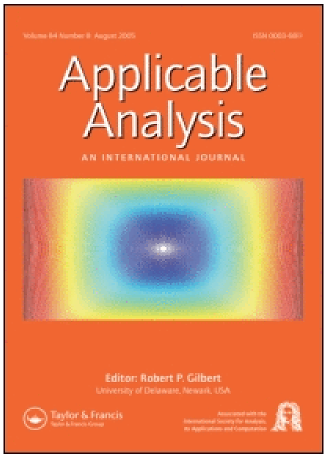

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Long-time behavior of solution for rosenau-burgers equation (i)

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Long-Time Behavior of Solution for Rosenau-Burgers Equation (I)

Communicated by R.P. Gilbert

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ABSTRACT

The aim of this paper is to consider the time-decay properties of the solution for the Rosenau-Burgers equation in the form $u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0$. In particular, we prove some algebraic time decay rates of the solution within some spatial Sobolev spaces. The asymptotic stability the solution of the corresponding of linear equation is also obtained. To prove all of these, we make using of the method of Fourier transform together with the energy method.

AMS: 35Q53, 35B40, 35L65

KEY WORDS: Rosenau-Burgers equation, asymptotic behavioe, stability

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1. Introduction

We consider the Cauchy problem for the generalized Rosenau-Burgers equation (R-B) in the form

$$u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0, \quad x \in R^1, t > 0, \tag{1.1}$$

with initial data

$$u|_{t=0} = u_0(x), \tag{1.2}$$

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where $\alpha > 0$, $\beta \in R^1$ are the given constants, the nonlinear function $\phi(u)$ is smooth and satisfies

$$|\phi'(u)| \leq C|u|^{p-1}, \tag{1.3}$$

for some $C > 0$ and $p \geq 1$.

When $\alpha = 0$, equation (1.1) is called as Rosenau equation proposed by P. Rosenau [Ro] for treating the dynamics of dense discrete systems. Since the KdV equation describes a unidimensional propagation of waves, wave-wave and wave-wall interactions cannot be treated by it. Further, the slope and the behavior of high amplitude waves may not be well predicted by the KdV equation, because it was modelled under the assumption of weak anharmonicity. In order to overcome the shortcomings of the KdV equation, P. Rosenau has developed the model of (1.1) with $\alpha = 0$. This model has been studied by Park [Pa] and by Chung and Ha [CH]. Both of them deal with the global existence and the uniqueness of the solution for IBVP by using of Galerkin's method. When $\alpha > 0$, we call equation (1.1) as the Rosenau-Burgers equation (R-B) somehow corresponding to the KdV-B and the BBM-B equations, but it is given neither by Rosenau nor by Burgers. The asymptotic properties of the solutions for the KdV-B equations and for the BBM-B equations have been studied by many persons (see Bona and Luo [BL], Zhang [Zh], and also the references therein). In this paper, we are going to show the global existence and the time-decay rates of the solution for (1.1) and (1.2), and the stability of the solution to the corresponding linearized equation of (1.1) by means of the method of Fourier transforms with the energy method. The method we adopt is similar to the skill used by Zhang [Zh], but more technical than Zhang's. The nonlinear stability of travelling wave solutions of (1.1) will be discussed by the author in another paper [Me].

We first state some notations for simplicity. C always denotes some positive constant, but never depends on t. H^k ($k \geq 0$ integer) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, L^2 denotes the square integrable space with the norm $\|\cdot\|$, and L^∞ is the essentially bounded space with the norm $\|\cdot\|_\infty$. Suppose that

$f(x) \in L^1 \cap L^2(R)$, we define the Fourier transforms of $f(x)$ as follows:

$$F[f](\xi) \equiv \hat{f}(\xi) = \int_R f(x)e^{-ix\xi}dx.$$

Our plan in this paper is the following: after stating our main results and some elementary preliminaries in Section 2, we give the proofs of the main theorems in Section 3.

2. Main Results and Preliminaries

We first state the following two well-known lemmas which can be found in Segal [Se] (see also Matsumura [Ma]) and Kreiss and Lorenz [KL], respectively.

