M. J. Gander, M. Mei and E. J. P. Georg Schmidt. (2007) "Phase Transitions in a Relaxation Model of Mixed Type with Periodic Boundary Condition,"
Applied Mathematics Research eXpress, Vol. 2007, Article ID abm006, 34 pages.
doi:10.1093/amrx/abm006

Phase Transitions in a Relaxation Model of Mixed Type with Periodic Boundary Condition

Martin J. Gander[1], Ming Mei[2,3,*] and E. J. P. Georg Schmidt[3]

[1]Department of Mathematics, University of Geneva CH-1211 Geneva, Switzerland and [2]Department of Mathematics, Champlain College, St.-Lambert, Quebec, Canada J4Y 2P3 and [3]Department of Mathematics and Statistics, McGill University, Montreal, Quebec, Canada H3A 2G6

Correspondence to be sent to: Ming Mei, Department of Mathematics, Champlain College, St.-Lambert, Quebec, Canada J4Y 2P3. e-mail: mei@mathstat.concordia.ca

We study the asymptotic behavior of solutions for a 2×2 relaxation model of mixed type with periodic initial and boundary conditions. We prove that the asymptotic behavior of the solutions and their phase transitions are dependent on the location of the initial data and the size of the viscosity. If the average of the initial data is in the hyperbolic region and the initial data does not deviate too much from its average, we prove that there exists a unique global solution and that it converges time-asymptotically to the average in the same hyperbolic region. No phase transition occurs after initial oscillations. If the average of the initial data is in the elliptic region and the initial data does not deviate too much from its average, and in addition if the viscosity is big, then the solution converges to the average in the same elliptic region, and does not exhibit phase transitions after initial oscillations. If, however, the viscosity is small, numerical evidence indicates that the solution oscillates across the hyperbolic and elliptic regions for all time, exhibiting phase transitions. In this case, we conjecture that the solution converges to an oscillatory standing wave (steady-state solution).

Received September 14, 2006; Revised April 7, 2007; Accepted June 3, 2007

See http://www.oxfordjournals.org/our_journals/amrx/ for proper citation instructions.

1 Introduction and main results

Relaxation phenomena arise in many physical situations, such as gases which are not in thermodynamic equilibrium, river flows, traffic flows, kinetic theory, viscoelasticity with memory and general waves, see for example [3, 5, 15–17, 19–22, 24, 27, 30] and the references therein, in particular, the textbook by Whitham [37]. The most basic features of relaxation phenomena are captured by the following 2×2 relaxation model

$$\begin{cases} v_t - u_x = 0, \\ u_t - \sigma(v)_x = \frac{f(v)-u}{\tau}, \end{cases} \quad x \in (-\infty, \infty), \quad t > 0, \tag{1.1}$$

where $v(t,x)$ is some conserved physical quantity, like the specific volume in the case of gases that are not in thermodynamic equilibrium, and $u(t,x)$ is some rate variable like the velocity in that case. The parameter $\sigma(v)$ denotes the nonlinear pressure, $f(v)$ represents the flux function, and τ is the so-called relaxation time. Usually, $\tau = \tau(v,u)$ depends on the specific volume v and the velocity u, see [37]. We, however do not consider the variability of τ here, and thus assume $\tau = 1$ throughout this paper.

It is well known that the characteristic roots of (1.1) are

$$\lambda_\pm = \pm\sqrt{\sigma'(v)}.$$

The usual assumption on $\sigma(v)$ is that $\sigma'(v) > 0$ for all v under consideration, which means that the roots $\lambda_\pm$ are real so that the system (1.1) is a hyperbolic conservation law. In [17], Jin and Xin do a detailed discretization study for the important special case in which $\sigma(v) = av$, with a a positive constant. The first investigation of the nonlinear stability of diffusion waves, traveling waves and rarefaction waves as time $t \to +\infty$ is due to T.-P. Liu [20] in 1987. Since then the subject has been broadly and deeply developed by many people: see, for example, [5, 19, 21, 22, 24, 27, 30, 38–41]. The asymptotic behavior of solutions to the equilibria as relaxation time $\tau \to 0^+$ was well studied in [3, 15–18, 35], etc. When $\sigma'(v)$ changes sign, then the system (1.1) is of mixed type. Its behavior is elliptic in some regions where $\sigma'(v) < 0$, and hyperbolic in other regions where $\sigma'(v) > 0$. Such a problem can exhibit phase transitions. Two prototypes of this situation are the case where the strain $\sigma(v) = v^3 - v$ in viscoelastic dynamics, and the pressure $p(v)(= -\sigma(v)) = \frac{R\theta}{v-b} - \frac{a}{v^2}$ with positive constants R, θ, a and b satisfying $R\theta b/a < (2/3)^3$ and $v > b$ in van der Waals fluid dynamics. In such a mixed case, the study becomes more difficult.

In this paper, as in the case of the above two examples of $\sigma(v)$, we assume that there exist two constants v_1 and v_2, $v_1 < v_2$, such that

$$\begin{cases} \sigma'(v_1) = \sigma'(v_2) = 0, \\ \sigma'(v) > 0 \quad \text{for } v \in (-\infty, v_1) \cup (v_2, \infty), \\ \sigma'(v) < 0 \quad \text{for } v \in (v_1, v_2). \end{cases} \tag{1.2}$$

The nonlinear function $f(v)$ is assumed to be smooth enough for all $v \in R$. With (1.2), the system (1.1) is hyperbolic in the region $(-\infty, v_1)$ and (v_2, ∞), and elliptic in the region (v_1, v_2).

The Cauchy problem for (1.1) is ill posed due to the sign changes of $\sigma'(v)$: in the elliptic region, this would require one to specify the value of $v(t, x)$ at $t = T$ as another "boundary" condition for some positive constant T. Because of this, artificial viscous terms are often added to make the system well posed. For example this has been done for the phase transition problems of the viscous-capillary p-system ([1, 4, 11, 25, 28, 42]; M. Mei, Y. S. Wong, L. Liu, unpublished data). Another way to study the phase transition phenomena within a strongly hyperbolic background is the construction of shock and wave curves, as well as the vanishing viscosity approach (see [2, 6–10, 12, 28, 29, 31–34]). Regarding the numerical computation for phase transition phenomena to the other models, we refer to, for example, [13, 36] and the references therein.

1.1 Periodic initial-boundary value problem

In this paper, we study the following system, in which an artificial viscous term εv_{xx} is added to the first equation in (1.1):

$$\begin{cases} v_t - u_x = \varepsilon v_{xx}, \\ u_t - \sigma(v)_x = f(v) - u, \end{cases} \quad x \in (-\infty, \infty), \quad t > 0, \tag{1.3}$$

with the periodic initial-boundary value conditions

$$\begin{cases} (v, u)(0, x) = (v_0, u_0)(x), \\ v(t, x) = v(t, x + 2L), \end{cases} \quad x \in (-\infty, \infty), \tag{1.4}$$

where $L > 0$ is a given constant. The initial data $(v_0, u_0)(x)$ satisfy the compatibility condition $(v_0, u_0)(x) = (v_0, u_0)(x + 2L)$.

Equilibrium solutions will play an important role in what follows. One can easily establish that all equilibria of the above system (or of the original system (1.1)) have the form $(v, u) \equiv (m_0, m_1)$, where m_0 is an arbitrary constant and $m_1 = f(m_0)$. An equilibrium (m_0, m_1) is said to be hyperbolic if $m_0 \in (-\infty, v_1) \cup (v_2, \infty)$ and elliptic if $m_0 \in (v_1, v_2)$.

We now show that every solution of the system (1.3) and (1.4) is associated with a certain equilibrium. Integrating the first equation in (1.3) with respect to x over $[0, 2L]$ and using the periodicity $u(t,x) = u(t, x+2L)$ and $v_x(t,x) = v_x(t, x+2L)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2L} v(t,x)dx = \int_0^{2L}[u_x(t,x) + \varepsilon v_{xx}(t,x)]dx = 0,$$

and thus the integral of $v(t,x)$ stays constant in time,

$$\int_0^{2L} v(t,x)dx = \int_0^{2L} v_0(x)dx.$$

Hence there is a natural association between the initial data (v_0, u_0) and the equilibrium (m_0, m_1) with

$$m_0 = \frac{1}{2L}\int_0^{2L} v_0(x)dx \quad \text{and} \quad m_1 = f(m_0). \tag{1.5}$$

We note that

$$\int_0^{2L}[v(t,x) - m_0]dx = 0. \tag{1.6}$$

The main purpose of this paper is to study the asymptotic behavior of solutions $(v,u)(t,x)$ for the periodic initial-boundary problem (1.3) and (1.4). We prove that the asymptotic behavior of the solutions $(v,u)(t,x)$ and their phase transitions are dependent on the location of the initial data $(v_0, u_0)(x)$ and the size of viscosity ε.

First of all, we investigate the criteria for linear stability or instability. We linearize the system (1.3) around the associated equilibrium (m_0, m_1), and set $(V, U) = (v - m_0, u - m_1)$. Then (V, U) satisfies

$$\begin{cases} V_t - U_x = \varepsilon V_{xx}, \\ U_t - \sigma'(m_0)V_x = f'(m_0)V - U, \end{cases} \tag{1.7}$$

where we used $m_1 = f(m_0)$, see (1.5). Differentiating the first equation of (1.7) with respect to t and the second one with respect to x, and substituting the first equation into the second equation, we obtain

$$V_{tt} + V_t - \varepsilon V_{xxt} - (\varepsilon + \sigma'(m_0))V_{xx} - f'(m_0)V_x = 0. \tag{1.8}$$

This equation admits a solution of the form

$$V(x,t) = \tilde{V} e^{\alpha t + i\beta x}, \tag{1.9}$$

where $\tilde{V}$ is a constant, α is the frequency and is complex, and β is the wave number satisfying the periodicity condition $e^{i\beta x} = e^{i\beta(x+2L)}$, which implies $\beta = \frac{k\pi}{L}$ for $k = 0, 1, 2, \ldots$. Substituting (1.9) into (1.8) yields

$$\alpha^2 + (1 + \varepsilon\beta^2)\alpha + (\varepsilon + \sigma'(m_0))\beta^2 - if'(m_0)\beta = 0,$$

which has two modes α_+ and α_- given by

$$\alpha_\pm = \frac{-(1+\varepsilon\beta^2) \pm \sqrt{(1+\varepsilon\beta^2)^2 - 4[(\varepsilon + \sigma'(m_0))\beta^2 - if'(m_0)\beta]}}{2} = \frac{-(1+\varepsilon\beta^2) \pm \sqrt{a + bi}}{2},$$

where

$$a = (1 + \varepsilon\beta^2)^2 - 4(\varepsilon + \sigma'(m_0))\beta^2, \qquad b = 4f'(m_0)\beta. \tag{1.10}$$

A straightforward, but tedious computation shows that the real part of $\sqrt{a+bi}$ is

$$\left| Re\left(\sqrt{a+bi}\right)\right| = \sqrt{\frac{a+r}{2}}, \qquad r = \sqrt{a^2 + b^2}.$$

Thus, we have

$$Re(\alpha_-) < Re(\alpha_+) = -\frac{1}{2}\left[(1+\varepsilon\beta^2) - \sqrt{\frac{a+r}{2}}\right].$$

If $Re(\alpha_+) < 0$, then from (1.9) we have $|(V,U)| = |(v - m_0, u - m_1)| \le O(1)e^{-|Re(\alpha_+)|t}$. In order to obtain $Re(\alpha_+) < 0$, we must have

$$1 + \varepsilon\beta^2 > \sqrt{\frac{a+r}{2}}.$$

Using (1.10) and $r^2 = a^2 + b^2$, another straightforward but lengthy calculation reveals

$$(1+\varepsilon\beta^2)^2(\varepsilon+\sigma'(m_0)) > f'(m_0)^2.$$

Since $\beta = \frac{k\pi}{L} \geq 0$ for $k = 0, 1, 2, \ldots$, which implies $(1+\varepsilon\beta^2)^2 \geq 1$, we must assume for stability

$$\varepsilon + \sigma'(m_0) > f'(m_0)^2. \tag{1.11}$$

This is an *optimal* sufficient condition for stability whenever m_0 is in the hyperbolic or elliptic region. If m_0 is in the hyperbolic region, i.e., $\sigma'(m_0) > 0$, and if $\sigma'(m_0)$ is big enough, so that

$$\sigma'(m_0) > f'(m_0)^2, \tag{1.12}$$

we may take $\varepsilon = 0$. The condition (1.12) is the so-called sub-characteristic condition introduced first by T.-P. Liu in [20] for the study of stability of elementary waves in hyperbolic conservation laws with relaxation. If m_0 is in the elliptic region, i.e., $\sigma'(m_0) < 0$, the sufficient condition (1.11) can be rewritten as

$$\varepsilon > |\sigma'(m_0)| + f'(m_0)^2. \tag{1.13}$$

For instability, we need at least $Re(\alpha_+) \geq 0$. As shown before, we can similarly have

$$\varepsilon < |\sigma'(m_0)| + f'(m_0)^2 \tag{1.14}$$

at least as a necessary condition for a possible instability.

