

Revisiting the coherence theory $\llcorner$ of bicategories and tricategories

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§1. Introduction

Mac Lane 1963 & [CWM] 1971:

monoidal category = tensor category

$(\mathbb{C}; I, \otimes, \alpha, \lambda, \rho)$

$\uparrow$
tensor $\uparrow$

(etc)

subject to: pentagon, etc associativity iso

Strict tensor category

$(\mathbb{C}; I, \otimes)$

special case: α, λ, ρ : identities

Let: S : a set of 'objects' $\underline{L^2}$

$\mathcal{F}_{\text{ten}}(S)$: the free tensor

category on S

(objects of $\mathcal{F}_{\text{ten}}(S)$:

example: $(X \otimes (Y \otimes X)) \otimes Y$

$(X, Y, \dots \in S) \quad \dots \quad)$

$\mathcal{F}_{\text{sten}}(S)$: the free strict tensor
category on S

(objects of $\mathcal{F}_{\text{sten}}(S)$:

$X_1 \otimes X_2 \otimes \dots \otimes X_n \quad (n \geq 0)$

$X_k \in S)$

$\Rightarrow \mathcal{F}_{\text{sten}}(S)$ is (obviously) a discrete
category

Mac Lane coherence theorem:

[3]

$$P: \mathcal{F}_{ten}(S) \xrightarrow{\text{can.}} \mathcal{F}_{sten}(S)$$

is an equivalence of categories:

in $\mathcal{F}_{ten}(S)$, all diagrams commute.

Joyal & Street coherence theorem
(1993):

For any category $\mathcal{S}$

$$P: \mathcal{F}_{ten}(\mathcal{S}) \xrightarrow{\text{can.}} \mathcal{F}_{sten}(\mathcal{S})$$

is an equivalence of categories.

Tensor category = one-object bicategory [4]

Strict tensor category = one-object 2-category

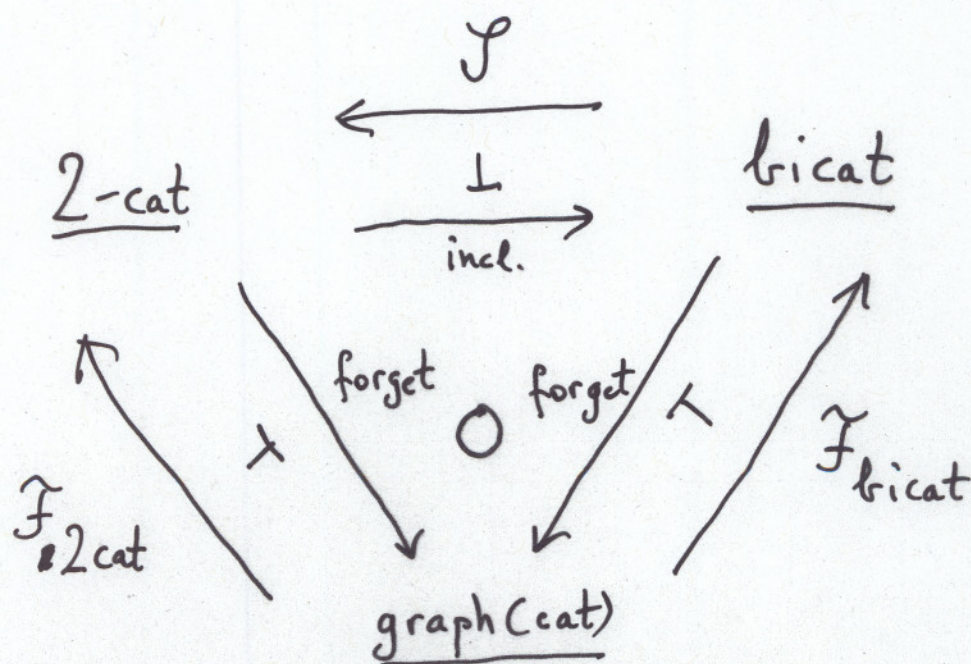
Nick Gurski [Chicago thesis 2007]:

$$\Gamma: \mathcal{F}_{\text{bicat}}(G) \xrightarrow{\text{can.}} \mathcal{F}_{\text{2-cat}}(G)$$