Lemma 2.1. *Suppose that $a > 0$ and $b > 0$, then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{-min(a,b)}, \quad \text{for } \max(a,b) > 1, \tag{2.1}$$

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{1-a-b}, \quad \text{for } \max(a,b) < 1. \tag{2.2}$$

Lemma 2.2. *If $u(x) \in H^l(R)$, then*

$$\|u\|_\infty^2 \le \|u\|\|u_x\|, \quad \|u_{x^k}\| \le \|u\|^{1-k/l}\|u_{x^l}\|^{k/l} \tag{2.3}$$

for $k \in [1, l]$, where k and l are integers.

Theorem 2.3 (Existence). *Suppose that (1.3) holds. If $u_0(x) \in H^4(R)$, then there exists a positive constant δ such that $\|u_0\|_4 < \delta$, then the initial value problem (1.1) and (1.2) has a unique global solution $u(t,x)$ with*

$$u(t,x) \in C(0,\infty; H^4), \quad u_x(t,x) \in L^2(0,\infty; H^2).$$

Moreover, $u(t,x)$ satisfies

$$\|u(t)\|_4 \le \|u_0\|_4, \quad \int_0^\infty \|u_x(t)\|_2^2 dt \le C\|u_0\|_4^2, \tag{2.4}$$

for all $t \ge 0$.

Proof. By the standard energy method, combining the local existence with the *a priori* estimates somewhat like (2.4), we can prove the global existence. To prove (2.4), we multiply (1.1) by $2\partial^i u/\partial x^i$ $(i = 0, 1, 2, 3, 4)$, respectively, and integrate over $[0, \infty) \times R$, those yield (2.4).

Theorem 2.4 (Decay Rates). *When $p > 7/2$, and $u_0(x) \in W^{4,1} \cap H^4$, then $u(t,x)$ satisfies the following for all $t \geq 0$*

$$\|u(t)\|_\infty \leq C(1+t)^{-1/2}, \quad \|u_x(t)\|_\infty \leq C(1+t)^{-1}. \tag{2.5}$$

When $p > 2$, and $u_0(x) \in W^{3,1} \cap H^4$, then $u(t,x)$ satisfies the followings for all $t \geq 0$

$$\|u(t)\| \leq C(1+t)^{-1/4}, \quad \|u_x(t)\| \leq C(1+t)^{-3/4}, \tag{2.6}$$

which imply

$$\|u(t)\|_\infty \leq C(1+t)^{-1/4}. \tag{2.7}$$

Remark. To get the decay rates likes (2.5), Zhang[Zh] needs $p \geq 5$ for the BBM-B equation. However, without any difficulty, Zhang's result ($p \geq 5$) can be improved as $p > 7/2$ by our scheme in this paper. For the details, see below Section 3.

Let $v(t,x)$ be the solution of the linear equation corresponding to (1.1) by dropping the nonlinear term $\phi(u)$, namely,

$$v_t + v_{xxxxt} - \alpha v_{xx} + \beta v_x = 0, \quad x \in R^1, t > 0, \tag{2.8}$$

with the same initial data

$$v|_{t=0} = u_0(x), \tag{2.9}$$

we have the asymptotic stability as follows.

Theorem 2.5 (Asymptotic Stability). *When $p > 7/2$ and $u_0(x) \in W^{4,1} \cap H^4$, then we have*

$$\|u(t) - v(t)\|_\infty \leq C(1+t)^{-1/2}, \quad \|u_x(t) - v_x(t)\|_\infty \leq C(1+t)^{-1}. \tag{2.10}$$

When $p > 2$ and $u_0(x) \in W^{3,1} \cap H^4$, then we have

$$\|u(t) - v(t)\| \le C(1+t)^{-1/4}, \quad \|u_x(t) - v_x(t)\| \le C(1+t)^{-3/4}, \tag{2.11}$$

$$\|u(t) - v(t)\|_\infty \le C(1+t)^{-1/4}. \tag{2.12}$$

3. Proofs of Theorem 2.4 and Theorem 2.5

Taking the Fourier transform to (1.1), we have

$$\hat{u}_t + (i\xi)^4 \hat{u}_t - \alpha(i\xi)^2 \hat{u} + i\beta\xi\hat{u} + \widehat{\phi(u)}_x = 0, \tag{3.1}$$

which means

$$\hat{u}(\xi, t) = e^{-A(\xi)t}\hat{u}_0(\xi) - \int_0^t e^{-A(\xi)(t-s)} \frac{\widehat{\phi(u)}_x(\xi, s)}{1+\xi^4} ds, \tag{3.2}$$

where

$$A(\xi) = B(\xi) + \frac{i\beta\xi}{1+\xi^4} \equiv \frac{\alpha\xi^2 + i\beta\xi}{1+\xi^4}. \tag{3.3}$$