We turn to the nonlinear stability and instability of solutions to the system (1.3) and (1.4). First of all, we introduce the steady-state solutions to the system (1.3) and (1.4). Such steady-state solutions are in the form $(v, u) = (\mathcal{V}, \mathcal{U})(x)$ satisfying the following system

$$\begin{cases} -\mathcal{U}_x = \varepsilon\mathcal{V}_{xx}, \\ -\sigma(\mathcal{V})_x = f(\mathcal{V}) - \mathcal{U}, \\ (\mathcal{V}, \mathcal{U})(x+2L) = (\mathcal{V}, \mathcal{U})(x), \\ \left(\frac{1}{2L}\int_0^{2L}\mathcal{V}(x)dx, \frac{1}{2L}\int_0^{2L}\mathcal{U}(x)dx\right) = (m_0, m_1). \end{cases} \tag{1.15}$$

Obviously, the constant equilibrium (m_0, m_1) is a trivial steady-state solution to (1.3) and (1.4). When the initial average m_0 is in the elliptic region (v_1, v_2), and the artificial viscosity ε is small enough, then the steady-state solutions are usually nontrivial and nonunique. This will be studied carefully with some criteria for the location of the initial average mean m_0 and the size of the viscosity ε in the near future.

Integrating the first equation of (1.15) with respect to x, we obtain

$$-\mathcal{U} = \varepsilon \mathcal{V}_x - \mathcal{C}, \tag{1.16}$$

where $\mathcal{C}$ is an integral constant. Substituting (1.16) into the second equation of (1.15), we then have

$$-\sigma(\mathcal{V})_x = f(\mathcal{V}) + \varepsilon \mathcal{V}_x - \mathcal{C}. \tag{1.17}$$

Integrating (1.17) with respect to x over $[0, 2L]$, and noting the periodicity $\mathcal{V}(0) = \mathcal{V}(2L)$ which implies $\int_0^{2L} \sigma(\mathcal{V})_x dx = \sigma(\mathcal{V}(2L)) - \sigma(\mathcal{V}(0)) = 0$ and $\int_0^{2L} \varepsilon \mathcal{V}_x dx = \varepsilon \mathcal{V}(2L) - \varepsilon \mathcal{V}(0) = 0$, we obtain the integral constant:

$$\mathcal{C} = \frac{1}{2L} \int_0^{2L} f(\mathcal{V}) dx. \tag{1.18}$$

Thus, the steady-state system (1.15) for $(\mathcal{V}, \mathcal{U})(x)$ is reduced to a single steady-state equation for $\mathcal{V}(x)$ as follows

$$\begin{cases} \varepsilon \mathcal{V}_x + \sigma(\mathcal{V})_x + f(\mathcal{V}) - \mathcal{C} = 0, \\ \mathcal{V}(x + 2L) = \mathcal{V}(x), \\ \frac{1}{2L} \int_0^{2L} \mathcal{V}(x) dx = m_0, \\ \mathcal{C} = \frac{1}{2L} \int_0^{2L} f(\mathcal{V}) dx. \end{cases} \tag{1.19}$$

For a hyperbolic equilibrium (m_0, m_1) and a viscosity ε sufficiently large to satisfy the optimal condition (1.11), we prove that the system has a unique global solution for all associated initial data close enough to the equilibrium. Moreover, that solution converges asymptotically to the equilibrium and no phase transitions occur after some possible initial oscillations. A similar result holds for elliptic equilibria. In that case ε has to be large enough for (1.13) to hold and again there are no phase transitions except possibly in an initial time period. When the viscosity ε is small, numerical experiments show that the solution $(v, u)(t, x)$ oscillates across the hyperbolic and elliptic regions for all time, exhibiting phase transitions. In accordance with our numerical study in the last section of this paper, we conjecture that the solution converges to an oscillatory standing wave (one nontrivial steady-state solution $(\mathcal{V},\mathcal{U})(x)$).

Notation. Before stating our main results, we need to set up some notation. Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0$ $(i = 0, 1, 2, \ldots)$ represents a specific constant, $R = (-\infty,\infty)$. Since solutions $(v, u)(t, x)$ of (1.3) and (1.4) are periodic, we introduce spaces of periodic functions which will be used in our analysis. Letting $p = 2L$ denote the period, and we first introduce the Hilbert space $L^2_{per}(R)$ of locally square integrable functions which are periodic of period p,

$$L^2_{per}(R) = \Big\{ v(x) \mid v(x) = v(x+2L) \text{ for all } x \in R, \text{ and } v(x) \in L^2(0,p) \text{ for } x \in [0,p] \Big\},$$

with the norm given by integration over $[0, p]$ (or over any other interval of length p),

$$\|v\|_{L^2_{per}} = \Big(\int_0^p v^2(x)dx \Big)^{1/2}.$$

We also define the Sobolev space $H^k_{per}(R)$ $(k \geq 0)$ to be the space of functions $v(x)$ in $L^2_{per}(R)$ whose derivatives $\partial_x^i v$, $i = 1, \ldots, k$ also belong to $L^2_{per}(R)$ with the norm

$$\|v\|_{H^k_{per}} = \Big(\sum_{i=0}^{k} \int_0^p |\partial_x^i v(x)|^2 dx \Big)^{1/2}.$$

Similarly, the periodic spaces $L^1_{per}(R)$ and $L^\infty_{per}(R)$ are defined, too. We also often use the simplified notation $\|(f, g)\|^2_{L^2_{per}} = \|f\|^2_{L^2_{per}} + \|g\|^2_{L^2_{per}}$ and $\|(f, g)\|^2_{H^k_{per}} = \|f\|^2_{H^k_{per}} + \|g\|^2_{H^k_{per}}$. Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C^0([0, T]; \mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0, T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

1.2 Main results

We are now ready to state our main results.

Theorem 1.1 (Convergence in Hyperbolic Phase)**.** Let (m_0, m_1) be a hyperbolic equilibrium and suppose that (1.11) holds. Then there exist positive constants δ_1, γ_1 and C_1 depending on m_0 and ε such that, for associated initial data $v_0(x) \in H^3_{per}(R)$ with mean m_0 and $u_0(x) \in H^2_{per}(R)$, satisfying $\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}} \leq \delta_1$, there exists a unique global solution $(v, u)(t, x)$ of the periodic initial-boundary value problem (1.3) and (1.4) which satisfies

$$v(t,x) - m_0 \in C^0(0,+\infty; H^2_{per}(R)), \quad u(t,x) - m_1 \in C^0(0,+\infty; H^2_{per}(R)),$$

and

$$\sup_{x\in R} |(v,u)(t,x) - (m_0,m_1)| \leq C_1 e^{-\gamma_1 t}(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}}), \quad 0 \leq t \leq \infty. \tag{1.20}$$

Moreover, there exists

$$t_* = \max\left\{0,\ \frac{1}{\gamma_1}\ln\frac{C_1(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}})}{v_1 - m_0}\right\}$$

for $m_0 < v_1$, or

$$t_* = \max\left\{0,\ \frac{1}{\gamma_1}\ln\frac{C_1(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}})}{m_0 - v_2}\right\}$$

for $m_0 > v_2$, such that, when $t > t_*$, the solution of system (1.3) and (1.4) does not exhibit phase transitions,

$$v(x,t) > v_1 \text{ (or } < v_2) \text{ for } x \in (-\infty,\infty), t > t_*. \tag{1.21}$$

□

Theorem 1.2 (Convergence in Elliptic Phase)**.** Let (m_0, m_1) be an elliptic equilibrium and suppose that (1.13) holds. Then there exist positive constants δ_2, C_2 and γ_2 depending on m_0 and ε such that, for associated initial data $v_0(x) \in H^3_{per}(R)$ with mean m_0 and $u_0(x) \in H^2_{per}(R)$, if $\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}} \leq \delta_2$, there exists a unique global solution $(v, u)(t, x)$ of the periodic initial-boundary value problem (1.3) and (1.4) which satisfies

$$v(t,x) - m_0 \in C^0(0,+\infty; H^2_{per}(R)), \quad u(t,x) - m_1 \in C^0(0,+\infty; H^2_{per}(R)),$$

and

$$\sup_{x\in R} |(v,u)(t,x) - (m_0,m_1)| \leq C_2 e^{-\gamma_2 t}(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}}), \ 0 \leq t \leq \infty. \tag{1.22}$$

Moreover, there exists

$$\bar{t}_* = \max\left\{0,\ \frac{1}{\gamma_2}\ln\frac{C_2(\|v_0-m_0\|_{H^3_{per}}+\|u_0-m_1\|_{H^2_{per}})}{m_0-v_2},\right.$$
$$\left.\frac{1}{\gamma_2}\ln\frac{C_2(\|v_0-m_0\|_{H^3_{per}}+\|u_0-m_1\|_{H^2_{per}})}{v_1-m_0}\right\}$$

such that, if $t > \bar{t}_*$, then the solution of system (1.3) and (1.4) does not exhibit phase transitions,

$$v_1 < v(x,t) < v_2 \text{ for } x \in (-\infty,\infty),\ t > \bar{t}_*. \tag{1.23}$$

□

Remark 1.3.

1. As analyzed before, the sufficient condition (1.11) (see also (1.13) in the elliptic case of m_0) is also optimal for the nonlinear stability.
2. The condition (1.13) means that the artificial viscosity ε must be big. The large viscosity ensures strong parabolicity of the system (1.3).
3. In both Theorems 1.1 and 1.2, we need that the initial perturbation $(v_0(x) - m_0, u_0(x) - m_1)$ is small to prove global existence of the solution, but according to our numerical study (see Section 4), this may not be necessary. We still observe global existence and convergence with large initial perturbations. The essential criterion seems to be the location of the equilibrium (m_0, m_1) and the size of the viscosity ε needed to ensure $\varepsilon + \sigma'(m_0) > f'(m_0)^2$. □

Conjecture 1.4 (Oscillatory Phase Transitions)**.** If m_0 is in the elliptic region (v_1, v_2), and if the artificial viscosity ε is small enough such that the condition (1.13) does not hold, then based on the numerical experiments shown in Sections 4.4 and 4.5 we conjecture that the solution $(v,u)(t,x)$ is oscillating for all time and converges time-asymptotically to a periodic, oscillatory standing wave (nontrivial steady-state solution) $(\mathcal{V},\mathcal{U})(x)$, which exhibits phase transitions through three different phases from the hyperbolic phase to the elliptic phase and then to the second hyperbolic phase. Here $(\mathcal{V},\mathcal{U})(x)$ is a steady-state solution of (1.3), i.e., the solution of (1.15). □

This paper is organized as follows: in the next section, we first reduce the original periodic initial-boundary value problem to an equivalent system, and show two key a priori estimates using the energy method. In Section 3, we prove the main theorems depending on the a priori estimates by a continuity argument. Finally, we carry out numerical computations in Section 4, corresponding to all the cases described above.

