Introduction

The papers [H/M/P] and [M8] describe a concept of "weak higher dimensional category", the *multitopic categories* and their structural environment (their morphisms, and higher "transfors".) The concept is based on *multitopic sets*.

The category of small multitopic sets is denoted by $\mathtt{mltSet}$.

The "multitopic" concepts were inspired, in the first place, by J. Baez's and J. Dolan's "opetopic" notions [BD1], [BD2]. The multitopic end-product as it appears in the papers cited above is independent of, and different from, its opetopic counterpart.

The present paper makes a connection between multitopic sets and Ross Street's fundamental concept of *computad*; see [S1], [S2], [B2].

The main assertion of the present paper is that "multitopic set" can be given a short and conceptually simple definition as "many-to-one computad".

A computad is a structure based on a set (call it S) of symbols, each of which is designated to be a cell of a definite dimension $0, 1, 2, ...$ in an ω-category. The structure of the computad records how the elements of S "fit together". It specifies, for each $n>0$ and for each n-cell $a\in S$ what the domain $\mathrm{d}a$ and the codomain $\mathrm{c}a$ of a are; here, $\mathrm{d}a$ and $\mathrm{c}a$ are elements of the $(n-1)$-category *freely generated* by the $\le(n-1)$-cells in S .

The precise definition of "computad" is given in section 6 in a way that slightly, and only inessentially, differs from the cited sources. We define computads as ω-categories with certain properties; that is, according to our definition, it makes sense to ask, of any given ω-category, whether or not it is a computad. Moreover, any ω-category isomorphic to a computad is itself a computad.

A computad is an ω-category freely generated by certain cells called *indeterminates*. It turns out the indeterminates can be identified in the computad as an ω-category as the so-called indecomposable elements; see section 6.

The notions of " ω-category" and " n-category" we use is the ones usually called "strict

ω-category" and "strict n-category", respectively. We use the formulations given on page 8 of [Le]. Every ω-category has a n-truncation for all $n\in\mathbb{N}$, and every n-category may be regarded as an ω-category in which every $\geq(n+1)$-cell is an identity.

A morphism of ω-categories is one that preserves the ω-category structure strictly ("strict ω-functor" in *loc.cit.*).

$\omega\mathtt{Cat}$ denotes the category of all small ω-categories.

A morphism of computads is an ω-category morphism that maps every indeterminate in its domain into an indeterminate in its codomain.

The category of all small computads is denoted by $\mathtt{Comp}$.

A *many-to-one computad* is one in which the codomain of every indeterminate of positive dimension is again an indeterminate. $\mathtt{Comp}_{\mathtt{m/1}}$ denotes the category of many-to-one computads, the full subcategory of $\mathtt{Comp}$ whose objects are the many-to-one computads.

A(n ordinary) functor $F:\boldsymbol{X}\to\boldsymbol{A}$ is said to be *full on isomorphisms* if for any objects $X, Y\in\mathrm{Ob}(\boldsymbol{X})$ and any isomorphism $j:FX\xrightarrow{\cong}FY$ in $\boldsymbol{A}$, there is a morphism (not necessarily an isomorphism) $i:X\xrightarrow{\cong}Y$ in $\boldsymbol{X}$ such that $Fi=j$.

Assume that $F:\boldsymbol{X}\to\boldsymbol{A}$ is full on isomorphisms.

The following definition gives a category $\boldsymbol{M}$, in fact a subcategory of $\boldsymbol{A}$. $\mathrm{Ob}(\boldsymbol{M})$ is a subclass of $\mathrm{Ob}(\boldsymbol{A})$; $A\in\mathrm{Ob}(\boldsymbol{A})$ belongs to $\mathrm{Ob}(\boldsymbol{M})$ iff there is $X\in\mathrm{Ob}(\boldsymbol{X})$ such that $A\cong FX$. For $A, B\in\mathrm{Ob}(\boldsymbol{M})$, $\mathrm{hom}_{\boldsymbol{M}}(A, B)$ is a subset of $\mathrm{hom}_{\boldsymbol{A}}(A, B)$; $f\in\mathrm{hom}_{\boldsymbol{A}}(A, B)$ belongs to $\mathrm{hom}_{\boldsymbol{M}}(A, B)$ iff there are $X\xrightarrow{u}Y\in\mathrm{Arr}(\boldsymbol{X})$ and a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i", "\cong"'] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & FX \arrow[d, "Fu"] \\ B \arrow[r, "\cong", "j"'] & FY \end{tikzcd}$$

in $\boldsymbol{A}$. Indeed, as a consequence of F being full on isomorphisms, $f\in\mathrm{hom}_{\boldsymbol{M}}(A, B)$ and