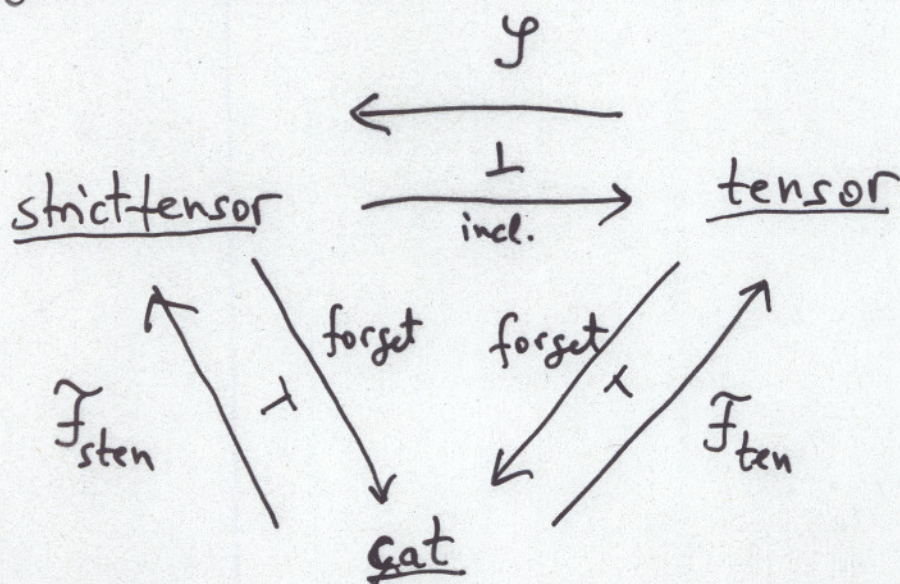
is a biequivalence

for any category-enriched graph G

(straight forward after Joyal/Street;
for Gurski, this is introduction
to tricategories)



One-object case:



Adjoints compose:

$$\gamma \circ \tilde{\gamma} \cong \tilde{\gamma}_s$$

The Joyal / Street / Gurski morphism

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$$\mathcal{F}(G) \xrightarrow{\quad \eta \quad} \mathcal{F}_s(G)$$

is the same as

$$\mathcal{F}(G) \xrightarrow{\quad \eta_{\mathcal{F}(G)} \quad} \mathcal{I}(\mathcal{F}(G))$$

with the unit η for $\mathcal{I} \dashv \text{incl.}$

Steve Lack's coherence theorem (2004):

$$\mathbb{X} \xrightarrow{\quad \eta_{\mathbb{X}} \quad} \mathcal{I}(\mathbb{X})$$

is a biequivalence

for any bicategory $\mathbb{X}$ that is 1-free:

the underlying 1-magma (compositional graph) is a free 1-magma (on a graph)

$\Rightarrow \mathcal{F}(G)$ obviously is 1-free: Lack's is stronger.

All approaches (here) are
based on the Yoneda lemma
for bicategories (Street 1980)

Corollary (John Power)

(Weak coherence theorem)

of Yoneda

Every bicategory is biequivalent
to a 2-category.

§2 The straight graph of a
functor of bicategories

A straight map $F: \mathbb{X} \xrightarrow{\square} \mathbb{A}$
(= strict
in $\dim = 2$)

is a strong equivalence if

1) $F: \mathbb{X}_0 \rightarrow \mathbb{A}_0$ is surjective

2) for all $X, Y \in \mathbb{X}_0$:

$F: \text{hom}_{\mathbb{X}_0}(X, Y) \longrightarrow \text{hom}_{\mathbb{A}_0}(FX, FY)$
is surjective

3) for all $X \overset{x}{\underset{y}{\rightrightarrows}} Y \in \mathbb{X}_1$:

$F: \text{hom}_{\mathbb{X}_0}^{(2)}(x, y) \longrightarrow \text{hom}_{\mathbb{A}_0}^{(2)}(Fx, Fy)$

3a)

is surjective

and

3b)

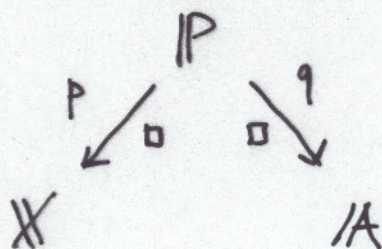
is injective

$\Rightarrow$ 3): ... is bijective

Proposition ($\mathbb{X}, \mathbb{A} : \text{bicat's}$)

$\mathbb{X} \simeq \mathbb{A} \iff \text{there exist :}$

$\uparrow$
bivalent



P, Q (straight) strong equivalences.

$\uparrow$
'If' : obvious

'Only if' : by a construction

Note: $[\mathbb{X}, \mathbb{A}]$ is functorial:

$$\underline{\text{bicat}}^{\circ P} \times \underline{\text{bicat}} \xrightarrow{[-, -]} \underline{\text{bicat}}$$

Actions of (straight) arrows: pointwise.