2 Reformulation of the original problem

2.1 New system of equations

First of all, we reformulate the original periodic initial-boundary problem (1.3) and (1.4) into a new system of equations. Differentiating the second equation in (1.3) with respect to x and substituting the first equation of (1.3) into it, we get a scalar pseudo-hyperbolic wave equation for $v(t,x)$,

$$\begin{cases} v_{tt}+v_t-\varepsilon v_{xxt}-\varepsilon v_{xx}-\sigma(v)_{xx}-f(v)_x=0, \quad x\in(-\infty,\infty), \quad t>0, \\ (v,v_t)(0,x)=(v_0,u_{0x}+\varepsilon v_{0xx})(x), \\ v(t,x)=v(t,x+2L), \\ \frac{1}{2L}\int_0^{2L} v(t,x)dx=m_0, \quad t\geq 0, \end{cases} \tag{2.1}$$

where the last condition in (2.1) is from (1.6). Letting

$$\phi(t,x):=v(t,x)-m_0, \tag{2.2}$$

Eqn. (2.1) can be reduced to

$$\begin{cases} \phi_{tt}+\phi_t-\varepsilon\phi_{xxt}-(\varepsilon+\sigma'(m_0))\phi_{xx}-f'(m_0)\phi_x=F_{1xx}+F_{2x}, \ x\in(-\infty,\infty), \quad t>0, \\ (\phi,\phi_t)(0,x)=(v_0-m_0,u_{0x}+\varepsilon v_{0xx})(x)=:(\phi_0,\phi_1)(x), \\ \phi(t,x)=\phi(t,x+2L), \\ \int_0^{2L}\phi(t,x)dx=0, \end{cases} \tag{2.3}$$

where

$$F_1=\sigma(\phi+m_0)-\sigma(m_0)-\sigma'(m_0)\phi, \quad F_2=f(\phi+m_0)-f(m_0)-f'(m_0)\phi. \tag{2.4}$$

Due to its periodicity $\phi(x,t)=\phi(x+2L,t)$, we define for any given $T>0$ the solution space

$$X(0,T)=\{\phi(t,x)|\phi\in C^0(0,T;H^2_{per}(R)),\phi_t(t,x)\in C^0(0,T;H^1_{per}(R))\}$$

with the norm

$$M(T)=\sup_{t\in[0,T]}\{\|\phi(t)\|_{H^2_{per}}+\|\phi_t(t)\|_{H^1_{per}}\}.$$

Now we have the following results.

Proposition 2.1. Under the assumptions in Theorem 1.1, there exist positive constants δ_1, γ_1 and $C_3 > 1$ such that, if $\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}} \le \delta_1$, then the periodic initial-boundary value problem (2.3) has a unique global solution $\phi(t,x) \in X(0,\infty)$ satisfying

$$\|\phi(t)\|_{H^2_{per}} + \|\phi_t(t)\|_{H^1_{per}} \le C_3 e^{-\gamma_1 t}(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}}), \quad 0 \le t \le \infty. \tag{2.5}$$

□

Proposition 2.2. Under the assumptions in Theorem 1.2, there exist positive constants δ_2, γ_2 and $\bar{C}_3 > 1$ such that, if $\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}} \le \delta_1$, then the periodic initial-boundary value problem (2.3) has a unique global solution $\phi(t,x) \in X(0,\infty)$ satisfying

$$\|\phi(t)\|_{H^2_{per}} + \|\phi_t(t)\|_{H^1_{per}} \le \bar{C}_3 e^{-\gamma_2 t}(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}}), \quad 0 \le t \le \infty. \tag{2.6}$$

□

2.2 A priori estimates

To prove Propositions 2.1 and 2.2, we need several lemmas to establish the *a priori* estimates for the solution $\phi(t,x)$ of (2.3). First, we prove a so-called Poincaré inequality.

Lemma 2.3. Let $\phi(t,x) \in X(0,T)$ for $T > 0$ be a solution of (2.3). Then, for any given $t \ge 0$, there exists at least one point $x_* = x_*(t) \in [0,2L]$ such that $\phi(t,x_*) = 0$. Moreover,

$$\|\phi(t)\|_{L^\infty_{per}} \le 2L\|\phi_x(t)\|_{L^\infty_{per}}, \tag{2.7}$$

$$\|\phi(t)\|_{L^2_{per}} \le 2L\|\phi_x(t)\|_{L^2_{per}}, \tag{2.8}$$

$$\|\phi(t)\|_{L^1_{per}} \le 2L\|\phi_x(t)\|_{L^1_{per}} \tag{2.9}$$

holds for $t \in [0,T]$. □

Proof. For any given $t \ge 0$, since $\phi(t,x)$ is periodic and $\int_0^{2L}\phi(t,x)dx = 0$, there must be at least one point, say $x_* = x_*(t) \in [0,2L]$, such that $\phi(t,x_*) = 0$. If this is not true, then either $\phi(t,x) > 0$ or $\phi(t,x) < 0$ for all $x \in [0,2L]$, which leads to $\int_0^{2L}\phi(t,x)dx > 0$ or $\int_0^{2L}\phi(t,x)dx < 0$, which is in contradiction with the condition $\int_0^{2L}\phi(t,x)dx = 0$. Thus,

$$\phi(t,x) = \phi(t,x_*) + \int_{x_*}^x \phi_x(t,y)dy = \int_{x_*}^x \phi_x(t,y)dy$$

implies

$$|\phi(t,x)| \le \int_{x_*}^x |\phi_x(t,y)|dy \le \int_0^{2L} |\phi_x(t,x)|dx. \tag{2.10}$$

Immediately, (2.10) implies (2.7) as follows

$$\|\phi(t)\|_{L^\infty_{per}} \leq \sup_{x\in[0,2L]} \int_0^{2L} |\phi_x(t,x)|dx \leq 2L\|\phi_x(t)\|_{L^\infty_{per}},$$

and (2.9) as follows

$$\|\phi(t)\|_{L^1_{per}} \leq \int_0^{2L} \Big(\int_0^{2L} |\phi_y(t,y)|dy\Big) dx \leq 2L\|\phi_x(t)\|_{L^1_{per}}.$$

Furthermore, squaring both sides of (2.10) and using the Cauchy-Schwarz inequality leads to

$$|\phi(t,x)|^2 \leq \left(\int_0^{2L} |\phi_x(t,x)|dx\right)^2 \leq 2L\int_0^{2L} |\phi_x(t,x)|^2 dx.$$

Integrating now with respect to x over $[0, 2L]$ yields

$$\|\phi(t)\|^2_{L^2_{per}} \leq 4L^2\|\phi_x(t)\|^2_{L^2_{per}},$$

which completes the proof. □

Second, we show the local existence of the solution $\phi(t,x)$ of (2.3).

Lemma 2.4 (Local Existence)**.** For any given initial data $(\phi_0, \phi_1)(x) \in H^2_{per}(R) \times H^1_{per}(R)$, letting $\bar{\delta}$ be a positive constant such that $\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}} \leq \bar{\delta}$, there exists a time $t_0 = t_0(\bar{\delta}) > 0$ such that (2.3) has a unique solution $\phi(t,x) \in X(0,t_0)$ and $M(t_0) \leq \sqrt{2}\bar{C}\bar{\delta}$, where

$$\bar{C} = \left(\max\left\{\frac{3}{4}, \frac{3\varepsilon+\sigma'(m_0)}{4}\right\}\Big/ \min\left\{\frac{1}{4}, \frac{3\varepsilon+\sigma'(m_0)}{4}\right\}\right)^{1/2} \geq 1. \tag{2.11}$$

□

Proof: This local result can be shown using a fixed point iteration, (cf. [23]). We omit the details here.□

Finally, we have the following a priori estimates.

Lemma 2.5 (A Priori Estimate in the Hyperbolic Case)**.** Under the assumptions in Theorem 1.1, let $\phi(t,x)$ be a local solution of (2.3) in $X(0,T)$ for a given constant $T > 0$. Then there exist positive constants $C_3 > 1$ and δ_3 independent of T such that, if $M(T) \leq \delta_3$, then

$$\|\phi(t)\|_{H^2_{per}} + \|\phi_t(t)\|_{H^1_{per}} \leq C_3(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}})e^{-\gamma_1 t}, \quad 0 \leq t < T. \tag{2.12}$$

□

Lemma 2.6 (A Priori Estimate in the Elliptic Case)**.** Under the assumptions in Theorem 1.2, let $\phi(t,x)$ be a local solution of (2.3) in $X(0,T)$ for a given constant $T>0$. Then there exist positive constants $\bar{C}_3>1$ and δ_4 independent of T such that, if $M(T)\le\delta_4$, then

$$\|\phi(t)\|_{H^2_{per}}+\|\phi_t(t)\|_{H^1_{per}}\le\bar{C}_3(\|\phi_0\|_{H^2_{per}}+\|\phi_1\|_{H^1_{per}})e^{-\gamma_2 t},\quad 0\le t<T. \tag{2.13}$$

□

Proof of Lemma 2.5. Multiplying (2.3) by $(\frac{1}{2}\phi+\phi_t)e^{2\gamma_1 t}$ and doing a simple but tedious computation, we obtain

$$\begin{aligned}&\{e^{2\gamma_1 t}A_1(\phi,\phi_t,\phi_x)\}_t+e^{2\gamma_1 t}A_2(\phi,\phi_t,\phi_x,\phi_{xt})-e^{2\gamma_1 t}\{A_3(\phi,\phi_t,\phi_x,\phi_{xt})\}_x\\&=e^{2\gamma_1 t}(\frac{1}{2}\phi+\phi_t)(F_{1xx}+F_{2x}),\end{aligned} \tag{2.14}$$

where

$$A_1(\phi,\phi_t,\phi_x)=(\frac{1}{4}-\frac{\gamma_1}{2})\phi^2+\frac{1}{2}\phi\phi_t+\frac{1}{2}\phi_t^2+\frac{3\varepsilon+\sigma'(m_0)}{4}\phi_x^2, \tag{2.15}$$

$$\begin{aligned}A_2(\phi,\phi_t,\phi_x,\phi_{xt})&=(\frac{1}{2}-\gamma_1)\phi_t^2-2\gamma_1(\frac{1}{2}-\gamma_1)\phi^2-f'(m_0)\phi_x\phi_t\\&\quad+\Big[(\frac{1}{2}-\gamma_1)(\varepsilon+\sigma'(m_0))-\frac{\varepsilon\gamma}{2}\Big]\phi_x^2+\varepsilon\phi_{xt}^2,\end{aligned} \tag{2.16}$$

$$\begin{aligned}A_3(\phi,\phi_t,\phi_x,\phi_{xt})&=\varepsilon(\frac{1}{2}\phi+\phi_t)\phi_{xt}+\frac{1}{4}f'(m_0)\phi^2\\&\quad+(\varepsilon+\sigma'(m_0))(\frac{1}{2}\phi+\phi_t)\phi_x.\end{aligned} \tag{2.17}$$

Let $0<\gamma_1<\frac{1}{4}$. Using the inequality $|ab|\le\eta a^2+\frac{1}{4\eta}b^2$ for any $\eta>0$, we get

$$\frac{1}{2}\phi\phi_t\le\frac{\eta_0}{2}\phi_t^2+\frac{1}{8\eta_0}\phi^2\quad\text{for any }\eta_0>0,$$

and selecting η_0 such that

$$\frac{1}{2-4\gamma_1}<\eta_0<1,\ \text{i.e.,}\ \frac{1}{8\eta_0}<\frac{1}{4}-\frac{\gamma_1}{2},$$

we obtain

$$(\frac{1}{4}-\frac{\gamma_1}{2})\phi^2+\frac{1}{2}\phi\phi_t+\frac{1}{2}\phi_t^2\ge\frac{1-\eta_0}{2}\phi_t^2+(\frac{1}{4}-\frac{\gamma_1}{2}-\frac{1}{8\eta_0})\phi^2\ge C_4(\phi^2+\phi_t^2)$$

with the constant C_4 given by

$$C_4=\min\left\{\frac{1-\eta_0}{2},\ \frac{1}{4}-\frac{\gamma_1}{2}-\frac{1}{8\eta_0}\right\}.$$

We therefore obtain the estimate

$$C_5(\phi^2 + \phi_t^2 + \phi_x^2) \le A_1(\phi, \phi_t, \phi_x) \le C_6(\phi^2 + \phi_t^2 + \phi_x^2), \tag{2.18}$$

where the constants C_5 and C_6 are given by

$$C_5 = \min\left\{C_4,\ \frac{3\varepsilon + \sigma'(m_0)}{4}\right\}, \quad C_6 = \max\left\{\frac{3}{4},\ \frac{1}{2}(1-\gamma_1),\ \frac{3\varepsilon + \sigma'(m_0)}{4}\right\},$$

for $\gamma_1 < 1/4$.

