Appendix E: More on equivalence and interpolation

In this section, S and T are small Heyting categories, $\boldsymbol{L}$ is a DSV, $\boldsymbol{K}$ its category of kinds, and $F:\boldsymbol{L}\to S$, $G:\boldsymbol{L}\to T$ are S-, resp. T-valued $\boldsymbol{L}$-structures. $\mathrm{Mod}(S)$ denotes the category of coherent functors $S\to\mathrm{Set}$, a full subcategory of Set^S; similarly for $\mathrm{Mod}(T)$.

Primarily, we have in mind T (also, S) obtained as the Lindenbaum-Tarski category $[T_0]$ of a theory T_0 in intuitionistic logic. We will be looking at *Kripke-models of* T; that is, Heyting functors $\Phi:T\to Set^{\boldsymbol{C}}$, with various exponent categories $\boldsymbol{C}$; we write $\Phi\models T$ for "Φ is a Kripke model of T". "σ is a sentence of T", "$\Phi\models\sigma$" and other unexplained notation have the meanings analogous to the ones used in §5.

We have the following intuitionistic version of the interpolation theorem 5.(7)(a).

(1) Assume that σ,τ are sentences of T, and for all Kripke models $\Phi,\Psi\models T$,

$$\Phi\models\sigma \;\;\&\;\; \Phi\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}\Psi\restriction\boldsymbol{L} \;\Longrightarrow\; \Psi\models\tau\;.$$

Then there is an $\boldsymbol{L}$-sentence θ in logic with dependent sorts without equality such that for all $\Phi\models T$,

$$\Phi\models\sigma\Longrightarrow\Phi\restriction\boldsymbol{L}\models\theta \qquad\text{and}\qquad \Phi\restriction\boldsymbol{L}\models\theta\Longrightarrow\Phi\models\tau\;.$$

In (5) below, we will reformulate (and strengthen) the theorem in a purely syntactical fashion, by removing references to Kripke semantics.

We will imitate [M4] in the proof of (1).

When $I:T\to Q$ is a Heyting functor, and $F:\boldsymbol{L}\to T$, we have an obvious composite $IF:\boldsymbol{L}\to Q$.

Recall that for $\boldsymbol{L} \underset{I}{\overset{H}{\rightrightarrows}} Q$, $\alpha : H \underset{\boldsymbol{L}}{\leftrightarrow} I$ (called an *$\boldsymbol{L}$-equivalence*) is $\alpha = (A, \alpha_0, \alpha_1)$, with $A : \boldsymbol{K} \to Q$ and $\alpha_0 : A \to H \restriction \boldsymbol{K}$, $\alpha_1 : A \to I \restriction \boldsymbol{K}$ with suitable properties. Given also $J : Q \to R$, we have the composite $J\alpha \underset{\text{def}}{=} (JA, J\alpha_0, J\alpha_1) : JH \underset{\boldsymbol{L}}{\leftrightarrow} JI$; the requisite properties are easily checked.

Consider data as in

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ F\uparrow & & \uparrow I \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \alpha : HF \underset{\boldsymbol{L}}{\overset{\cong}{\longleftrightarrow}} IG \tag{2}$$

with H, I Heyting functors. Fixing the items $\boldsymbol{L}, S, T, F, G$, and for Q a Heyting category, let C_Q be the groupoid whose objects are triples (H, I, α) as in (2), and whose arrows $(H, I, \alpha) \xrightarrow{\cong} (H', I', \alpha')$ (where $\alpha = (A, \alpha_0, \alpha_1)$, $\alpha' = (A', \alpha'_0, \alpha'_1)$) are triples $(\varphi : H \xrightarrow{\cong} H', \psi : I \xrightarrow{\cong} I', \gamma : A \xrightarrow{\cong} A')$ of natural isomorphisms such that

$$\begin{array}{ccc} HF & \xrightarrow{\varphi F} & H'K \\ \alpha_0 \uparrow & \circ & \uparrow \alpha'_0 \\ A & \xrightarrow{\gamma} & A' \\ \alpha_1 \downarrow & \circ & \downarrow \alpha'_1 \\ IG & \xrightarrow[\psi G]{} & I'G \end{array} \tag{2'}$$

Composition in C_Q is defined in the obvious way. We may write $(Q; H, I, \alpha)$ for (H, I, α) to emphasize Q.

Given an object $\Gamma = (Q; H, I, \alpha)$ of C_Q, and $L : Q \to R$, a Heyting functor, we have the composite object $L\Gamma = (R; LH, LI, L\alpha)$ (with $L\alpha$ described above) of C_R. Moreover, we have the functor

$$\Gamma^*_R = \Gamma^* : \mathrm{Hom}(Q, R) \longrightarrow C_R$$

where $\mathrm{Hom}(Q, R)$ is the category (groupoid) of Heyting functors $Q \to R$ with isomorphisms as arrows; the object-function of γ^* is $L \mapsto L\gamma$ as described, the arrow-function being similarly defined by composition.

There are $Q=S+_{\boldsymbol{L}}T$, a Heyting category, and $\Gamma\in\mathrm{C}_Q$, given by the data

$$\begin{array}{ccc} S & \xrightarrow{I_0} & S+_{\boldsymbol{L}}T \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I_1} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \alpha: I_0F \xleftrightarrow[\boldsymbol{L}]{\cong} I_1G \tag{3}$$

,

such that $(Q;\Gamma)$ enjoys the universal property that for any Heyting category R, Γ^*_R is a *surjective* (on objects) equivalence of categories (groupoids).

The description of $Q = S+_{\boldsymbol{L}}T$ is as follows. Q is the Lindenbaum-Tarski category $[Q_0]$ of a theory Q_0 in intuitionistic logic. L_{Q_0} consists of $\mathrm{L}_S\bigsqcup\mathrm{L}_T$, the disjoint union of the underlying graphs of S and T, together with new objects $\mathrm{A}K$, one for each $K\in\boldsymbol{K}$, arrows $\mathrm{A}p:\mathrm{A}K\to\mathrm{A}K_p$, one for each $K\in\boldsymbol{K}$ and $p\in K|\boldsymbol{K}$, and arrows $\alpha_{0K}:\mathrm{A}K\to FK$, $\alpha_{1K}:\mathrm{A}K\to GK$. The axioms of Σ_{Q_0} are those of S and T (formulated for the symbols that are the images of the original symbols of S and T in $\mathrm{L}_S\bigsqcup\mathrm{L}_T$), together with axioms amounting to the assertion that $(A,\alpha_0,\alpha_1)=(\mathrm{A}K,\alpha_{0K},\alpha_{1K})_{K\in\boldsymbol{K}}$ is an $\boldsymbol{L}$-equivalence between the S-model and the T-model involved. The object $\Gamma\in\mathrm{C}_Q$ is the evident one. Kripke-models of $S+_{\boldsymbol{L}}T$ are essentially the same as triples $(M\models S, N\models T, \alpha: M\xleftrightarrow[\boldsymbol{L}]{} N)$; this fact is essentially the universal property of $(S+_{\boldsymbol{L}}T,\gamma)$ with respect to R a presheaf category $\mathrm{Set}^{\boldsymbol{C}}$.

