar{v}_x)_x = 0, \\ \bar{v}(0,x) \to v_\pm \text{ as } x \to \pm\infty, \end{cases}$$

where $(\bar{v}, \bar{u}) = (\bar{v}, \bar{u})(\frac{x}{\sqrt{1+t}})$, the so-called diffusion waves, are self-similar solutions. They showed the convergence of the original solution for p-system to the diffusion waves in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(t^{-1/2}, t^{-1/2})$, when the wave strength $|v_+ - v_-| + |u_+ - u_-|$ and the initial perturbation around the diffusion waves both are small enough. Such a convergence was then improved to $O(t^{-3/4}, t^{-5/4})$ by Nishihara [32] when the initial perturbations are in L^2-sense, and improved to $(t^{-1}, t^{-3/2})$ by Nishihara-Wang-Yang [33] for the L^1 integrable initial perturbations, and further improved to $O(t^{-3/2}\ln t, t^{-2}\ln t)$ by Mei [31] by finding the best asymptotic profile, where Mei's technique is to carry out twice anti-derivatives for the equation of mass conservation and the second equation of momentum conservation, and finally the convergence rates were improved to $O(t^{-3/2}, t^{-2})$ by Geng and Wang [10]. When $v_+ = v_-$

and $u_- = u_+ = 0$, the global existence in 3D case was also significantly obtained by Sideris-Thomases-Wang in [38]. For the case with boundary effect, we refer to [28,29].

Note that, the global existence of the solutions mentioned in the above works requires the initial data to be small. However, even for the damped Euler-Poisson system, when the derivatives of initial data are big, Wang and Chen remarkably showed in the pioneering work [43] that the solutions are still bounded but their derivatives must blow up at finite time. Recently, Liu and Fang [26] and Li and Wang [21] extended such a blow-up result to the 3D case. For other interesting contributions, we refer to [9,16–18,30,47] and the references therein.

When $\mu > 0$ and $\lambda > 0$, the effect of damping $\alpha(x,t) = \frac{\mu}{(1+t)^\lambda}$ is time-asymptotically vanishing, which makes the fantastic variety of the p-system (1.1). For the linear wave equations with time-asymptotically vanishing damping,

$$v_{tt} + \frac{\mu}{(1+t)^\lambda} v_t - \Delta v = 0,$$

Wirth [44–46] first derived the optimal decay rates of the solutions related to the sizes of λ and μ. For the p-system case, when the state constants satisfy $v_+ = v_-$, Pan [35,36] gave the thresholds of μ and λ separating the existence and nonexistence of global solution in small data regime. That is, if $0 \le \lambda < 1$ and $\mu > 0$ or $\lambda = 1$ and $\mu > 2$, then solutions of (1.1) globally exist for small initial perturbation around the constant state $(v_+, 0)$; if $\lambda > 1$ and $\mu > 0$ or $\lambda = 1$ but $0 < \mu \le 2$, then the solutions of (1.1) will blow up at finite time for some large initial data. Here, $\lambda = 1$ and $\mu = 2$ is the critical case. Recently, by using Riemann invariants method, Sugiyama [41] further obtained the sharp upper and lower estimates of lifespan for solutions with the small initial data when $\lambda > 1$ and $\mu > 0$ or $\lambda = 1$ and $0 < \mu \le 2$. And then in [42], Sugiyama improved the global existence of the solutions for the case of $0 < \lambda < 1$ and $\mu > 0$ or $\lambda = 1$ and $\mu > 2$ by removing the restriction on the initial data at far fields: $\lim_{x\to\pm\infty} |v_0(x) - v_+| + |u_0(x)| = 0$, but the initial perturbation still needs to be small, i.e. $|v_0(x) - v_+| + |u_0(x)| \ll 1$. For 3D damped Euler equations, Hou-Yin [14] and Hou-Witt-Yin [13] proved the global existence of the solutions for $0 < \lambda < 1$ and the blow-up of solutions for $\lambda > 1$ with life-span, but they did not specify whether the blow-up phenomena are for the solutions themselves or their gradients.

When $0 < \lambda < 1$ and $\mu > 0$ but $v_+ \ne v_-$, the asymptotic profiles are expected to be diffusion waves that satisfy the time-dependent porous media equation:

$$\begin{cases} \bar v_t = \bar u_x, \\ p(\bar v)_x = -\dfrac{\mu}{(1+t)^\lambda}\bar u, \end{cases} \quad \text{equivalently,} \quad \begin{cases} \dfrac{\mu}{(1+t)^\lambda}\bar v_t + (p'(\bar v)\bar v_x)_x = 0, \\ \bar v(0,x) \to v_\pm \text{ as } x \to \pm\infty, \end{cases}$$

with $(\bar v, \bar u) = (\bar v, \bar u)(\frac{x}{\sqrt{(1+t)^{1+\lambda}}})$. Two groups of Li-Li-Mei-Zhang [23] and Cui-Yin-Zhang-Zhu [7] both initially paid attention to this topic, and almost at the same time but independently obtained the convergence of the original solutions for (1.1) to the diffusion waves when the initial perturbation around the diffusion waves $(\bar v, \bar u)$ and the wave strength $|v_+ - v_-| + |u_+ - u_-|$ both are sufficiently small. The convergence rates presented in [7] are

$$\|(v - \bar v, u - \bar u)(t)\|_{L^\infty(R)} = \begin{cases} O(t^{-\frac{3}{4}(1+\lambda)}, t^{-\frac{\lambda+5}{4}}), & 0 < \lambda < \frac{1}{7}, \\ O(t^{-\frac{6}{7}+\varepsilon}, t^{-\frac{9}{7}+\varepsilon}), & \lambda = \frac{1}{7}, \\ O(t^{\lambda-1}, t^{\frac{3(\lambda-1)}{2}}), & \frac{1}{7} < \lambda < 1, \end{cases} \quad \text{for any } 0 < \varepsilon < 1,$$

which are better than that showed in [23]. These rates were then improved by Li-Li-Mei-Zhang [24] for Euler-Poisson equations and by Li [22] for the p-system to the following optimal rates:

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty(R)} = \begin{cases} O(t^{-\frac{3}{4}(1+\lambda)}, t^{-\frac{\lambda+5}{4}}), & 0<\lambda<\frac{1}{7}, \\ O(t^{-\frac{6}{7}}\ln t, t^{-\frac{9}{7}}\ln t), & \lambda=\frac{1}{7}, \\ O(t^{\lambda-1}, t^{\frac{3(\lambda-1)}{2}}), & \frac{1}{7}<\lambda<1. \end{cases}$$

Clearly, for the case $0<\lambda<1$, the system (1.1) eventually behaves like a degenerate parabolic system with diffusion phenomena.

When $\lambda=1$ and $\mu>2$ (the critical case) with $v_- \neq v_+$, then the story is different from before. In fact, from the rates showed in the above that, the diffusion waves for the porous media equation are no longer the asymptotic profile for the original system (1.1), because $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty(R)} \approx (O(1),O(1))$. Recently, Geng-Lin-Mei [11] observed that, in this critical case, the effect of v_{tt} cannot be neglected, and technically confirmed that the asymptotic profile is the solution of the linear wave equation with time-vanishing damping

$$\bar{v}_{tt} + \frac{\mu}{1+t}\bar{v}_t + (p'(\hat{v})\bar{v}_x)_x = 0,$$

with an artful construction for the featured function $\hat{v}(x,t)$. By constructing the best asymptotic profile, they show a much better convergence.

We note that, in the previous studies, when $0<\lambda<1$ and $\mu>0$, or $\lambda=1$ and $\mu>2$, the global existence of the solutions is verified only for small initial perturbation, but it is not clear if the global solutions exist in the case of large initial perturbation; while, when $\lambda>1$ and $\mu>0$ or $\lambda=1$ and $0<\mu\le 2$, the solutions will blow up once the initial derivatives are sufficiently large, but it is also not clear if the solutions still blow up when the initial perturbations are small. These open questions will be our main concerns in this paper. In fact, roughly speaking, we will prove that:

1. When $0<\lambda<1$ and $\mu>0$, or $\lambda=1$ and $\mu>2$, even if the initial perturbation and the wave strength are large, once the derivatives of the initial data are not large, then the solutions of (1.1) still globally exist. This problem is the so-called global-in-time solution with large initial data.
2. In the same cases for λ and μ as mentioned before, the derivatives of solutions of (1.1) will blow up when the derivatives of the initial values are sufficiently large at some points.
3. When $\lambda>1$ and $\mu>0$, or $\lambda=1$ and $0<\mu\le 1$, we shall prove that, the solutions of (1.1) will blow up for all initial data with monotonic initial Riemann invariants, in particular, the small initial data with monotonic initial Riemann invariants. This also essentially improves the existing results of blow-up for fluid dynamic systems, where the sufficient conditions are usually with some large initial data.

Regarding blow-up phenomena, in general, there are three types of blow-up for the system (1.1). The first one is for the solutions themselves at a finite time: $\lim_{t\to t^*}|v(x,t)|+|u(x,t)|=\infty$; the second one is for the derivatives of the solutions: $\lim_{t\to t^*}|v_x(x,t)|+|u_x(x,t)|=\infty$; and the third one is for the pressure function $p(v)=\frac{v^{-\gamma}}{\gamma}$: $\lim_{t\to t^*}p(v)=\infty$, namely,

$\lim_{t\to t^*} v(x,t) = 0$. However, we will prove that, for the system (1.1), both the solutions themselves (v,u) and the pressure function $p(v)$ will be bounded and never blow up, but the derivatives of the solutions (v_x, u_x) will blow up at finite time for some cases of λ and μ, as well as the size of derivatives of initial data.

In order to verify these new phenomena, we combine the energy method with the Riemann invariants method, as developed in [37,43,1,14,35,48,34,40,3]. But the technical point in this paper is to construct a new functional for Riemann invariants to obtain upper and lower bound for the solutions of (1.1), and to build up the maximum principle for the damped Euler system (1.1) in any bounded domain. This is a crucial key to guarantee the global existence of the solutions for (1.1) with large initial data, in the case of $0 \le \lambda < 1$ and $\mu > 0$, or $\lambda = 1$ and $\mu > 2$. This new functional method is powerful to get a priori estimates for elliptic and parabolic equations (see [4,5]). We also use the similar method as in [41] to obtain the upper bound for the derivatives of solutions by estimating derivatives of Riemann invariants. Remarkably, the hyperbolic system of PDEs usually does not hold the maximum principle. But, under certain setting frame, the new system may possess the maximum principle for the solutions in the new frame. See the significant studies by J. Hong [12] and Huang-Pan-Wang [19].

We first consider the following Cauchy problem:

$$\begin{cases} v_t - u_x = 0, \\ u_t = -p(v)_x - \dfrac{\mu}{(1+t)^\lambda} u, & x \in R,\ t \in R_+, \\ (v,u)|_{t=0} = (v_0(x), u_0(x)) \to (v_\pm, u_\pm), & \text{as } x \to \pm\infty. \end{cases} \tag{1.2}$$

Let $c = \sqrt{|p'(v)|} = v^{-(\gamma+1)/2}$, the so-called sound speed in fluid terminology, and note $\int_v^\infty c(s)ds = \frac{2}{\gamma-1} v^{-\frac{\gamma-1}{2}}$, we introduce the Riemann invariants to (1.2):

$$\xi(x,t) := \frac{2}{\gamma-1} v^{-(\gamma-1)/2} - u, \quad \eta(x,t) := \frac{2}{\gamma-1} v^{-(\gamma-1)/2} + u. \tag{1.3}$$

Then (1.2) is reduced to

$$\begin{cases} (\partial_t - c\partial_x)\xi = -\dfrac{\mu}{2(1+t)^\lambda}(\xi - \eta), \\ (\partial_t + c\partial_x)\eta = -\dfrac{\mu}{2(1+t)^\lambda}(\eta - \xi), \\ \xi|_{t=0} = \frac{2}{\gamma-1}[v_0(x)]^{-(\gamma-1)/2} - u_0(x) =: \xi_0(x), \\ \eta|_{t=0} = \frac{2}{\gamma-1}[v_0(x)]^{-(\gamma-1)/2} + u_0(x) =: \eta_0(x). \end{cases} \tag{1.4}$$

We denote, for $\sigma > 0$, $0 < \lambda < 1$ and $\mu > 0$, or $\lambda = 1$ and $\mu > 2$,

$$A_\sigma(t) := \exp\left(\int_0^t \frac{\sigma\mu}{(1+\tau)^\lambda} d\tau \right), \tag{1.5}$$

$$c_\sigma := \sup_t A_\sigma^{-1}(t)(1+t)^\lambda, \tag{1.6}$$

$$d_\sigma := \sup_t A_\sigma^{-1}(t)(1+t)^\lambda \int_0^t \frac{\mu A_\sigma(\tau)}{2(1+\tau)^\lambda} \left| \frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau} \right| d\tau, \tag{1.7}$$

$$\overline{c} := \sup_t A_{1/2}^{-1}(t)(1+t)^\lambda \int_0^t \frac{A_{1/2}(\tau)}{(1+\tau)^{2\lambda}} d\tau, \tag{1.8}$$

$$\delta_\sigma := \left(\frac{1}{2} - \sigma\right) \frac{2\mu}{\gamma+1}, \ \sigma < 1/2, \tag{1.9}$$

$$\max_{a \le x \le b} (f(x), g(x)) := \max(\max_{a \le x \le b} f(x), \max_{a \le x \le b} g(x)), \tag{1.10}$$

$$\min_{a \le x \le b} (f(x), g(x)) := \min(\min_{a \le x \le b} f(x), \min_{a \le x \le b} g(x)), \tag{1.11}$$

where σ depends on λ and μ. We choose $\sigma \in [0.1, 0.4]$ such that d_σ is as small as possible and δ_σ is as large as possible. Numerical computations show that, for $2 \le \mu \le 4$, we can set

$$\sigma = \begin{cases} 0.1, & \text{if} \quad 0 < \lambda \le 0.5, \\ (\lambda - 0.3)/2, & \text{if} \quad 0.5 < \lambda \le 0.9, \\ 0.3, & \text{if} \quad 0.9 < \lambda < 1. \end{cases}$$

For example, if $\mu = 2, \lambda = 0.5$, then $\sigma = 0.1, c_\sigma = 1.37, d_\sigma = 0.97$; if $\mu = 3, \lambda = 0.9$, then $\sigma = 0.3, c_\sigma = 1, d_\sigma = 1.17$. Our main results are as follows.