$g \in \hom_{\boldsymbol{M}}(B, C)$ imply that $gf \in \hom_{\boldsymbol{M}}(A, C)$; and if $A \in \mathrm{Ob}(\boldsymbol{M})$, then $\mathrm{id}_A \in \hom_{\boldsymbol{M}}(A, A)$.

$\boldsymbol{M}$ just described is called the *essential image* of F , and it is denoted by $\mathrm{EssIm}(F)$. We have a factorization

$$\begin{tikzcd} \boldsymbol{X} \arrow[rr, "F"] \arrow[dr, "G"'] \arrow[rr, phantom, shift right=4ex, "\circ"] & & \boldsymbol{A} \\ & \mathrm{EssIm}(F) \arrow[ur, "H"'] & \end{tikzcd}$$

where G is essentially surjective on objects and full, and H is a not necessarily full, but replete, inclusion.

If, in addition, F is faithful, then G is an equivalence of categories.

Our main result can be put in this way.

Theorem There is a pair of adjoint functors

$$\mathrm{mltSet} \underset{[-]}{\overset{\langle - \rangle}{\rightleftarrows}} \omega\mathrm{Cat} \qquad\qquad \langle - \rangle \dashv [-]$$

such that $\langle - \rangle$ is faithful and full on isomorphisms, and the essential image of $\langle - \rangle$ is identical to the $\mathrm{Comp}_{\mathrm{m}/1}$.

As a consequence,

Corollary 1 $\mathrm{mltSet} \simeq \mathrm{Comp}_{\mathrm{m}/1}$.

For an ω-category $\boldsymbol{X} \in \omega\mathrm{Cat}$, $[\boldsymbol{X}]$, the result of the functor $[-]$ applied to $\boldsymbol{X}$, is called the *multitopic nerve* of $\boldsymbol{X}$. For $\boldsymbol{R} \in \mathrm{mltSet}$, $\langle \boldsymbol{R} \rangle$ is the *free* ω*-category generated by* the multitopic set $\boldsymbol{R}$.

Let us note that Corollary 1 contains the essence of the Theorem. This is because it is fairly immediate that the inclusion

$$\langle\text{-}\rangle:\mathtt{Comp}_{\mathtt{m/1}}\longrightarrow\omega\mathtt{Cat}$$

(which would be called the "free functor" by R. Street) has a right adjoint

$$[\text{-}]:\omega\mathtt{Cat}\longrightarrow\mathtt{Comp}_{\mathtt{m/1}}\,.$$

Thus, by Corollary 1 identifying $\mathtt{Comp}_{\mathtt{m/1}}$ with $\mathtt{mltSet}$, we recapture the "free-ω-category versus multitopic-nerve" adjunction of the theorem.

On the other hand, the proof of Corollary 1 given in this paper is completely tied up with the details of the adjunction of the theorem. It remains to be seen if a proof of Corollary 1 can be given without going through the setting up of an explicit adjunction as in the theorem.

In [H/M/P3], it is proved that $\mathtt{mltSet}$ is a presheaf category. In fact, it is proved that there is a category $\mathtt{Mlt}$, the category of multitopes, for which

$$\mathtt{mltSet}\simeq\mathtt{Set}^{\mathtt{Mlt}^{\mathtt{op}}} \qquad (1)$$

[we write here $\mathtt{Mlt}$ for the opposite of the category we called $\mathtt{Mlt}$ in [H/M/P3]], and which, in addition, has the following properties: each hom-set $\mathtt{hom}_{\mathtt{Mlt}}(A,A)$ is a singleton (there are no nontrivial endomorphisms), and, assuming (as we of course may) that $\mathtt{Mlt}$ is skeletal, each slice-category $\mathtt{Mlt}/A$ is finite. Such categories are called *FOLDS-signatures* in [M8], [M7] (FOLDS is an abbreviation for "First Order Logic with Dependent Sorts").