Integrating (2.14) over $[0, 2L] \times [0, t]$ and using (2.18) leads to

$$\begin{aligned} &C_5 e^{2\gamma_1 t}\|(\phi, \phi_t, \phi_x)(t)\|^2_{L^2_{per}} + \int_0^t e^{2\gamma_1 s}\int_0^{2L} A_2(\phi, \phi_t, \phi_x, \phi_{xt})(s,x)dxds \\ &\quad \le C_6\|(\phi_0, \phi_1, \phi_{0,x})\|^2_{L^2_{per}} + \int_0^t \int_0^{2L} e^{2\gamma_1 s}(\frac{1}{2}\phi + \phi_t)(F_{1xx} + F_{2x})dxds. \end{aligned} \tag{2.19}$$

Now we are estimating the second term on the left-hand-side of (2.19). Using Lemma 2.3, we first have

$$-2\gamma_1(\frac{1}{2} - \gamma_1)\|\phi(t)\|^2_{L^2_{per}} \ge -8L^2\gamma_1(\frac{1}{2} - \gamma_1)\|\phi_x(t)\|^2_{L^2_{per}}. \tag{2.20}$$

On the other hand, the inequality $|ab| \le \eta a^2 + \frac{1}{4\eta}b^2$ for any $\eta > 0$ gives

$$|f'(m_0)\phi_x\phi_t| \le \frac{\eta}{2}\phi_t^2 + \frac{1}{2\eta}f'(m_0)^2\phi_x^2. \tag{2.21}$$

Thus, using (2.16), we obtain

$$\begin{aligned} &\int_0^{2L} A_2(\phi, \phi_t, \phi_x, \phi_{xt})(s,x)dx \\ &\quad = (\frac{1}{2} - \gamma_1)\|\phi_t(s)\|^2_{L^2_{per}} - 2\gamma_1(\frac{1}{2} - \gamma_1)\|\phi(s)\|^2_{L^2_{per}} + \varepsilon\|\phi_{xt}(s)\|^2_{L^2_{per}} \\ &\quad + \Big[(\frac{1}{2} - \gamma_1)(\varepsilon + \sigma'(m_0)) - \frac{\varepsilon\gamma_1}{2}\Big]\|\phi_x(s)\|^2_{L^2_{per}} - \int_0^{2L} f'(m_0)\phi_x\phi_t dx \\ &\quad \ge (\frac{1-\eta}{2} - \gamma_1)\|\phi_t(s)\|^2_{L^2_{per}} + \varepsilon\|\phi_{xt}(s)\|^2_{L^2_{per}} + C_7\|\phi_x(s)\|^2_{L^2_{per}}, \end{aligned} \tag{2.22}$$

where the constant C_7 is given by

$$C_7 := \frac{1}{2}\Big\{(\varepsilon + \sigma'(m_0)) - \frac{1}{\eta}f'(m_0)^2 - \gamma_1\Big[2(\varepsilon + \sigma'(m_0)) + \varepsilon + 16L^2(\frac{1}{2} - \gamma_1)\Big]\Big\}. \tag{2.23}$$

Applying (1.11), i.e.,

$$0 < \frac{f'(m_0)^2}{\varepsilon + \sigma'(m_0)} < 1,$$

and selecting η to be such that

$$\frac{f'(m_0)^2}{\varepsilon + \sigma'(m_0)} < \eta < 1,$$

we obtain

$$\varepsilon + \sigma'(m_0) - \frac{1}{\eta} f'(m_0)^2 > 0.$$

Furthermore, letting γ_1 be such that $\gamma_1 \ll 1$, the constant C_7 satisfies

$$C_7 = \frac{1}{2}\left\{(\varepsilon + \sigma'(m_0)) - \frac{1}{\eta} f'(m_0)^2 - \gamma_1 \left[2(\varepsilon + \sigma'(m_0)) + \varepsilon + 16L^2(\frac{1}{2} - \gamma_1)\right]\right\} > 0. \tag{2.24}$$

Thus, we can estimate the second term on the left-hand-side of (2.22) by

$$\int_0^t e^{2\gamma_1 s} \int_0^{2L} A_2(\phi, \phi_t, \phi_x, \phi_{xt})(s,x) dx ds \geq C_8 \int_0^t e^{2\gamma_1 s} \|(\phi_t, \phi_x, \phi_{xt})(s)\|^2_{L^2_{per}} ds, \tag{2.25}$$

where the constant C_8 is given by

$$C_8 = \min\left\{\frac{1-\eta}{2} - \gamma_1, \varepsilon, C_7\right\} > 0.$$

The next step is to estimate the nonlinear term in (2.19). Using integration by parts, $|F_{1x}| = O(1)|\phi\phi_x|$ and $|F_{2x}| = O(1)|\phi\phi_x|$, the Sobolev inequality $|\phi(t,x)|, |\phi_x(t,x)| \leq C\|\phi(t)\|_{H^2_{per}} \leq CM(t)$, as well as Lemma 2.3, we have

$$\begin{aligned} \left|\int_0^{2L} F_{1xx}(\frac{1}{2}\phi + \phi_t) dx\right| &= \left|\int_0^{2L} F_{1x}(\frac{1}{2}\phi_x + \phi_{xt}) dx\right| \\ &\leq C \int_0^{2L} |\phi\phi_x|(\frac{1}{2}|\phi_x| + |\phi_{xt}|) dx \leq C_9 M(t) \|(\phi_x, \phi_{xt})(t)\|^2_{L^2_{per}} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &\left|\int_0^{2L} F_{2x}(\frac{1}{2}\phi + \phi_t) dx\right| \leq C \int_0^{2L} |\phi\phi_x|(\frac{1}{2}|\phi| + |\phi_t|) dx \\ &\leq CM(t)\|(\phi, \phi_x, \phi_t)(t)\|^2_{L^2_{per}} \leq C_{10} M(t) \|(\phi_x, \phi_t)(t)\|^2_{L^2_{per}} \end{aligned} \tag{2.27}$$

for some positive constants C_9 and C_{10}. Substituting (2.25)–(2.27) back into (2.19) yields

$$e^{2\gamma_1 t}\|(\phi,\phi_t,\phi_x)(t)\|^2_{L^2_{per}} + \frac{C_8-(C_9+C_{10})M(t)}{C_5}\int_0^t e^{2\gamma_1 s}\|(\phi_t,\phi_x,\phi_{xt})(s)\|^2_{L^2_{per}}ds \\ \le \frac{C_6}{C_5}\|(\phi_0,\phi_1,\phi_{0x})\|^2_{L^2_{per}}. \tag{2.28}$$

Let δ_3 be such that

$$0<\delta_3<\frac{C_8}{C_9+C_{10}}.$$

If $M(t)\le\delta_3$, we obtain from (2.28)

$$\|(\phi,\phi_t,\phi_x)(t)\|^2_{L^2_{per}} \le \frac{C_6}{C_5}e^{-2\gamma_1 t}\|(\phi_0,\phi_1,\phi_{0x})\|^2_{L^2_{per}}. \tag{2.29}$$

To estimate the higher order derivatives of the solution, we differentiate (2.3) with respect to x and multiply by $(\frac{1}{2}\phi_x+\phi_{xt})e^{2\gamma_1 t}$ to obtain

$$\{e^{2\gamma_1 t}A_1(\phi_x,\phi_{xt},\phi_{xx})\}_t + e^{2\gamma_1 t}A_2(\phi_x,\phi_{xt},\phi_{xx},\phi_{xxt}) - e^{2\gamma_1 t}\{A_3(\phi_x,\phi_{xt},\phi_{xx},\phi_{xxt})\}_x \\ = e^{2\gamma_1 t}(\frac{1}{2}\phi_x+\phi_{xt})(F_{1xxx}+F_{2xx}). \tag{2.30}$$

Now integrating (2.30) over $[0,2L]\times[0,t]$ and using (2.28), as we did in (2.29), we obtain

$$\|(\phi_x,\phi_{xt},\phi_{xx})(t)\|^2_{L^2_{per}} \le C_{11}e^{-2\gamma_1 t}\|(\phi_0,\phi_1,\phi_{0x})\|^2_{H^1_{per}} \tag{2.31}$$

for some positive constant C_{11}, provided $M(t)\le\delta_3$.

Combining (2.29) and (2.31) yields (2.12) for some positive constant $C_3>1$ and the proof is complete. □

Proof of Lemma 2.6. This lemma can be proved similar to the one above. We omit the details here. □

Based on Lemmas 2.4, 2.5, and 2.6, respectively, we can prove Propositions 2.1 and 2.2 as follows.

2.3 Proof of proposition 2.1 and proposition 2.2

Now we are going to prove Proposition 2.1 and Proposition 2.2. We focus only on the proof of Proposition 2.1, because Proposition 2.2 can be proved similarly. Let

$$\delta_1 = \frac{\delta_3}{\sqrt{2}\bar{C}C_3}, \quad \bar{\bar{\delta}} = C_3\delta_1 = \frac{\delta_3}{\sqrt{2}\bar{C}}$$

and the initial data (ϕ_0, ϕ_1) satisfy $\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}} \le \delta_1 < \bar{\bar{\delta}}$ due to $C_3 > 1$.

By Lemma 2.4, there exists $t_0 = t_0(\bar{\bar{\delta}}) > 0$ such that the solution of (2.3) exists in $X(0, t_0)$ and satisfies $M(t_0) \le \sqrt{2}\bar{C}\bar{\bar{\delta}}$. On the interval $[0, t_0]$, since $M(t_0) \le \sqrt{2}\bar{C}\bar{\bar{\delta}} = \delta_3$, applying the a priori estimate (Lemma 2.5), we further have

$$\begin{aligned} \|\phi(t)\|_{H^2_{per}} &+ \|\phi_t(t)\|_{H^1_{per}} \le C_3(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}})e^{-\gamma_1 t} \\ &\le C_3\delta_1 = \bar{\bar{\delta}} = \frac{\delta_3}{\sqrt{2}\bar{C}}, \quad t \in [0, t_0], \end{aligned} \tag{2.32}$$

i.e., $M(t_0) \le \bar{\bar{\delta}} = \delta_3/(\sqrt{2}\bar{C})$. Now, we consider the periodic initial-boundary value problem (2.3) with the new "initial data" $(\phi(t_0, x), \phi_t(t_0, x))$ at the new "initial time" $t = t_0$, since $\|\phi(t_0)\|_{H^2_{per}} + \|\phi_t(t_0)\|_{H^1_{per}} \le \bar{\bar{\delta}}$ (see (2.32)), then Lemma 2.4 gives $\phi(t, x) \in X(t_0, 2t_0)$ and

$$M(2t_0) \le \sqrt{2}\bar{C}\bar{\bar{\delta}} = \delta_3.$$

Thus, we can apply Lemma 2.5 again on $[0, 2t_0]$ to have

$$\begin{aligned} \|\phi(t)\|_{H^2_{per}} &+ \|\phi_t(t)\|_{H^1_{per}} \le C_3(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}})e^{-\gamma_1 t} \\ &\le C_3\delta_1 = \bar{\bar{\delta}} = \frac{\delta_3}{\sqrt{2}\bar{C}}, \quad t \in [0, 2t_0]. \end{aligned} \tag{2.33}$$

Repeating the previous procedure, we extend the existence interval of $\phi(t, x)$ step by step to $[0, nt_0]$ $(n \in \mathbf{N}_+)$ and finally to $[0, \infty)$, i.e., $\phi(t, x) \in X(0, \infty)$, as well as

$$\|\phi(t)\|_{H^2_{per}} + \|\phi_t(t)\|_{H^1_{per}} \le C_3(\|\phi_0\|_{H^2_{per}} + \|\phi_1\|_{H^1_{per}})e^{-\gamma_1 t}, \quad \text{for all } t \in [0, \infty),$$

and the proof is complete. □

3 Proof of theorems 1.1 and 1.2

We first prove Theorem 1.1. Since in Proposition 2.1, we need $\phi_0(x) = v_0(x) - m_0 \in H^2_{per}(R)$ and $\phi_1(x) = u_{0x}(x) + \varepsilon v_{0xx} \in H^1_{per}(R)$, we have to assume $v_0(x) - m_0 \in H^3_{per}(R)$ and $u_0(x) - m_1 \in H^2_{per}(R)$ in Theorem 1.1. With $m_1 = f(m_0)$ from (1.6), we obtain from the second equation of (1.3)

$$(u - m_1)_t - (\sigma(v) - \sigma(m_0))_x = f(v) - f(m_0) - (u - m_1),$$

which is equivalent to

$$u(t,x) = u_0(x) + \int_0^t e^{-\tau}[(\sigma(v) - \sigma(m_0))_x + f(v) - f(m_0)](\tau,x)d\tau.$$

Since $v(t,x) = \phi(t,x) + m_0$, Proposition 2.1 gives existence and uniqueness of the global solution $(v,u)(t,x)$ to the periodic initial-boundary value problem (1.3) and (1.4), as well as the estimate (1.20) from (2.5) by Sobolev's inequality for a positive constant C_1.