We call (3) the $\boldsymbol{L}$-*pushout of* $(F:\boldsymbol{L}\to S, G:\boldsymbol{L}\to T)$.

Next, we introduce some auxiliary concepts.

Suppose that in

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array}$$

Q is a coherent category, H and I are coherent functors (however, S and T are still the same Heyting categories as before). Let $A:\mathbf{K}\to Q$, $\alpha_0 :A\to HF\restriction\mathbf{K}$, $\alpha_1 :A\to IG\restriction\mathbf{K}$. We write $\alpha=(A,\ \alpha_0,\ \alpha_1):H\underset{*}{\longrightarrow}I$ if the following holds:

(3') for every finite $\mathbf{K}$-context $\mathcal{X}$, and any $\mathbf{L}$-formula θ of FOLDS,

$$(\alpha_0)^*_{[\mathcal{X}]}(F[\mathcal{X}:\theta])\leq_{A[\mathcal{X}]}(\alpha_1)^*_{[\mathcal{X}]}(G[\mathcal{X}:\theta]) .$$

This refers to the arrows $HF[\mathcal{X}]\xleftarrow{(\alpha_0)[\mathcal{X}]}A[\mathcal{X}]\xrightarrow{(\alpha_1)[\mathcal{X}]}IG[\mathcal{X}]$ induced by α_0 and α_1. We write $(A,\ \alpha_0,\ \alpha_1):H\underset{*}{\longleftrightarrow}I$ if both $(A,\ \alpha_0,\ \alpha_1):H\underset{*}{\longrightarrow}I$ and $(A,\ \alpha_1,\ \alpha_0):I\underset{*}{\longrightarrow}H$; of course, this just means an equality in place of $\leq_{A[\mathcal{X}]}$ in (3'). Finally, we write $(A,\ \alpha_0,\ \alpha_1):H\underset{\#}{\longleftrightarrow}I$ if $\alpha=(A,\ \alpha_0,\ \alpha_1):H\underset{*}{\longleftrightarrow}I$ and α_0 and α_1 are very surjective.

Notice that if $(A,\ \alpha_0,\ \alpha_1):H\underset{\#}{\longleftrightarrow}I$, then $\alpha=(A,\ \alpha_0,\ \alpha_1):H\underset{\mathbf{L}}{\longleftrightarrow}I$; the latter involves preserving atomic $\mathbf{L}$-formulas only.

Let us explain the meaning of the last-mentioned concepts when $Q=\mathrm{Set}$, and $H=M\in\mathrm{Mod}(S)$, $I=N\in\mathrm{Mod}(T)$.

With $\mathcal{X}$ and φ as above, let $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}\in(M\restriction\mathbf{K})[\mathcal{X}]$. We write $M\models_{\mathrm{w}}\varphi[\vec{a}]$ for $\langle\vec{a}\rangle\in M(F[\mathcal{X}:\varphi])$ $(\subset M(F[\mathcal{X}]))$; here, the notation $\langle\vec{a}\rangle$ is used in the sense given to it in the line after 5.(7'). The subscript w is to serve as a warning that this is a "non-standard" meaning for truth ($\models$); the coherent functor $M:S\to\mathrm{Set}$ is not supposed to respect the full logical structure of S, hence it does not necessarily "recognize" the full meaning of φ; M is a "weak model for $\mathbf{L}$-formulas". We have that for $U:\mathbf{K}\to\mathrm{Set}$, and $M\xleftarrow{m}U\xrightarrow{n}N$, $(U,\ m,\ n):M\underset{*}{\longrightarrow}N$ iff for all $\mathcal{X}$ and φ as above, and for any $\langle c_x\rangle_{x\in\mathcal{X}}\in U[\mathcal{X}]$,

$$M\models_{\mathrm{w}}\varphi[\langle mc_x\rangle_{x\in\mathcal{X}}]\implies N\models_{\mathrm{w}}\varphi[\langle nc_x\rangle_{x\in\mathcal{X}}] .$$

Note that when $U=\emptyset$, $(\emptyset,\ \emptyset,\ \emptyset):M\underset{*}{\longrightarrow}N$ means that $M(F[\emptyset:\varphi])=1\Rightarrow N(G[\emptyset:\varphi])=1$.

Let

$$\begin{array}{ccc} S & \xrightarrow{\bar{I}_0} & S+_{\#}T \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle \bar{I}_1} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \bar{\alpha}=(\bar{A},\ \bar{\alpha}_0,\ \bar{\alpha}_1):\bar{I}_0F \underset{\#}{\leftrightarrows} \bar{I}_1G \tag{4}$$

be the entity that is "initial" among all

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \alpha=(A,\ \alpha_0,\ \alpha_1):HF \underset{\#}{\leftrightarrows} IG\ , \tag{4'}$$

in the following natural sense, amounting to a modification of the definition of $S+_{\boldsymbol{L}}T$. The category $\mathrm{C}^{\#}_{Q}$, for Q a coherent category, has objects (4'), and arrows

$$(\varphi:H\to H',\ \psi:I\to I',\ \gamma:A\to A'):(H,\ I,\ \alpha)\longrightarrow(H',\ I',\ \alpha')$$

($\alpha=(A,\ \alpha_0,\ \alpha_1)$, $\alpha'=(A',\ \alpha'_0,\ \alpha'_1)$) such that (2') holds; it is important that here φ, ψ and γ are not restricted to be isomorphisms. For any coherent category R, and $\Gamma\in\mathrm{C}^{\#}_{Q}$, we have

$$\Gamma^{*}:\mathrm{Coh}(Q,\ R)\longrightarrow\mathrm{C}^{\#}_{R}$$

where $\mathrm{Coh}(Q,\ R)$ is the category of coherent functors $Q\to R$, a full subcategory of R^{Q}. The universal property of $S+_{\#}T$ is that, for Γ given by (4), for any coherent R, Γ^{*} is a surjective equivalence of categories.