Theorem 1.1 *(Maximum principle). Let $(v, u)(x, t) \in C^1(R \times [0, T])$ be the solutions to the system (1.1) for $T > 0$ with a general form of $\alpha(x, t) > 0$. Then, for any $a < b$,*

$$\max_{a \le x \le b} \{(|\xi|, |\eta|)(x, t)\} \le \max\{ \max_{a \le x \le b} (|\xi_0|, |\eta_0|)(x), \max_{0 \le t \le T} (|\xi|, |\eta|)(b, t), \max_{0 \le t \le T} (|\xi|, |\eta|)(a, t)\}. \tag{1.12}$$

Theorem 1.2 *(Uniform boundedness). Let $(v, u)(x, t) \in C^1(R \times [0, T])$ be the solutions to the system (1.2) for $T > 0$ and the initial Riemann invariants satisfy*

$$\xi_0(x), \eta_0(x) \ge \varepsilon_0 > 0, \quad x \in R, \tag{1.13}$$

for some constant ε_0, then $(v, u)(x, t)$ are uniformly bounded in $R \times [0, T]$:

$$\left[\frac{2}{(\gamma-1)\sup_{x \in R}(\xi_0, \eta_0)(x)} \right]^{\frac{2}{\gamma-1}} \le v(x, t) \le \left[\frac{2}{(\gamma-1)\inf_{x \in R}(\xi_0, \eta_0)(x)} \right]^{\frac{2}{\gamma-1}}, \tag{1.14}$$

$$|u(x, t)| \le \sup_{x \in R} \{\xi_0(x), \eta_0(x)\}. \tag{1.15}$$

Theorem 1.3 *(Monotonicity). Let $(v, u)(x, t) \in C^1(R \times [0, T])$ be the solutions to the system (1.2) for $T > 0$ and the initial Riemann invariants $\xi_0(x)$ and $\eta_0(x)$ both be decreasing (or increasing) for all $x \in R$. Then the solution $v(x, t)$ of (1.2) is increasing (or decreasing).*

Theorem 1.4 *(Global existence with monotonic initial Riemann invariants). Let the initial data of Riemann invariants* $(\xi, \eta)(x, 0)$ *both be decreasing (or increasing) for all* $x \in R$*. When* $0 < \lambda < 1$ *with* $\mu > 0$ *and*

$$\left[c_\sigma \sup_{x\in R}(|\xi_{0x}(x)|, |\eta_{0x}(x)|) + (2d_\sigma + \mu)|v_+ - v_-|/4\right] / \min(v_-, v_+) < \delta_\sigma, \tag{1.16}$$

or $\lambda = 1$ *with* $\mu > 2, \sigma = 1/\mu$ *and*

$$\left[\sup_{x\in R}(|\xi_{0x}(x)|, |\eta_{0x}(x)|) + \mu|v_+ - v_-|/4\right] / \min(v_-, v_+) < \delta_\sigma, \tag{1.17}$$

then (1.2) has a unique pair of solutions globally in time satisfying the uniform boundedness (1.14) *and* (1.15)*, and* $v(x, t)$ *is increasing (or decreasing) in* x*. In particular, the following decay estimate holds:*

$$\sup_{x\in R}(|v_x(x, t)|, |u_x(x, t)|) \le C\delta_\sigma(1 + t)^{-\lambda} \min(v_-, v_+). \tag{1.18}$$

Theorem 1.5 *(Global existence for non-monotonic initial Riemann invariants in the case of* $0 < \lambda < 1$*,* $\mu > 0$*, or* $\lambda = 1$ *but* $\mu > 2$*). Assume that* $v_- > v_+$*,* $u_- = u_+ = 0$ *and*

$$\frac{2}{\gamma - 1} v_-^{-(\gamma-1)/2} \le \inf_{x\in R}(\xi_0, \eta_0)(x) < \sup_{x\in R}(\xi_0, \eta_0)(x) \le \frac{2}{\gamma - 1} v_+^{-(\gamma-1)/2}. \tag{1.19}$$

If either $0 < \lambda < 1$*,* $\mu > 0$*, and*

$$c_{1/2} \sup_{x\in R}[(|\xi_{0x}(x)|, |\eta_{0x}(x)|) v_0^{-1}(x)] + (d_{1/2} + \mu/2)(v_- - v_+)/v_+ < \frac{1}{2\overline{c}(\gamma + 1)}, \tag{1.20}$$

or $\lambda = 1$*,* $\mu > 2$*, and*

$$\sup_{x\in R}[(|\xi_{0x}(x)|, |\eta_{0x}(x)|) v_0^{-1}(x)] + \mu(v_- - v_+)/v_+ < \frac{(\mu - 2)}{2(\gamma + 1)}, \tag{1.21}$$

then (1.2) has a unique pair of global solutions $(v, u)(x, t)$*, whose derivatives satisfy the decay estimate* (1.18)*.*

Theorem 1.6 *(Global existence for general initial data in the case of* $0 < \lambda < 1$*,* $\mu > 0$*, or* $\lambda = 1$ *but* $\mu > 2$*). Let* $\underline{v} := \inf_{x\in R} v_0(x) > 1$*,* $\overline{v} := \sup_{x\in R} v_0(x)$*. Assume that there exists a number* $\varepsilon > 0$ *such that the initial data satisfy:*

$$(\gamma - 1)|u_0(x)|v_0(x)^{(\gamma-1)/2} \le \varepsilon, \tag{1.22}$$

and, for $0 < \lambda < 1$*,* $\mu > 0$*,*

$$c_{1/2} \sup_{x\in R}\{|\xi_{0x}|, |\eta_{0x}|\} + \left(\frac{\mu}{2} + d_{1/2}\right) b_1(\overline{v} - \underline{v}) < \left[\frac{b_2}{2\overline{c}(\gamma + 1)} - \left(\frac{\mu}{2} + d_{1/2}\right)\frac{2\varepsilon}{\gamma - 1}\right]\underline{v},$$

or, for $\lambda = 1$, $\mu > 2$,

$$\sup_{x\in R}\{|\xi_{0x}|, |\eta_{0x}|\} + \mu b_1(\overline{v} - \underline{v}) < \left[\frac{(\mu-2)b_2}{2(\gamma+1)} - \frac{2\mu\varepsilon}{\gamma-1}\right]\underline{v},$$

where $b_1 := [2/(2-\varepsilon)]^{2/(\gamma-1)}$ *and* $b_2 := [2/(2+\varepsilon)]^{2/(\gamma-1)}$. *Then (1.2) has a unique pair of solutions globally in time, whose derivatives satisfy the decay estimate (1.18).*

Theorem 1.7 *(Blow up in the case of* $\lambda > 0$ *and* $\mu > 0$*). For all cases* $\lambda > 0$, $\mu > 0$, *let the initial data of Riemann invariants* $(\xi, \eta)(x, 0)$ *both be monotonic for all* $x \in R$. *If either* $\eta_x(x_0, 0) \ll -1$ *or* $\xi_x(x_0, 0) \gg 1$, *for some point* x_0, *then, there exists a finite time* $t_* > 0$ *such that,* $(v, u)(x, t)$ *are still bounded in* $R \times [0, t_*)$, *but*

$$\lim_{t\uparrow t_*^-} \|(v_x, u_x)(t)\|_{L^\infty(R)} = +\infty. \tag{1.23}$$

Theorem 1.8 *(Blow up in the case of* $\lambda > 1$, $\mu > 0$, *or* $\lambda = 1$ *but* $0 < \mu \le 1$*). For the cases either* $\lambda > 1$ *with* $\mu > 0$, *or* $\lambda = 1$ *but* $0 < \mu \le 1$, *let the initial Riemann invariants both be monotonic. Then, for all initial data, there exists a finite number* $t^* > 0$ *such that the solutions* $(v, u)(x, t)$ *of (1.2) are still bounded in* $R \times [0, t^*)$, *but*

$$\lim_{t\uparrow t^*} \|(v_x, u_x)(t)\|_{L^\infty(R)} = +\infty.$$

Corollary 1.9. *When* $\mu > 0$ *and* $\lambda = 0$, *then (1.2) is reduced to the well-known* p*-system with damping*

$$\begin{cases} v_t - u_x = 0, & \\ u_t + p(v)_x = -\mu u, & x \in R,\ t \in R_+, \\ (v, u)|_{t=0} = (v_0(x), u_0(x)) \to (v_\pm, u_\pm), & \text{as } x \to \pm\infty. \end{cases} \tag{1.24}$$

Suppose that the initial Riemann invariants are monotonic.

1. *If* $\sup_{x\in R}(|\xi_{0x}(x)|, |\eta_{0x}(x)|) + \mu|v_+ - v_-|/4 < \mu \min(v_-, v_+)/(\gamma+1)$, *then system (1.24) admits a unique pair of global solutions* $(v, u) \in C^1(R \times R_+)$.
2. *If either* $\eta_x(x_0, 0) \ll -1$ *when* $v_{0x}(x) > 0$ *or* $\xi_x(x_0, 0) \gg 1$ *when* $v_{0x} < 0$, *for some point* x_0, *then the solution of (1.24) will blow up at a finite time* t_**:*

$$\lim_{t\uparrow t_*^-} \|(v_x, u_x)(t)\|_{L^\infty(R)} = +\infty.$$

For a more general $\alpha(x, t)$ we can obtain similar results as Theorems 1.5–1.7. Please see Section 5 for details.

Remark 1.10.

1. The maximum principle (Theorem 1.1) and the uniform boundedness (Theorem 1.2) for the local solutions play the key roles to get the global existence with some large initial data. To establish the maximum principle, it is crucial to set up the new energy functional related to the Riemann invariants. See (2.1) later.
2. For $0 < \lambda < 1$ and $\mu > 0$, or $\lambda = 1$ and $\mu > 2$, when the derivatives of the initial data are small, but the initial data themselves can be arbitrarily large, as showed in Theorem 1.4-Theorem 1.6, then the global solutions exist. Here we prove the challenging case for the global existence with large initial data. While, when the absolute value of the derivatives for the initial data are large at some points, then the derivatives of the solutions as showed in Theorem 1.7 will blow up in finite time.
3. In fact, as showed in Theorem 1.7, for all cases of $\lambda > 0$ and $\mu > 0$, once the initial Riemann invariants are monotonic, and the initial derivatives are big at some points, the solution's gradients will blow up. This is somewhat like the formation of shocks when the initial data are steeper (Riemann-data-like).
4. For $\lambda = 1$ with $0 < \mu < 1$, or $\lambda > 1$ and $\mu > 0$, the derivatives of the solutions for (1.2) always blow up for all initial data. In this case, the damping effect $\alpha(x,t) = \frac{\mu}{1+t}$ is so weak that the system (1.2) is more like the standard Euler equations without damping, and the blow-up phenomena cannot be excluded. Such a blow-up phenomenon is determined by the mechanism of the system itself. For the case of $\lambda = 1$ with $1 < \mu \leq 2$, we would expect the derivatives of solutions also blow up for all initial data, but we could not prove it yet in this paper due to a technic reason. Note that, the life-span of solutions was also obtained in [14,13] for $\lambda > 1$ and $\mu > 0$ even for the small initial data, but they did not specify whether the blow-up phenomenon happens for the solutions themselves or their derivatives. Here, we confirm that the solutions are still bounded, but their derivatives blow up in finite time.
5. When $\lambda = 0$ and $\mu > 0$, remarkably, in Corollary 1.9 for the regular p-system with damping, we obtain the global existence of the solutions with large initial data, once their derivatives are small, and the blow-up of the derivatives of the solutions once the derivatives of the initial data are sufficiently large. This blow-up phenomenon matches the previous study in [43].

The rest of the paper is organized as follows. Section 2 is devoted to establishing the *a priori* estimates for local solutions of (1.1) and (1.2). We use the new functional method to show that $v, 1/v, u$ are bounded as long as $\alpha(x,t) \geq 0$ and then prove v is monotone if the initial Riemann invariants are monotone. In Section 3, we prove the existence of global solutions for (1.2) under the assumptions that $0 < \lambda < 1$ or $\lambda = 1$ but $\mu > 2$. We settle the problem of so-called global existence of large solutions. In Section 4, we show the occurrence of derivative blow-up in finite time. For all case $\lambda > 0$, once $\eta_x(x_0, 0) \ll -1$ or $\xi_x(x_0, 0) \gg 1$ at a point x_0, the solutions (v, u) of (1.2) are bounded, but the derivatives $(v_x, u_x)(x,t)$ will blow up in finite time. In particular, if $\lambda > 1$ or $\lambda = 1$ but $\mu < 1$, the derivatives $(v_x, u_x)(x,t)$ for (1.2) blow up, no matter how small the initial data are. In Section 5, we discuss global and blow-up solutions for general $\alpha(x,t)$ in system (1.1). In Section 6, we present numerical simulations in many cases which confirm our theoretical results.

2. A priori estimates: maximum principle, uniform boundedness and monotonicity

This section is devoted to establishing the *a priori* estimates for the local solutions. Let $(v, u)(x, t) \in C^1(R \times [0, T])$ be the solutions to the system (1.1) and (1.2) for a given $T > 0$. We first prove the maximum principle (Theorem 1.1) by constructing a new energy functional.

Proof of Theorem 1.1. Let $(\xi, \eta)(x, t)$ be the pair of Riemann invariants. We define a new energy functional:

$$\int_a^b [\xi^n(x,t) + \eta^n(x,t)] v^{(\gamma+1)/4} dx, \tag{2.1}$$

where n is an even integer. Let $\beta = (\gamma + 1)/4$, we first have

$$\begin{aligned}
\frac{d}{dt}\int_a^b \xi^n v^\beta dx &= n\int_a^b \xi^{n-1} v^\beta (-v^{-(\gamma+1)/2} v_t - u_t) dx + \beta \int_a^b \xi^n v^{\beta-1} v_t dx \\
&= n\int_a^b \xi^{n-1} v^\beta [-v^{-(\gamma+1)/2} u_x - v^{-\gamma-1} v_x + \alpha(x,t) u] dx + \beta \int_a^b \xi^n v^{\beta-1} u_x dx \\
&= n\int_a^b \xi^{n-1} v^{\beta-(\gamma+1)/2} (-u_x - v^{-(\gamma+1)/2} v_x) dx \\
&\quad + n\int_a^b \xi^{n-1} v^\beta \alpha(x,t) u dx + \beta \int_a^b \xi^n v^{\beta-1} u_x dx \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{2.2}$$

Note that

$$\xi_x = -v^{-(\gamma+1)/2} v_x - u_x.$$

Integrating by parts for I_1, we get

$$\begin{aligned}
I_1 &= n\int_a^b \xi^{n-1} v^{\beta-(\gamma+1)/2} \xi_x dx \\
&= \int_a^b (\xi^n)_x v^{-(\gamma+1)/4} dx
\end{aligned}$$

$$= \xi^n v^{-\beta} \Big|_a^b + \beta \int_a^b \xi^n v^{-\beta-1} v_x dx. \tag{2.3}$$

Substituting (2.3) into (2.2) yields

$$\begin{aligned}
\frac{d}{dt}\int_a^b \xi^n v^\beta dx &= \xi^n v^{-\beta} \Big|_a^b + \beta \int_a^b \xi^n (v^{-\beta-1} v_x + v^{\beta-1} u_x) dx + I_2 \\
&= \xi^n v^{-\beta} \Big|_a^b - \beta \int_a^b \xi^n v^{\beta-1} (-v^{-2\beta} v_x - u_x) dx + I_2 \\
&= \xi^n v^{-\beta} \Big|_a^b - \beta \int_a^b \xi^n v^{\beta-1} \xi_x dx + I_2 \\
&= \xi^n v^{-\beta} \Big|_a^b - \frac{\beta}{n+1} \xi^{n+1} v^{\beta-1} \Big|_a^b + \frac{\beta(\beta-1)}{n+1} \int_a^b \xi^{n+1} v^{\beta-2} v_x dx + I_2.
\end{aligned} \tag{2.4}$$

Similarly, we find

$$\begin{aligned}
\frac{d}{dt}\int_a^b \eta^n v^\beta dx &= n \int_a^b \eta^{n-1} v^\beta (-v^{-(\gamma+1)/2} v_t + u_t) dx + \beta \int_a^b \eta^n v^{\beta-1} v_t dx \\
&= -n \int_a^b \eta^{n-1} v^{\beta-(\gamma+1)/2} (u_x - v^{-(\gamma+1)/2} v_x) dx \\
&\quad - n \int_a^b \eta^{n-1} v^\beta \alpha(x,t) u dx + \beta \int_a^b \eta^n v^{\beta-1} u_x dx \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Then

$$J_1 = -n \int_a^b \eta^{n-1} v^{-\beta} \eta_x dx = -\eta^n v^{-\beta} \Big|_a^b - \beta \int_a^b \eta^n v^{-\beta-1} v_x dx,$$

and

$$
\begin{aligned}
\frac{d}{dt}\int_a^b \eta^n v^\beta dx &= -\eta^n v^{-\beta}\Big|_a^b + \beta\int_a^b \eta^n v^{\beta-1}(u_x - v^{-2\beta}v_x)dx + J_2 \\
&= -\eta^n v^{-\beta}\Big|_a^b + \frac{\beta}{n+1}\eta^{n+1}v^{\beta-1}\Big|_a^b - \frac{\beta(\beta-1)}{n+1}\int_a^b \eta^{n+1}v^{\beta-2}v_x dx + J_2.
\end{aligned}
\tag{2.5}
$$

Summing up (2.4) and (2.5), we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_a^b (\xi^n + \eta^n)v^\beta dx = &\left[(\xi^n - \eta^n)v^{-\beta} - \frac{\beta}{n+1}(\xi^{n+1} - \eta^{n+1})v^{\beta-1}\right]\Bigg|_a^b \\
&- \frac{\beta(\beta-1)}{n+1}\int_a^b (\eta^{n+1} - \xi^{n+1})v^{\beta-2}v_x dx \\
&- n\int_a^b (\eta^{n-1} - \xi^{n-1})v^\beta \alpha(x,t)u dx.
\end{aligned}
\tag{2.6}
$$