Taking the inverse Fourier transform to (3.2) yields

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)t}\hat{u}_0(\xi) d\xi - \frac{1}{2\pi}\int_0^t \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{\phi(u)}_x}{1+\xi^4} d\xi ds. \tag{3.4}$$

Lemma 3.1. *There hold*

$$\int_{-\infty}^{\infty} \frac{e^{-B(\xi)t}}{1+\xi^4} d\xi \le C(1+t)^{-1/2}, \tag{3.5}$$

$$\left| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)t}\hat{u}_0(\xi) d\xi \right| \le C\|u_0\|_{W^{4,1}}(1+t)^{-1/2}, \tag{3.6}$$

$$\left| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{\phi(u)}_x}{1+\xi^4} d\xi \right| \le C\|u(s)\|_\infty^{p-3} \|u_x(s)\|_\infty (1+t-s)^{-1/2}, \tag{3.7}$$

for all $t \ge 0$.

Proof. Since

$$\int_{-\infty}^{\infty}\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi=2\int_{0}^{\infty}\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi=2\Big(\int_0^2+\int_2^{\infty}\Big)\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi, \tag{3.8}$$

and that

$$\begin{aligned}&\int_0^2\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi\le\int_0^2\frac{1}{1+\xi^4}e^{-\frac{\alpha\xi^2}{1+2^4}t}d\xi\\&\le e^{\frac{\alpha 2^2}{1+2^4}}\int_0^2 e^{-\frac{\alpha\xi^2}{1+2^4}(1+t)}d\xi\le C(1+t)^{-1/2},\end{aligned}\tag{3.9}$$

on the other hand, let $\lambda=\frac{B(\xi)}{\alpha}=\frac{\xi^2}{1+\xi^4}$, and note that $\frac{1+\xi^4}{2\xi^5-2\xi}$ is positive and bounded for $\xi\in[2,\infty)$, we have

$$\begin{aligned}&\int_2^{\infty}\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi=\int_0^{4/17}\frac{1+\xi^4}{2\xi^5-2\xi}e^{-\alpha\lambda t}d\lambda\\&\le Ce^{4\alpha/17}\int_0^{4/17}e^{-\alpha\lambda(1+t)}d\lambda\le C(1+t)^{-1},\end{aligned}\tag{3.10}$$

then we obtain (3.5) from (3.8)–(3.10).

To prove (3.6), we first note

$$\begin{aligned}&(1+\xi^4)|\hat{u}_0(\xi)|=\Big|(1+\xi^4)\int_{-\infty}^{\infty}e^{-ix\xi}u_0(x)dx\Big|\\&=\Big|\int_{-\infty}^{\infty}e^{-ix\xi}(u_0(x)+u_{0xxxx}(x))dx\Big|\\&\le\int_{-\infty}^{\infty}(|u_0(x)|+|u_{0xxxx}(x)|)dx,\end{aligned}$$

by using of (3.5) and above-mentioned, we then have

$$\begin{aligned}&\Big|\int_{-\infty}^{\infty}e^{i\xi x}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big|\le\int_{-\infty}^{\infty}e^{-B(\xi)t}|\hat{u}_0(\xi)|d\xi\\&\le\sup_{\xi\in R}\{(1+\xi^4)|\hat{u}_0(\xi)|\}\int_{-\infty}^{\infty}\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi\\&\le C(1+t)^{-1/2}\int_{-\infty}^{\infty}(|u_0(x)|+|u_{0xxxx}|)dx\\&\le C\|u_0\|_{W^{4,1}}(1+t)^{-1/2}.\end{aligned}$$

Finally, to prove (3.7), noting (3.5) and (1.3), we have

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\Big| \le \int_{-\infty}^{\infty}|\widehat{\phi(u)_x}|\frac{e^{-B(\xi)(t-s)}}{1+\xi^4}d\xi \\
&\le \sup_{\xi\in R}|\widehat{\phi(u)_x}|\int_{-\infty}^{\infty}\frac{e^{-B(\xi)(t-s)}}{1+\xi^4}d\xi \le C(1+t-s)^{-1/2}\int_{-\infty}^{\infty}|\phi(u)_x|dx \\
&\le C(1+t-s)^{-1/2}\int_{-\infty}^{\infty}|\phi'(u)||u_x|dx \\
&\le C(1+t-s)^{-1/2}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty\|u(s)\|^2 \\
&\le C(1+t-s)^{-1/2}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty,
\end{aligned}$$

where we used $\|u(s)\| \le \|u_0\|_4$ (see (2.4)).