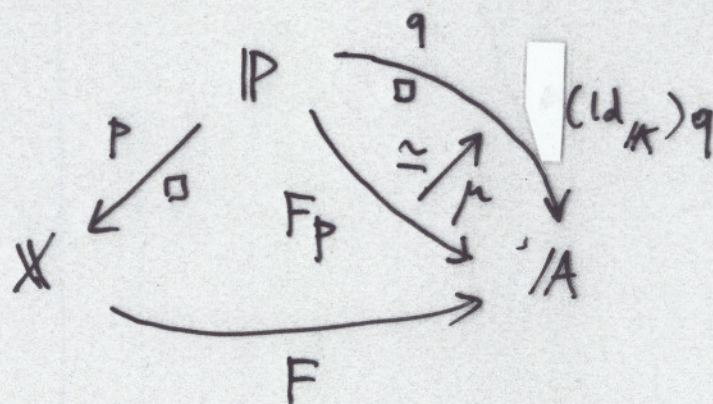
$$\Rightarrow \text{Fix } F: \mathbb{X} \longrightarrow \mathbb{A} \quad \underline{10}$$

functor (homomorphism) of bicategories

Definition A straight span on F

with vertex IP (a bicat) is:

$$\mathcal{P} = (IP; p, q, \mu) :$$



$$\begin{aligned} p: IP &\xrightarrow{\square} X && \in \underline{\text{bicat}}_1 \\ q: IP &\xrightarrow{\square} A && \in \underline{\text{bicat}}_1 \end{aligned}$$

$$F_p = [p, A](F)$$

$$\Rightarrow [[p, A]: [X, A] \longrightarrow [IP, A]]$$

$$(Id_A)q = [q, A](Id_A)$$

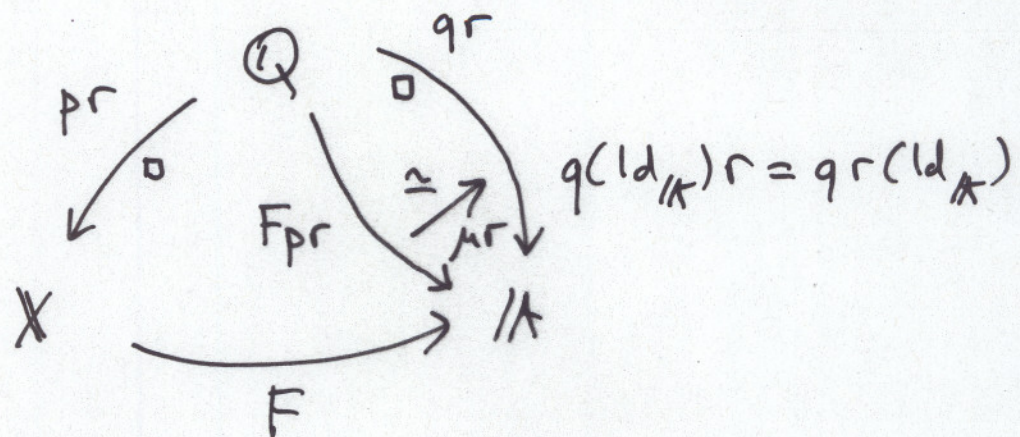
$$\mu: F_p \xrightarrow{\cong} (Id_A)q$$

↑
equivalence 1-cell in the bicat $[IP, A]$

Note: Any straight map $r: \mathcal{Q} \xrightarrow{\square} \mathbb{P}$ $\sqsubseteq$

gives rise to $\mathcal{P} \mapsto \mathcal{P}r =$

$$\mathcal{P}r = (\mathcal{Q}; pr, qr, \mu_r):$$



Get

functor

$$\text{bicat}^{\text{op}} \xrightarrow{\text{StSp}} \text{Set}$$

$$\begin{array}{ccc}
 \mathbb{P} & \mapsto & \{ \mathcal{P} = (\mathbb{P}, \dots) \} \\
 \uparrow r & & \downarrow \text{StSp}(r) \\
 \mathcal{Q} & \mapsto & \{ \mathcal{Q} = (\mathcal{Q}, \dots) \}
 \end{array}$$

$\begin{array}{c} \mathcal{P} \\ \hline \downarrow \\ \mathcal{P}r \end{array}$

Theorem (1) $St Sp$ is representable 12

The representing object is called
the straight graph of F

$\Rightarrow$ (let $\mathcal{P} = (IP, p, q, \mu)$ be
the straight graph)

(2) The left leg p of $\mathcal{P}$ is
a strong equivalence

(3) The right leg q of $\mathcal{P}$ is
a strong equivalence iff

$F: X \rightarrow A$ is a biequivalence

$\Rightarrow$ Proposition above (on 'span-form' of
equivalence) is a corollary.

$\Rightarrow$ The weak coherence theorem for bicats 13
 is an 'immediate' consequence of the
 strong coherence theorem
 (= Steve Lack's thm)

as follows:

(forget) $\underbrace{\text{Bicat}}_{\varphi \downarrow}$ is a $\downarrow$ Grothendieck fibration
 $\underbrace{\text{1-magma}}$

$\Rightarrow$ Let: $\mathbb{X}$ be an arbitrary bicat;

$\mathcal{U} \xrightarrow{f} \mathbb{X} \Pi$ a map from a free

1-magma $\mathcal{U}$ suggestive on objects & full

($\mathcal{U}$ = free on underlying graph of $\mathbb{X} \Pi$).