Note that

$$
\begin{aligned}
-n\int_a^b (\eta^{n-1} - \xi^{n-1})v^\beta\alpha(x,t)u dx &= -n\int_a^b\int_0^1 \frac{d}{dr}[r\eta + (1-r)\xi]^{n-1} dr v^\beta \alpha u dx \\
&= -n(n-1)\int_a^b\int_0^1 [r\eta + (1-r)\xi]^{n-2}(\eta - \xi) dr v^\beta \alpha u dx \\
&= -2n(n-1)\int_a^b\int_0^1 [r\eta + (1-r)\xi]^{n-2} dr v^\beta \alpha u^2 dx \\
&\le 0.
\end{aligned}
\tag{2.7}
$$

Since $(v,u) \in C^1([a,b]\times[0,T])$, then $|\xi(x,t)|, |\eta(x,t)|, |v_x|, 0 < v \le C$ for some constant $C = C(a,b,T)$. Let us choose even numbers n sufficiently large such that

$$
\left|\frac{\beta(1-\beta)}{n+1}\right|\left(\max_{a\le x\le b}|\eta(x,t)| + \max_{a\le x\le b}|\xi(x,t)|\right)\max_{a\le x\le b}|v_x v^{-2}| \le 1
$$

for $t \le T$. Then

$$\left|-\frac{\beta(\beta-1)}{n+1}\int_a^b(\eta^{n+1}-\xi^{n+1})v^{\beta-2}v_x dx\right|$$

$$\leq \frac{|\beta(\beta-1)|}{n+1}(\max|\eta|+\max|\xi|)\max|v_x v^{-2}|\int_a^b(\eta^n+\xi^n)v^\beta dx$$

$$\leq \int_a^b(\eta^n+\xi^n)v^\beta dx, \tag{2.8}$$

and (2.6) can be estimated as

$$\frac{d}{dt}\int_a^b(\xi^n+\eta^n)v^\beta dx \leq \left[(\xi^n-\eta^n)v^{-\beta}-\frac{\beta}{n+1}(\xi^{n+1}-\eta^{n+1})v^{\beta-1}\right]\Big|_a^b+\int_a^b(\xi^n+\eta^n)v^\beta dx. \tag{2.9}$$

Multiplying both sides of (2.9) by e^{-t}, we have

$$\frac{d}{dt}\left[e^{-t}\int_a^b(\xi^n+\eta^n)v^\beta dx\right]\leq e^{-t}\left[(\xi^n-\eta^n)v^{-\beta}-\frac{\beta}{n+1}(\xi^{n+1}-\eta^{n+1})v^{\beta-1}\right]\Big|_a^b,$$

or

$$\int_a^b(\xi^n+\eta^n)v^\beta dx \leq e^t\int_a^b[\xi^n(x,0)+\eta^n(x,0)]v_0^\beta dx$$
$$+\int_0^t e^{t-\tau}\left[(\xi^n-\eta^n)v^{-\beta}-\frac{\beta}{n+1}(\xi^{n+1}-\eta^{n+1})v^{\beta-1}\right]\Big|_a^b d\tau. \tag{2.10}$$

Now, we use the fact

$$\lim_{n\to\infty}\left(\int_a^b|F(x)|^n dx\right)^{\frac{1}{n}}=\max_{x\in[a,b]}|F(x)|,$$

where $F(x)$ is a continuous function on $[a,b]$. Taking nth roots and letting $n\to\infty$ in (2.10), we obtain (1.12). □

We next establish the uniform boundedness of (v,u).

Lemma 2.1. *Let* $(v, u) \in C^1(R \times [0, T])$ *be the solutions to* (1.2). *Then*

$$\lim_{x\to\pm\infty} v(x,t) = v_\pm \quad \textit{and} \quad \lim_{x\to\pm\infty} u(x,t) = u_\pm A_1^{-1}(t), \tag{2.11}$$

and

$$|u|,\ \frac{2}{\gamma-1} v^{-(\gamma-1)/2} \le \max\left(\sup_{x\in R} |\xi_0(x)|, \sup_{x\in R} |\eta_0(x)|\right), \tag{2.12}$$

where $A_1(t)$ *is defined by* (1.5) *with* $\sigma = 1$.

Proof. We first prove $\lim_{x\to+\infty} v(x,t) = v_+$. Let $x_\pm(t)$ be the plus and minus characteristic curves which are solutions to the following differential equations:

$$\frac{dx_\pm(t)}{dt} = \pm v^{-(\gamma+1)/2}(x_\pm(t), t). \tag{2.13}$$

Taking derivatives of $\xi(x,t)A_1(t)$ along the minus characteristic curve, we have

$$\begin{aligned}
&\frac{d}{dt}[\xi(x_(t),t)A_1(t)] \\
&= \frac{d}{dt}\left[\left(\frac{2}{\gamma-1} v^{-(\gamma-1)/2}(x_(t),t) - u(x_(t),t)\right) A_1(t)\right] \\
&= \left(-v^{-(\gamma+1)/2}(v_t - v_x v^{-(\gamma+1)/2}) - u_t + u_x v^{-(\gamma+1)/2}\right) A_1(t) \\
&\quad + \left(\frac{2}{\gamma-1} v^{-(\gamma-1)/2} - u\right) \frac{\mu A_1(t)}{(1+t)^\lambda} \\
&= \left[-v^{-(\gamma+1)/2} u_x + v^{-\gamma-1} v_x - v^{-\gamma-1} v_x + \frac{\mu u}{(1+t)^\lambda} + u_x v^{-(\gamma+1)/2}\right] A_1(t) \\
&\quad + \left(\frac{2}{\gamma-1} v^{-(\gamma-1)/2} - u\right) \frac{\mu A_1(t)}{(1+t)^\lambda} \\
&= \frac{2}{\gamma-1} v^{-(\gamma-1)/2} \frac{\mu A_1(t)}{(1+t)^\lambda}.
\end{aligned}$$

Similarly,

$$\frac{d}{dt}\left[\eta(x_+(t),t)A_1(t)\right] = \frac{2}{\gamma-1} v^{-(\gamma-1)/2}(x_+(t),t) A_1(t) \frac{\mu}{(1+t)^\lambda}.$$

Define

$$h_\pm(t) = \frac{2}{\gamma-1} v^{-(\gamma-1)/2}(x_\pm(t),t) \quad \text{and} \quad \bar{h}_\pm = \frac{2}{\gamma-1} v_\pm^{-(\gamma-1)/2}. \tag{2.14}$$

Then

$$\xi(x,t) = A_1^{-1}(t)\xi(x\_(0),0) + A_1^{-1}(t)\int_0^t h\_(\tau)A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}d\tau, \tag{2.15}$$

$$\eta(x,t) = A_1^{-1}(t)\eta(x_+(0),0) + A_1^{-1}(t)\int_0^t h_+(\tau)A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}d\tau. \tag{2.16}$$

Note that, at each point (x,t), we have two characteristic curves intersecting at (x,t), that is $h_-(t) = h_+(t)$ for fixed t. Adding (2.15) to (2.16) and solving for $h_+(t)$, we get

$$\begin{aligned} h_+(t) = &\frac{1}{2}A_1^{-1}(t)[\xi(x\_(0),0) + \eta(x_+(0),0)] \\ &+ \frac{1}{2}A_1^{-1}(t)\int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}\big[h_-(\tau) + h_+(\tau)\big]d\tau. \end{aligned} \tag{2.17}$$

It is easy to see that

$$\begin{aligned} \bar{h}_+ &= A_1^{-1}(t)\bar{h}_+ + A_1^{-1}(t)[A_1(t) - A_1(0)]\bar{h}_+ \\ &= A_1^{-1}(t)\bar{h}_+ + A_1^{-1}(t)\int_0^t \frac{d}{d\tau}A_1(\tau)d\tau\bar{h}_+ \\ &= A_1^{-1}(t)\bar{h}_+ + A_1^{-1}(t)\int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}\bar{h}_+ d\tau. \end{aligned} \tag{2.18}$$

Subtracting (2.18) from (2.17) yields

$$\begin{aligned} h_+(t) - \bar{h}_+ = &\frac{1}{2}A_1^{-1}(t)[h_-(0) - \bar{h}_+ + h_+(0) - \bar{h}_+ + u_0(x_+(0)) - u_0(x_-(0))] \\ &+ \frac{1}{2}A_1^{-1}(t)\int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}\big[(h_-(\tau) - \bar{h}_+) + (h_+(\tau) - \bar{h}_+)\big]d\tau. \end{aligned} \tag{2.19}$$

For any $\varepsilon > 0$ and fixed t, choose $M = M(t)$ sufficiently large such that

$$|h_-(0) - \bar{h}_+ + h_+(0) - \bar{h}_+ + u_0(x_+(0)) - u_0(x_-(0))| < 2\varepsilon, \tag{2.20}$$

as long as $x_-(t) > x_+(t) > M$. Denote

$$s(\tau) = \max_{x(\tau)\in[x_+(\tau),x_-(\tau)]}[h_-(\tau) - \bar{h}_+] = \max_{x(\tau)\in[x_+(\tau),x_-(\tau)]}[h_+(\tau) - \bar{h}_+], \tag{2.21}$$

for $0 < \tau \le t$, where $x_\pm(\tau)$ are two characteristic curves intersecting at (x,t) and $x(\tau)$ is any characteristic curve between $x_+(\tau)$ and $x_-(\tau)$. From (2.19), we have

$$s(t) \le \varepsilon A_1^{-1}(t) + A_1^{-1}(t)\int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}s(\tau)d\tau =: f_1(t).$$

Since

$$f_1'(t) = -A_1^{-1}(t)\frac{\mu}{(1+t)^\lambda}\left[\varepsilon + \int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}s(\tau)d\tau\right] + A_1^{-1}(t)A_1(t)\frac{\mu}{(1+t)^\lambda}s(t) \le 0,$$

we find

$$s(t) \le f_1(t) \le f_1(0) = \varepsilon, \tag{2.22}$$

which implies that

$$\left|\frac{2}{\gamma-1}v^{-(\gamma-1)/2}(x_\pm(t),t) - \frac{2}{\gamma-1}v_+^{-(\gamma-1)/2}\right| < \varepsilon,$$

or $\lim_{x\to+\infty} v(x,t) = v_+$.

To prove that $u(x,t)$ has limits as $x \to +\infty$, we subtract (2.15) from (2.16) to get

$$\begin{aligned} 2u(x,t)A_1(t) = {} & \eta(x_+(0),0) - \xi(x_-(0),0) \\ & + \int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}\big[h_+(\tau) - h_-(\tau)\big]d\tau, \end{aligned}$$

or

$$\begin{aligned} u(x,t) = {} & \frac{1}{2}A_1^{-1}(t)[\eta(x_+(0),0) - \xi(x_-(0),0)] \\ & + \frac{1}{2}A_1^{-1}(t)\int_0^t A_1(\tau)\frac{\mu}{(1+\tau)^\lambda}\big[h_+(\tau) - h_-(\tau)\big]d\tau, \end{aligned}$$

which implies that $\lim_{x\to+\infty} u(x,t) = u_+A_1^{-1}(t)$ for fixed t. Similarly, we can prove $\lim_{x\to-\infty} v(x,t) = v_-$ and $\lim_{x\to-\infty} u(x,t) = u_-A_1^{-1}(t)$.

Note that it is easy to see from (2.21) and (2.22) that the above limits are uniform in t in the following sense: if, for any $\varepsilon > 0$ and any fixed $t > 0$, there is an $M > 0$, such that $|v(x,t) - v_+| < \varepsilon$ whenever $x > M$, then $|v(x,\tau) - v_+| < \varepsilon$ for $x > M$ and $\tau \le t$.

Finally, for any $\varepsilon > 0$, there are $a < 0$ and $b > 0$, such that

$$\left|\xi(x,t) - \bar{h}_+ + u_+A_1^{-1}(t)\right| < \varepsilon \quad \text{and} \quad \left|\eta(x,t) - \bar{h}_+ - u_+A_1^{-1}(t)\right| < \varepsilon,$$

whenever $x \ge b$, $0 \le t \le T$, and

$$\left|\xi(x,t) - \bar{h}_- + u_- A_1^{-1}(t)\right| < \varepsilon \quad \text{and} \quad \left|\eta(x,t) - \bar{h}_- - u_- A_1^{-1}(t)\right| < \varepsilon,$$

whenever $x \le a, 0 \le t \le T$. From Theorem 1.1, we obtain

$$\begin{aligned}
&\max\left(\max_{a\le x\le b}|\xi(x,t)|,\ \max_{a\le x\le b}|\eta(x,t)|\right)\\
&\le \max\Bigg(\sup_{x\in R}|\xi(x,0)|, \sup_{x\in R}|\eta(x,0)|, \max_{0\le t\le T}|\xi(b,t)|,\ \max_{0\le t\le T}|\eta(b,t)|,\\
&\qquad \max_{0\le t\le T}|\xi(a,t)|, \max_{0\le t\le T}|\eta(a,t)|\Bigg)\\
&\le \max\Bigg(\sup_{x\in R}|\xi(x,0)|, \sup_{x\in R}|\eta(x,0)|, |\bar{h}_- - u_- A_1^{-1}(t)| + \varepsilon, |\bar{h}_- + u_- A_1^{-1}(t)| + \varepsilon,\\
&\qquad |\bar{h}_+ - u_+ A_1^{-1}(t)| + \varepsilon, |\bar{h}_+ + u_+ A_1^{-1}(t)| + \varepsilon\Bigg)\\
&\le \max\left(\sup_{x\in R}|\xi(x,0)| + \varepsilon, \sup_{x\in R}|\eta(x,0)| + \varepsilon\right). \qquad (2.23)
\end{aligned}$$

Since ε is arbitrarily small, (2.23) is true for $\varepsilon = 0$. Then

$$\frac{4}{\gamma-1}v^{-(\gamma-1)/2} = \xi + \eta \le |\xi| + |\eta| \quad \text{and} \quad 2|u| = |\eta - \xi| \le |\xi| + |\eta|,$$

which implies that (2.12) is true. □

Lemma 2.2. *Let* $(v,u) \in C^1(R \times [0,T])$ *be the solutions to* (1.2)*, and*

$$\xi_0(x), \eta_0(x) \ge \varepsilon_0 > 0 \quad \textit{for all} \quad x \in \mathrm{R}. \qquad (2.24)$$

Then

$$0 < v \le \left[\frac{2}{(\gamma-1)\min\left[\inf_{x\in R}\xi_0(x), \inf_{x\in R}\eta_0(x)\right]}\right]^{2/(\gamma-1)}, \qquad (2.25)$$

for all t *such that* u_x *and* v_x *are continuous on* $[0,T]$.

