Revisiting the coherence theory of bicategories and tricategories

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§1. Introduction

Mac Lane 1963 & [CWM] 1971:

monoidal category = tensor category

$$(\mathbb{C};\ I, \otimes, \alpha, \lambda, \rho)$$

tensor

(etc)

subject to: pentagon, etc   associativity iso

Strict tensor category

$$(\mathbb{C};\ I, \otimes)$$

special case: α, λ, ρ : identities

Let: S : a set of 'objects'

$\mathcal{F}_{ten}(S)$: the free tensor category on S

(objects of $\mathcal{F}_{ten}(S)$:

example: $(X \otimes (Y \otimes X)) \otimes Y$

$(X, Y, \dots \in S)$ ...)

$\mathcal{F}_{sten}(S)$: the free strict tensor category on S

(objects of $\mathcal{F}_{sten}(S)$:

$X_1 \otimes X_2 \otimes \cdots \otimes X_n$ $(n \geq 0)$

$X_k \in S$)

$\Rightarrow$ $\mathcal{F}_{sten}(S)$ is (obviously) a <u>discrete</u> category

Mac Lane coherence theorem:

$$\Gamma : \mathcal{F}_{ten}(S) \xrightarrow[\text{can.}]{} \tilde{\mathcal{F}}_{sten}(S)$$

is an equivalence of categories:

in $\tilde{\mathcal{F}}_{ten}(S)$, all diagrams commute.

Joyal & Street coherence theorem (1993):

For any <u>category</u> $\mathbb{S}$

$$\Gamma : \tilde{\mathcal{F}}_{ten}(\mathbb{S}) \xrightarrow[\text{can.}]{} \mathcal{F}_{sten}(\mathbb{S})$$

is an equivalence of ccategories.

Tensor category = one-object bicategory

Strict tensor category = one-object 2-category

Nick Gurski [Chicago thesis 2007]:

$$\Gamma : \tilde{\mathcal{F}}_{\text{bicat}}(G) \xrightarrow[\text{can.}]{} \tilde{\mathcal{F}}_{\text{2-cat}}(G)$$

is a biequivalence

for any category-enriched graph G

(<u>straightforward</u> after Joyal/Street; for Gurski, this is introduction to tricategories)

$$\begin{tikzcd}[column sep=large]
\underline{2\text{-cat}} \arrow[rr, shift right=1.5, "\text{incl.}"'] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left=1.5, "\text{forget}"] & & \underline{\text{bicat}} \arrow[ll, shift right=1.5, "\mathcal{S}"'] \arrow[dl, shift right=1.5, "\text{forget}"'] \\
& \underline{\text{graph(cat)}} \arrow[ul, shift left=1.5, "\mathcal{F}_{2cat}"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift right=1.5, "\mathcal{F}_{bicat}"'] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

One-object case:

$$\begin{tikzcd}[column sep=large]
\underline{\text{strict-tensor}} \arrow[rr, shift right=1.5, "\text{incl.}"'] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left=1.5, "\text{forget}"] & & \underline{\text{tensor}} \arrow[ll, shift right=1.5, "\mathcal{S}"'] \arrow[dl, shift right=1.5, "\text{forget}"'] \\
& \underline{\text{Cat}} \arrow[ul, shift left=1.5, "\mathcal{F}_{sten}"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift right=1.5, "\mathcal{F}_{ten}"'] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

Adjoints compose: $\mathcal{S} \circ \tilde{\mathcal{F}} \cong \mathcal{F}_s$

The Joyal / Street / Gurski morphism

$$\mathcal{F}(G) \xrightarrow{\Gamma} \mathcal{F}_s(G)$$

is the same as

$$\mathcal{F}(G) \xrightarrow[\eta_{\mathcal{F}(G)}]{} \mathcal{S}(\mathcal{F}(G))$$

with the unit η for $\mathcal{S} \dashv \text{incl.}$

Steve Lack's coherence theorem (2004):

$$\mathbb{X} \xrightarrow[\eta_{\mathbb{X}}]{} \mathcal{S}(\mathbb{X})$$

is a biequivalence
for any bicategory $\mathbb{X}$ that is 1-free:
the underlying 1-magma (compositional
graph) is a free 1-magma (on a graph)

$\Rightarrow$ $\mathcal{F}(G)$ obviously is 1-free: Lack's is <u>stronger</u>.

All approaches (here) are based on the Yoneda lemma for bicategories (Street 1980)

Corollary (John Power) (Weak coherence theorem)

of Yoneda

Every bicategory is biequivalent to a 2-category.