For

$$\begin{array}{ccc}
 \mathcal{Y} = f^*(\mathcal{U}) & \xrightarrow{g = cf} & \mathbb{X} \\
 \varphi \downarrow & & \downarrow \varphi \\
 \mathcal{U} & \xrightarrow{f} & \mathbb{X} \Pi
 \end{array}$$

$g: \mathcal{Y} \rightarrow \mathbb{X}$ is a strong equivalence (easy);

$\mathcal{Y}$ is 1-free; $\mathcal{Y} \simeq /A$: $\mathbb{X} \xleftarrow{g} \mathcal{Y} \xleftarrow{f} /P \xrightarrow{q} /A$.
 $\uparrow$ 2-cat $\therefore \mathbb{X} \simeq /A$

$\Rightarrow$ For the converse:

(weak coherence $\Rightarrow$ strong coherence):

Steve Lack's observation:

For any bicat $\mathbb{X}$,

$$\mathbb{X} \xrightarrow{\eta_{\mathbb{X}}} \mathcal{Y}(\mathbb{X})$$

has the surjectivity properties

1), 2) and 3a)

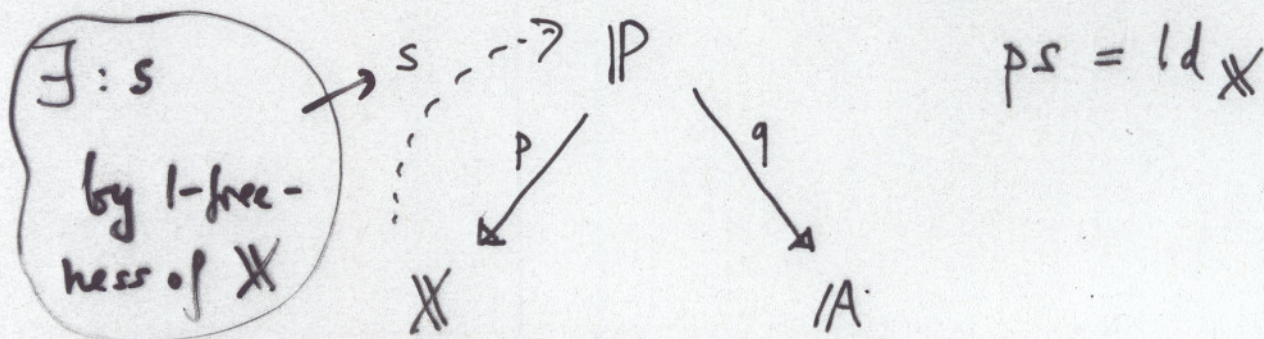
(but not in general the injectivity property

3b): 2-faithfulness)

$\Rightarrow$ Assume: $\mathbb{X}$ is 1-free; get

by Proposition & Weak coherence:

2-cat $\mathcal{A}$, strong equiv's P, Q :



As a section, s has

property 3b) (2-faithful).

As a strong equivalence, q has
property 3b).

$$\therefore qs : X \longrightarrow A$$

is 2-faithful

$\Rightarrow$ Use the universal property of

$(\mathcal{Y}(X), \eta_X)$ against qs

& conclude that η_X is 2-faithful

— . —

§3 Tricategories and Gray categories 16

$$\begin{array}{ccc}
 & \xleftarrow{\gamma_{\text{Gray}}} & \\
 \text{Gray-cat} & \xrightleftharpoons[\text{incl.}]{\perp} & \text{tricat}
 \end{array}$$

X : tricategory

$$X \xrightarrow{\eta_X} \gamma_{\text{Gray}}(X)$$

Theorem For any X that is 2-free,
 η_X is a strong equivalence

Proof based on: [1] weak coherence : Every tricategory
 is equivalent to a Gray category (Gordon/Power/Street)
 1995

[2] "Steve Lack's observation" and

[3] the Proposition (based on the Straight Graph
 theorem)
 generalize to $\dim = 3$

GURSKI has, in his thesis, the form

Gurski coherence for tricats:

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for a category-enriched 2-graph G
the canonical

$$P: \mathcal{T}_{\text{tricat}}(G) \longrightarrow \mathcal{T}_{\text{Gray}}(G)$$



$\eta^{(G)}: \mathcal{T}_{\text{Gray}}(G) \hookrightarrow \mathcal{T}_{\text{tricat}}(G)$

is a strong equivalence of
tricategories

$\Rightarrow$ Follows from above Theorem