Proof. By (2.11), for any $\varepsilon \in (0, \varepsilon_0/4)$, there are two numbers a and b such that

$$\xi(x,t), \eta(x,t) \ge \varepsilon_0 - \varepsilon,$$

for $x < a$ or $x > b$ and $0 \le t \le T$. By (2.24) and the continuity of $\xi(x,t), \eta(x,t)$, there is a $t_1 > 0$, such that

$$\xi(x,t), \eta(x,t) \ge \varepsilon_0/2 \quad \text{for all} \quad (x,t) \in [a,b] \times [0,t_1].$$

We first claim that $\xi(x,t), \eta(x,t) \geq \varepsilon_0/2$ on $[a,b]$ for all $t \leq T$. Suppose this is not true. Then there is a point (x_0, t_2) with $x_0 \in [a,b]$ and $t_2 \geq t_1$, such that either $\xi(x_0,t_2) < \varepsilon_0/2$ or $\eta(x_0,t_2) < \varepsilon_0/2$. It is easy to see that (2.4) and (2.5) are still true if n is replaced by $-n$, that is

$$\frac{d}{dt}\int_a^b \frac{v^\beta}{\xi^n}dx = \left(\frac{v^{-\beta}}{\xi^n} + \frac{\beta}{n-1}\frac{v^{\beta-1}}{\xi^{n-1}}\right)\Bigg|_a^b - \frac{\beta(\beta-1)}{n-1}\int_a^b \frac{v^{\beta-2}}{\xi^{n-1}}v_x dx - n\int_a^b \frac{v^\beta u}{\xi^{n+1}}\frac{\mu}{(1+t)^\lambda}dx$$

and

$$\frac{d}{dt}\int_a^b \frac{v^\beta}{\eta^n}dx = \left(-\frac{v^{-\beta}}{\eta^n} - \frac{\beta}{n-1}\frac{v^{\beta-1}}{\eta^{n-1}}\right)\Bigg|_a^b + \frac{\beta(\beta-1)}{n-1}\int_a^b \frac{v^{\beta-2}}{\eta^{n-1}}v_x dx + n\int_a^b \frac{v^\beta u}{\eta^{n+1}}\frac{\mu}{(1+t)^\lambda}dx$$

for $t \leq t_2$. Thus

$$\begin{aligned}\frac{d}{dt}\int_a^b \left(\frac{1}{\xi^n}+\frac{1}{\eta^n}\right)v^\beta dx &= \left[\left(\frac{1}{\xi^n}-\frac{1}{\eta^n}\right)v^{-\beta} + \frac{\beta}{n-1}\left(\frac{1}{\xi^{n-1}}-\frac{1}{\eta^{n-1}}\right)v^{\beta-1}\right]\Bigg|_a^b \\ &\quad - \frac{\beta(\beta-1)}{n-1}\int_a^b \left(\frac{1}{\xi^{n-1}}-\frac{1}{\eta^{n-1}}\right)v^{\beta-2}v_x dx \\ &\quad - n\int_a^b \left(\frac{1}{\xi^{n+1}}-\frac{1}{\eta^{n+1}}\right)v^\beta u\frac{\mu}{(1+t)^\lambda}dx.\end{aligned}$$

Again

$$\begin{aligned}&-n\int_a^b \left(\frac{1}{\xi^{n+1}}-\frac{1}{\eta^{n+1}}\right)v^\beta u\frac{\mu}{(1+t)^\lambda}dx \\ &= -n\int_a^b\int_0^1 \frac{d}{dr}\frac{1}{[r\xi+(1-r)\eta]^{n+1}}dr\, v^\beta u\frac{\mu}{(1+t)^\lambda}dx \\ &= -2n(n+1)\int_a^b\int_0^1 \frac{1}{[r\xi+(1-r)\eta]^{n+2}}dr\, v^\beta u^2\frac{\mu}{(1+t)^\lambda}dx \\ &\leq 0.\end{aligned}$$

Then, similar to (2.8)–(2.10), we have

$$\max\left(\max_{a\leq x\leq b}\frac{1}{\xi}, \max_{a\leq x\leq b}\frac{1}{\eta}\right) \leq \max\left[\max_{a\leq x\leq b}\frac{1}{\xi(x,0)}, \max_{a\leq x\leq b}\frac{1}{\eta(x,0)},\right.$$

$$\max_{0\le t\le t_2}\frac{1}{\xi(b,t)},\ \max_{0\le t\le t_2}\frac{1}{\eta(b,t)},\ \max_{0\le t\le t_2}\frac{1}{\xi(a,t)},\ \max_{0\le t\le t_2}\frac{1}{\eta(a,t)}\Bigg]$$

$$\le \frac{1}{\varepsilon_0-\varepsilon}, \tag{2.26}$$

which implies that $\xi(x,t), \eta(x,t) \ge \varepsilon_0 - \varepsilon > \varepsilon_0/2$ for all $x \in [a,b]$ and $t \le t_2$. This is a contradiction. Since ε is arbitrarily small, (2.26) is true for $\varepsilon = 0$. Finally, similar to (2.23), we have

$$\frac{1}{\xi(x,t)},\ \frac{1}{\eta(x,t)} \le \max\left[\max_{a\le x\le b}\frac{1}{\xi(x,0)},\ \max_{a\le x\le b}\frac{1}{\eta(x,0)}\right],$$

or

$$\xi(x,t),\ \eta(x,t) \ge \min\left(\inf_{x\in R}\xi(x,0),\ \inf_{x\in R}\eta(x,0)\right),$$

which implies that

$$\frac{4}{\gamma-1}v^{-(\gamma-1)/2} = \xi+\eta \ge 2\min\left(\inf_{x\in R}\xi_0(x),\ \inf_{x\in R}\eta_0(x)\right).$$

Hence (2.25) is true. □

Proof of Theorem 1.2. Based on Lemma 2.1 and Lemma 2.2, we immediately obtain the uniform boundedness (1.14) and (1.15) for the solutions. The proof is complete. □

Now, we set up some useful identities along two characteristic curves, similar to those in [41]. From (1.3), let

$$w = -\xi_x = v^{-(\gamma+1)/2}v_x + u_x \quad \text{and} \quad z = -\eta_x = v^{-(\gamma+1)/2}v_x - u_x. \tag{2.27}$$

For any β and $\sigma > 0$, denote

$$g_1(t) = A_\sigma(t)w(x_-(t),t)v^\beta(x_-(t),t), \quad g_2(t) = A_\sigma(t)z(x_+(t),t)v^\beta(x_+(t),t), \tag{2.28}$$

where $A_\sigma(t)$ is defined by (1.5).

Lemma 2.3. *If u_x and v_x are continuous on $[0,T]$ for some $T>0$, then*

$$\frac{d}{dt}g_1(t) = \frac{\mu}{(1+t)^\lambda}A_\sigma(t)[\sigma v^{-(\gamma+1)/2}v_x + (\sigma-1)u_x]v^\beta - A_\sigma(t)wv^{\beta-1}\left[\left(\beta+\frac{\gamma+1}{2}\right)v^{-(\gamma+1)/2}v_x - \beta u_x\right], \tag{2.29}$$

and

$$\begin{aligned}\frac{d}{dt}g_2(t) &= \frac{\mu}{(1+t)^\lambda}A_\sigma(t)[\sigma v^{-(\gamma+1)/2}v_x + (1-\sigma)u_x]v^\beta \\ &\quad + A_\sigma(t)zv^{\beta-1}\left[\left(\frac{\gamma+1}{2}+\beta\right)v^{-(\gamma+1)/2}v_x + \beta u_x\right], \end{aligned} \tag{2.30}$$

for $t \le T$.

Proof. Taking derivatives of $g_1(t)$ along the minus characteristic curve, we have

$$\begin{aligned}\frac{d}{dt}g_1(t) &= \frac{\sigma\mu}{(1+t)^\lambda}A_\sigma(t)wv^\beta + A_\sigma(t)\Big[v^{-(\gamma+1)/2}(-v^{-(\gamma+1)/2}v_{xx} + v_{xt}) \\ &\quad - \frac{\gamma+1}{2}v^{-(\gamma+3)/2}v_x\left(-v^{-(\gamma+1)/2}v_x + v_t\right) - u_{xx}v^{-(\gamma+1)/2} + u_{xt}\Big]v^\beta \\ &\quad + \beta A_\sigma(t)wv^{\beta-1}(-v^{-(\gamma+1)/2}v_x + v_t) \\ &= \frac{\sigma\mu}{(1+t)^\lambda}A_\sigma(t)wv^\beta + A_\sigma(t)\Big[-v^{-\gamma-1}v_{xx} + v^{-(\gamma+1)/2}u_{xx} \\ &\quad - \frac{\gamma+1}{2}v^{-(\gamma+3)/2}v_x\left(-v^{-(\gamma+1)/2}v_x + u_x\right) - u_{xx}v^{-(\gamma+1)/2} + v^{-\gamma-1}v_{xx} \\ &\quad - (\gamma+1)v^{-\gamma-2}v_x^2 - \frac{\mu}{(1+t)^\lambda}u_x\Big]v^\beta + \beta A_\sigma(t)wv^{\beta-1}(-v^{-(\gamma+1)/2}v_x + u_x) \\ &= \frac{\mu}{(1+t)^\lambda}A_\sigma(t)[\sigma v^{-(\gamma+1)/2}v_x + (\sigma-1)u_x]v^\beta \\ &\quad - \frac{\gamma+1}{2}A_\sigma(t)v^{\beta-1-(\gamma+1)/2}v_x w + \beta A_\sigma(t)wv^{\beta-1}(-v_x v^{-(\gamma+1)/2} + u_x) \\ &= \frac{\mu}{(1+t)^\lambda}A_\sigma(t)[\sigma v^{-(\gamma+1)/2}v_x + (\sigma-1)u_x]v^\beta \\ &\quad - A_\sigma(t)wv^{\beta-1}\left[\left(\frac{\gamma+1}{2}+\beta\right)v^{-(\gamma+1)/2}v_x - \beta u_x\right]. \end{aligned} \tag{2.31}$$

Expression (2.30) can be proved in the similar way. This completes the proof. □

Proof of Theorem 1.3. We prove it by contradiction. Suppose that $\xi_{0x}(x), \eta_{0x}(x) < 0$ for all $x \in R$, namely the initial Riemann invariants $\xi_0(x)$ and $\eta_0(x)$ are decreasing for $x \in R$, and suppose that there is a point (x_3, t_3) such that $v_x(x_3, t_3) < 0$. We can find two characteristic curves $\hat{x}_\pm(t)$ which intersect at (x_3, t_3). Let E be the set enclosed by the two characteristic curves and the line $t = 0$. We can find a point $(x_4, t_4) \in E$ such that $v_x(x_4, t_4) = 0$ and $v_x(x, t) > 0$ for $t < t_4$ in E. Let $x_\pm(t)$ denote the two characteristic curves which intersect at (x_4, t_4). Then $v_x > 0$ along the two curves for $t < t_4$. Using (2.29) with $\sigma = 1$ and $\beta = 0$, we get

$$\begin{aligned} g_1'(t) &= \frac{\mu}{(1+t)^\lambda}A_1(t)v^{-(\gamma+1)/2}v_x - \frac{\gamma+1}{2}A_1(t)v^{-(\gamma+3)/2}v_x w \\ &\ge -\frac{\gamma+1}{2}A_1(t)v^{-(\gamma+3)/2}v_x w \end{aligned}$$

$$\geq -Cg_1(t) \tag{2.32}$$

for some constant $C > 0$, which implies that $g_1(t) \geq g_1(0)e^{-Ct}$ for $0 < t \leq t_4$. Similarly, using (2.30) with $\sigma = 1$ and $\beta = 0$, we have

$$\begin{aligned} g_2'(t) &= \frac{\mu}{(1+t)^\lambda} A_1(t) v^{-(\gamma+1)/2} v_x + \frac{\gamma+1}{2} A_1(t) v^{-(\gamma+3)/2} v_x z \\ &\geq \frac{\gamma+1}{2} v^{-(\gamma+3)/2} v_x g_2(t) \\ &\geq 0, \end{aligned} \tag{2.33}$$

which implies that $g_2(t) \geq g_2(0)$ for $0 < t \leq t_4$. Then,

$$A_1^{-1}(t)g_1(t) + A_1^{-1}(t)g_2(t) = 2v^{-(\gamma+1)/2}v_x \geq [A_1^{-1}(t)g_1(0)e^{-Ct} + A_1^{-1}(t)g_2(0)] > 0,$$

for $t \leq t_4$, which contradicts our assumption that $v_x(x_4, t_4) = 0$. Hence, $v_x > 0$ for $(x, t) \in R \times [0, T]$. □

3. Global solutions

After preparation in the last section, now we turn to prove global existence of solutions for (1.2) if either $0 < \lambda < 1$ and $\mu > 0$, or $\lambda = 1$ and $\mu > 2$. As shown in the standard way of the textbook [25], we can first prove the local existence of the solutions to (1.2) as follows. The detail of proof is omitted.

Proposition 3.1 *(Local existence). Suppose that the initial data satisfy* $(v_0, u_0) \in C_b^1(R)$ *and* $v_0 > 0$*, then there exists a number* $t_0 > 0$ *such that the solutions of* (1.2) *uniquely exist and satisfy*

$$(v, u) \in C_b^1(R \times [0, t_0]).$$

Next, we derive the *a priori* estimates for the local solutions of (1.2).

Proposition 3.2 *(A priori estimates with monotonic initial data). Let* $(v, u) \in C^1(R \times [0, T])$ *be the solutions of* (1.2) *for* $T > 0$ *and the initial Riemann invariants* $\xi_0(x)$ *and* $\eta_0(x)$ *both be decreasing (or increasing). When* $0 < \lambda < 1$ *and* $\mu > 0$ *and the initial data satisfy* (1.16)*, or when* $\lambda = 1$, $\mu > 2$ *and the initial data satisfy* (1.17)*, then the solution* $v(x, t)$ *of* (1.2) *is increasing (or decreasing) with respect to* x*, and* $(v, u)(x, t)$ *satisfy the uniform boundedness* (1.14) *and* (1.15)*, and the following decay rates*

$$v^{-(\gamma+1)/2}|v_x| < \delta_\sigma(1+t)^{-\lambda}\min(v_-, v_+) \quad \textit{and} \quad |u_x| < 2\delta_\sigma(1+t)^{-\lambda}\min(v_-, v_+), \tag{3.1}$$

which implies

$$\sup_{x \in R}(|v_x(x,t)| + |u_x(x,t)|) \leq C\delta_\sigma(1+t)^{-\lambda}\min(v_-, v_+).$$

Proof. First of all, in Theorem 1.3 we have proved that $v(x,t)$ is monotonic with respect to x. Next we are going to prove (3.1). We prove it by contradiction. Without loss of generality we assume $v_- < v_+$ and $v_x > 0$. Suppose that there is a point (x_5, t_5) with $0 < t_5 < T$ such that

$$v_x(x_5,t_5)v^{-(\gamma+1)/2}(x_5,t_5) = \delta_\sigma(1+t_5)^{-\lambda}v_-,$$

but the first inequality of (3.1) is true for $t < t_5$, where δ_σ is defined in (1.9). We first assume $\lambda < 1$. Using (2.30) with $\beta = 0$, we have, on the characteristic curve $x_+(t)$,

$$\begin{aligned}
\frac{d}{dt}[A_\sigma(t)z(x_+(t),t)] &= \frac{\mu}{(1+t)^\lambda}A_\sigma(t)[\sigma v^{-(\gamma+1)/2}v_x + (1-\sigma)u_x] \\
&\quad + \frac{\gamma+1}{2}A_\sigma(t)zv^{-1}v_x v^{-(\gamma+1)/2} \\
&= -\left(\frac{1}{2}-\sigma\right)\frac{\mu}{(1+t)^\lambda}A_\sigma(t)z + \frac{\mu}{2(1+t)^\lambda}A_\sigma(t)[v^{-(\gamma+1)/2}v_x + u_x] \\
&\quad + \frac{\gamma+1}{2}A_\sigma(t)zv^{-1}v_x v^{-(\gamma+1)/2} \\
&= -\frac{1}{(1+t)^\lambda}A_\sigma(t)z\left[\left(\frac{1}{2}-\sigma\right)\mu - \frac{\gamma+1}{2}(1+t)^\lambda v^{-1}v_x v^{-(\gamma+1)/2}\right] \\
&\quad + \frac{\mu}{2(1+t)^\lambda}A_\sigma(t)[v^{-(\gamma+1)/2}v_x + u_x],
\end{aligned} \tag{3.2}$$

for $t \le t_5$. By (2.33), $g_2(t) \ge 0$, which implies $z(x_+(t),t) \ge 0$ on the characteristic curve $x_+(t)$. According to our assumptions,

$$\left(\frac{1}{2}-\sigma\right)\mu - \frac{\gamma+1}{2}(1+t)^\lambda v^{-1}v_x v^{-(\gamma+1)/2} \ge \left(\frac{1}{2}-\sigma\right)\mu - \frac{\gamma+1}{2}\delta_\sigma = 0.$$

Then

$$\frac{d}{dt}[A_\sigma(t)z(x_+(t),t)] \le \frac{\mu}{2(1+t)^\lambda}A_\sigma(t)[v^{-(\gamma+1)/2}v_x + u_x]. \tag{3.3}$$