§2) The straight graph of a functor of bicategories

A straight map $F : \mathbb{X} \xrightarrow{\square} \mathbb{A}$
(= strict in dim = 2)
is a strong equivalence if

1) $F : \mathbb{X}_0 \to \mathbb{A}_0$ is surjective

2) for all $X, Y \in \mathbb{X}_0$:
$$F : \hom_{\mathbb{X}_0}(X, Y) \longrightarrow \hom_{\mathbb{A}_0}(FX, FY)$$
is surjective

3) for all $X \underset{y}{\overset{x}{\rightrightarrows}} Y \in \mathbb{X}_1$:
$$F : \hom^{(2)}_{\mathbb{X}_0}(x, y) \longrightarrow \hom^{(2)}_{\mathbb{A}_0}(Fx, Fy)$$

3a) is surjective

and

3b) is injective

$\Rightarrow$ 3): … is bijective

$\boxed{\text{Proposition}}$ ($\mathbb{X}$, $\mathbb{A}$: bicat's)

$\mathbb{X} \simeq \mathbb{A}$ $\Longleftrightarrow$ there exist :

$\uparrow$
biequivalent

$$\begin{tikzcd} & \mathbb{P} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{X} & & \mathbb{A} \end{tikzcd}$$

p, q (straight) strong equivalences.

'If' : obvious

'Only if' : by a construction

Note: $[\mathbb{X}, \mathbb{A}]$ is functorial :

$$\underline{\text{bicat}}^{op} \times \underline{\text{bicat}} \xrightarrow[{[-,-]}]{} \underline{\text{bicat}}$$

Actions of (straight) arrows: pointwise.

$\Rightarrow$ Fix $F : \mathbb{X} \longrightarrow \mathbb{A}$

functor (homomorphism) of bicategories

Definition A straight span on F

with vertex $\mathbb{P}$ (a bicat) is:

$\mathcal{P} = (\mathbb{P};\ p, q, \mu)$:

$$\begin{tikzcd}
 & \mathbb{P} \arrow[dl, "p"', "\square"] \arrow[dr, "F_p"'] \arrow[dr, bend left=50, "q", "\square"', ""{name=Q}] & \\
\mathbb{X} \arrow[rr, bend right, "F"'] & & \mathbb{A}
\arrow[from=2-3, to=1-2, phantom, shift left=3, "\overset{\simeq}{\Nearrow}\mu"]
\end{tikzcd}$$

$(\mathrm{Id}_{\mathbb{A}})q$

$p : \mathbb{P} \xrightarrow{\square} \mathbb{X} \quad \in \underline{\text{bicat}}_1$

$q : \mathbb{P} \xrightarrow{\square} \mathbb{A} \quad \in \underline{\text{bicat}}_1$

$F_p = [p, \mathbb{A}](F)$

$\Rightarrow \Big[\quad [p, \mathbb{A}] : [\mathbb{X}, \mathbb{A}] \longrightarrow [\mathbb{P}, \mathbb{A}] \Big]$

$(\mathrm{Id}_{\mathbb{A}})q = [q, \mathbb{A}](\mathrm{Id}_{\mathbb{A}})$

$\mu : F_p \xrightarrow{\simeq} (\mathrm{Id}_{\mathbb{A}})q$

equivalence 1-cell in the bicat $[\mathbb{P}, \mathbb{A}]$

<u>Note:</u> Any straight map $r : \mathbb{Q} \xrightarrow{\square} \mathbb{P}$

gives rise to $\mathcal{P} \longmapsto \mathcal{P}r =$

$$\mathcal{P}r = (\mathbb{Q};\ pr, qr, \mu r):$$

$$\begin{tikzcd}
 & \mathbb{Q} \arrow[dl, bend right, "pr"', "\square"] \arrow[dr, "Fpr"'] \arrow[dr, bend left=50, "qr", "\square"'] \arrow[dr, phantom, bend left=25, "\overset{\simeq}{\nearrow}\,\mu r"] & \\
\mathbb{X} \arrow[rr, bend right, "F"'] & & \mathbb{A}
\end{tikzcd}$$

$q(\mathrm{Id}_{\mathbb{A}})r = qr(\mathrm{Id}_{\mathbb{A}})$

Get functor

$$\mathrm{bicat}^{op} \xrightarrow{StSp} Set$$

$$\begin{tikzcd}
\mathbb{P} \arrow[r, mapsto] & \{\ \mathcal{P} = (\mathbb{P}, \dots)\ \} \arrow[d, "StSp(r)"'] & \mathcal{P} \arrow[d, mapsto] \\
\mathbb{Q} \arrow[u, "r"] \arrow[r, mapsto] & \{\ \underset{\sim}{Q} = (\mathbb{Q}, \dots) & \mathcal{P}r
\end{tikzcd}$$

Theorem (1) $St\,S_P$ is representable

The representing object is called the straight graph of F

$\Rightarrow$ (let $\mathcal{P} = (\mathbb{P}, p, q, \mu)$ be the straight graph)

(2) The left leg p of $\mathcal{P}$ is a strong equivalence

(3) The right leg q of $\mathcal{P}$ is a strong equivalence iff

$F : \mathbb{X} \to \mathbb{A}$ is a biequivalence

—·—

$\Rightarrow$ $\boxed{\text{Proposition}}$ above (on 'span-form' of equivalence) is a corollary.