From (3.4), by using of (3.6) and (3.7), we immediately get the following estimate.

Lemma 3.2. *It holds that*

$$\|u(t)\|_\infty \le C\|u_0\|_{W^{4,1}}(1+t)^{-1/2}+C\int_0^t(1+t-s)^{-1/2}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty ds, \quad (3.11)$$

for all $t \ge 0$.

Now we also have from (3.4)

$$u_x(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi - \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi ds. \quad (3.12)$$

The next Lemma 3.3 can be proved by the same scheme to Lemma 3.1, we here state it as follows without the detail proof.

Lemma 3.3. *There hold that*

$$\int_{-\infty}^{\infty}|\xi|\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi \le C(1+t)^{-1}, \quad (3.13)$$

$$\Big|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big| \le C\|u_0\|_{W^{4,1}}(1+t)^{-1}, \quad (3.14)$$

$$\Big|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\Big| \le C\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty(1+t-s)^{-1}, \quad (3.15)$$

for all $t \ge 0$.

Proof. We first prove (3.13). Similar to the proof of (3.5), we have

$$\int_{-\infty}^{\infty}|\xi|\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi = 2\Big(\int_0^2+\int_2^{\infty}\Big)\xi\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi,$$

and that

$$\begin{aligned}&\int_0^2 \xi\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi \le \int_0^2\frac{\xi}{1+\xi^4}e^{-\frac{\alpha\xi^2}{1+2^4}t}d\xi\\ &\le Ce^{\frac{\alpha 2^2}{1+2^4}}\int_0^2 e^{-\frac{\alpha\xi^2}{1+2^4}(1+t)}d\xi^2 \le C(1+t)^{-1},\\ &\int_2^{\infty}\xi\frac{e^{-B(\xi)t}}{1+\xi^4}d\xi = \int_0^{4/17}\frac{1+\xi^4}{2\xi^4-2}e^{-\alpha\lambda t}d\lambda\\ &\le Ce^{4\alpha/17}\int_0^{4/17}e^{-\alpha\lambda(1+t)}d\lambda \le C(1+t)^{-1},\end{aligned}$$

where we used the fact that $\frac{1+\xi^4}{2\xi^4-2}$ is positive and bounded for $\xi \in [2,\infty)$, and $\lambda = \frac{\xi^2}{1+\xi^4} \in [0, \frac{4}{17}]$. Then, all of these facts imply (3.13).

Using (3.13) and by the same schemes in the proofs of (3.6) and (3.7), we easily prove (3.14) and (3.15).

Thus, (3.12) and Lemma 3.3 immediately yield the following energy estimate.

Lemma 3.4. *It holds that*

$$\|u_x(t)\|_\infty \le C\|u_0\|_{W^{4,1}}(1+t)^{-1}+C\int_0^t(1+t-s)^{-1}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty ds. \quad (3.16)$$

To prove (2.5) in Theorem 2.4, we can use the continuity argument based on the local existence with the *a priori* estimates as follows. Therefore, to prove the following lemma is our main purpose.

Lemma 3.5. *Suppose that $p > 7/2$, $u_0(x) \in W^{4,1} \cap H^4$ and $u(x,t)$ is a local solution of (1.1) and (1.2) for $t \in [0,T]$, where $T > 0$ is a any given constant. Let*

$$M_1(t) = \sup_{0\le s\le t} \{(1+s)^{1/2}\|u(s)\|_\infty + (1+s)\|u_x(s)\|_\infty\},$$

then there exist positive constants C and δ_1, such that when $M_1(T) < \delta_1$, then

$$\|u(t)\|_\infty \le C(1+t)^{-1/2}, \quad \|u_x(t)\|_\infty \le C(1+t)^{-1}, \tag{3.17}$$

for $t \in [0,T]$.