It remains to prove that there is no phase transition after time t_*. For $m_0 \in (v_2,\infty)$ (the case $m_0 \in (-\infty, v_1)$ can be treated similarly), using (1.20), where $\phi_0 = v_0 - m_0$ and $\phi_1 = v_1$, we obtain

$$\begin{aligned} v(t,x) &= m_0 + (v(t,x) - m_0) \\ &\geq m_0 - \sup_{x\in[0,2L]} |v(t,x) - m_0| \\ &\geq m_0 - C_1 e^{-\gamma_1 t}(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}}) \\ &= v_2 + (m_0 - v_2) - C_1 e^{-\gamma_1 t}(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}}) \\ &\geq v_2 \end{aligned}$$

for $t \geq t_*$, where

$$t_* = \max\left\{0,\ \frac{1}{\gamma_1}\ln\frac{C_1(\|v_0 - m_0\|_{H^3_{per}} + \|u_0 - m_1\|_{H^2_{per}})}{m_0 - v_2}\right\}.$$

Theorem 1.2 can be proved similarly. We omit the details here. □

4 Numerical computations

We solve the periodic initial-boundary value problem (1.3)–(1.4) using the pressure function $\sigma(v) = v^3 - v$, the flux function $f(v) = \frac{1}{2}v^2$ and the periodicity condition $v(x-\pi,t) = v(x+\pi,t)$, i.e. $L = \pi$. According to the sign of $\sigma'(v)$, the phases are divided into a first hyperbolic region $(-\infty, -\frac{1}{\sqrt{3}})$, an elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and a second hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$, where $v_1 = -\frac{1}{\sqrt{3}}$ and $v_2 = \frac{1}{\sqrt{3}}$ are the two phase boundaries.

4.1 Two numerical schemes

The numerical schemes adopted here are the finite difference method and the Fourier pseudo-spectral method, according to different cases.

4.1.1 Finite Difference Method. If the initial average m_0 is located in the hyperbolic regions $(-\infty, v_1) \cup (v_2, \infty)$ and not close to the phase boundary $v = v_1$ or $v = v_2$, respectively, the system (1.3) and (1.4) is strongly hyperbolic. If the initial average m_0 is located in the elliptic regions (v_1, v_2) and the artificial viscosity ε is big, the system (1.3) and (1.4) is strongly parabolic. In these two cases, we use the central finite-difference method to carry out the numerical experiments, because in these cases, the adopted scheme is fast. Let N denote the number of time steps and M the number of spatial steps, T be the length of the time interval we simulate, and $\Delta x := \frac{2\pi}{M}$ be the spatial discretization step and $\Delta t := \frac{T}{N}$ be the time discretization step. To discretize the system (1.3) and (1.4), we use the centered finite difference scheme

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} - \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} = \varepsilon \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{(\Delta x)^2}, \tag{4.1}$$

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - \frac{\sigma(v_{m+1}^n) - \sigma(v_{m-1}^n)}{2\Delta x} = f(v_m^n) - u_m^n, \tag{4.2}$$

with the initial condition

$$(v_m^0, u_m^0) = (v_0, u_0)(m\Delta x), \tag{4.3}$$

and the 2π-periodic boundary condition

$$(v_j^n, u_j^n) = (v_{j+M}^n, u_{j+M}^n), \ \ j = -1, 0, \tag{4.4}$$

and thus (v_m^n, u_m^n) are approximations to $(v, u)(n\Delta t, m\Delta x)$. Since our scheme is explicit in time, there is a stability constraint on the time step; we use as a guideline the linear stability condition

$$\varepsilon \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}, \tag{4.5}$$

but often need to choose the time step slightly smaller due to the nonlinear nature of the problem. In the following numerical experiments for supporting Theorem 1.1 and Theorem 1.2, we use $M = 100$, which in several of our simulations over long time intervals requires many time steps for stability. We therefore often use less resolution when plotting the results, than we used for the actual computations, to avoid cluttering the graphs with too many data points.

4.1.2 Pseudo-Spectral Method. If the initial average m_0 is in the unstable elliptic region (v_1, v_2), and the viscosity ε is small, we adopt the Fourier pseudo-spectral method to carry out our numerical experiments. We reduce (1.3) and (1.4) to

$$\begin{cases} v_{tt} + v_t - \varepsilon v_{xxt} - \varepsilon v_{xx} - \sigma(v)_{xx} - f(v)_x = 0, \\ (v, v_t)|_{t=0} = (v_0, v_1)(x), \\ \frac{1}{2L}\int_0^{2L} v(x)dx = m_0, \end{cases} \tag{4.6}$$

where $v_1(x) = u_{0x}(x) + \varepsilon v_{0xx}(x)$. Let $N = 2^s$, where $s > 0$ is an integer (in what follows, we set $N = 2^8 = 258$ for our numerical experiments), and let $\Delta x = \frac{2\pi}{N}$, $\Delta t = \frac{1}{\varepsilon N^2}$ such that the stability condition (4.5) holds. Let $\hat{v}(t,k)$ be the Fourier transforma of $v(t,x)$. Taking a Fourier transform of (4.6), we obtain

$$\begin{cases} \hat{v}_{tt} + (1+\varepsilon k^2)\hat{v}_t + \varepsilon k^2 \hat{v} + k^2\widehat{\sigma(v)} + ik\widehat{f(v)} = 0, \\ (\hat{v}, \hat{v}_t)|_{t=0} = (\hat{v}_0, \hat{v}_1)(k), \end{cases} \tag{4.7}$$

and then discretize (4.7) in time to obtain

$$\frac{\widehat{v}^{n+1} - 2\widehat{v}^n + \widehat{v}^{n-1}}{(\Delta t)^2} + (1+\varepsilon k^2)\frac{\widehat{v}^{n+1} - \widehat{v}^{n-1}}{2\Delta t} + \varepsilon k^2 \widehat{v}^n + k^2\widehat{\sigma(v)^n} + ik\widehat{f(v)^n} = 0.$$

Taking the inverse Fourier transform of the above equation, we obtain v_m^n. This is the so-called Fourier pseudo-spectral method, which is adopted to carry out some numerical experiments for Conjecture 1.4, see the numerical results in Sections 4.4 and 4.5. Again, in order to avoid cluttering the graphs with too many data points, as mentioned before, we usually use less resolution for plotting the results.

4.2 Numerical simulations for theorem 1.1

To illustrate Theorem 1.1, we select the initial values to be $v_0(x) = 0.8 + 0.5\sin 2x$ and $u_0(x) = 3\sin 4x$, so that some parts of $v_0(x)$ are in the hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$, and some parts are in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The average of this initial data, $m_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_0(x)dx = 0.8 > \frac{1}{\sqrt{3}}$, is in the hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$. The viscosity ε is chosen to be $\varepsilon = 0.5$, ensuring that the condition (1.11), i.e., $\varepsilon + \sigma'(m_0) > f'(m_0)^2$, is satisfied. Figure 1 shows on the right that the solution $v(x,t)$ of (1.3) and (1.4) converges to the average $m_0 = 0.8$, and stays in the same hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$ as the average of

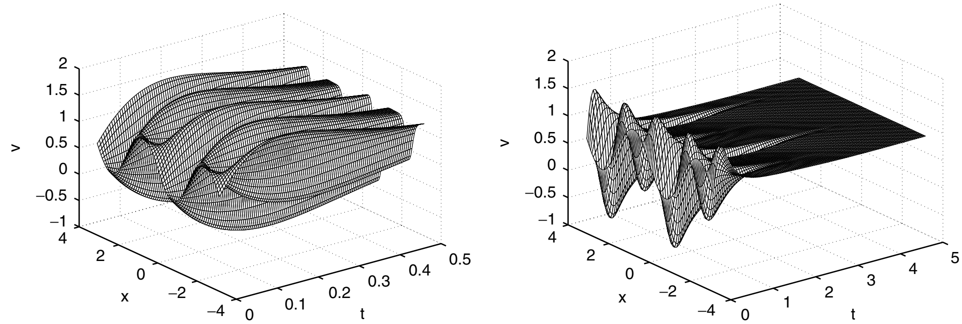

Figure 1 On the left initial, transient behavior for an example where the average of the initial data is in the hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$, and on the right the asymptotic long time behavior, where the solution converges to the average of the initial data, as predicted by Theorem 1.1.

the initial condition, after a short time of initial oscillation. There is no phase transition after this short time of initial oscillation, as predicted by Theorem 1.1. On the left in Figure 1 we show a closeup view in time of how the initial condition with two peaks transits into a growing solution with four peaks, before decaying to the average of the initial data.

4.3 Numerical simulations for theorem 1.2

We now choose $v_0(x) = 0.5 \sin 2x$ and $u_0(x) = 3 \sin 4x$, such that the average of the initial data is in the elliptic region, $m_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_0(x)dx = 0 \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Choosing the viscosity to be $\varepsilon = 2$, the sufficient condition (1.13), i.e. $\varepsilon = 2 > |\sigma'(m_0)| + f'(m_0)^2 = 1$, holds, and hence Theorem 1.2 applies, which again predicts asymptotic convergence to the average of the initial condition. In Figure 2, on the right, we show the convergence of the solution $v(t, x)$ to $m_0 = 0$. No phase transition occurs, as predicted by our analysis. In this case, although the average of the initial value m_0 is in the elliptic region, with the viscosity big enough, the system (1.3) and (1.4) behaves strongly like a parabolic equation, and we obtain asymptotic convergence. In Figure 2, on the left, we show again a closeup in time of the solution, where one can see that the initial condition with 2 peaks shows a rapid transition to a solution with 4 peaks, before the asymptotic decay starts to set in.

We now vary the viscosity ε, leaving all the remaining parameters and the initial conditions the same. We choose first three values of the viscosity, $\varepsilon = 1$, $\varepsilon = 0.9$, and $\varepsilon = 0.8$. Note that the condition (1.13) implies that $\varepsilon = 1$ is right on the boundary, while the condition is violated for smaller ε. In Figure 3 we show in the three columns,

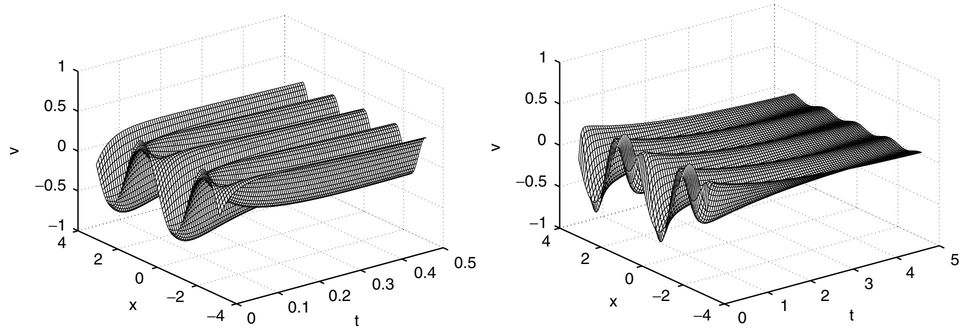

Figure 2 On the left initial, transient behavior for an example where the average of the initial data is in the elliptic region, $m_0 \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and on the right the asymptotic long time behavior, where the solution converges to the average of the initial data, as predicted by Theorem 1.2.

corresponding to the three values of the viscosity ε, snapshots at the times $t_1 = 0.05$, $t_2 = 0.4$, $t_3 = 2$, $t_4 = 50$, and $t_5 = 100$ of the solution v. While for short time the solutions still look very similar for the different values of the viscosity ε, one can clearly see for larger t a fundamental change in the behavior of the system: while for $\varepsilon = 1$ the solution still seems to decay toward its average, this process is slowed down very much for $\varepsilon = 0.9$ and $\varepsilon = 0.8$. Comparing the last two rows at times $t = 50$ and $t = 100$ for $\varepsilon = 0.9$ and $\varepsilon = 0.8$, the solutions do not decay any further and seem to stay oscillatory. This indicates that the sufficient condition (1.13) could also be necessary. One also sees other interesting behavior: the oscillations that appeared to be very regular at the beginning start to change, and some oscillations have become wider at the cost of their neighbors, which becomes especially apparent in the last row of Figure 3. We investigate this phenomenon further numerically in the next subsection.