As a consequence, each object A of $\mathtt{Mlt}$ has a finite dimension: $\mathtt{dim}(A)$ is the least $n\in\mathbb{N}$ such that for all $B\to A$ with $B\neq A$, $\mathtt{dim}(B)<n$.

A presheaf $X:\boldsymbol{C}^{\mathtt{op}}\to\mathtt{Set}$ over a FOLDS-signature $\boldsymbol{C}$ may be regarded as an (abstract) geometrical object given by a *face-structure*, without any *degeneracies*. The faces of a "cell" $a\in X(A)$ with $A\in\mathtt{Ob}(\boldsymbol{C})$, $\mathtt{dim}(A)=n$ are in a one-to-one correspondence with, and therefore identified with, the arrows $p:B\to A$ into the given object A. The *shape* of the face p of $a\in X(A)$ is the cell $X(p)(a)\in X(B)$. The faces, with one exception, are all of dimensions less than n. The face structure of a given cell $a \in X(A)$ is entirely encapsulated by the (finite) slice-category $\mathbf{C}/A$.

We have, by Corollary 1 and (1):

Corollary 2 $\mathrm{Comp}_{m/1}$ is a presheaf category. In fact, $\mathrm{Comp}_{m/1}$ is equivalent to the category of presheaves over a FOLDS-signature.

In contrast, $\mathtt{Comp}$ is not a presheaf-category. In fact, already the category of 3-computads is not a presheaf category. (In [C/J], Example 3.6, S. H. Schanuel's result to the effect that 2-computads (the "computads" of [S1]) form a presheaf category is explained.)

The concept of multitopic set in [H/M/P] is based on the notion of *multicategory*.

Multicategories were first introduced by J. Lambek [La]. The multicategories [H/M/P] introduce an essential modification of Lambek's concept. The modified notion of multicategory is a multicategory with *abstract places*, in contrast to the Lambek multicategories in which the places are "concrete". One sign of the fact that the places are abstract is that we need *coherence commutativities* for the handling of places.

Lambek style multicategories are closely related to operads, the main tool in [B/D1] and [B/D2]. However, the connection of operads and the abstract-place multicategories is less clear to us.

The multicategories with abstract places are an essential generalization of those with concrete places: "essential" because the basic construction of "multicategories of function replacement" used in [H/M/P] yields, in general, one of the generalized kind, not isomorphic to one of the special kind.

It should be pointed out that there is no available simple "constructive" definition of the category $\mathtt{Mlt}$ itself that would be independent of the development of the theory of multitopic sets through multicategories. In fact, [H/M/P] first defines multitopic sets before defining $\mathtt{Mlt}$; $\mathtt{Mlt}$ is obtained from the terminal multitopic set.

(There is a "simple", but decidedly non-constructive, definition of $\mathtt{Mlt}$ based on fact (1), obtained by the general method of characterizing, up to Cauchy completion, the image of the Yoneda functor $Y:\boldsymbol{C}\longrightarrow\mathtt{Set}^{\boldsymbol{C}^{\mathrm{op}}}$ inside the category $\mathtt{Set}^{\boldsymbol{C}^{\mathrm{op}}}$; in fact, because of the special nature of FOLDS signatures, this will be a characterization up to isomorphism.)

Multicategories are the essential tool in this paper as well, in ways that go beyond their uses in [H/M/P]. In the construction of the multitopic nerve $[\boldsymbol{X}]$ of an ω-category $\boldsymbol{X}$, we need a series of new multicategories $\boldsymbol{E}_n$, $n\in\mathbb{N}-\{0\}$, attached to $\boldsymbol{X}$. These are "new" in the sense that they appear over and above the multicategories that come with the various auxiliary multitopic sets we need in the construction.