The construction of $S+_{\#}T$ is similar to that of $S+_{\boldsymbol{L}}T$. $S+_{\#}T$ is the Lindenbaum-Tarski category of a *coherent* theory $Q^{\#}_{0}$; the language of $Q^{\#}_{0}$ is the same as that for Q_0 given above for $S+_{\boldsymbol{L}}T$. We include (coherent) axioms to ensure

$$(\bar{\alpha}_0)^{*}_{[\mathcal{X}]}(F[\mathcal{X}:\theta]) =_{\bar{A}[\mathcal{X}]} (\bar{\alpha}_1)^{*}_{[\mathcal{X}]}(G[\mathcal{X}:\theta])$$

for each $\mathcal{X}$, θ as above. Note that the (ordinary, Set-valued) models of $S+_{\#}T$ are essentially the same as triples (M, N, u), with $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $u: M \underset{\#}{\longleftrightarrow} N$.

(4) may be referred to as the $\#$-*pushout of* $(F:\mathbf{L}\to S,\ G:\mathbf{L}\to T)$.

Notice that there is a coherent comparison functor $J: S+_{\#}T \longrightarrow S+_{\mathbf{L}}T$ for which $J\bar{I}_0 = I_0$, $J\bar{I}_1 = I_1$ and $J\bar{\alpha} = \alpha$. The reason is the universal property of $S+_{\#}T$, and the fact that, for Heyting functors $\Phi: S\to R$, $\Psi: T\to R$, $\alpha: \Phi \underset{\mathbf{L}}{\longleftrightarrow} \Psi$ implies $\alpha: \Phi \underset{\#}{\longleftrightarrow} \Psi$.

Any diagram

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \mathbf{L} & \xrightarrow[G]{} & T \end{array} \qquad\qquad HF \xleftarrow{\alpha_0} A \xrightarrow{\alpha_1} IG$$

involving (at least) coherent categories and coherent functors, is said to have the *interpolation property* if the following holds: whenever $\mathcal{X}$ is a finite context for $\mathbf{L}$, $\sigma\in \mathrm{S}_S(F[\mathcal{X}])$, $\tau\in\mathrm{S}_T(G[\mathcal{X}])$ and $(\alpha_0)^*_{[\mathcal{X}]}(H\sigma) \le_{A[\mathcal{X}]} (\alpha_1)^*_{[\mathcal{X}]}(I\tau)$, then there is an $\mathbf{L}$-formula θ (of FOLDS) such that $\sigma \le_{F[\mathcal{X}]} F[\mathcal{X}:\theta]$ and $G[\mathcal{X}:\theta] \le_{G[\mathcal{X}]} \tau$.

Using the (Kripke) completeness theorem for intuitionistic logic (for any small Heyting category S, there is a conservative Heyting functor $S\to \mathrm{Set}^{\mathbf{C}}$), it is easy to see that (1) is a weakened form of saying that the $\mathbf{L}$-pushout diagrams have the interpolation property. Thus, (1) will follow from

(5) Both the $\#$-pushout and the $\mathbf{L}$-pushout of a pair $(F:\mathbf{L}\to S,\ G:\mathbf{L}\to T)$, with S and T small Heyting categories, have the interpolation property. Moreover, the comparison map $J: S+_{\#}T \longrightarrow S+_{\mathbf{L}}T$ is conservative; thus, the assertion for the $\mathbf{L}$-pushout is a consequence of that for the $\#$-$\mathbf{L}$-pushout.

For the proof of (5), we will employ the method described in [M4] (and adapted there from [G]).

Let $M\in\mathrm{Mod}(S)$. $M\restriction L \underset{\mathrm{def}}{=} M\circ F$, and $M\restriction \boldsymbol{K} \underset{\mathrm{def}}{=} M\circ F\circ j$, for the inclusion $j:\boldsymbol{K}\to\boldsymbol{L}$. For $W\in\mathrm{Set}^{\boldsymbol{K}}$, an arrow $m:W\to M$ means an arrow $m:W\to M\restriction\boldsymbol{K}$.

We write L_S for the underlying graph of the category S , and regard it as a vocabulary for intuitionistic first-order logic. (Now, S is a general small Heyting category; in particular, what follows will also be applied to T .) For a finite sequence $\vec{x}=\langle x_i\rangle_{i<n}$ of distinct variables , by $[\vec{x}]$ we mean a chosen product $X_0\times X_1\times\ldots\times X_{n-1}$, where $x_i:X_i$. For any (first-order) formula φ over L_S , with free variables in $\vec{x}$, we have $[\vec{x}:\varphi]$, a subobject of $[\vec{x}]$, the "internal interpretation of φ in the context $\vec{x}$ in S "; see [MR1].

We will use the coherent theory $T_S^{\mathrm{coh}}=(L_S,\Sigma_S^{\mathrm{coh}})$, the internal theory of S as a coherent category introduced in [MR1]. $\mathrm{Mod}(S)$ is identical to $\mathrm{Mod}(T_S^{\mathrm{coh}})$, the category of models of the theory T_S^{coh} with ordinary homomorphisms as arrows. For a coherent formula φ with free variables in $\vec{x}$, $M([\vec{x}:\varphi])$, a subset of $M([\vec{x}])$, is identical to the ordinary interpretation of φ , $\{\vec{a}:M\models\varphi[\vec{a}/\vec{x}]\}$, modulo the canonical isomorphism $j:\prod_{i<n}X_i\to M([\vec{x}])$ ($\vec{x}=\langle x_i\rangle_{i<n}$, $x_i:X_i$) ; that is, $M([\vec{x}:\varphi])$ = $j(\{\vec{a}:M\models\varphi[\vec{a}/\vec{x}]\})$. For coherent formulas φ and ψ over L_S , with free variables included in $\vec{x}$,

$$T_S^{\mathrm{coh}}\models\varphi\underset{\vec{x}}{\Longrightarrow}\psi \text{ (that is, for all } M\in\mathrm{Mod}(S)\text{ , } M\models\forall\vec{x}(\varphi\to\psi)\text{) iff}$$

$$[\vec{x}:\varphi]\le_{[\vec{x}]}[\vec{x}:\psi] \ ;$$

in other words, a coherent entailment is an ordinary semantic consequence of T_S^{coh} iff it is internally true in S ; this is but a form of the (Gödel) completeness theorem for coherent logic.