Since, along the characteristic curve $x_+(t)$,

$$v^{-(\gamma+1)/2}v_x + u_x = \frac{d}{dt}v(x_+(t),t) = \frac{d}{dt}\left[v(x_+(t),t) - v_0(x_+(0))\right], \tag{3.4}$$

we have,

$$\begin{aligned}
z(x,t) &\le A_\sigma^{-1}(t)z(x_+(0),0) + A_\sigma^{-1}(t)\int_0^t \frac{\mu A_\sigma(\tau)}{2(1+\tau)^\lambda}\frac{d}{d\tau}\left[v(x_+(\tau),\tau) - v_0(x_+(0))\right]d\tau \\
&= A_\sigma^{-1}(t)z(x_+(0),0) + \frac{\mu\left[v(x_+(t),t) - v_0(x_+(0))\right]}{2(1+t)^\lambda}
\end{aligned}$$

$$- A_\sigma^{-1}(t) \int_0^t \frac{\mu A_\sigma(\tau)}{2(1+\tau)^\lambda} \left(\frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau} \right) \left[v(x_+(\tau), \tau) - v_0(x_+(0)) \right] d\tau, \quad (3.5)$$

for $t \le t_5$. Similarly, on the characteristic curve $x_-(t)$, we have, $w > 0$, and

$$\begin{aligned}
\frac{d}{dt}[A_\sigma(t) w(x_-(t), t)] &= \frac{\mu}{(1+t)^\lambda} A_\sigma(t)[\sigma v^{-(\gamma+1)/2} v_x + (\sigma - 1) u_x] \\
&\quad - \frac{\gamma+1}{2} A_\sigma(t) w v^{-1} v_x v^{-(\gamma+1)/2} \\
&= - \left(\frac{1}{2} - \sigma \right) \frac{\mu}{(1+t)^\lambda} A_\sigma(t) w + \frac{\mu}{2(1+t)^\lambda} A_\sigma(t)[v^{-(\gamma+1)/2} v_x - u_x] \\
&\quad - \frac{\gamma+1}{2} A_\sigma(t) w v^{-1} v_x v^{-(\gamma+1)/2} \\
&\le \frac{\mu}{2(1+t)^\lambda} A_\sigma(t)[v^{-(\gamma+1)/2} v_x - u_x] \\
&= \frac{\mu}{2(1+t)^\lambda} A_\sigma(t) \frac{d}{dt}[v_0(x_-(0)) - v(x_-(t), t)]. \qquad (3.6)
\end{aligned}$$

Solving for w yields

$$\begin{aligned}
w(x,t) &\le A_\sigma^{-1}(t) w(x_-(0), 0) + \frac{\mu \left[v_0(x_-(0)) - v(x_-(t), t) \right]}{2(1+t)^\lambda} \\
&\quad - A_\sigma^{-1}(t) \int_0^t \frac{\mu A_\sigma(\tau)}{2(1+\tau)^\lambda} \left(\frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau} \right) \left[v_0(x_-(0)) - v(x_-(\tau), \tau) \right] d\tau,
\end{aligned} \quad (3.7)$$

for $t \le t_5$. Adding (3.5) to (3.7) and dividing the result by 2, we get (noting that $v(x_-(t), t) = v(x_+(t), t)$ at the intersection of two characteristic curves),

$$\begin{aligned}
v^{-(\gamma+1)/2} v_x &\le \frac{1}{2} A_\sigma^{-1}(t)[z(x_+(0), 0) + w(x_-(0), 0)] + \frac{\mu \left[v_0(x_-(0)) - v_0(x_+(0)) \right]}{4(1+t)^\lambda} \\
&\quad - A_\sigma^{-1}(t) \int_0^t \frac{\mu A_\sigma(\tau)}{4(1+\tau)^\lambda} \left(\frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau} \right) \\
&\quad \times \left[v(x_+(\tau), \tau) - v_0(x_+(0)) + v_0(x_-(0)) - v(x_-(\tau), \tau) \right] d\tau. \qquad (3.8)
\end{aligned}$$

Since v is monotone increasing, we find

$$v_0(x_-(0)) - v_0(x_+(0)) > 0 \quad \text{and} \quad v(x_-(\tau), \tau) - v(x_+(\tau), \tau) > 0.$$

The difference of two positive numbers must satisfy

$$
\begin{aligned}
&|(v_0(x_-(0)) - v_0(x_+(0))) - (v(x_-(\tau),\tau) - v(x_+(\tau),\tau))| \\
&\le \max(v_0(x_-(0)) - v_0(x_+(0)), v(x_-(\tau),\tau) - v(x_+(\tau),\tau)) \\
&\le v_+ - v_-.
\end{aligned}
$$

Then, (3.8) can be estimated as

$$
\begin{aligned}
v^{-(\gamma+1)/2}v_x &\le \frac{1}{2}A_\sigma^{-1}(t)[z(x_+(0),0) + w(x_-(0),0)] + \frac{\mu(v_+ - v_-)}{4(1+t)^\lambda} \\
&\quad + A_\sigma^{-1}(t)\int_0^t \frac{\mu A_\sigma(\tau)}{4(1+\tau)^\lambda}\left|\frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right|(v_+ - v_-)d\tau \\
&\le \frac{1}{(1+t)^\lambda}[c_\sigma \sup_{x\in R}(|\xi_{0x}(x)|, |\eta_{0x}(x)|) + (\mu/4 + d_\sigma/2)(v_+ - v_-)] \\
&< \frac{\delta_\sigma v_-}{(1+t)^\lambda},
\end{aligned}
\tag{3.9}
$$

for $t \le t_5$, which is a contradiction. Furthermore, from (3.5) and (3.7),

$$
\begin{aligned}
|u_x| &\le \frac{1}{2}(|w| + |z|) \\
&\le \frac{1}{2}A_\sigma^{-1}(t)[z(x_+(0),0) + w(x_-(0),0)] \\
&\quad + \frac{\mu\left[|v_0(x_-(0)) - v(x_-(t),t)| + |v(x_+(t),t) - v_0(x_+(0))|\right]}{4(1+t)^\lambda} \\
&\quad + A_\sigma^{-1}(t)\int_0^t \frac{\mu A_\sigma(\tau)}{4(1+\tau)^\lambda}\left|\frac{\sigma\mu}{(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right| \\
&\qquad \times \left[|v_0(x_+(0)) - v(x_+(\tau),\tau)| + |v(x_-(\tau),\tau) - v_0(x_-(0))|\right]d\tau \\
&\le \frac{1}{(1+t)^\lambda}[c_\sigma \sup_{x\in R}(|\xi_{0x}(x)|, |\eta_{0x}(x)|) + 2(\mu/4 + d_\sigma/2)(v_+ - v_-)] \\
&< \frac{2\delta_\sigma v_-}{(1+t)^\lambda}.
\end{aligned}
$$

Hence (3.1) is true for all $t > 0$.

If $\lambda = 1$ and $\mu > 2$, choose $\sigma = 1/\mu$. Then $A_\sigma(t) = (1+t)$ and

$$
v^{-(\gamma+1)/2}v_x \le \frac{1}{2(1+t)}|z(x_+(0),0) + w(x_-(0),0)| + \frac{\mu(v_+ - v_-)}{4(1+t)} < \frac{\delta_\sigma v_-}{(1+t)}.
$$

So (3.1) is still true. □

Proof of Theorem 1.4. Based on the local existence of the solutions in Proposition 3.1 and the *a priori* estimates in Proposition 3.2, by the continuity extension argument, we can prove the global existence of the solutions for (1.2). The details are omitted. □

Proof of Theorem 1.5. We also prove it by contradiction. We first assume $\lambda < 1$. Suppose that there is a point (x_6, t_6) with $0 < t_6 < T$ such that either

$$|\xi_x(x_6,t_6)|v^{-1}(x_6,t_6) = \delta_1(1+t_6)^{-\lambda} \quad \text{or} \quad |\eta_x(x_6,t_6)|v^{-1}(x_6,t_6) = \delta_1(1+t_6)^{-\lambda},$$

or both, but

$$\sup_{x\in R}[|\xi_x|v^{-1}] < \delta_1(1+t)^{-\lambda} \quad \text{or} \quad \sup_{x\in R}[|\eta_x|v^{-1}] < \delta_1(1+t)^{-\lambda}, \tag{3.10}$$

is true for $t < t_6$, where $\delta_1 = 1/[\overline{c}(\gamma+1)]$. By (1.19), Theorem 1.1 and Theorem 1.2, we get $v_+ \le v(x,t) < v_-$. Using (2.29) with $\beta = -1, \sigma = 1/2$ and $A_{1/2}(t) := A(t)$, we have, on the characteristic curve $x_-(t)$,

$$\begin{aligned}\frac{d}{dt}[A(t)w(x_-(t),t)v^{-1}(x_-(t),t)] &= \frac{\mu}{2(1+t)^\lambda}A(t)[v^{-(\gamma+1)/2}v_x - u_x]v^{-1}\\ &\quad - A(t)wv^{-2}\left(\frac{\gamma-1}{2}v_x v^{-(\gamma+1)/2} + u_x\right).\end{aligned} \tag{3.11}$$

Since, along the characteristic curve $x_-(t)$,

$$(v^{-(\gamma+1)/2}v_x - u_x)v^{-1} = -\frac{d}{dt}\ln(v(x_-(t),t)) = \frac{d}{dt}\left[\ln(v_0(x_-(0))) - \ln(v(x_-(t),t))\right], \tag{3.12}$$

we have,

$$\begin{aligned} w(x,t)v^{-1} &= A^{-1}(t)w(x_-(0),0)v_0^{-1}(x_-(0)) + \frac{\mu\left[\ln(v_0(x_+(0))) - \ln(v)\right]}{2(1+t)^\lambda}\\ &\quad - A^{-1}(t)\int_0^t \frac{\mu A(\tau)}{2(1+\tau)^\lambda}\left(\frac{\mu}{2(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right)\left[\ln(v_0(x_+(0))) - \ln(v)\right]d\tau\\ &\quad - A^{-1}(t)\int_0^t A(\tau)wv^{-2}\left(\frac{\gamma-1}{2}v_x v^{-(\gamma+1)/2} + u_x\right)d\tau, \end{aligned} \tag{3.13}$$

for $t \le t_6$. Note that, from Mean-value Theorem,

$$|\ln(v_0(x_-(0))) - \ln(v)| = \frac{|v_0(x_-(0)) - v|}{v^*} \le \frac{|v_- - v_+|}{v_+}$$

where v^* is located between $v_0(x_-(0))$ and v. Also note that

$$|v_x|v^{-(\gamma+1)/2-1} = |w+z|v^{-1}/2 \le \delta_1(1+t)^{-\lambda},\ |u_x|v^{-1} = |w-z|v^{-1}/2 \le \delta_1(1+t)^{-\lambda}.$$

By definitions of $c_{1/2}, d_{1/2}, \overline{c}$ in (1.6)–(1.8), we get

$$\begin{aligned}
|w(x,t)|v^{-1} &\le \frac{c_{1/2}}{(1+t)^\lambda}\frac{|w(x_-(0),0)|}{v_0(x_-(0))} + \frac{\mu}{2(1+t)^\lambda}\frac{v_- - v_+}{v_+} \\
&\quad + \frac{d_{1/2}}{(1+t)^\lambda}\frac{v_- - v_+}{v_+} + \frac{(\gamma+1)\overline{c}}{2(1+t)^\lambda}\delta_1^2 \\
&\le \left[c_{1/2}\frac{|w(x_-(0),0)|}{v_0(x_-(0))} + \left(\frac{\mu}{2} + d_{1/2}\right)\frac{v_- - v_+}{v_+} + \frac{\delta_1}{2}\right](1+t)^{-\lambda} \\
&< \delta_1(1+t)^{-\lambda},
\end{aligned} \tag{3.14}$$

for $t \le t_6$. Similarly, we can obtain $|z(x,t)|v^{-1} < \delta_1(1+t)^{-\lambda}$, which is a contradiction with (3.10). Hence $|w|v^{-1}, |z|v^{-1} < \delta_1(1+t)^{-\lambda}$ for all $t > 0$.

If $\lambda = 1$ and $\mu > 2$, then $A(t) = (1+t)^{\mu/2}, c_{1/2} = 1, d_{1/2} = \mu/2, \overline{c} = 1/(\mu-2)$. Similar to (3.14), we can obtain the result. □

Remark 3.3. In the proof of Theorem 1.5, we let $\beta = -1$ in order to obtain the term $\sup_{x\in R}[|\xi_{0x}(x)|v_0^{-1}(x)]$. If we let $\beta = 0$, we would get a larger term $\sup_{x\in R}|\xi_{0x}(x)|\inf_{x\in R} v_0^{-1}(x)$ but we need to introduce the constant $\overline{c}$ and use a larger number $(\mu/2 + d_{1/2})(v_- - v_+)/v_+$ compared with the conditions in Theorem 1.4. We can choose $\beta = -1$ in the proof of Theorem 1.4 with a better term $\sup_{x\in R}[|\xi_{0x}(x)|v_0^{-1}(x)]$ but require to double the number $(\mu + 2d_{1/2})(v_- - v_+)/(4v_+)$ and to calculate the number $\overline{c}$.

Proof of Theorem 1.6. Suppose that there is a $t_7 > 0$ such that

$$|w(x,t)| \le \delta_2(1+t)^{-\lambda} \quad \text{and} \quad |z(x,t)| \le \delta_2(1+t)^{-\lambda}, \tag{3.15}$$

for $0 < t \le t_7$, where $\delta_2 = \underline{v}b_2/[\overline{c}(\gamma+1)]$. By (1.22), we have

$$\begin{aligned}
\inf_{x\in R}\{\xi_0,\eta_0\} &\ge \inf_{x\in R}\left(\frac{2}{\gamma-1}v_0^{-(\gamma-1)/2} - \frac{\varepsilon}{\gamma-1}v_0^{-(\gamma-1)/2}\right) \\
&= \frac{2-\varepsilon}{\gamma-1}\inf_{x\in R} v_0^{-(\gamma-1)/2} = \frac{2-\varepsilon}{\gamma-1}\overline{v}^{-(\gamma-1)/2}, \\
\sup_{x\in R}\{\xi_0,\eta_0\} &\le \sup_{x\in R}\left(\frac{2}{\gamma-1}v_0^{-(\gamma-1)/2} + \frac{\varepsilon}{\gamma-1}v_0^{-(\gamma-1)/2}\right) \\
&= \frac{2+\varepsilon}{\gamma-1}\sup_{x\in R} v_0^{-(\gamma-1)/2} = \frac{2+\varepsilon}{\gamma-1}\underline{v}^{-(\gamma-1)/2}.
\end{aligned}$$

The first inequality implies that (1.13) holds. Then by (1.14), we get

$$\left(\frac{2}{2+\varepsilon}\right)^{2/(\gamma-1)}\inf_{x\in R} v_0(x) \le v(x,t) \le \left(\frac{2}{2-\varepsilon}\right)^{2/(\gamma-1)}\sup_{x\in R} v_0(x),$$

and

$$
\begin{aligned}
|v_0(x_-(0)) - v(x_-(t),t)| &\le \sup_{x\in R} v(x,t) - \inf_{x\in R} v_0(x) \\
&\le \left(\frac{2}{2-\varepsilon}\right)^{2/(\gamma-1)}(\overline{v}-\underline{v}) + \left[\left(\frac{2}{2-\varepsilon}\right)^{2/(\gamma-1)} - 1\right]\inf_{x\in R} v_0(x) \\
&\le b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}.
\end{aligned} \tag{3.16}
$$

Using (2.29) with $\beta = 0, \sigma = 1/2$ and $A_{1/2}(t) := A(t)$, we have, on the characteristic curve $x_-(t)$,

$$
\begin{aligned}
\frac{d}{dt}[A(t)w(x_-(t),t)] &= \frac{\mu}{2(1+t)^\lambda}A(t)[v^{-(\gamma+1)/2}v_x - u_x] - \frac{\gamma+1}{2}A(t)wv^{-1}v_x v^{-(\gamma+1)/2} \\
&= -\frac{\mu}{2(1+t)^\lambda}A(t)\frac{d}{dt}[v(x_-(t),t) - v_0(x_-(0))] \\
&\quad - \frac{\gamma+1}{2}A(t)wv^{-1}v_x v^{-(\gamma+1)/2}.
\end{aligned}
$$

Solving for w yields

$$
\begin{aligned}
|w(x,t)| &\le A^{-1}(t)|w(x_-(0),0)| + \frac{\mu|v - v_0(x_-(0))|}{2(1+t)^\lambda} \\
&\quad + A^{-1}(t)\int_0^t \frac{\mu A(\tau)}{2(1+\tau)^\lambda}\left|\frac{\mu}{2(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right| |v - v_0(x_-(0))| d\tau \\
&\quad + \frac{\gamma+1}{2}A^{-1}(t)\int_0^t A(\tau)|w||v^{-1}|v_x|v^{-(\gamma+1)/2}d\tau \\
&\le \frac{c_{1/2}}{(1+t)^\lambda}|w(x_-(0),0)| + \left(\frac{\mu}{2} + d_{1/2}\right)\left(b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}\right)\frac{1}{(1+t)^\lambda} \\
&\quad + \frac{(\gamma+1)\overline{c}}{2(1+t)^\lambda}\underline{v}^{-1}b_2^{-1}\delta_2^2 \\
&\le \left[c_{1/2}\sup_{x\in R}|w(x,0)| + \left(\frac{\mu}{2} + d_{1/2}\right)\left(b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}\right) + \frac{\delta_2}{2}\right]\frac{1}{(1+t)^\lambda} \\
&< \frac{\delta_2}{(1+t)^\lambda}.
\end{aligned}
$$

Similarly, we can obtain $|z(x,t)| < \delta_2(1+t)^{-\lambda}$. Hence, (3.15) won't become equal at any point and the solution is global. □

4. Blow-up solutions

In this section, we show that the derivative blow-up occurs in finite time with solutions themselves bounded if the derivatives of initial data are sufficiently large at a point. We obtain the same result even for some initial data with small derivatives if either $\lambda > 1$ or $\lambda = 1$ and $0 < \mu \leq 1$.