Revising the concept of multicategory

We change, in fact, generalize, the concept of "multicategory" given in [H/M/P]. However, the change will be small: in fact, all new multicategories will be *isomorphic* to old ones. In fact, the generalization is a matter of style only.

Having said that, we must say that the new, generalized, concept brings out a feature that seems "philosophically" important, and which is not shown explicitly enough in the original formulation. This is the fact that the concept of "place" is, in essence, something purely abstract. For instance, places do not have to be integers (natural numbers) as they are stipulated to be in [H/M/P]. In particular,

the (natural) order on the integers plays no role whatsoever in the concept.

In fact, upon reflection, one sees that this is the main conceptual point in the generalization from the original Lambek notion of multicategory to the one given in [H/M/P]. In particular, although there is some talk of "standard amalgamation" in [H/M/P], that is, Lambek multicategory, no essential use is made of it.

[Notation 1/2 refer to section 1 in Part 2 of [H/M/P]; similarly for other such references.]

To obtain the present version, to begin with, in 1/2, forget the initial talk about "tuples"; or rather, to minimize changes, let "tuple of elements of O " mean a function $s\colon |s| \longrightarrow O$ where $|s|$ is an *arbitrary finite set*. Of course, the original "tuples" are still available; just that there are more. Let O^* denote the "set" (class) of all tuples of elements in O ; and $O^\#$ the category of tuples, with object-class O^* , defined just as in [H/M/P] but with the new tuples in mind.

Now, read the definition of "multicategory" (clauses (i) to (xi)) with the (slightly) changed meanings of the terms and symbols just mentioned. Since each place in a multicategory was originally meant to be an integer, variables such as p and q in that definition were originally denoting integers. Fortunately, this fact is not stated in the text; therefore the wording does not need any change! In fact, no change is needed in the first three sections, beyond those mentioned above.

Let us note that the place-sets $|s|$ can be changed at will to others that are in a bijective correspondence with the original sets; when doing so the result will be another multicategory which is isomorphic to the original one. In detail: suppose $\mathbf{C}$ is a multicategory, and we are given, for each $f\in \mathrm{A}(\mathbf{C})$, a set D_f and a bijection $\theta_f\colon |s(f)| \xrightarrow{\cong} D_f$. We define the multicategory $\mathbf{C}'$ as follows. We put $\mathrm{O}(\mathbf{C}')=\mathrm{O}(\mathbf{C})$, $\mathrm{A}(\mathbf{C}')=\mathrm{A}(\mathbf{C})$, $t_{\mathbf{C}'}(f)=t_{\mathbf{C}}(f)$ for $f\in A$. Next, as the point of the thing, we define $s_{\mathbf{C}'}(f) = D_f$. We do not have to give any more detail, since it suffices to say that the structure of $\mathbf{C}'$ is uniquely determined by the stipulation that we have an isomorphism of multicategories $F\colon\mathbf{C} \xrightarrow{\cong} \mathbf{C}'$ which is the identity on O and A , and whose transition bijections are the given $\theta_f\colon |s_{\mathbf{C}}(f)| \xrightarrow{\cong} D_f = |s_{\mathbf{C}'}(f)|$. In brief, we can uniquely transport the structure of $\mathbf{C}$ along the θ_f to form another multicategory structure.

The last-stated fact contains the assertion that every multicategory in the present sense, with arbitrary places, is isomorphic to one in the exact sense given in [H/M/P]. For this, with a given "new" $\mathbf{C}$ one only needs to specify a bijection $\theta_f\colon |s(f)| \xrightarrow{\cong} D_f$ where D_f is of the form $[1, n] = \{i\in\mathbb{N}:1\le i\le n\}$; since each $|s(f)|$ was stipulated to be a finite set, this is possible.

A notational point: henceforth, when we have a morphism $F\colon\mathbf{C} \longrightarrow \mathbf{C}'$ of multicategories, and $\theta^F_f \colon |s(f)| \xrightarrow{\cong} |s(Ff)|$, we write simply $F(p)$ for $\theta^F_f(p)$.