Proof. Combining Lemmas 3.2 and 3.4, and by Schwartz's inequality, we have

$$\begin{aligned}
&(1+t)^{1/2}\|u(t)\|_\infty + (1+t)\|u_x(t)\|_\infty \\
&\le C\Big\{\|u_0\|_{W^{4,1}} + (1+t)^{1/2}\int_0^t (1+t-s)^{-1/2}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty ds \\
&\quad + (1+t)\int_0^t (1+t-s)^{-1}\|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty ds\Big\} \\
&= C\Big\{\|u_0\|_{W^{4,1}} + (1+t)^{1/2}\int_0^t (1+t-s)^{-1/2}(1+s)^{-(p-1)/2} \\
&\qquad \cdot \Big((1+s)^{1/2}\|u(s)\|_\infty\Big)^{p-3}\Big((1+s)\|u_x(s)\|_\infty\Big) ds \\
&\quad + (1+t)\int_0^t (1+t-s)^{-1}(1+s)^{-(p-1)/2} \cdot \\
&\qquad \cdot \Big((1+s)^{1/2}\|u(s)\|_\infty\Big)^{p-3}\Big((1+s)\|u_x(s)\|_\infty\Big) ds\Big\} \\
&\le C\Big\{\|u_0\|_{W^{4,1}} + (1+t)^{1/2}\int_0^t (1+t-s)^{-1/2}(1+s)^{-(p-1)/2} \\
&\qquad \cdot \Big[\Big((1+s)^{1/2}\|u(s)\|_\infty\Big)^{2(p-3)} + \Big((1+s)\|u_x(s)\|_\infty\Big)^2\Big] ds \\
&\quad + (1+t)\int_0^t (1+t-s)^{-1}(1+s)^{-(p-1)/2} \\
&\qquad \cdot \Big[\Big((1+s)^{1/2}\|u(s)\|_\infty\Big)^{2(p-3)} + \Big((1+s)\|u_x(s)\|_\infty\Big)^2\Big] ds\Big\}.
\end{aligned} \tag{3.18}$$

Let $M_1(t)$ be small, say $M_1(t) < \delta_1 < 1$, and let $q := \min\{2(p-3), 2\} > 1$ due to

$p > 7/2$. Using Lemma 2.1, we obtain from (3.18)

$$\begin{aligned} M_1(t) &\le C\Big\{\|u_0\|_{W^{4,1}} + (1+t)^{1/2}M_1(t)^q\int_0^t (1+t-s)^{-1/2}(1+s)^{-(p-1)/2}ds \\ &\quad + (1+t)M_1(t)^q\int_0^t (1+t-s)^{-1}(1+s)^{-(p-1)/2}ds\Big\} \\ &\le C\|u_0\|_{W^{4,1}} + CM_1(t)^q, \end{aligned}$$

which implies (3.17) for some small $M_1(t)$. So, we complete the proof of this lemma.

To prove (2.6) in Theorem 2.4, same to above-mentioned, the following energy estimates must also be necessary. Like as Lemmas 3.1 and 3.3, we firstly establish some estimates as follows.

Lemma 3.6.

$$\Big\|\int_{-\infty}^{\infty} e^{ix\xi}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big\| \le C\|u_0\|_{W^{2,1}}(1+t)^{-1/4}, \tag{3.19}$$

$$\Big\|\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\Big\| \le C\|u(s)\|^{p/2}\|u_x(s)\|^{p/2}(1+t-s)^{-1/4}, \tag{3.20}$$

$$\Big\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big\| \le C\|u_0\|_{W^{3,1}}(1+t)^{-3/4}, \tag{3.21}$$

$$\Big\|\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\Big\| \le C\|u(s)\|^{p/2}\|u_x(s)\|^{p/2}(1+t-s)^{-3/4}. \tag{3.22}$$

Proof. Since

$$\begin{aligned} &\sup_{\xi\in R}(|\hat{u}_0|^2(1+\xi^4)) \le (\sup_{\xi\in R}(|\hat{u}_0|(1+\xi^2))^2 \\ &= \Big(\int_{-\infty}^{\infty}|u_0(x)| + |u_{0xx}(x)|d\xi\Big)^2 \le \|u_0\|^2_{W^{2,1}}, \end{aligned}$$

using the Parseval's equality and (3.5), we can prove (3.19)

$$\begin{aligned} &\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big\| \\ &= \|F^{-1}(e^{-A(\xi)t}\hat{u}_0(\xi))\| = \|e^{-A(\xi)t}\hat{u}_0\| \\ &= \Big\{\int_{-\infty}^{\infty} e^{-2B(\xi)t}|\hat{u}_0|^2 d\xi\Big\}^{1/2} \\ &\le \Big\{\sup_{\xi\in R}(|\hat{u}_0|^2(1+\xi^4))\int_{-\infty}^{\infty}\frac{e^{-2B(\xi)t}}{1+\xi^4}d\xi\Big\}^{1/2} \\ &\le C\|u_0\|_{W^{2,1}}(1+t)^{-1/4}. \end{aligned}$$

