Revisiting the coherence theory of bicategories and tricategories

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§1. Introduction

Mac Lane 1963 & [CWM] 1971:

monoidal category = tensor category

\[(C; I, \otimes, \alpha, \lambda, s)\]

\[
\begin{array}{c}
\uparrow \\
\text{tensor} \\
\end{array}
\]

(subject to: pentagon, etc. associativity iso)

Strict tensor category

\[(C; I, \otimes)\]

special case: \(\alpha, \lambda, s\) : identities
Let: $S$: a set of objects

$\mathcal{F}(S):$ the free tensor category on $S$

(objects of $\mathcal{F}_{\text{ten}}(S)$):

example: $(X \otimes (Y \otimes X)) \otimes Y$

$(X, Y, \ldots \in S)$

$\mathcal{F}_{\text{sten}}(S):$ the free strict tensor category on $S$

(objects of $\mathcal{F}_{\text{sten}}(S)$):

$X_1 \otimes X_2 \otimes \cdots \otimes X_n$ ($n \geq 0$)

$X_n \in S$

$\Rightarrow \mathcal{F}_{\text{sten}}(S)$ is (obviously) a discrete category
Mac Lane coherence theorem:

\[ P: \mathcal{F}_{\text{ten}}(S) \longrightarrow \mathcal{F}_{\text{sten}}(S) \]

can.

is an equivalence of categories:
in $\mathcal{F}_{\text{ten}}(S)$, all diagrams commute.

Joyal & Street coherence theorem (1993):

For any category $S$

\[ P: \mathcal{F}_{\text{ten}}(S) \longrightarrow \mathcal{F}_{\text{sten}}(S) \]

can.

is an equivalence of categories.
Tensor category = one-object bicategory
Strict tensor category = one-object 2-category

Nick Gurski [Chicago thesis 2007]:

\( \Pi : \mathcal{F}_{\text{fricat}} (G) \overset{\text{can.}}{\longrightarrow} \mathcal{F}_{\text{2-cat}} (G) \)

is a biequivalence

for any category-enriched graph \( G \)

(straightforward after Joyal/Street;
for Gurski, this is introduction to tricategories)
One-object case:

\[ \text{strict-tensor} \xleftarrow{\text{incl.}} \mathcal{F} \xrightarrow{\text{forget}} \mathcal{C} \xleftarrow{\text{incl.}} \text{tensor} \]

Adjoints compose: \( \eta \circ F = F_s \)
The Joyal/Street/Gurski morphism is the same as

\[ F(G) \xrightarrow{P} \mathcal{F}(G) \]

with the unit \( \eta \) for \( \mathcal{F} + 1 \) incl.

Steve Lack's coherence theorem (2004):

\[ X \xrightarrow{\eta_X} \mathcal{F}(X) \]

is a biequivalence for any bicategory \( X \) that is 1-free: the underlying 1-magma (compositional graph) is a free 1-magma (on a graph) \( \Rightarrow F(G) \) obviously is 1-free: Lack's is stronger.
All approaches (here) are based on the Yoneda lemma for bicategories (Street 1980).

Corollary (John Power)
- (Weak coherence theorem)

Every bicategory is biequivalent to a 2-category.
The straight graph of a functor of bicategories

A straight map $F : \mathbf{X} \to \mathbf{A}$

($= \text{shift in dim } 2$)

is a strong equivalence if

1) $F : \mathbf{X}_0 \to \mathbf{A}_0$ is surjective

2) for all $X, Y \in \mathbf{X}_0$:
   $F : \text{hom}_{\mathbf{X}_0}(X, Y) \to \text{hom}_{\mathbf{A}_0}(FX, FY)$
   is surjective

3) for all $X \xRightarrow{y} Y \in \mathbf{X}_1$:
   $F : \text{hom}^{(2)}_{\mathbf{X}_0}(x, y) \to \text{hom}^{(2)}_{\mathbf{A}_0}(FX, FY)$
   is surjective
   and
   is injective

$\Rightarrow 3)$: ... is bijective
Proposition \((X, A : \text{bicat's})\)

\[ X \cong A \iff \text{there exist:} \]

\[ \begin{array}{ccc}
\exists & p & q \\
\downarrow & \downarrow & \downarrow \\
X & \cong & A
\end{array} \]

\[
p > q \quad \text{(straight) strong equivalences.} \]

\[
\text{If}: \quad \text{obvious} \\
\text{Only if}: \quad \text{by a construction}
\]

Note: \([X, A]\) is functorial:

\[
\text{bicat}^p \times \text{bicat} \rightarrow \text{bicat} \\
[-, -]
\]