We first prove that, for all cases of $\lambda > 0$ and $\mu > 0$, including the case of $0 < \lambda < 1$ and $\mu > 0$ mentioned in the last section for global existence of the solutions, once the derivatives of the initial data are sufficiently large at a point x_0, then the derivatives of the solutions for (1.2) still blow up as follows.

Proof of Theorem 1.7. If $\xi_0(x)$ and $\eta_0(x)$ both be monotone, then by Theorem 1.3, we have $\min\{v_+, v_-\} \leq v(x,t) \leq \max\{v_+, v_-\}$. Suppose $z(x_0, 0)$ is sufficiently large, using (2.30) with $\sigma = 1/2$, $\beta = -(\gamma+1)/4$ and $A_\sigma(t) = A_{1/2}(t) := A(t)$, we have, on the characteristic curve $x_+(t)$,

$$\begin{aligned}\frac{d}{dt}g_2(t) &= \frac{\mu}{2(1+t)^\lambda}A(t)[v^{-(\gamma+1)/2}v_x + u_x]v^\beta + \frac{\gamma+1}{4}A(t)zv^{\beta-1}\left(v^{-(\gamma+1)/2}v_x - u_x\right) \\ &= \frac{\mu}{2(1+t)^\lambda}A(t)[v^{-(\gamma+1)/2}v_x + u_x]v^\beta + \frac{\gamma+1}{4}A^{-1}(t)v^{(\gamma-3)/4}g_2^2(t). \end{aligned} \tag{4.1}$$

Since, along the characteristic curve $x_+(t)$,

$$(v^{-(\gamma+1)/2}v_x + u_x)v^{-(\gamma+1)/4} = \frac{d}{dt}\theta[v(x_+(t),t)], \quad \text{where} \quad \theta(v) = \begin{cases} \frac{4}{3-\gamma}v^{(3-\gamma)/4}, & \text{if } \gamma \neq 3, \\ \ln v, & \text{if } \gamma = 3, \end{cases}$$

we find

$$\begin{aligned} g_2(t) = g_2(0) &+ \frac{\mu}{2(1+t)^\lambda}A(t)\theta(v) - \frac{\mu}{2}\theta[v_0(x_+(0))] \\ &- \frac{1}{2}\int_0^t \frac{\mu A(\tau)}{(1+\tau)^\lambda}\left[\frac{\mu}{2(1+\tau)^\lambda} - \frac{\lambda}{1+\tau}\right]\theta(v)d\tau + \frac{\gamma+1}{4}\int_0^t A^{-1}(\tau)v^{(\gamma-3)/4}g_2^2(\tau)d\tau, \end{aligned}$$

where $x_+(0) = x_0$. If $0 < \lambda \leq 1$, we can find positive constants c_1, c_2, c_3, c_4 such that

$$A(t) \leq c_1 e^{c_2 t} \quad \text{and} \quad A^{-1}(t) \geq c_3 e^{-c_4 t}.$$

Denote

$$\begin{aligned} c_5 &= \mu c_1 e^{c_2}\max\theta(v)/2 + \mu\max\theta(v_0)/2 + d_{1/2}\max\theta(v) \quad \text{and} \\ c_6 &= \frac{\gamma+1}{4}c_3\min v^{(\gamma-3)/4}e^{-c_4}. \end{aligned}$$

Then, for $t \leq 1$, we have

$$g_2(t) \geq g_2(0) - c_5 + c_6 \int_0^t g_2^2(\tau) d\tau. \tag{4.2}$$

We will show that (4.2) blows up before $t = 1$ if $g_2(0)$ is sufficiently large. In fact, consider the integral equation

$$q(t) = 1/c_6 + c_6 \int_0^t q^2(\tau) d\tau,$$

which is equivalent to

$$q'(t) = c_6 q^2(t), \; q(0) = 1/c_6.$$

The solution satisfies

$$c_6 - \frac{1}{q(t)} = c_6 t,$$

which will blow up for $t \leq 1$. Now, if we choose $g_2(0) = 1/c_6 + c_5 + 1$, then

$$g_2(t) - q(t) \geq 1 + c_6 \int_0^t [g_2(\tau) - q(\tau)][g_2(\tau) + q(\tau)] d\tau,$$

which implies that $g_2(t) > q(t)$. Hence, $g_2(t)$ will also blow up for $t \leq 1$. If $\lambda > 1$, then both $A(t)$ and $A^{-1}(t)$ are bounded and (4.2) holds for all $t > 0$. Choosing $g_2(0) > c_5$, we find $g_2(t)$ must blow up in finite time. Similarly, if $-w(x_0, 0)$ is sufficiently large, we can prove that $-g_1(t)$ will blow up for $t \leq 1$. □

Next we prove Theorem 1.8, namely, for $\lambda > 1, \mu > 0$ or $\lambda = 1, 0 < \mu \leq 1$, $v_x(x,t)$ will blow up at finite time for all initial data including the ones with small derivatives, if $\xi_{0x}(x)$ and $\eta_{0x}(x)$ are monotonic.

Proof of Theorem 1.8. Let $\xi_{0x}(x) < 0$ and $\eta_{0x}(x) < 0$, then by Theorem 1.3, $v(x,t)$ is increasing for all x, namely, $v_x \geq 0$. Using (2.30) with $\sigma = 1$ and $\beta = -(\gamma+1)/4$, we have on the characteristic curve $x_+(t)$,

$$\begin{aligned} \frac{d}{dt} g_2(t) &= \frac{\mu}{(1+t)^\lambda} A_1(t) v_x v^{\beta - (\gamma+1)/2} + \frac{\gamma+1}{4} A_1(t) v^{\beta-1} z^2 \\ &\geq \frac{\gamma+1}{4} A_1^{-1}(t) v^{(\gamma-3)/4} g_2^2(t). \end{aligned} \tag{4.3}$$

If $\lambda > 1$, then

$$\frac{\gamma+1}{4} A_1^{-1}(t) v^{(\gamma-3)/4} \geq \frac{\gamma+1}{4} \exp(-\mu/(\lambda-1)) \min\left(v_-^{(\gamma-3)/4}, v_+^{(\gamma-3)/4}\right) =: c_7,$$

and

$$\frac{d}{dt}g_2(t) \geq c_7 g_2^2(t).$$

Hence, $g_2(t)$ must blow up in finite time. If $\lambda = 1$ and $0 < \mu \leq 1$, then

$$\begin{aligned}\frac{\gamma+1}{4}A_1^{-1}(t)v^{(\gamma-3)/4} &= \frac{\gamma+1}{4}(1+t)^{-\mu}v^{(\gamma-3)/4}\\ &\geq \frac{\gamma+1}{4}(1+t)^{-\mu}\min\left(v_-^{(\gamma-3)/4}, v_+^{(\gamma-3)/4}\right)\\ &=: c_8(1+t)^{-\mu},\end{aligned}$$

and

$$\frac{d}{dt}g_2(t) \geq c_8(1+t)^{-\mu}g_2^2(t). \tag{4.4}$$

Solving (4.4) yields

$$\frac{1}{g_2(0)} - \frac{1}{g_2(t)} \geq \begin{cases}\frac{c_8}{1-\mu}\left[(1+t)^{1-\mu}-1\right], & \text{if} \quad 0 \leq \mu < 1,\\ c_8\ln(1+t), & \text{if} \quad \mu = 1,\end{cases}$$

which implies that $g_2(t)$ still blows up in finite time. Similarly, if $\xi_{0x}(x) > 0$ and $\eta_{0x}(x) > 0$ for all x, from Theorem 1.3, then $v_x \leq 0$. Using (2.29) with $\sigma = 1$ and $\beta = -(\gamma+1)/4$, we have on the characteristic curve $x_-(t)$,

$$\begin{aligned}\frac{d}{dt}[-g_1(t)] &= -\frac{\mu}{(1+t)^\lambda}A_1(t)v^{\beta-(\gamma+1)/2}v_x + \frac{\gamma+1}{4}A_1(t)v^{\beta-1}w^2\\ &\geq \frac{\gamma+1}{4}A_1^{-1}(t)v^{(\gamma-3)/4}g_1^2(t),\end{aligned}$$

which means $g_1(t)$ also blow up in finite time. □

5. Time and space dependent damping

For general $\alpha(x,t)$, we can obtain similar results except Theorems 1.5 and 1.8 because we cannot prove monotonicity for v if the initial data are monotonic. We first show that $(v,u) \to (v_\pm, u_\pm)$ as $x \to \pm\infty$.

Lemma 5.1. *Let $(v,u) \in C^1(R\times[0,T])$ be the solutions to (1.1) and $\lim_{x\to\pm\infty}\alpha(x,t) = \alpha_\pm(t)$. Then*

$$\lim_{x\to\pm\infty} v(x,t) = v_\pm \quad \textit{and} \quad \lim_{x\to\pm\infty} u(x,t) = u_\pm \bar{B}_\pm^{-1}(t), \tag{5.1}$$

and (2.12) holds, where $\bar{B}_\pm(t) = \exp\left(\int_0^t \alpha_\pm(\tau)d\tau\right)$.

Proof. Denote

$$B_{\pm}(t) = \exp\left(\int_0^t \alpha(x_{\pm}(\tau), \tau)d\tau\right). \tag{5.2}$$

Taking derivatives of $\xi(x,t)B_-(t)$ along the minus characteristic curve, we have

$$\begin{aligned}
&\frac{d}{dt}[\xi(x_(t),t)B_-(t)] \\
&= \frac{d}{dt}\left[\left(\frac{2}{\gamma-1}v^{-(\gamma-1)/2}(x_(t),t) - u(x_(t),t)\right)B_-(t)\right] \\
&= \left(-v^{-(\gamma+1)/2}(v_t - v_x v^{-(\gamma+1)/2}) - u_t + u_x v^{-(\gamma+1)/2}\right)B_-(t) \\
&\quad + \left(\frac{2}{\gamma-1}v^{-(\gamma-1)/2} - u\right)B_-(t)\alpha(x_-(t),t) \\
&= \left[-v^{-(\gamma+1)/2}u_x + v^{-\gamma-1}v_x - v^{-\gamma-1}v_x + \alpha(x_-(t),t)u + u_x v^{-(\gamma+1)/2}\right]B_-(t) \\
&\quad + \left(\frac{2}{\gamma-1}v^{-(\gamma-1)/2} - u\right)B_-(t)\alpha(x_-(t),t) \\
&= \frac{2}{\gamma-1}v^{-(\gamma-1)/2}B_-(t)\alpha(x_-(t),t).
\end{aligned}$$

Similarly,

$$\frac{d}{dt}\left[\eta(x_+(t),t)B_+(t)\right] = \frac{2}{\gamma-1}u^{-(\gamma-1)/2}(x_+(t),t)B_+(t)\alpha(x_+(t),t).$$

Then

$$\xi(x,t) = B_-^{-1}(t)\xi(x\_(0),0) + B_-^{-1}(t)\int_0^t h\_(\tau)B_-(\tau)\alpha(x\_(\tau),\tau)d\tau, \tag{5.3}$$

$$\eta(x,t) = B_+^{-1}(t)\eta(x_+(0),0) + B_+^{-1}(t)\int_0^t h_+(\tau)B_+(\tau)\alpha(x_+(\tau),\tau)d\tau. \tag{5.4}$$

Define $h_{\pm}(t)$ and $\bar{h}_{\pm}$ as in (2.14). Adding (5.3) to (5.4) and solving for $h_+(t)$, we derive

$$\begin{aligned}
h_+(t) = \frac{1}{2}\Bigg[&B_-^{-1}(t)\xi(x_(0),0) + B_+^{-1}(t)\eta(x_+(0),0) \\
&+ B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_(\tau),\tau)h_-(\tau)d\tau + B_+^{-1}(t)\int_0^t B_+(\tau)\alpha(x_+(\tau),\tau)h_+(\tau)d\tau\Bigg].
\end{aligned} \tag{5.5}$$

We rewrite (2.18) as

$$\bar{h}_+ = \frac{1}{2}\Big[B_-^{-1}(t)\bar{h}_+ + B_+^{-1}(t)\bar{h}_+\Big] + \frac{1}{2}B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)\bar{h}_+ d\tau$$
$$+ \frac{1}{2}B_+^{-1}(t)\int_0^t B_+(\tau)\alpha(x_+(\tau),\tau)\bar{h}_+ d\tau. \tag{5.6}$$

Subtracting (5.6) from (5.5) yields

$$\begin{aligned} h_+(t) - \bar{h}_+ = {} & \frac{1}{2}\Big\{B_-^{-1}(t)[h_-(0) - \bar{h}_+] + B_+^{-1}(t)[h_+(0) - \bar{h}_+] \\ & + B_+^{-1}(t)u_0(x_+(0)) - B_-^{-1}(t)u_0(x_-(0))\Big\} \\ & + \frac{1}{2}B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)(h_-(\tau) - \bar{h}_+)d\tau \\ & + \frac{1}{2}B_+^{-1}(t)\int_0^t B_+(\tau)\alpha(x_+(\tau),\tau)(h_+(\tau) - \bar{h}_+)d\tau. \end{aligned} \tag{5.7}$$

Note that

$$\begin{aligned} & B_+^{-1}(t)\int_0^t B_+(\tau)\alpha(x_+(\tau),\tau)(h_+(\tau) - \bar{h}_+)d\tau \\ = {} & B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)\left[\frac{B_+^{-1}(t)B_+(\tau)\alpha(x_+(\tau),\tau)}{B_-^{-1}(t)B_-(\tau)\alpha(x_-(\tau),\tau)} - 1\right](h_+(\tau) - \bar{h}_+)d\tau \\ & + B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)(h_+(\tau) - \bar{h}_+)d\tau, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} |B_+^{-1}(t)u_0(x_+(0)) - B_-^{-1}(t)u_0(x_-(0))| \le {} & B_-^{-1}(t)\Big[B_+^{-1}(t)B_-(t) - 1\Big]u_0(x_+(0)) \\ & + B_-^{-1}(t)[u_0(x_+(0)) - u_0(x_-(0))]. \end{aligned} \tag{5.9}$$