Now, we refer to $F:\boldsymbol{L}\to S$ as well. Let $x\mapsto\underline{x}$ a 1-1 mapping of variables of FOLDS over $\boldsymbol{L}$ into variables over L_S so that $\underline{x}:F(K_x)$. Let, for any finite context $\mathcal{X}$ of $\boldsymbol{L}$-variables, $E(\underline{\mathcal{X}})$ denote the formula

$$\bigwedge \{(Fp)(\underline{x}) =_{FK_p} \underline{x_{x,p}} : x\in\mathcal{X},\ p\in K_x|\boldsymbol{K}\} .$$

This formula describes that the $\underline{x}$ for $x\in\mathcal{X}$ fit together via the maps Fp, $p\in K_x|\boldsymbol{K}$, as dictated by the structure of the context $\mathcal{X}$.

Recall $F[\mathcal{X}]$ defined as a certain pullback; we have a monomorphism $m:F[\mathcal{X}] \rightarrowtail [\underline{\mathcal{X}}]$ for which $\pi_{\underline{x}}\circ m = \pi_x$ ($x\in\mathcal{X}$); here, we refer to the evident projections. In fact, m represents the subobject $[\underline{\mathcal{X}}:\mathrm{E}(\underline{\mathcal{X}})]$ of $[\underline{\mathcal{X}}]$. If $\Phi = [n: |\Phi| \rightarrowtail F[\mathcal{X}]]$ is any subobject of $F[\mathcal{X}]$, then $\underline{\Phi} \underset{\text{def}}{=} [mn: |\Phi| \rightarrowtail [\underline{\mathcal{X}}]]$ is a subobject of $[\underline{\mathcal{X}}]$. We have a formula $\underline{\Phi}(\underline{\mathcal{X}})$ with free variables in $\underline{\mathcal{X}}$ such that $[\underline{\mathcal{X}}:\underline{\Phi}(\underline{\mathcal{X}})] = \underline{\Phi}$;

$$\underline{\Phi}(\underline{\mathcal{X}}) \underset{\text{def}}{=} \exists z\in |\Phi| \bigwedge_{x\in\mathcal{X}} (\pi_x\circ n)(z)=\underline{x}$$

($\pi_x:F[\mathcal{X}] \longrightarrow FK_x$). When φ is an $\boldsymbol{L}$-formula in FOLDS, with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, and we take $\Phi=F[\mathcal{X}:\varphi]\in \mathrm{S}(F[\mathcal{X}])$, we get $\underline{\varphi}(\underline{\mathcal{X}}) \underset{\text{def}}{=} \underline{F[\mathcal{X}:\varphi]}(\underline{\mathcal{X}})$.

Note that if $M\in\mathrm{Mod}(S)$, then for $\langle a_x\rangle_{x\in\mathcal{X}}\in F[\mathcal{X}]$,

$$M\models_{\mathrm{w}}\varphi[\langle a_x\rangle_{x\in\mathcal{X}}] \iff M\models\underline{\varphi}(\underline{\mathcal{X}})[a_x/\underline{x}]_{x\in\mathcal{X}} . \qquad (5')$$

If $\mathrm{Var}(\varphi)\subset\mathcal{X}\subset\mathcal{X}'$, then

$$[\underline{\mathcal{X}}':\mathrm{E}(\underline{\mathcal{X}}')\wedge\underline{\varphi}(\underline{\mathcal{X}})] = [\underline{\mathcal{X}}':\underline{\varphi}(\underline{\mathcal{X}}')] \qquad (6)$$

as is easily seen.

Let $\mathcal{X}\subset\mathcal{Y}$ be finite contexts over $\boldsymbol{L}$; assume $\mathrm{Var}(\varphi)\subset\mathcal{Y}$. Let us write $\forall(\mathcal{Y}-\mathcal{X})\varphi$ for the formula $\forall z_1\forall z_2\ldots\forall z_n\varphi$, where $\langle z_i\rangle_{i=1}^{n}$ is a repetition-free enumeration of the set $\mathcal{Y}$-$\mathcal{X}$ such that for all $j\le n$, $\mathcal{X}\cup\{x_i:i\le j\}$ is a context (an enumeration in a non-decreasing order of the level of K_z will ensure this; the formula $\forall z_1\forall z_2\ldots\forall z_n\varphi$ is well-formed as a consequence. $\forall(\mathcal{Y}-\mathcal{X})\varphi$ is not quite uniquely determined, but it is, up to logical equivalence). We have the equality:

$$[\underline{\mathcal{X}}:\underline{(\forall(\mathcal{Y}-\mathcal{X})\varphi)}(\underline{\mathcal{X}})] = [\underline{\mathcal{X}}:\mathrm{E}(\underline{\mathcal{X}})\wedge\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}})(\mathrm{E}(\underline{\mathcal{Y}})\rightarrow\underline{\varphi}(\underline{\mathcal{Y}}))] ; \qquad (7)$$

here, $\forall(\underline{\mathcal{Y}-\mathcal{X}})$ stands for $\forall \underline{z_1} \forall \underline{z_2} \ldots \forall \underline{z_n}$ for $\mathcal{Y}-\mathcal{X} = \{z_1, \ldots z_n\}$ as above. This is easy to show by induction on the cardinality of $\mathcal{Y}$-$\mathcal{X}$.

Other easily seen equalities we will use are

$$[\mathcal{Y}: \underline{\bigwedge_{i<m} \varphi_i}(\underline{\mathcal{Y}})] = [\mathcal{Y}: \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{Y}})] \ , \tag{8}$$

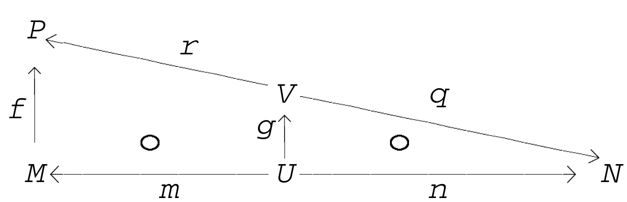
$$[\mathcal{Y}: \underline{\bigvee_{j<n} \psi_j}(\underline{\mathcal{Y}})] = [\mathcal{Y}: \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{Y}})] \ , \tag{9}$$

$$[\underline{\mathcal{Y}}: (\underline{\varphi \to \psi})(\underline{\mathcal{Y}})] = [\underline{\mathcal{Y}}: \mathrm{E}(\underline{\mathcal{Y}}) \wedge (\underline{\varphi}(\underline{\mathcal{Y}}) \to \underline{\psi}(\underline{\mathcal{Y}}))] \ , \tag{10}$$

under the natural conditions on the parameters involved.