4.4 Numerical simulations for conjecture 1.4: (I) continuous initial data

We investigate now numerically the behavior of solutions for viscosities $\varepsilon < 1$. Keeping all the other parameters the same as before, but $\varepsilon = 0.5$ so that the condition (1.13) does not hold, we first show the behavior of the solution $v(t, x)$ in Figure 4. As shown in Figure 4, when $t = 0$, the initial data $v_0(x)$ oscillates with two peaks and two valleys between -0.5 and 0.5 which is in the elliptic region, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and there are no phase transitions. Then soon the solution $v(t, x)$ goes through a transition, four peaks appear and the solution increases until it exhibits phase transitions through all three phases.

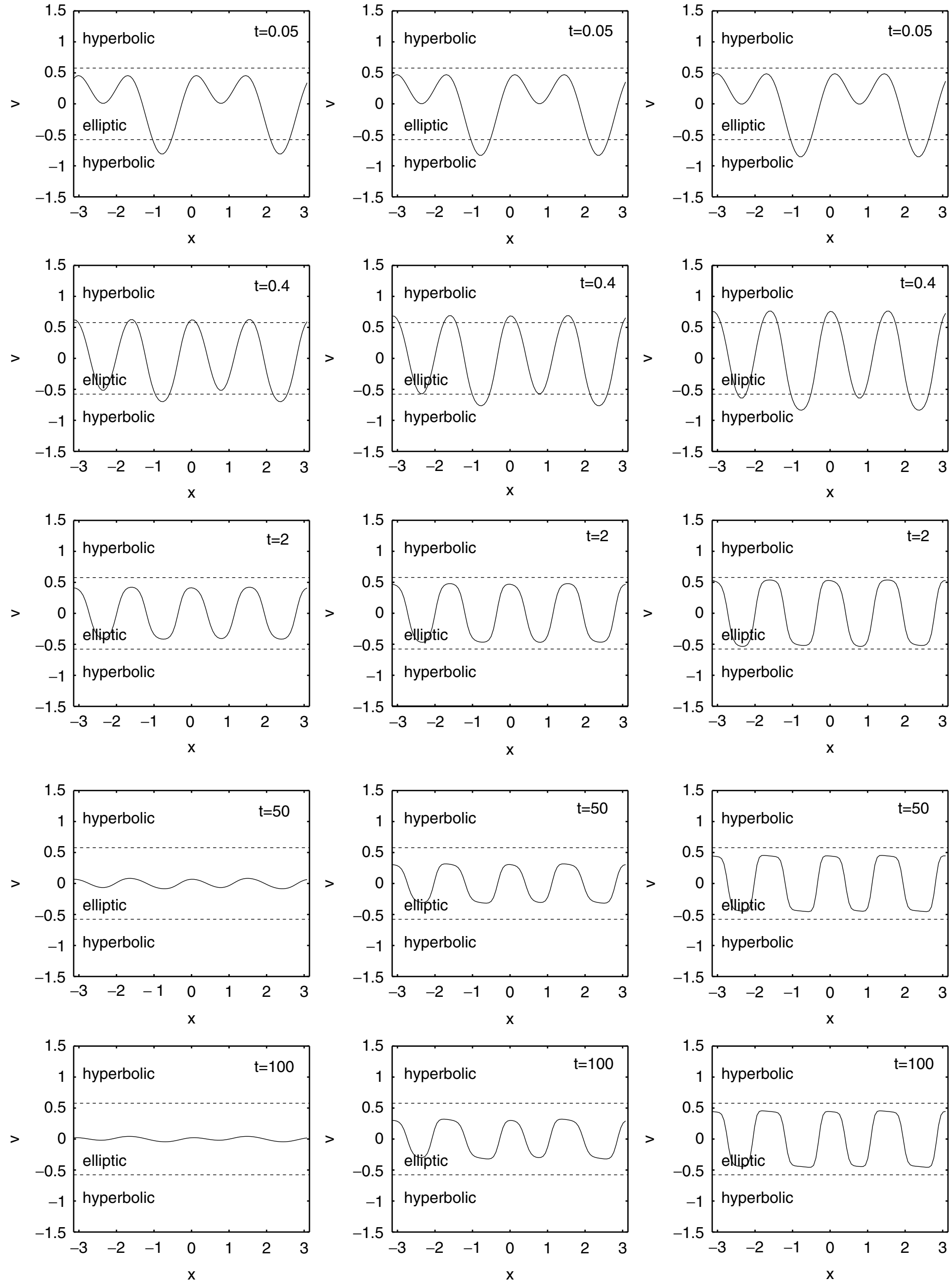

Figure 3 Snapshots of $v(x, t)$ at $t = 0.05, 0.4, 2, 50, 100$ for $\varepsilon = 1$ in the left column, for $\varepsilon = 0.9$ in the middle column, and for $\varepsilon = 0.8$ in the right column.

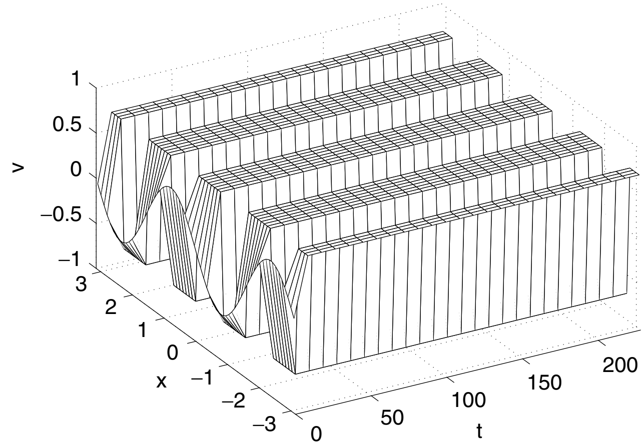

Figure 4 Continuous initial data in the elliptic region, $m_0 \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, with a small viscosity $\varepsilon = 0.5$. The solution $v(t,x)$ behaves like a standing wave with phase transitions.

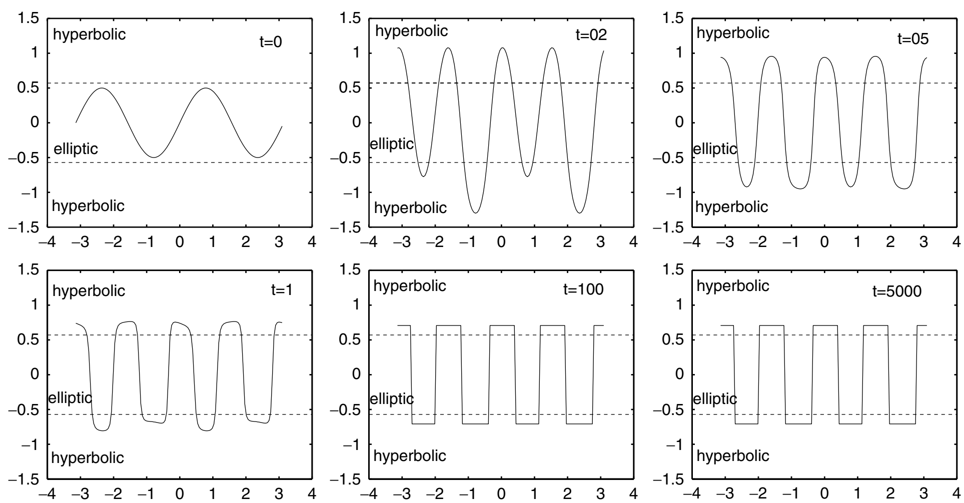

Figure 5 Snapshots of $v(x,t)$ at $t = 0, 0.2, 0.5, 1, 100, 5000$ for $\varepsilon = 0.5$ and $m_0 = 0$. The solution $v(t,x)$ exhibits phase transitions and behaves like a standing wave, as predicted in Conjecture 1.4.

Then, as one can see in Figure 5, a steady state seems to form, with four regular flat peaks and valleys.

We show also two other experiments, where the average of the initial data v_0 is nonzero. We keep u_0 as before, but set for the first experiment $v_0 = \frac{1}{2}(1 + \sin(2x))$, such

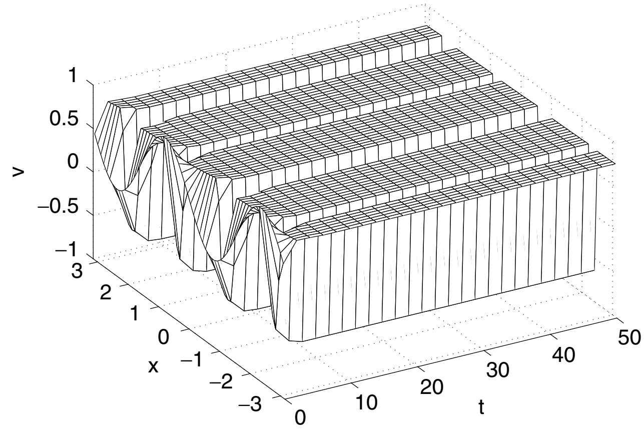

Figure 6 Continuous initial data in the elliptic region, $m_0 = \frac{1}{2} \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, with a small viscosity $\varepsilon = 0.5$. The solution converges to a stationary wave.

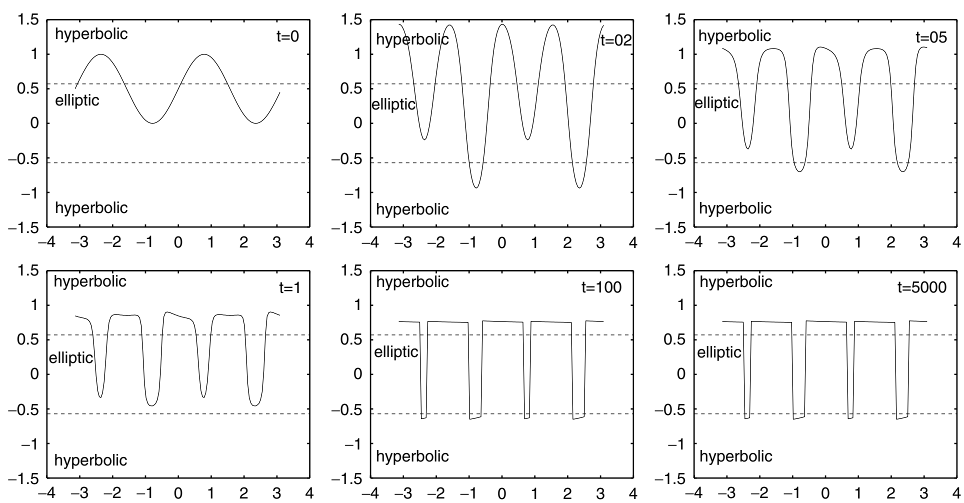

Figure 7 Snapshots of $v(x,t)$ at $t = 0, 0.2, 1, 5, 100, 5000$ for $\varepsilon = 0.5$ and $m_0 = 0.5$. The solution $v(t,x)$ exhibits phase transitions and behaves like a standing wave, as predicted in Conjecture 1.4.

that the average $m_0 = \frac{1}{2}$ is still in the elliptic region. In Figure 6 the result $v(t,x)$ is shown up to $t = 50$, and in Figure 7 we show the graph of the solution $v(t,x)$ for time $t = 0, 0.2, 1, 5, 100, 5000$, respectively. The solution converges to a stationary wave which has four big peaks and four small valleys, and after a short initial time, the solution exhibits phase transitions through all three phases from the hyperbolic phase to the elliptic phase and then to the other hyperbolic phase.