Similarly, making use of the Parseval's equality and (3.5), we have

$$\begin{aligned}
&\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\Big\| \\
&= \Big\|e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}\Big\| \\
&\le \sup_{\xi\in R}|\widehat{\phi(u)_x}|\Big\{\int_{-\infty}^{\infty}\frac{e^{-2B(\xi)(t-s)}}{1+\xi^4}d\xi\Big\}^{1/2} \\
&\le C(1+t-s)^{-1/4}\sup_{\xi\in R}|\widehat{\phi(u)_x}|. \qquad (3.23)
\end{aligned}$$

From (1.3) and using Schwartz's inequality and Lemma 2.2 yield

$$\begin{aligned}
&\sup_{\xi\in R}|\widehat{\phi(u)_x}| \le \int_{-\infty}^{\infty}|\phi'(u)||u_x|dx \\
&\le \|u(s)\|_{\infty}^{p-2}\|u(s)\|\|u_x(s)\| \\
&\le C\|u(s)\|^{p/2}\|u_x(s)\|^{p/2}, \qquad (3.24)
\end{aligned}$$

where we used Lemma 2.2 to get the last term of (3.23). By (3.23) and (3.24), we can prove (3.20).

To prove (3.21) and (3.22), we first see the fact

$$\int_{-\infty}^{\infty}|\xi|^2\frac{e^{-2B(\xi)t}}{1+\xi^6}d\xi \le C(1+t)^{-3/2}. \qquad (3.25)$$

Since the approach is also same to the proofs of (3.5) and (3.13), we here omit the proof of (3.25). Then the Parseval's equality and (3.25) yield

$$\begin{aligned}
&\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{u}_0(\xi)d\xi\Big\| \\
&= \Big\|F^{-1}(i\xi e^{-A(\xi)t}\hat{u}_0(\xi))\Big\| = \Big\|i\xi e^{-A(\xi)t}\hat{u}_0(\xi)\Big\| \\
&= \Big\{\int_{-\infty}^{\infty}|\xi|^2 e^{-2B(\xi)t}|\hat{u}_0|^2 d\xi\Big\}^{1/2} \\
&\le \Big\{\sup_{\xi\in R}(|\hat{u}_0|^2(1+|\xi|^6))\int_{-\infty}^{\infty}|\xi|^2\frac{e^{-2B(\xi)t}}{1+\xi^6}d\xi\Big\}^{1/2} \\
&\le \sup_{\xi\in R}(|\hat{u}_0|(1+|\xi|^3))\Big\{\int_{-\infty}^{\infty}|\xi|^2\frac{e^{-2B(\xi)t}}{1+\xi^6}d\xi\Big\}^{1/2} \\
&\le C(1+t)^{-3/4}\int_{-\infty}^{\infty}|u_0|+|u_{0xxx}|dx \\
&\le C(1+t)^{-3/4}\|u_0\|_{W^{3,1}}. \qquad (3.26)
\end{aligned}$$

Finally, to prove (3.22), we make use of the Parseval's equality, (3.25) and (3.24), we then have

$$
\begin{aligned}
&\Big\| \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{\phi(u)_x}}{1+\xi^4} d\xi \Big\| \\
&= \Big\| i\xi e^{-A(\xi)(t-s)} \frac{\widehat{\phi(u)_x}}{1+\xi^4} \Big\| \\
&= \Big\{ \int_{-\infty}^{\infty} |\xi|^2 e^{-2B(\xi)(t-s)} \frac{|\widehat{\phi(u)_x}|^2}{(1+\xi^4)^2} d\xi \Big\}^{1/2} \\
&\le \sup_{\xi\in R} |\widehat{\phi(u)_x}| \Big\{ \int_{-\infty}^{\infty} |\xi|^2 \frac{e^{-2B(\xi)(t-s)}}{1+\xi^6} d\xi \Big\}^{1/2} \\
&\le C(1+t-s)^{-3/4} \int_{-\infty}^{\infty} |\phi'(u)||u_x| dx \\
&\le C(1+t-s)^{-3/4} \|u(s)\|^{p/2} \|u_x(s)\|^{p/2}. \qquad (3.27)
\end{aligned}
$$

We have finished the proof of this lemma.