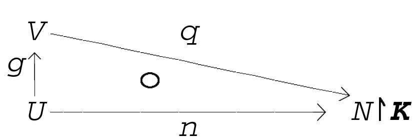
The following is the analog of Lemma 3 of [M4].

(11) Suppose $M \in \mathrm{Mod}(S)$, $N \in \mathrm{Mod}(T)$ and $(U, m, n): M \underset{*}{\longrightarrow} N$. Then we have $P \in \mathrm{Mod}(S)$, $(f: M \to P) \in \mathrm{Mod}(S)$, $g: U \to V$ and $(V, r, q): P \underset{*}{\longleftrightarrow} N$ such that q is very surjective, and



(12)

Proof. We first construct



in $\mathrm{Set}^{\boldsymbol{K}}$ such that $q: V \twoheadrightarrow N \restriction \boldsymbol{K}$ is very surjective and g is a monomorphism. We put

$U_0 \xrightarrow{n_0} N\restriction\boldsymbol{K} = U \xrightarrow{n} N\restriction\boldsymbol{K}$, and by recursion on $i<k$, the height of $\boldsymbol{K}$, assuming $U_i \xrightarrow{n_i} N\restriction\boldsymbol{K}$ defined, we define $U_{i+1} \xrightarrow{n_{i+1}} N\restriction\boldsymbol{K}$ as follows. For all $K\in\boldsymbol{K}$ except when the level of K equals i, we put $U_{i+1}K = U_iK$. When $K\in\boldsymbol{K}_i$, we put, for all $\vec{a}\in U_i[K]=U_{i+1}[K]$, $U_{i+1}K(\vec{a}) = U_iK(\vec{a})\bigsqcup(N\restriction\boldsymbol{K})K(n_i(\vec{a}))$; here, we use the notation of 1.(3). (This means that $U_{i+1}K = \bigsqcup_{\vec{a}\in U_i[K]} (U_iK(\vec{a})\bigsqcup(N\restriction\boldsymbol{K})K(n_i(\vec{a})))$.) We have the map $g_{i,\ i+1}: U_i \to U_{i+1}$ whose component at each $K\notin\boldsymbol{K}_i$ is the identity, and whose component at $K\in\boldsymbol{K}_i$ on the fiber over $\vec{a}\in U_i[K]$ is the coproduct coprojection $U_iK(\vec{a}) \longrightarrow U_{i+1}K(\vec{a})$. The component of n_{i+1} at each $K\notin\boldsymbol{K}_i$ is that of n_i. For $K\in\boldsymbol{K}_i$, $(n_{i+1})_K: U_{i+1}K \to (N\restriction\boldsymbol{K})K$ maps the image of $b\in U_iK(\vec{a})$ in $U_{i+1}K(\vec{a})$ under the first coprojection to $(n_i)_K(b)$, and the image of $b\in(N\restriction\boldsymbol{K})K(n_i(\vec{a}))$ in $U_{i+1}K(\vec{a})$ under the second coprojection to b itself. We have that $n_{i+1}\circ g_{i,\ i+1}=n_i$. Having defined all $U_i \xrightarrow{n_i} N\restriction\boldsymbol{K}$, we let $V = \operatorname{colim}_{i<k} U_{i+1}$ ($= U_{k-1}$ when $k<\omega$), with the $g_{i,\ i+1}$ as connecting maps, and $q = \operatorname{colim} n_i$. g is the coprojection $U_0 \to V$. It is fairly clear that V, g and q so constructed are appropriate.

We may assume that g is an inclusion (that is, each of its components g_K is an inclusion of sets).

Consider the (infinite) contexts $\mathcal{Y}_U\subset\mathcal{Y}_V$ associated with U and V as in §4. For $x\in\mathcal{Y}_V-\mathcal{Y}_U$, let $\underline{x}$ denote a variable for ordinary multisorted logic over L_S, of the sort $F(\mathrm{K}_x)$; the mapping $x\mapsto\underline{x}$ is 1-1. For any $A\in S$, $a\in M(A)$, let $\underline{(A,\ a)}$, abbreviated as $\underline{a}$, be a variable of sort A; assume that the $\underline{a}$ are different from the $\underline{x}$. With

$$C \underset{\text{def}}{=} \{\underline{x}: x\in\mathcal{Y}_V-\mathcal{Y}_U\} \,\dot{\cup}\, \{\underline{a}: A\in S,\ a\in M(A)\}\ ,$$

by a C-formula we mean one over L_S whose free variables all belong to C.

For $x\in\mathcal{Y}_U$, $m(\mathrm{a}(x))$ is an element of M, thus $\underline{m(\mathrm{a}(x))}$ belongs to the second term in C. When $x\in\mathcal{Y}_U$, let $\underline{x}$ stand for $\underline{m(\mathrm{a}(x))}$. (Recall the correspondence between the elements of $\mathcal{Y}_V$ and those of V; for any fixed $K\in\boldsymbol{K}$, $d\mapsto y^V_{K,\ d}$ is a bijection

$V(K) \xrightarrow{\cong} \{x\in\mathcal{Y}_V : K_x=K\}$, with inverse $x\mapsto \mathrm{a}(x)$). Now, $\underline{x}\in C$ is defined for all $x\in\mathcal{Y}_V$, and we have $\underline{x}: F\mathrm{K}_x$.