Actions of (straight) arrows: pointwise.
\[ \Rightarrow \text{Fix } F : X \rightarrow A \]

functor (homomorphism) of bicategories

**Definition**

A straight span on \( F \) with vertex \( IP \) (a bicat) is:

\[ \mathcal{P} = (IP; p, q, \lambda) : IP \rightarrow X \in \text{bicat}_1 \rightarrow IP \rightarrow F \rightarrow A \in \text{bicat}_1 \]

\[ F_p = [p, A](F) \]

\[ \Rightarrow [p, A] : [X, A] \rightarrow [IP, A] \]

\[ (\text{Id}_A)q = [q, A](\text{Id}_A) \]

\[ \mu : F_p \xrightarrow{\sim} (\text{Id}_A)q \quad \text{equivalence 1-cell in the bicat } [IP, A] \]
Note: Any straight map \( r: Q \to P \) gives rise to \( P \mapsto Pr = (Q; pr, qs, mr) \):

\[
\begin{array}{c}
\text{Get functor} \\
\text{bicat}^{op} \xrightarrow{\text{StSp}} \text{Set} \\
P \mapsto \{ P = (P, ...) \} \\
\text{StSp}(r) \downarrow \\
Q \mapsto \{ Q = (Q, ...) \}
\end{array}
\]
Theorem 1) \( \text{St} \text{Sp} \) is representable.

The representing object is called the straight graph of \( F \).

\( \Rightarrow \) let \( \mathcal{P} = (\mathcal{P}, p, q, \lambda) \) be the straight graph.

2) The left leg \( p \) of \( \mathcal{P} \) is a strong equivalence.

3) The right leg \( q \) of \( \mathcal{P} \) is a strong equivalence iff \( F: \mathcal{X} \rightarrow \mathcal{A} \) is a biequivalence.

\( \Rightarrow \) Proposition above (on 'span-form' of equivalence) is a corollary.
The weak coherence theorem for bicats is an 'immediate' consequence of the strong coherence theorem (= Steve Lack's thm) as follows:

\[
\begin{array}{cc}
\text{Bicat} & \text{Grothendieck} \\
\text{\(q\) (forget)} & \text{is a fibration}
\end{array}
\]

\[
\begin{array}{ccc}
\text{1-magma} & \text{Lat: \(X\) be an arbitrary bicat;} \\
\text{\(U\)} & \text{\(f\)} & \text{\(X^{\mathsf{Nil}}\)}
\end{array}
\]

\(U\) a map from a free 1-magma \(U\) surjective on objects & full
\((U = \text{free on underlying graph of } X^{\mathsf{Nil}})\).

\(\text{For } Y = f^*(U) \xrightarrow{g = cf} X\)

\[
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow & & \downarrow \\
U & \overset{f}{\rightarrow} & X^{\mathsf{Nil}}
\end{array}
\]

\(g: Y \rightarrow X\) is a strong equivalence (easy);
\(Y\) is 1-free;
\(\vdash Y \simeq X: X \overset{Y}{\leftarrow} Y \iff \exists P \overset{q}{\rightarrow} A\).

\(\vdash X \simeq A\)
For the converse:

(Weak coherence $\Rightarrow$ Strong coherence):

Steve Lack's observation:

For any bicat $\mathcal{X}$,

$$\mathcal{X} \rightarrow \mathcal{Y}(\mathcal{X})$$

has the surjectivity properties

1), 2) and 3a)

(but not in general the injectivity property

3b): 2-faithfulness)

Assume: $\mathcal{X}$ is 1-free; get

by Proposition & Weak coherence:

2-cat $\mathcal{I} \mathcal{A}$, strong equiv's $P \xrightarrow{q} Q$:

$s \quad \Rightarrow \quad IP$

by 1-freeness of $\mathcal{X}$

$ps = 1d_\mathcal{X}$
As a section, $s$ has property 3b) (2-faithful).

As a strong equivalence, $q$ has property 3b).

$q_s : X \to A$

is 2-faithful

⇒ Use the universal property of $(\eta(x), \eta_x)$ against $q_s$

⇒ conclude that $\eta_x$ is 2-faithful

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§3  Tricategories and Gray categories

\[ \begin{array}{c}
\text{Gray-cat} \\
\downarrow \text{incl.} \\
\uparrow \text{Gray} \\
\text{tricat}
\end{array} \]

\[ Y_{\text{Gray}} \]

\[ X : \text{tricategory} \]

\[ X \xrightarrow{\eta_X} Y_{\text{Gray}}(X) \]

Theorem: For any \( X \) that is 2-free, \( \eta_X \) is a strong equivalence.

Proof based on:

1. Weak coherence: Every tricategory is equivalent to a Gray category (Gordon/Powers/Street) 1995

2. "Steve Lack's Observation" and the Proposition (based on the Strict Graph theorem)

3. Gurski has, in his thesis, the form
Gurski coherence for tricats:

for a category-enriched 2-graph \( G \)

the canonical

\[
P : \mathcal{F}_{\text{tricat}}(G) \to \mathcal{F}_{\text{Gray}}(G)
\]

\( \eta \)

\[
\eta : \mathcal{F}_{\text{tricat}}(G) \to \mathcal{F}_{\text{Gray}}(G)
\]

is a strong equivalence of tricategories

\( \Rightarrow \) follows from above theorem