For any $\varepsilon > 0$ and fixed t, we can choose $M = M(t)$ sufficiently large such that

$$|h_-(0) - \bar{h}_+| \le \varepsilon, \quad |h_+(0) - \bar{h}_+| \le \varepsilon, \quad |u_0(x_+(0)) - u_0(x_-(0))| \le \varepsilon,$$

$$|B_+^{-1}(t)B_-(t)-1| = \left| \exp\left[\int_0^t (\alpha(x_-(\tau,\tau) - \alpha(x_+(\tau),\tau))d\tau \right] - 1 \right| \le \varepsilon,$$

and

$$\left| \frac{B_+^{-1}(t)B_+(\tau)\alpha(x_+(\tau),\tau)}{B_-^{-1}(t)B_-(\tau)\alpha(x_-(\tau),\tau)} - 1 \right| \le (1+t)^{-2},$$

as long as $x_-(t) > x_+(t) > M(t)$. Define $s(\tau)$ as in (2.21). Combining (5.7) - (5.9), we have

$$\begin{aligned} s(t) &\le \varepsilon(3+v_+ + |u_+|)B_-^{-1}(t) + B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)s(\tau)d\tau \\ &\quad + B_-^{-1}(t)\int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)s(\tau)(1+\tau)^{-2}d\tau \\ &=: f_2(t), \end{aligned}$$

where we have used the fact that $|u_0(x)| \le |u_+| + 1$ for $x > M(t)$. Since

$$\begin{aligned} f_2'(t) = &-B_-^{-1}(t)\alpha(x_-(t),t)\left[(3+v_+ + |u_+|)\varepsilon + \int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)s(\tau)d\tau \right. \\ &\left. + \int_0^t B_-(\tau)\alpha(x_-(\tau),\tau)s(\tau)(1+\tau)^{-2}d\tau \right] \\ &+ \alpha(x_-(t),t)s(t) + \alpha(x_-(t),t)s(t)(1+t)^{-2} \\ &\le c_9 f_2(t)(1+t)^{-2}, \end{aligned}$$

we find

$$s(t) \le f_2(t) \le f_2(0)e^{c_9} = \varepsilon(3+v_+ + |u_+|)e^{c_9},$$

where $c_9 = \sup_{(x,t)} \alpha(x_-(t),t)$, which implies that $\lim_{x\to+\infty} v(x,t) = v_+$. The rest of the proof is similar to that in Lemma 2.1. □

Now, we present two global existence results and a blow-up result. We assume that

$$\mu \le \alpha(x,t)(1+t)^\lambda \le d_1, \quad (|a_t(x,t)| + |a_x(x,t)|)(1+t)^{2\lambda} \le d_2, \tag{5.10}$$

where μ, λ, d_1, d_2 are positive constants and $\lambda > 0$. Define

$$
\begin{aligned}
&A_{\pm}(t) := \exp\left(\frac{1}{2}\int_0^t \alpha(x_{\pm}(\tau),\tau)d\tau\right),\\
&c_{10} := \sup_t A_{\pm}^{-1}(t)(1+t)^{\lambda},\\
&c_{11} := \sup_t A_{\pm}^{-1}(t)(1+t)^{\lambda}\int_0^t \frac{A_{\pm}(\tau)}{(1+\tau)^{2\lambda}}d\tau. \qquad (5.11)
\end{aligned}
$$

Theorem 5.2. *Assume that* $\lambda < 1$, $v_- > v_+$, $u_- = u_+ = 0$ *and*

$$
\frac{2}{\gamma-1}v_-^{-(\gamma-1)/2} \le \inf_{x\in R}(\xi_0,\eta_0)(x) < \sup_{x\in R}(\xi_0,\eta_0)(x) \le \frac{2}{\gamma-1}v_+^{-(\gamma-1)/2}. \qquad (5.12)
$$

If

$$
\begin{aligned}
&c_{10}\sup_{x\in R}(|\xi_{0x}(x)|,|\eta_{0x}(x)|) + [d_1/2 + c_{11}(d_1^2+d_2)](v_- - v_+) + \frac{2c_{11}d_2 v_+^{-(\gamma+1)/2}}{\gamma-1}\\
&< \frac{v_+}{2c_{11}(\gamma+1)}, \qquad (5.13)
\end{aligned}
$$

then (1.1) has a unique pair of global solutions $(v,u)(x,t)$ *satisfying the decay estimate* (1.18).

Proof. It is easy to see that Theorem 1.3 still holds if $\mu(1+t)^{-\lambda}$ is replaced by $\alpha(x,t)$. But (3.11) becomes

$$
\begin{aligned}
\frac{d}{dt}[A_-(t)w(x_-(t),t)] &= \frac{1}{2}\alpha(x_-(t),t)A_-(t)[v^{-(\gamma+1)/2}v_x - u_x] - A_-(t)\alpha_x(x_-(t),t)u\\
&\quad - \frac{\gamma+1}{2}A_-(t)wv^{-1}v^{-(\gamma+1)/2}v_x.
\end{aligned}
$$

Solving for w yields

$$
\begin{aligned}
w(x,t) &= A_-^{-1}(t)w(x_-(0),0) + \frac{1}{2}A_-^{-1}(t)\int_0^t \alpha(x_-(\tau),\tau)A_-(\tau)\frac{d}{dt}\left[v_0(x_-(0)) - v(x_-(\tau),\tau)\right]d\tau\\
&\quad - A_-^{-1}(t)\int_0^t A_-(\tau)\alpha_x(x_-(\tau),\tau)ud\tau - \frac{\gamma+1}{2}A_-^{-1}(t)\int_0^t A_-(\tau)wv^{-1}v^{-(\gamma+1)/2}v_x d\tau\\
&= A_-^{-1}(t)w(x_-(0),0) + \frac{\alpha(x_-(t),t)[v_0(x_-(0)) - v]}{2}\\
&\quad - \frac{1}{2}A_-^{-1}(t)\int_0^t A_-(\tau)\left[\frac{1}{2}\alpha^2(x_-(\tau),\tau) - \alpha_x(x_-(\tau),\tau)v^{-(\gamma+1)/2} + \alpha_t(x_-(\tau),\tau)\right]
\end{aligned}
$$

$$\times [v_0(x_-(0)) - v]d\tau - A_-^{-1}(t)\int_0^t \alpha_x(x_-(\tau),\tau)A_-(\tau)u d\tau$$

$$- \frac{\gamma+1}{2}A_-^{-1}(t)\int_0^t A_-(\tau)wv^{-1}v^{-(\gamma+1)/2}v_x d\tau.$$

Suppose that there is a $t_8 > 0$ such that

$$|w(x,t)| \le \delta_3(1+t)^{-\lambda} \quad \text{and} \quad |z(x,t)| \le \delta_3(1+t)^{-\lambda}, \tag{5.14}$$

for $0 < t \le t_8$, where $\delta_3 = v_+/[c_{11}(\gamma+1)]$. By Theorem 1.1 and (5.12), we have

$$|u| \le \frac{2}{\gamma-1}v_+^{-(\gamma-1)/2}.$$

Then, using (5.13) and notations defined in (5.11), we find

$$\begin{aligned}
|w(x,t)| \le{}& \frac{c_{10}}{(1+t)^\lambda}|w(x_-(0),0)| + \frac{d_1}{2(1+t)^\lambda}(v_- - v_+)\\
&+ \frac{c_{11}(d_1^2+d_2)}{(1+t)^\lambda}(v_- - v_+) + \frac{c_{11}d_2}{(1+t)^\lambda}\frac{2}{\gamma-1}v_+^{-(\gamma-1)/2} + \frac{c_{11}(\gamma+1)}{2(1+t)^\lambda}v_+^{-1}\delta_3^2\\
<{}& \frac{v_+}{2c_{11}(\gamma+1)(1+t)^\lambda} + \frac{\delta_3}{2(1+t)^\lambda}\\
={}& \delta_3(1+t)^{-\lambda},
\end{aligned} \tag{5.15}$$

for $t \le t_8$. Similarly, we can obtain $|z(x,t)| < \delta_3(1+t)^{-\lambda}$. Hence, (3.15) won't become equal at any point and the solution is global. □

Theorem 5.3. *Define $\underline{v}$, $\overline{v}$, b_1 and b_2 as in Theorem 1.6, and let the initial data satisfy the following conditions:*

(i) *For some small $\varepsilon > 0$, $(\gamma-1)|u_0(x)|v_0(x)^{(\gamma-1)/2} \le \varepsilon$.*
(ii) *$c_{10}\sup_{x\in R}\{|\xi_{0x}|, |\eta_{0x}|\} + d_3b_1(\overline{v} - \underline{v}) < \left[\frac{b_2}{2c_{11}(\gamma+1)} - d_3\frac{2\varepsilon}{\gamma-1}\right]\cdot \underline{v}$, where $d_3 = d_1/2 + c_{11}(d_1^2+d_2)$.*

If $0 < \lambda < 1$, then (1.1) has a unique pair of solutions globally-in-time, whose derivatives satisfy the decay rate $(1+t)^{-\lambda}$.

Proof. Suppose that there is a $t_9 > 0$ such that

$$|w(x,t)| \le \delta_4(1+t)^{-\lambda} \quad \text{and} \quad |z(x,t)| \le \delta_4(1+t)^{-\lambda}, \tag{5.16}$$

for $0 < t \le t_9$, where $\delta_4 = b_2\underline{v}/[c_{11}(\gamma+1)]$. It is easy to see that (3.16) is still true. From the first inequality of (5.15), we have

$$\begin{aligned}|w(x,t)| \le &\frac{c_{10}}{(1+t)^\lambda}|w(x_-(0),0)| + \frac{d_1}{2(1+t)^\lambda}\left[b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}\right]\\ &+ \frac{c_{11}(d_1^2+d_2)}{(1+t)^\lambda}\left[b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}\right]\\ &+ \frac{c_{11}d_2}{(1+t)^\lambda}\frac{2}{\gamma-1}\underline{v}^{-(\gamma-1)/2} + \frac{c_{11}(\gamma+1)}{2(1+t)^\lambda}b_2^{-1}\underline{v}^{-1}\delta_4^2\\ \le &\left[c_{10}\sup_{x\in R}|w(x,0)| + [d_1/2 + c_{11}(d_1^2+d_2)]\left(b_1(\overline{v}-\underline{v}) + \frac{2\varepsilon}{\gamma-1}\underline{v}\right) + \frac{\delta_4}{2}\right](1+t)^{-\lambda}\\ < &\,\delta_4(1+t)^{-\lambda}.\end{aligned}$$

Similarly, we can obtain $|z(x,t)| < \delta_4(1+t)^{-\lambda}$. Hence, (5.16) won't become equal at any point and the solution is global. □

Theorem 5.4. *Suppose that $\lambda > 0$ and (1.13) is satisfied. If either $\eta_x(x_0,0) \ll -1$ or $\xi_x(x_0,0) \gg 1$ at some point x_0, then, there exists a finite time $t_* > 0$ such that $\lim_{t\uparrow t_*^-}\|(v_x,u_x)(t)\|_{L^\infty(R)} = +\infty$.*

Proof. The proof is similar to that of Theorem 1.7. □

Remark 5.5. It is easy to see that the following functions satisfy Condition (5.10):

$$\alpha(x,t) = \frac{1}{(1+t)^\lambda}\left(1 + \frac{\beta(x)}{(1+t)^\lambda}\right) \quad \text{or} \quad \alpha(x,t) = \left[\exp\left(\frac{\beta(x)}{1+t}\right) - 1\right]^\lambda,$$

where $\beta(x) \ge 0.2$, $\beta(x)$ and $\beta'(x)$ are bounded.

6. Numerical simulations

In this section, we present numerical simulations to confirm our theoretical results and demonstrate the arising of blow-up solution at a point. To obtain a stable numerical solution, we differentiate $(1.2)_1$ with respect to t, and submit it to $(1.2)_2$, then we reduce the system (1.2) to a second-order wave equation with time-vanishing-damping:

$$\begin{cases} v_{tt} = (v^{-\gamma-1}v_x)_x - \dfrac{\mu}{(1+t)^\lambda}v_t,\\ v(x,0) = v_0(x),\\ v_t(x,0) = u_0'(x).\end{cases} \tag{6.1}$$

Here, we take the initial data as $v_0(x) = 3 + a_1\arctan(a_2x)$ and $u_0(x) = a_3 + a_4\exp(-0.5x^2)$, with the parameters a_1, a_2, a_3, a_4 to be specified later in different cases. The computational domain is $[-50, 50]$ with Neumann boundary conditions and 100001 uniform mesh points. We use

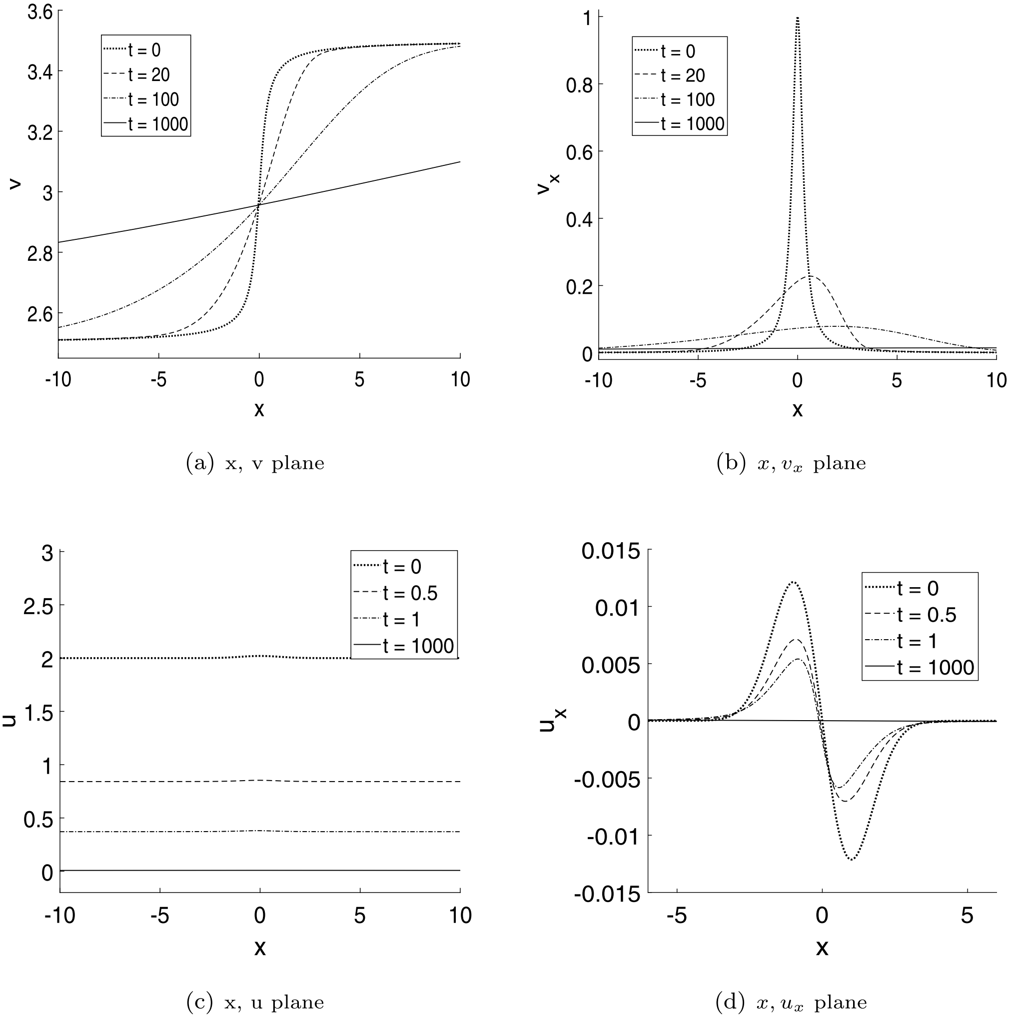


Fig. 1. Global solution for $\lambda = 0.5, \gamma = 2, \mu = 2$ with a monotonic initial data $v_0(x)$ and a small $u_{0x}(x)$.

the explicit central difference scheme in x and implicit central difference scheme in t to numerically study the following five examples. Note that the solution of (6.1) is independent of a_3 because $v_t(x, 0)$ only depends on $u_0'(x)$,

Example 1. Let $\lambda = 0.5, \mu = 2, \gamma = 2, a_1 = 1/\pi, a_2 = 1, a_3 = 2$ and $a_4 = 0.02$. Then the initial Riemann data are bounded, monotonically increasing, and their derivatives are not big. From Theorem 1.4 we expect that the solutions of (1.2) globally exist, and $v(x, t)$ is increasing in x. This is confirmed numerically in Fig. 1. Interestingly, in this example, we observe that, the derivative of the initial data $v_{0x}(x)$ is still steep near $x = 0$, but $v_x(x, t)$ goes more and more flat as time t increases, no blow-up for v_x occurs. We also observe that u has decay rate $a_3 \exp(4 - 4\sqrt{1 + t})$ for large $|x|$ as shown in (2.11).