We now write down a set Σ of formulas over the language L_S with free variables in the set C. Σ is the union of the following five sets of formulas in classical first order logic:

$$\Sigma_S^{\mathrm{coh}} \tag{13.1}$$

$$\{o(\underline{a})=_B\underline{b} \,:\, (o:A\to B)\in S,\ a\in MA,\ b\in MB,\ (Mo)(a)=b\} \tag{13.2}$$

$$\{(Fp)(\underline{x}) =_{F\mathrm{K}_p} \underline{\mathrm{x}_{x,\,p}} \,:\, x\in\mathcal{Y}_V,\ p\in\mathrm{K}_x|\boldsymbol{K}\} \tag{13.3}$$

$$\{\underline{\varphi}(\underline{\mathcal{X}}) \,:\, \mathrm{Var}(\varphi)\subset\mathcal{X}\subset\mathcal{Y}_V\,,\ N\models_W\varphi[\langle qx\rangle_{x\in\mathcal{X}}]\} \tag{13.4}$$

$$\{\neg(\underline{\psi}(\underline{\mathcal{X}})) \,:\, \mathrm{Var}(\psi)\subset\mathcal{X}\subset\mathcal{Y}_V\,,\ N\not\models_W\psi[\langle qx\rangle_{x\in\mathcal{X}}]\} \tag{13.5}$$

(note that $N\not\models_W\psi[\langle nx\rangle_{x\in\mathcal{X}}]$ is not the same as $N\models_W(\neg\psi)[\langle nx\rangle_{x\in\mathcal{X}}]$!).

Let us understand the free variables in Σ as individual constants. Assume that Σ is consistent (satisfiable); let $(P,\hat{c})_{c\in C}$ be a model of Σ. Then, by (13.1), $P\in\mathrm{Mod}(S)$. By (13.2), $f=\langle f_A\rangle_{A\in S}$ for which $f_A(a)=\hat{\underline{a}}$ $(A\in S,\ a\in MA)$ is a natural transformation $f:M\to P$. By (13.3), $r=\langle r_K\rangle_{K\in\boldsymbol{K}}$ for which $r_K(d) = (\underline{\mathrm{y}^V_{K,\,d}})^\wedge$ whenever $K\in\boldsymbol{K}$, $d\in VK$ is a natural transformation $r:V\to P\restriction\boldsymbol{K}$. Since for $c\in U$, $\underline{\mathrm{y}^V_{K,\,c}} = \underline{m(\mathrm{a}(\mathrm{y}^V_{K,\,c}))} = \underline{m(c)}$, we have the left-hand commutativity in (12). Finally, by (13.4) and (13.5), $(V,r,q):P\xleftrightarrow[*]{}N$ (see (5')). We have verified that the consistency of Σ establishes (11).

Let us prove that Σ is satisfiable. Assume that a finite subset Φ of Σ is not satisfiable. Φ involves a finite number of C-variables. There is a finite context $\mathcal{Y}\subset\mathcal{U}_V$ and a finite set $\mathcal{A}$ of elements $a=(A\in S,\ a\in MA)$ of M such that all formulas in Φ have free variables from $\underline{\mathcal{Y}}\cup\underline{\mathcal{A}}$; $\underline{\mathcal{Y}}=\{\underline{y}:y\in\mathcal{Y}\}$, $\underline{\mathcal{A}}=\{\underline{a}:a\in\mathcal{A}\}$. Let Θ denote the set $\Phi\cap$(13.2) ; for all formulas $\theta\in\Theta$, $\mathrm{Var}(\theta)\subset\underline{\mathcal{A}}$. By increasing Φ, we may assume that it is a subset of

$$\Sigma_S^{\mathrm{coh}} \cup \Theta \cup \mathrm{E}'(\underline{\mathcal{Y}}) \cup \{\underline{\varphi_i}(\underline{\mathcal{U}_i}) : i<m\} \cup \{\neg(\underline{\psi_j}(\underline{\mathcal{V}_j})) : j<n\} \tag{14}$$

where $\mathrm{E}'(\underline{\mathcal{Y}})$ is the set whose union is $\mathrm{E}(\underline{\mathcal{Y}})$, each $\underline{\varphi_i}(\underline{\mathcal{U}_i})$ belongs to (13.4), each $\neg(\underline{\psi_j}(\underline{\mathcal{V}_j}))$ belongs to (13.5), and $\mathcal{U}_i \subset \mathcal{Y}$, $\mathcal{V}_j \subset \mathcal{Y}$. Let $\theta = \bigwedge \Theta$.

The inconsistency of (14) is the same as saying that

$$\mathrm{T}_S^{\mathrm{coh}} \vDash \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{U}_i}) \xrightarrow[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}]{} \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{V}_j}) \quad .$$

By our remarks above (completeness), this is the same as

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \theta \wedge \mathrm{E}(\underline{\mathcal{X}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{U}_i})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{V}_j})] \quad .$$

By (6), this may be rewritten as

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{Y}})] \quad .$$

With $\varphi = \bigwedge_{i<m} \varphi_i$, $\psi = \bigvee_{j<n} \psi_j$, we see that $\underline{\varphi}(\underline{\mathcal{Y}}) \in$(13.4), $\neg(\underline{\psi}(\underline{\mathcal{Y}})) \in$(13.5).

Also using (8), (9), we have

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \underline{\varphi}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\psi}(\underline{\mathcal{Y}})]$$

In other words,

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \theta \wedge \mathrm{E}(\underline{\mathcal{Y}})] \wedge [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\varphi}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\psi}(\underline{\mathcal{Y}})] \ ,$$

and as a consequence, using the Heyting implication in $\mathrm{S}([\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}])$,

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \theta \wedge \mathrm{E}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\varphi}(\underline{\mathcal{Y}})] \longrightarrow [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\psi}(\underline{\mathcal{Y}})] = [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}:\ \underline{\varphi}(\underline{\mathcal{Y}}) \rightarrow \underline{\psi}(\underline{\mathcal{Y}})] \quad .$$

By (10), it follows that

$$[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \theta\wedge \mathrm{E}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ (\underline{\varphi\rightarrow\psi})(\underline{\mathcal{Y}})]$$

and

$$[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \theta] \leq [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\rightarrow\psi})(\underline{\mathcal{Y}})]\ . \tag{15}$$