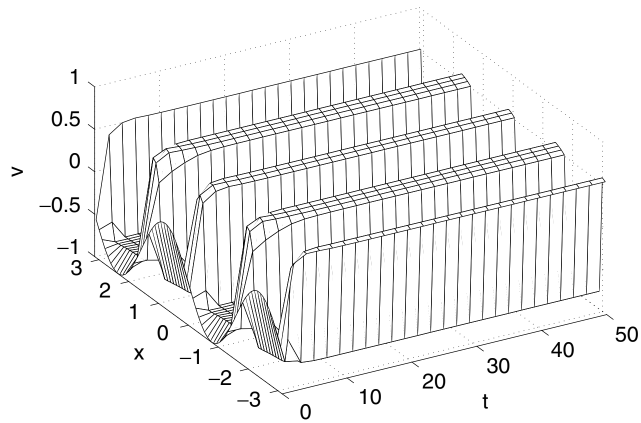

Figure 8 Continuous initial data in the elliptic region, $m_0 = -\frac{1}{2} \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, with a small viscosity $\varepsilon = 0.5$. The solution converges to a stationary wave.

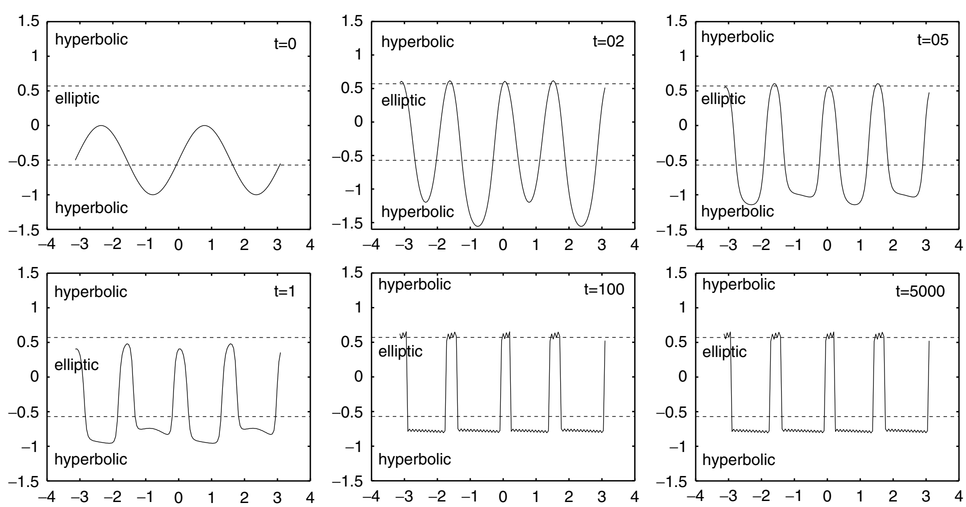

Figure 9 Snapshots of $v(x, t)$ at $t = 0, 0.2, 0.5, 1, 100, 5000$ for $\varepsilon = 0.5$ and $m_0 = -0.5$. The solution $v(t, x)$ exhibits phase transitions and behaves like a standing wave, as predicted in Conjecture 1.4.

Next we set $v_0 = \frac{1}{2}(-1 + \sin(2x))$, such that the average $m_0 = -\frac{1}{2}$ is still in the elliptic region, but now negative. In Figures 8 and 9, we show the corresponding results.

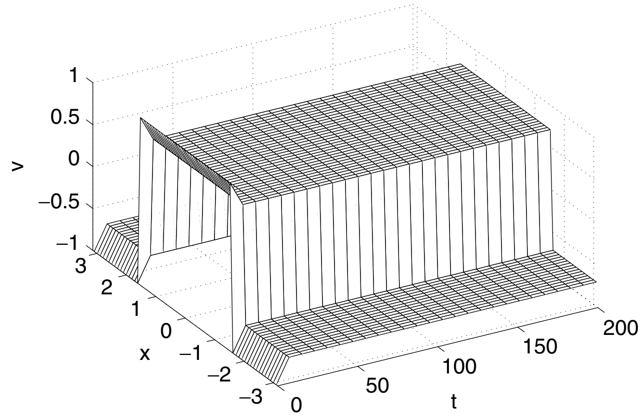

Figure 10 Riemann initial data in the elliptic region, $m_0 = 0 \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, with a small viscosity $\varepsilon = 0.5$. The solution converges to a stationary wave.

4.5 Numerical simulations for conjecture 1.4: (II) riemann initial data

In this section we keep $\varepsilon = 0.5$, but now we change the initial data to the discontinuous Riemann data

$$v_0(x) = \begin{cases} -\alpha, & x \in [-\pi, -\gamma), \\ \beta, & x \in [-\gamma, \gamma), \\ -\alpha, & x \in [\gamma, \pi], \end{cases} \tag{4.8}$$

and $u_0(x) = 0$. Here α and β are real parameters, and $0 < \gamma < \pi$.

For the first experiment, we choose a symmetric configuration, $\gamma = \pi/2$, and set $\alpha = \beta = 1$. Then the initial average is $m_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_0(x)dx = 0$, which is in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and the condition (1.13) does not hold. In Figure 10, we show a graph of the solution $v(t, x)$ up to time $t = 200$, and in Figure 11 we show snapshots in time of the evolution of v. The solution converges to a stationary wave which has one regular flat peak and valley, and after time $t = 5$ the solution exhibits phase transitions through all three phases.

For the second experiment, we keep the symmetric configuration, $\gamma = \pi/2$, and set $\alpha = \beta = 0.1$. The initial average $m_0 = 0$ is again in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and the condition (1.13) does not hold. But now there are no phase transitions initially, in contrast to the first experiment. In Figure 12 we show snapshots in time of the evolution of v. One can see that the Riemann data is first rapidly smoothed and strongly attracted toward the hyperbolic regions: three peaks reach this region and thus six transient phase

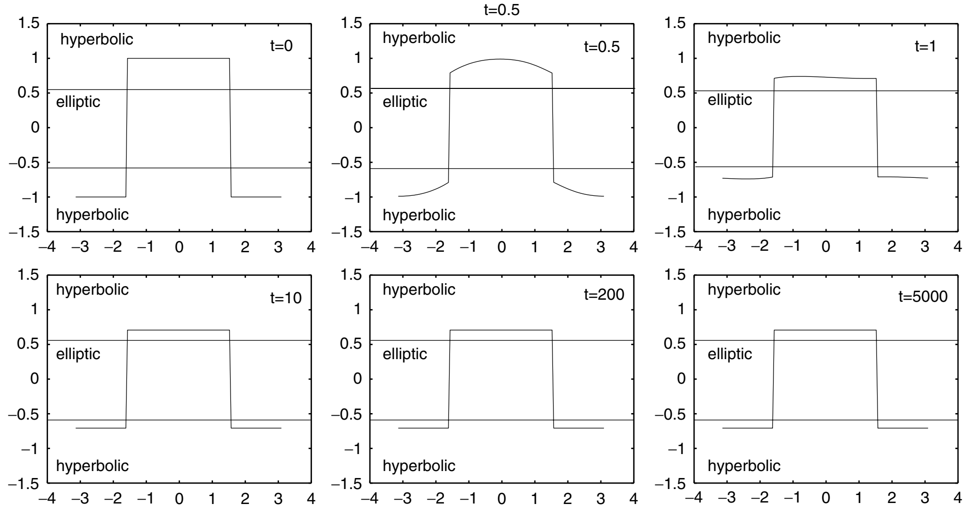

Figure 11 Snapshots in time at $t = 0, 0.1, 0.5, 10, 200, 5000$ of the solution $v(t,x)$ using symmetric Riemann initial data over all three phases. The solution $v(t,x)$ exhibits phase transitions and behaves like a standing wave, as predicted in Conjecture 1.4.

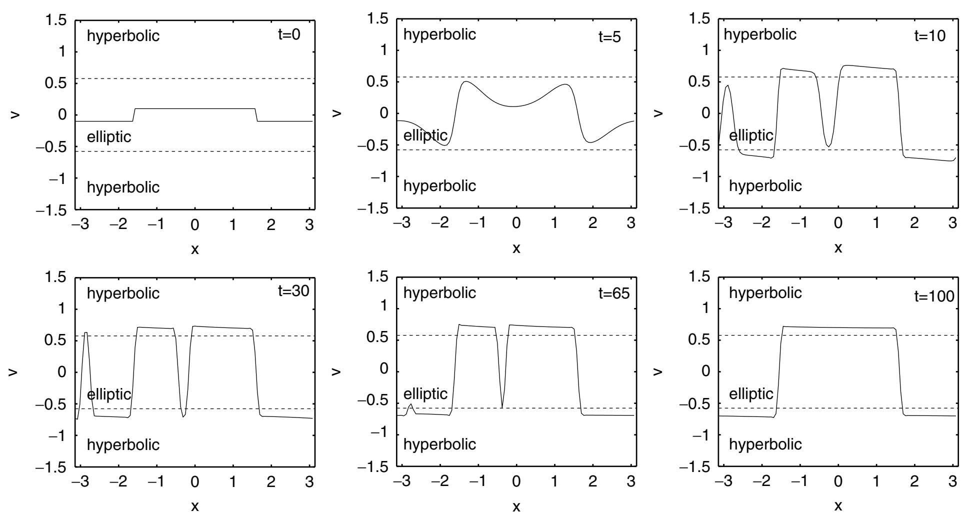

Figure 12 Snapshots in time at $t = 0, 5, 10, 30, 65, 100$ of the solution $v(t,x)$ using symmetric Riemann initial data which is in the elliptic region only.

transitions form. These however disappear fairly quickly, and at $t = 100$ one can see a very similar solution, as in the case of large Riemann initial data: the numerical solution reaches a numerical steady state at $t = 10$.

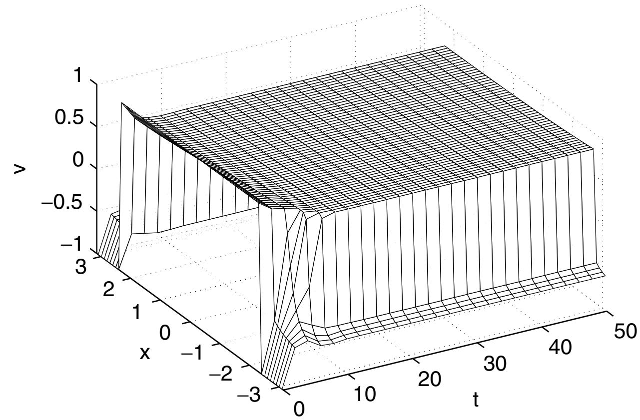

Figure 13 The solution $v(t, x)$ with wide Riemann data with average $m_0 = \frac{1}{2}$, over all the phases, where a standing wave has formed.

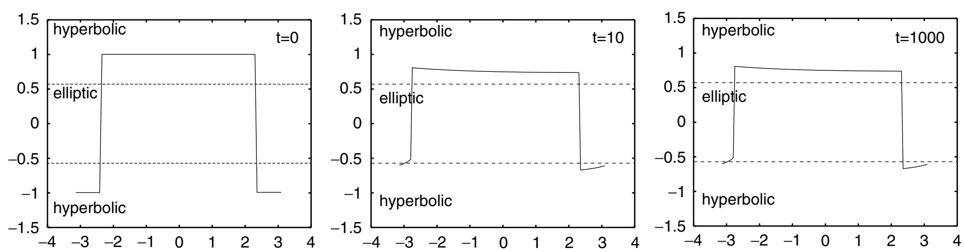

Figure 14 Snapshots in time at $t = 0, 10, 1000$ of the solution $v(t, x)$ using symmetric Riemann initial data whose average is in the elliptic region.

We now abandon the configuration with average initial data equal to zero and show two numerical experiments, both with large initial data in both the hyperbolic and elliptic region. For the first experiment we choose $\gamma = \frac{3\pi}{4}$, so the average $m_0 = \frac{1}{2}$ is positive and in the elliptic region. The results we obtain are shown in Figures 13 and 14. Again we see that very quickly two phase transitions are established, and soon the solution behaves like a standing wave with phase transitions.

For the second experiment with nonzero initial average m_0, we choose $\gamma = \frac{\pi}{4}$, so that the average of v_0 is negative, $m_0 = -\frac{1}{2}$, but still in the elliptic region. As shown in Figures 15 and 16, we note that the solution $v(t, x)$ behaves like a standing wave with phase transitions.