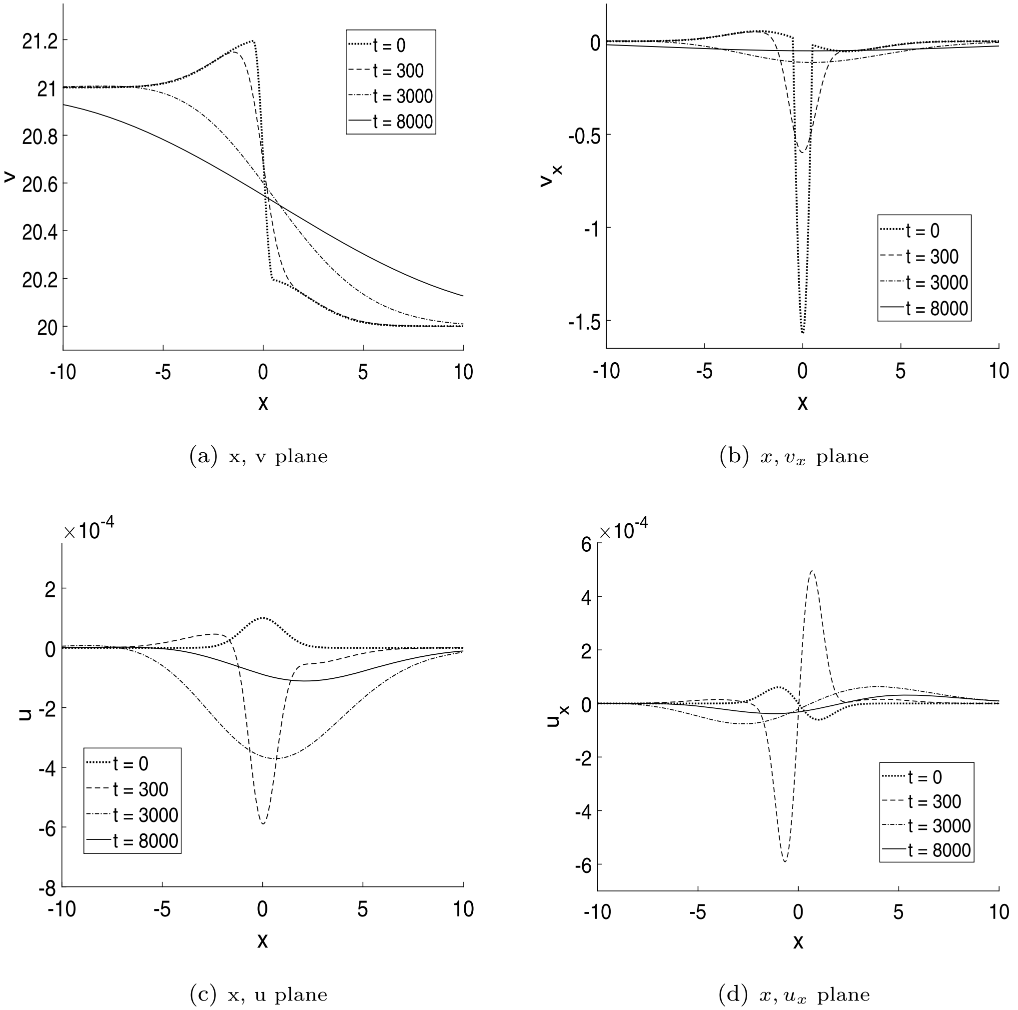
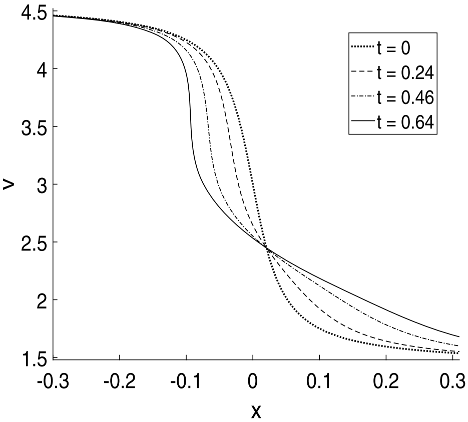


Fig. 2. Global solution for $\lambda = 0.5$, $\gamma = 2$, $\mu = 2$ with a non-monotonic initial data $v_0(x)$ and small $u_0(x), u_{0x}(x)$.

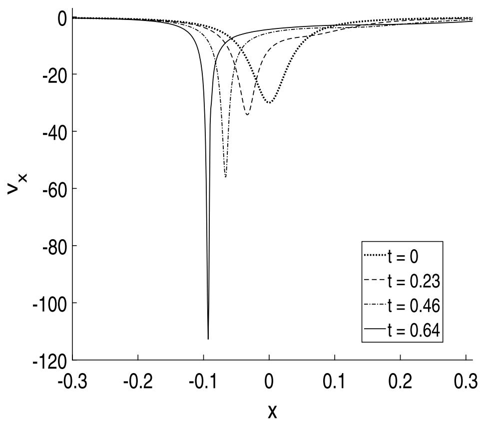
Example 2. On the other hand, we choose non-monotonic initial data for $v_0(x)$:

$$v_0(x) = 20 + 0.2e^{0.1x^2} + \begin{cases} 1 & \text{if } x < -0.5, \\ 0.5(1 + \cos(\pi x + 0.5\pi)) & \text{if } -0.5 \le x \le 0.5, \\ 0 & \text{if } x > 0.5, \end{cases}$$

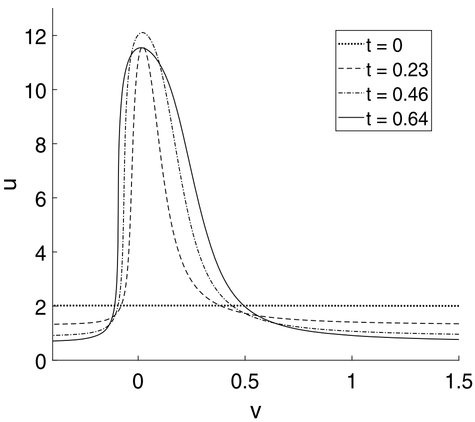
and keep the same form for $u_0(x) = a_3 + a_4 \exp(-0.5x^2)$, with $\lambda = 0.5, \mu = 2, \gamma = 2, a_3 = 0, a_4 = 0.0001$. Then all conditions in Theorem 1.6 are satisfied. So the behaviors of the solution are similar to those in Example 1 (see Fig. 2).



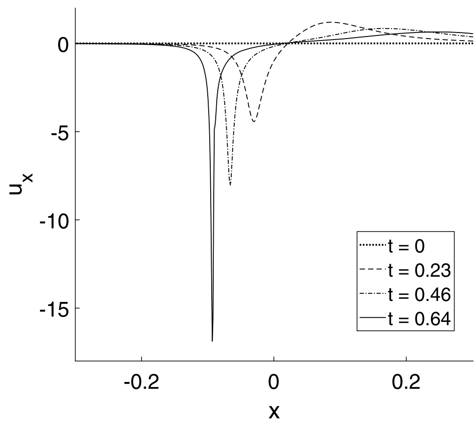
(a) x, v plane



(b) x, v_x plane



(c) x, u plane

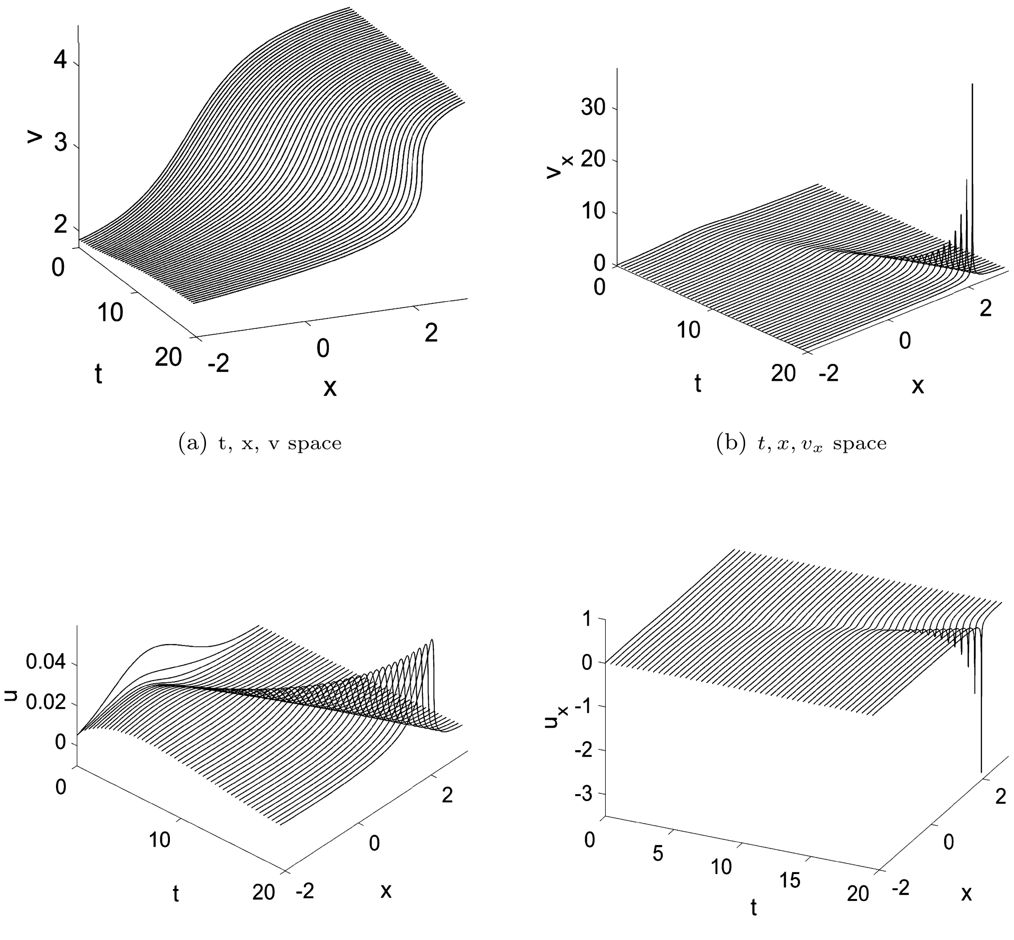


(d) x, u_x plane

Fig. 3. Blow-up solution for $\lambda = 0.5$, $\gamma = 2$, $\mu = 2$ with a monotonic initial data $v_0(x)$ but large $v_{0x}(x)$ and $u_{0x}(x)$.

Example 3. Let $\lambda = 0.5, \mu = 2, \gamma = 2, a_1 = -1, a_2 = 30, a_3 = 2, a_4 = 0.004$. In this case, the initial data $v_0(x)$ is decreasing, but the derivative of the initial data $v_{0x}(x)$ is big, then, from Theorem 1.7, the solutions of (1.2) $(v, u)(x, t)$ are still bounded, but their derivatives $(v_x, u_x)(x, t)$ will blow up at a finite time. As showed in Fig. 3, we see that $v(x, t)$ and $u_{0x}(x)$ both are bounded, where $v(x, t)$ is monotonic deceasing in x, and $u(x, t)$ is non-monotonic, but both $v_x(x, t)$ and $u_x(x, t)$ blow up near $x = -0.1$ as time t goes up approximately to $t \approx 0.67$.

Example 4. Let $\lambda = 1.5 > 1, \mu = 2, \gamma = 3, a_1 = 1, a_2 = 1, a_3 = 0, a_4 = 0.03$. In this case, from Theorem 1.8, the derivatives of the solutions will blow up for all initial data including those small data, once $\xi_0(x)$ and $\eta_0(x)$ are monotonic. As showed in Fig. 4, the solution $v_x(x, t)$ blows up at $t \approx 19.7$. Note that the initial Riemann invariants $\xi_0(x)$ and $\eta_0(x)$ both are increasing, in fact, $0 < v_0^{-(\gamma+1)/2} v_{0x} \pm u_{0x} < 0.12$.



(a) t, x, v space

(b) t, x, v_x space

(c) t, x, u space

(d) t, x, u_x space

Fig. 4. Blow-up solution for $\lambda = 1.5, \gamma = 3, \mu = 2$ with initial data $v_0(x)$ monotone and $u_{0x}(x)$ small.

Example 5. We test the critical case with $\lambda = 1$, and take the other parameters as $\mu = 0.8, \gamma = 2, a_1 = 1, a_2 = 0.1, a_3 = 0, a_4 = 0.003$. The initial Riemann invariants $\xi_0(x)$ and $\eta_0(x)$ both are increasing, verified by $0 < v_0^{-(\gamma+1)/2} v_{0x} \pm u_{0x} < 0.014$. From Theorem 1.8, the derivatives of the solutions are still expected to blow up. In fact, as showed in Fig. 5, $v_x(x,t)$ blows up at $t \approx 310$.

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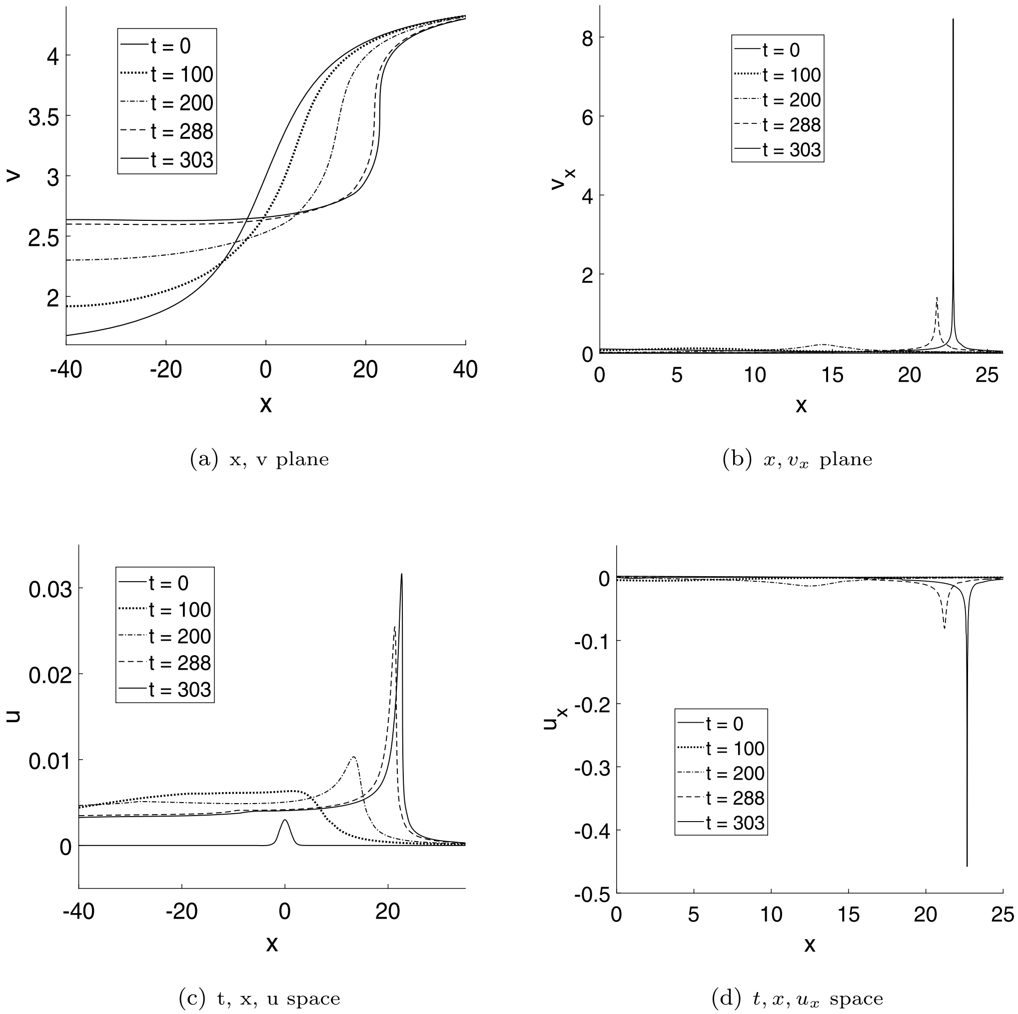

Fig. 5. Blow-up solution for the critical case of $\lambda = 1$, where $\gamma = 2$, $\mu = 0.8$ with initial data $v_0(x)$ monotone and $u_{0x}(x)$ quite small.

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