Let $\mathcal{X}=\mathcal{Y}\cap\mathcal{Y}_{\mathrm{U}}$. We have that $\underline{\mathcal{X}} = \underline{\mathcal{A}}\cap\mathcal{Y}\subset\underline{\mathcal{A}}$, and $\underline{\mathcal{A}}\cup\underline{\mathcal{Y}} = \underline{\mathcal{A}}\ \dot{\cup}\ (\underline{\mathcal{Y}}-\underline{\mathcal{X}})$. Let $\pi:[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}]\rightarrow[\underline{\mathcal{A}}]$ be the projection, let $\tau = \mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\rightarrow\psi})(\underline{\mathcal{Y}})$. As was mentioned above, $\mathrm{Var}(\theta)\subset\underline{\mathcal{A}}$. Using $\pi^{*}\dashv\forall_{\pi}$,

$$\pi^{*}[\underline{\mathcal{A}}:\theta] \leq [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau] \iff [\underline{\mathcal{A}}:\theta] \leq \forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau]\ .$$

By (15), it follows that $[\underline{\mathcal{A}}:\theta] \leq \forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau]$. Now, $\forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau] = [\underline{\mathcal{A}}:\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}})\tau]$. We conclude

$$[\underline{\mathcal{A}}:\ \theta\wedge \mathrm{E}(\underline{\mathcal{X}})] \leq [\underline{\mathcal{A}}:\ \mathrm{E}(\underline{\mathcal{X}})\wedge\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}}).\mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\rightarrow\psi})(\underline{\mathcal{Y}}).]$$

and by (7) ,

$$[\underline{\mathcal{A}}:\ \theta\wedge \mathrm{E}(\underline{\mathcal{X}})] \leq [\underline{\mathcal{A}}:\ (\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\rightarrow\psi)}(\underline{\mathcal{X}})]\ . \tag{16}$$

By the definition of $\mathrm{E}(\underline{\mathcal{X}})$, $M\models\mathrm{E}(\underline{\mathcal{X}})(m(\mathrm{a}(x))/\underline{x})_{x\in\mathcal{X}}$. But, for $x\in\mathcal{X}$, $\underline{x}=\underline{a}$ for $a=m(\mathrm{a}(x))$; thus, $M\models\mathrm{E}(\underline{\mathcal{X}})[a/\underline{a}]_{a\in\mathcal{A}}$. By $\Theta\subset$(13.4) , $M\models\theta[a/\underline{a}]_{a\in\mathcal{A}}$. By (16), we conclude that $M\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\rightarrow\psi)}(\underline{\mathcal{X}})[a/\underline{a}]_{a\in\mathcal{A}}$, that is,

$$M\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\rightarrow\psi)}(\underline{\mathcal{X}})(m(\mathrm{a}(x))/\underline{x})_{x\in\mathcal{X}}\ .$$

By $(U,m,n):M\xrightarrow[*]{}N$, we conclude

$$N\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\rightarrow\psi)}(\underline{\mathcal{X}})[q(\mathrm{a}(x))/\underline{x}]_{x\in\mathcal{X}}$$

(q extends n). By the choice of φ and ψ ,

$$N\models\varphi[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}}\ ,$$

$$N\not\models\psi[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}}\ .$$

Also,

$$N\models\mathrm{E}(\underline{\mathcal{Y}})[[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}}\ .$$

However,

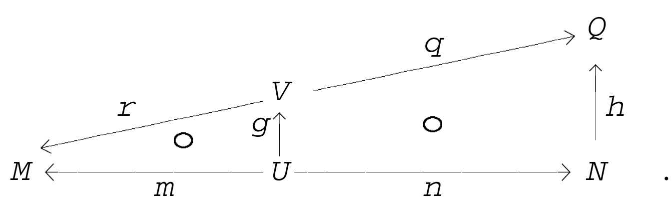
$$[\underline{\mathcal{Y}}:\ (\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}})\wedge\varphi\wedge\mathrm{E}(\underline{\mathcal{Y}})]\le[\underline{\mathcal{Y}}:\ \psi]\ .$$

The last five displays contain a direct contradiction.

This completes the proof of (11).

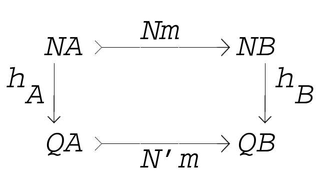
The following is essentially simpler than (11); it is the analog in our context of Lemma 4 of [M4].

(17) Suppose $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(U,m,n):M\overset{}{\underset{*}{\leftrightarrow}}N$. Then we have $Q\in\mathrm{Mod}(T)$, $(h:N\to Q)\in\mathrm{Mod}(T)$, $g:U\to V$ and $(V,r,q):M\underset{*}{\leftrightarrow}Q$ such that r is very surjective, h is pure, and



(12)

($h:N\to Q$ being *pure* means that the naturality squares



corresponding to monomorphisms $m\in T$ are pullbacks.)

Combining (11) and (17) in an "alternating chain" argument (see the proof of Lemma 2 in [M4]), we obtain

(18) Suppose $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(U, m, n):M\xrightarrow[*]{}N$. Then there are $M'\in\mathrm{Mod}(S)$, $N'\in\mathrm{Mod}(T)$, $g:U\to U'$, $f:M\to M'$, $h:N\to N'$ and $(U', m', n'):M'\xleftrightarrow[\#]{}N'$ (in particular, m' and n' are very surjective) such that h is pure, and

$$\begin{array}{ccccc} M' & \xleftarrow{\;m'\;} & U' & \xrightarrow{\;n'\;} & N' \\ f\uparrow & \circ & \uparrow g & \circ & \uparrow h \\ M & \xleftarrow[\;m\;]{} & U & \xrightarrow[\;n\;]{} & N \end{array} \quad . \qquad (18')$$

(Observe the asymmetry; $(U, m, n):M\xrightarrow[*]{}N$, and not the other way around; h, but not f, is required to be pure.)

Let us prove the assertion, contained in (5), that (4) has the interpolation property. Let σ and τ be as in the interpolation property, assume the hypotheses, and also that the conclusion fails. That is,

(19) $(\bar{\alpha}_0)^*_{[\mathcal{X}]}(\bar{I}_0\sigma) \le_{\bar{A}[\mathcal{X}]} (\bar{\alpha}_1)^*_{[\mathcal{X}]}(\bar{I}_1\tau)$;

however,

(20) there is no $\mathcal{L}$-formula θ (of FOLDS) such that $\sigma\le_{F[\mathcal{X}]} F[\mathcal{X}:\theta]$ and $G[\mathcal{X}:\theta]\le_{G[\mathcal{X}]}\tau$.