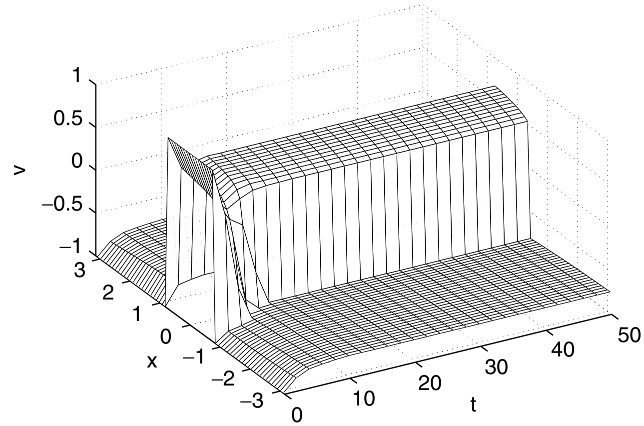

Figure 15 The solution $v(t, x)$ with narrow Riemann data and average $m_0 = -\frac{1}{2}$ is across all three phases, and again a standing wave forms.

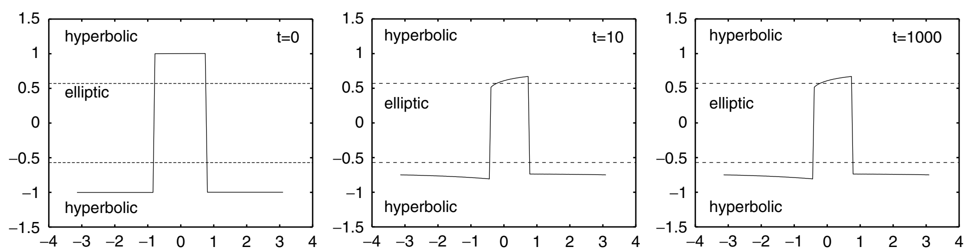

Figure 16 Snapshots in time at $t = 0, 10, 1000$ of the solution $v(t, x)$ using symmetric Riemann initial data whose average is in the elliptic region.

5 Conclusions

We have analyzed the long-time behavior of the solution of a 2×2 system of mixed type with a relaxation term and periodic boundary condition, in which the solution may exhibit phase transitions. We showed that the occurrence of phase transitions in the solution is dependent on the location of the initial data and the size of the artificial viscosity ε. More precisely, as long as the average of the initial data m_0 and the viscosity ε satisfy the optimal sufficient condition (1.11), i.e., $\varepsilon + \sigma'(m_0) > f'(m_0)^2$, after a short time oscillation, the solution $(v, u)(t, x)$ does not exhibit phase transitions but converges to its initial average (m_0, m_1), provided that the initial condition does not deviate too much from its average. In particular, if the initial average m_0 in the hyperbolic region is far away from the phase boundary $v = v_1$ or $v = v_2$, such that $\sigma'(m_0) > f'(m_0)^2$, then the system (1.3) is strongly hyperbolic, and even without artificial viscosity, i.e., $\varepsilon = 0$, the solution $(v, u)(t, x)$ has been proved to converge in the hyperbolic phase to the initial average (m_0, m_1) exponentially in time. If the initial average m_0 is in the unstable elliptic region, and if the viscosity ε is big enough, such that (1.13) holds, then the system (1.3) behaves strongly like a parabolic system, and we proved convergence of the solution $(v, u)(t, x)$ to its initial average (m_0, m_1) without phase transition. These are our main theoretical results.

For the case of small viscosity and with the initial average m_0 in the elliptic phase, i.e., (1.13) does not hold, our numerical study for $\sigma(v) = v^3 - v$ shows that the solution $v(t, x)$ is oscillatory for all time and exhibits phase transitions. In particular, the solution $v(t, x)$ behaves always like a periodic standing wave (steady-state solution). However, *these interesting problems of wave stability are open theoretically at this moment*; they are our conjectures in the present paper.

Acknowledgments

The authors would like to express their thanks to the referees for the valuable suggestion and comments. The authors also would like to thank Professor Liping Liu for her great help in providing with the code for the Fourier pseudo-spectral method. The research of MG was supported in part by NSERC 216871-03, the research of MM was supported in part by the CRM-fellowship, and the research of GS was supported in part by NSERC 228309.

References

[1] Affouf, M., and R. E. Caflisch. "A Numerical Study of Riemann Problem Solutions and Stability for a System of Viscous Conservation Laws of Mixed Type." *SIAM Journal on Applied Mathematics* 51 (1991): 605–634.

[2] Carr, J., M. Gurtin, and M. Slemrod. "Structured Phase Transitions on a Finite Interval." *Archive for Rational Mechanics and Analysis* 86 (1984): 317–351.

[3] Chen, G.-Q., C. D. Levermore, and T.-P. Liu. "Hyperbolic Conservation Laws with Stiff Relaxation Terms and Entropy." *Communications on Pure and Applied Mathematics* 47 (1994): 787–830.

[4] Chen, X., and X.-P. Wang. "Phase Transition Near a Liquid-Gas Coexistence Equilibrium." *SIAM Journal on Applied Mathematics* 61 (2000): 454–471.

[5] Chern, I.-L. "Long-Time Effect of Relaxation for Hyperbolic Conservation Laws." *Communications in Mathematical Physics* 172 (1995): 39–55.

[6] Colombo, R. M. "Hyperbolic Phase Transitions in Traffic Flow." *SIAM Journal on Applied Mathematics* 63 (2002): 708–721.

[7] Colombo, R. M., and F. S. Priuli. "Characterization of Riemann Solvers for the Two Phase *p*-system." *Communications in Partial Differential Equations* 28 (2003): 1371–1389.

[8] Fan, H.-T. "A Vanishing Viscosity Approach on the Dynamics of Phase Transitions in Van der Waals Fluids." *Journal of Differential Equations* 103 (1993): 179–204.

[9] Fan, H.-T. "A Limiting 'Viscosity' Approach to the Riemann Problem for Materials Exhibiting a Change of Phase. II.." *Archive for Rational Mechanics and Analysis* 116 (1992): 317–337.

[10] Fan, H.-T. "On a Model of the Dynamics of Liquid/Vapor Phase Transition." *SIAM Journal on Applied Mathematics* 60 (2000): 1270–1301.

[11] Fife, P. C., and X.-P. Wang. "Periodic Structures in a Van der Waals Fluid." *Proceedings of the Royal Society of Edinburgh* 128A (1998): 235–250.

[12] Hoff, D., and M. Khodja. "Stability of Coexisting Phases for Compressible Van der Waals Fluids." *SIAM Journal on Applied Mathematics* 53 (1993): 1–14.

[13] Hsieh, D.-Y., S. Tang, X.-P. Wang, and L.-X. Wu. "Dissipative Nonlinear Evolution Equations and Chaos." *Studies in Applied Mathematics* 101 (1998): 233–266.

[14] Hsieh, D.-Y., and X.-P. Wang. "Phase Transition in van der Waals Fluid." *SIAM Journal on Applied Mathematics* 57 (1997): 871–892.

[15] Jin, S. "A Convex Entropy for a Hyperbolic System with Stiff Relaxation." *Journal of Differential Equations* 127 (1996): 97–109.

[16] Jin, S., and C. D. Levermore. "Numerical Scheme for a Hyperbolic System with Stiff Relaxation Terms." *Journal of Computational Physics* 126 (1996): 449–467.

[17] Jin, S., and Z. Xin. "The Relaxing Schemes for System of Conservation Laws in Arbitrary Space Dimensions." *Communications on Pure and Applied Mathematics* 48 (1995): 555–563.

[18] Lattanzio, C., and P. Marcati. "The Zero Relaxation Limit for 2×2 Hyperbolic Systems." *Nonlinear Analysis* 38 (1999): 375–389.

[19] Liu, H. L., C. W. Woo, and T. Yang. "Decay Rates for Traveling Waves of a Relaxation Model." *Journal of Differential Equations* 134 (1997): 343–367.

[20] Liu, T.-P. "Hyperbolic Conservation Laws with Relaxation." *Communications in Mathematical Physics* 108 (1997): 153–175.

[21] Luo, T. "Asymptotic Stability of Planar Rarefaction Waves for the Relaxation Approximation of Conservation Laws in Serval Dimensions." *Journal of Differential Equations* 133 (1997): 255–279.

[22] Mascia, C., and R. Natalini. "L1 Nonlinear Stability of Traveling Waves for a Hyperbolic System with Relaxation." *Journal of Differential Equations* 132 (1996): 275–292.

[23] Matsumura, A. "On the Asymptotic Behavior of Solutions of Semilinear Wave Equations." *Publications of the Research Institute for Mathematical Sciences*, *Kyoto University* 12 (1976): 169–189.

[24] Mei, M., and B. Rubino. "Convergence to Traveling Waves with Decay Rates for Solutions of the Initial Boundary Problem to a Relaxation Model." *Journal of Differential Equations* 159 (1999): 138–185.

[25] Mei, M., Y. S. Wong, and L. Liu. "Phase Transition in a Coupled Viscoelastic System with Periodic Initial-Boundary Condition: (I) Existence and Uniform Boundedness." *Discrete and Continuous Dynamical Systems–Series B* 7 (2007): 825–837.

[26] Mei, M., Y. S. Wong, and L. Liu. "Phase Transition in a Coupled Viscoelastic System with Periodic Initial-Boundary Condition: (II) Convergence." *Discrete and Continuous Dynamical Systems–Series B* 7 (2007): 839–857.

[27] Mei, M., and T. Yang. "Convergence Rates to Traveling Waves for Nonconvex Relaxation Model." *Proceedings of the Royal Society of Edinburgh* 128A (1998): 1053–1068.

[28] Mercier, J.-M., and B. Piccoli. "Global Continuous Riemann Solver for Nonlinear Elasticity." *Archive for Rational Mechanics and Analysis* 156 (2001): 89–119.

[29] Natalini, R., and S. Tang. "Discrete Kinetic Models for Dynamical Phase Transitions." *Communications on Applied Nonlinear Analysis* 7 (2000): 1–32.

[30] Nishibata, S. "The Initial Boundary Value Problem for Hyperbolic Conservation Laws with Relaxation." *Journal of Differential Equations* 130 (1996): 100–126.

[31] Shearer, M. "Nonuniqueness of Admissible Solutions of Riemann Initial Value Problems for a System of Conservation Laws of Mixed Type." *Archive for Rational Mechanics and Analysis* 93 (1986): 45–59.

[32] Slemrod, M. "A Limiting "Viscosity" Approach to the Riemann Problem for Materials Exhibiting Change of Phase." *Archive for Rational Mechanics and Analysis* 105 (1989): 327–365.

[33] Slemrod, M. "Admissibility Criteria for Propagating Phase Boundaries in a Van der Waals Fluid." *Archive for Rational Mechanics and Analysis* 81 (1983): 301–315.

[34] Tang, S., and H. Zhao. "Stability of Suliciu Model for Phase Transitions." *Communications on Pure and Applied Analysis* 3 (2004): 545–556.

[35] Wang, W.-C., and Z.-P. Xin. "Asymptotic Limit of Initial-Boundary Value Problems for Conservation Laws with Relaxational Extensions." *Communications on Pure and Applied Mathematics* 51 (1998): 505–535.

[36] Wang, P., D.-Y. Hsieh, S. Tang, and J. Wu. "Pattern Selection in a Reaction-Diffusion Equation." *Acta Mechanica Sinica* 18 (2002): 652–660.

[37] Whitham, C. *Linear and Nonlinear Waves*. New York: Wiley-Interscience, 1974.

[38] Yao, Z.-A., and C. J. Zhu. "L^p-convergence Rate to Diffusion Waves for p-system with Relaxation." *Journal of Mathematical Analysis and Applications* 276 (2002): 497–515.

[39] Zhao, H. J. "Nonlinear Stability of Strong Planar Rarefaction Wave for the Relaxation Approximation of Conservation Laws in Serveral Space Dimensions." *Journal of Differential Equations* 163 (2000): 198–222.

[40] Zhu, C. J. "Asymptotic Behavior of Solutions for p-system with Relaxation." *Journal of Differential Equations* 180 (2002): 273–306.

[41] Zingano, P. "Nonlinear Stability with Decay Rate for Traveling Wave Solutions of a Hyperbolic System with Relaxation." *Journal of Differential Equations* 130 (1996): 36–58.

[42] Zumbrun, K. "Dynamical Stability of Phase Transitions in the p-system with Viscosity-Capillarity." *SIAM Journal on Applied Mathematics* 60 (2000): 1913–1924.