From (3.4) with (3.19) and (3.20), and (3.12) with (3.21) and (3.22), respectively, we can easily show the following lemma.

Lemma 3.7. *There hold that*

$$\|u(t)\| \le C\|u_0\|_{W^{2,1}}(1+t)^{-1/4} + C\int_0^t (1+t-s)^{-1/4} \|u(s)\|^{p/2} \|u_x(s)\|^{p/2} ds, \quad (3.28)$$

$$\|u_x(t)\| \le C\|u_0\|_{W^{3,1}}(1+t)^{-3/4} + C\int_0^t (1+t-s)^{-3/4} \|u(s)\|^{p/2} \|u_x(s)\|^{p/2} ds. \quad (3.29)$$

We prove (2.6) in Theorem 2.4 also by means of the continuity argument based on the local existence with the following *a priori* estimates.

Lemma 3.8. *Suppose that $p > 2$, $u_0(x) \in W^{3,1} \cap H^4$ and $u(x,t)$ is a local solution of (1.1) and (1.2) for $t \in [0,T]$, where $T > 0$ is a any given constant. Let*

$$M_2(t) = \sup_{0\le s\le t} \{(1+s)^{1/4}\|u(s)\| + (1+s)^{3/4}\|u_x(s)\|\},$$

then there exist positive constants C and δ_2, such that when $M_2(T) < \delta_2$, then

$$\|u(t)\| \le C(1+t)^{-1/4}, \quad \|u_x(t)\| \le C(1+t)^{-3/4}, \quad (3.30)$$

for $t \in [0, T]$.

Proof. By (3.28) and (3.29) in Lemma 3.6, and by Schwartz's inequality and Lemma 2.1, we have

$$
\begin{aligned}
&(1+t)^{1/4}\|u(t)\| + (1+t)^{3/4}\|u_x(t)\| \\
&\le C\Big\{\|u_0\|_{W^{3,1}} + (1+t)^{1/4}\int_0^t (1+t-s)^{-1/4}\|u(s)\|^{p/2}\|u_x(s)\|^{p/2}ds \\
&\quad + (1+t)^{3/4}\int_0^t (1+t-s)^{-3/4}\|u(s)\|^{p/2}\|u_x(s)\|^{p/2}ds\Big\} \\
&= C\Big\{\|u_0\|_{W^{3,1}} + (1+t)^{1/4}\int_0^t (1+t-s)^{-1/4}(1+s)^{-p/2} \\
&\quad \cdot \Big((1+s)^{1/4}\|u(s)\|\Big)^{p/2}\Big((1+s)^{3/4}\|u_x(s)\|\Big)^{p/2}ds \\
&\quad + (1+t)^{3/4}\int_0^t (1+t-s)^{-3/4}(1+s)^{-p/2} \\
&\quad \cdot \Big((1+s)^{1/4}\|u(s)\|\Big)^{p/2}\Big((1+s)^{3/4}\|u_x(s)\|\Big)^{p/2}ds\Big\} \\
&\le C\Big\{\|u_0\|_{W^{3,1}} + (1+t)^{1/4}\int_0^t (1+t-s)^{-1/4}(1+s)^{-p/2} \\
&\quad \cdot \Big[\Big((1+s)^{1/4}\|u(s)\|\Big)^{p} + \Big((1+s)^{3/4}\|u_x(s)\|\Big)^{p}\Big]ds \\
&\quad + (1+t)^{3/4}\int_0^t (1+t-s)^{-3/4}(1+s)^{-p/2} \\
&\quad \cdot \Big[\Big((1+s)^{1/4}\|u(s)\|\Big)^{p} + \Big((1+s)^{3/4}\|u_x(s)\|\Big)^{p}\Big]ds\Big\} \\
&\le C\Big\{\|u_0\|_{W^{3,1}} + (1+t)^{1/4}M_2(t)^p\int_0^t (1+t-s)^{-1/4}(1+s)^{-p/2}ds \\
&\quad + (1+t)^{3/4}M_2(t)^p\int_0^t (1+t-s)^{-3/4}(1+s)^{-p/2}ds \\
&\le C\|u_0\|_{W^{3,1}} + CM_2(t)^p.
\end{aligned} \tag{3.31}
$$

We then have (3.30) for some small $M_2(t) < \delta_2 < 1$. Here, the proof of Lemma 3.8 is complete.