$\Rightarrow$ The weak coherence theorem for bicats is an 'immediate' consequence of the strong coherence theorem

(= Steve Lack's thm)

as follows:

$$\begin{tikzcd} \underline{\mathrm{Bicat}} \arrow[d, "\varphi"'] \\ \underline{1\text{-magma}} \end{tikzcd}$$

(forget) is a Grothendieck fibration

$\Rightarrow$ Let: $\mathbb{X}$ be an arbitrary bicat;

$\mathbb{M} \xrightarrow{f} \mathbb{X}\restriction 1$ a map from a free 1-magma $\mathbb{M}$ surjective on objects & full

($\mathbb{M}$ = free on underlying graph of $\mathbb{X}\restriction 1$.

For

$$\begin{tikzcd} \mathbb{Y} = f^*(\mathbb{M}) \arrow[r, "g = {}^{c}f"] \arrow[d, "\varphi"'] & \mathbb{X} \arrow[d, "\varphi"] \\ \mathbb{M} \arrow[r, "f"'] & \mathbb{X}\restriction 1 \end{tikzcd}$$

$g : \mathbb{Y} \to \mathbb{X}$ is a strong equivalence (easy);

$\mathbb{Y}$ is 1-free; $\mathbb{Y} \simeq \mathbb{A}$: $\mathbb{X} \xleftarrow{g} \mathbb{Y} \xleftarrow{p} \mathbb{P} \xrightarrow{q} \mathbb{A}$.

2-cat

$\boxed{\therefore\ \mathbb{X} \simeq \mathbb{A}}$

$\Rightarrow$ For the converse:

(weak coherence $\Rightarrow$ strong coherence):

Steve Lack's observation:

For any bicat $\mathbb{X}$,

$$\mathbb{X} \xrightarrow[\eta_{\mathbb{X}}]{} \mathcal{S}(\mathbb{X})$$

has the surjectivity properties

1), 2) and 3a)

(but not in general the injectivity property

3b): 2-faithfulness)

$\Rightarrow$ Assume: $\mathbb{X}$ is 1-free; get

by $\boxed{\text{Proposition}}$ & Weak coherence:

2-cat $\mathbb{A}$, strong equiv's p, q:

∃: s by 1-freeness of $\mathbb{X}$

$$\begin{tikzcd} & \mathbb{P} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{X} \arrow[ur, dashed, bend left, "s"] & & \mathbb{A} \end{tikzcd} \qquad ps = \mathrm{Id}_{\mathbb{X}}$$

As a section, s has property 3b) (2-faithful).

As a strong equivalence, q has property 3b).

$\therefore$ $qs : \mathbb{X} \longrightarrow \mathbb{A}$ is 2-faithful

$\Rightarrow$ Use the universal property of $(\mathcal{S}(\mathbb{X}), \eta_{\mathbb{X}})$ against qs & conclude that $\eta_{\mathbb{X}}$ is 2-faithful

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§3 Tricategories and Gray categories

$$\begin{tikzcd} \text{Gray-cat} \arrow[r, bend right, "\text{incl.}"'] \arrow[r, phantom, "\perp"] & \text{tricat} \arrow[l, bend right, "\mathcal{G}_{Gray}"'] \end{tikzcd}$$

$\mathbb{X}$: tricategory

$$\mathbb{X} \xrightarrow[\eta_{\mathbb{X}}]{} \mathcal{G}_{Gray}(\mathbb{X})$$

Theorem For any $\mathbb{X}$ that is 2-free, $\eta_{\mathbb{X}}$ is a strong equivalence

Proof based on: $\boxed{1)}$ weak coherence : Every tricategory is equivalent to a Gray category (Gordon / Power / Street) 1995

$\boxed{2)}$ "Steve Lack's observation" and

$\boxed{3)}$ the $\boxed{\text{Proposition}}$ (based on the Straight graph theorem) generalize to dim = 3

GURSKI has, in his thesis, the form

Gurski coherence for tricats:

for a category-enriched 2-graph G

the canonical

$$\Gamma : \tilde{\mathcal{F}}_{tricat}(G) \longrightarrow \tilde{\mathcal{F}}_{Gray}(G)$$

$$\eta^{(\mathcal{F})}\, \mathcal{F}_{\cancel{Gray}\ tricat}(G)$$

is a strong equivalence of tricategories

$\Rightarrow$ Follows from above Theorem