Based on the Sobolev's inequality, both of two estimates of (2.6) can reduce (2.7). Therefore, up to now, we have proved Theorem 2.4 by above lemmas. The next is to prove Theorem 2.5.

The Proof of Theorem 2.5. Making Fourier transform to (2.8), we have

$$\hat{v}(\xi,t) = e^{-A(\xi)t}\hat{u}_0(\xi). \tag{3.32}$$

Then the inverse Fourier transform to be applied to (3.32) yields

$$v(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)t}\hat{u}_0(\xi)d\xi, \tag{3.33}$$

$$v_x(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)t}\hat{u}_0(\xi)d\xi. \tag{3.34}$$

Let (3.4) minus (3.32) and (3.12) minus (3.33), we have

$$u(x,t) - v(x,t) = -\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi ds, \tag{3.35}$$

$$u_x(x,t) - v_x(x,t) = -\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi ds. \tag{3.36}$$

When $p > 7/2$ and $u_0(x) \in W^{4,1}\cap H^4$, using (3.7),(3.15), (2.5) and Lemma 2.1, we can prove (2.10)

$$\begin{aligned}
|u(x,t) - v(x,t)| &\le \frac{1}{2\pi}\int_0^t \left|\int_{-\infty}^{\infty} e^{-B(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\right| ds \\
&\le C\int_0^t \|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty(1+t-s)^{-1/2}ds \\
&\le C\int_0^t (1+s)^{-(p-3)/2}(1+s)^{-1}(1+t-s)^{-1/2}ds \\
&\le C(1+t)^{-1/2},
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
|u_x(x,t) - v_x(x,t)| &\le \frac{1}{2\pi}\int_0^t \left|\int_{-\infty}^{\infty} i\xi e^{-B(\xi)(t-s)}\frac{\widehat{\phi(u)_x}}{1+\xi^4}d\xi\right| ds \\
&\le C\int_0^t \|u(s)\|_\infty^{p-3}\|u_x(s)\|_\infty(1+t-s)^{-1}ds \\
&\le C\int_0^t (1+s)^{-(p-3)/2}(1+s)^{-1}(1+t-s)^{-1}ds \\
&\le C(1+t)^{-1}.
\end{aligned} \tag{3.38}$$

When $p>2$ and $u_0(x)\in W^{3,1}\cap H^4$, using (3.20),(3.22), (2.6) and Lemma 2.1, we can prove (2.11)

$$\begin{aligned}
&\|u(x,t)-v(x,t)\| \le \frac{1}{2\pi}\int_0^t \Big\| \int_{-\infty}^{\infty} e^{-B(\xi)(t-s)} \frac{\widehat{\phi(u)_x}}{1+\xi^4} d\xi \Big\| ds \\
&\le C\int_0^t \|u(s)\|^{p/2}\|u_x(s)\|^{p/2}(1+t-s)^{-1/4}ds \\
&\le C\int_0^t (1+s)^{-p/8}(1+s)^{-3p/8}(1+t-s)^{-1/4}ds \\
&\le C(1+t)^{-1/4}, \qquad (3.39)
\end{aligned}$$

$$\begin{aligned}
&\|u_x(x,t)-v_x(x,t)\| \le \frac{1}{2\pi}\int_0^t \Big\| \int_{-\infty}^{\infty} i\xi e^{-B(\xi)(t-s)} \frac{\widehat{\phi(u)_x}}{1+\xi^4} d\xi \Big\| ds \\
&\le C\int_0^t \|u(s)\|^{p/2}\|u_x(s)\|^{p/2}(1+t-s)^{-3/4}ds \\
&\le C\int_0^t (1+s)^{-p/8}(1+s)^{-3p/8}(1+t-s)^{-3/4}ds \\
&\le C(1+t)^{-3/4}. \qquad (3.40)
\end{aligned}$$

This completes the proof of (2.11). By (2.11), and using the Sobolev's inequality, we can get (2.12).

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