I claim that (20) implies that

(21) there are $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(F_{\mathcal{X}}, \vec{a}, \vec{b}):M\xrightarrow[*]{}N$ such that $M\models_{\mathrm{w}}\sigma[\vec{a}]$ and $N\not\models_{\mathrm{w}}\tau[\vec{b}]$.

Let $x \mapsto \underline{x}$ be a 1-1 map of variables $x \in \mathcal{X}$ into variables over L_T, $\underline{x}: GK_X$. Let θ range over $\boldsymbol{L}$-formulas with $\mathrm{Var}(\theta) \subset \mathcal{X}$. $E'[\underline{\mathcal{X}}]$, $\underline{\theta}(\underline{\mathcal{X}})$ and $\underline{\tau}(\underline{\mathcal{X}})$ were defined before. Consider the set

$$\Sigma_T^{\mathrm{coh}} \cup E'[\underline{\mathcal{X}}] \cup \{\underline{\theta}(\underline{\mathcal{X}}) : \sigma \leq_{F[\mathcal{X}]} F[\mathcal{X}:\theta]\} \cup \{\neg(\underline{\tau}(\underline{\mathcal{X}}))\} \ . \tag{22}$$

If this were inconsistent, we would easily conclude that there is θ with $G[\mathcal{X}:\theta] \leq_{G[\mathcal{X}]} \tau$, contrary to (20). Let $(N; \bar{x}/\underline{x})_{x \in \mathcal{X}}$ be a model for (19). Next, let $x \mapsto \underset{\sim}{x}$ be a 1-1 map of variables $x \in \mathcal{X}$ into variables over L_S, $\underset{\sim}{x}: FK_X$, and consider

$$\Sigma_S^{\mathrm{coh}} \cup E'[\underset{\sim}{\mathcal{X}}] \cup \{\neg(\underline{\theta}(\underset{\sim}{\mathcal{X}})) : (N; \bar{x}/\underline{x})_{x \in \mathcal{X}} \not\models_{\mathrm{w}} \underline{\theta}(\underline{\mathcal{X}})\} \cup \{\underline{\sigma}(\underset{\sim}{\mathcal{X}})\} \ . \tag{23}$$

This is easily seen to be consistent by the fact that $(N; \bar{x}/\underline{x})_{x \in \mathcal{X}}$ is a model of (22). Now, if $(M; \tilde{x}/\underset{\sim}{x})_{x \in \mathcal{X}}$ is a model of (23), then with $\vec{a} = \langle \tilde{x} \rangle_{x \in \mathcal{X}}$, $\vec{b} = \langle \bar{x} \rangle_{x \in \mathcal{X}}$ we have (21).

Now, apply (18) to $(F_{\mathcal{X}}, \vec{a}, \vec{b}) : M \underset{*}{\longrightarrow} N$ as $(U, m, n) : M \underset{*}{\longrightarrow} N$; we obtain that

(24) there are $M' \in \mathrm{Mod}(S)$, $N' \in \mathrm{Mod}(T)$ and $(V, m, n) : M \underset{\#}{\leftrightarrow} N$ such that $M \models_{\mathrm{w}} \sigma[\vec{a}]$ and $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.

(Indeed, h being pure ensures that $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.) On the other hand, by (24) and the universal property of (4), there is $P: S +_{\#} T \to \mathrm{Set}$ such that $P\bar{I}_0 = M$, $P\bar{I}_1 = N$ and $P(\bar{A}, \bar{\alpha}_0, \bar{\alpha}_1) = (V, m, n)$. Applying these to (19), we get $m^*_{[\mathcal{X}]}(M\sigma) \leq_{V[\mathcal{X}]} n^*_{[\mathcal{X}]}(N\tau)$, which contradicts the conjunction of $M \models_{\mathrm{w}} \sigma[\vec{a}]$ and $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.

It remains to prove the other assertion of (4), namely that the comparison J is conservative.

For any small coherent category R, we have the evaluation functor $e: R \to \mathrm{Set}^{\mathrm{Mod}(R)}$, a conservative coherent functor, and if R is Heyting, e is Heyting (Kripke-Joyal theorem; see [M4]).

We show, in analogy to Proposition 7 of [M4], that

(25) For $R=S+_{\#}T$, the composites $T\xrightarrow{\bar{I}_1}R\xrightarrow{e}\mathrm{Set}^{\mathrm{Mod}(R)}$,

$S\xrightarrow{\bar{I}_0}R\xrightarrow{e}\mathrm{Set}^{\mathrm{Mod}(R)}$ are Heyting.

The argument is similar to that in *loc.cit.* We deal with the first composite; the second is symmetric. Upon an analysis similar to that in *loc.cit.*, we see that what we need is this:

given $k:A\to B$ in T, $X\in S(A)$, $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$, $u=(U,m,n):M\underset{\#}{\leftrightarrow}N$, $y\in NB$ such that $y\notin N(\forall_k X)$,

there are $M'\in\mathrm{Mod}(S)$, $N'\in\mathrm{Mod}(T)$, $u'=(U',m',n):M'\underset{\#}{\leftrightarrow}N'$ and $(f:M\to M', h:N\to N', g:U\to U'):(M,N,u)\to(M',N',u')$, an arrow in $\mathrm{C}^{\#}_{\mathrm{Set}}$, and $x\in N'(A)-N'(X)$ such that $h_B(y)=(N'k)(x)$.

As in *loc.cit.*, we have $N^*\in\mathrm{Mod}(T)$, $h^*:N\to N^*$, $x^*\in N^*(A)-N^*(X)$ such that $h^*_B(y)=(N^*k)(x^*)$. We build a commutative diagram

$$\begin{array}{ccccc}
M' & \xleftarrow{\;m'\;} & U' & \xrightarrow{\;n'\;} & N' \\
{\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h'} \\
M & \xleftarrow[\;m\;]{} & U & \xrightarrow[(h^*\restriction\boldsymbol{K})\circ n]{} & N^* \\
{\scriptstyle 1_M}\uparrow & & {\scriptstyle 1_U}\uparrow & & \uparrow{\scriptstyle h^*} \\
M & \xleftarrow[\;m\;]{} & U & \xrightarrow[\;n\;]{} & N
\end{array} .$$

The lower half is already constructed. The important remark is that $(U,m,n):M\underset{*}{\leftrightarrow}N$ implies that $(U,m,(h^*\restriction\boldsymbol{K})\circ n):M\underset{*}{\longrightarrow}N$. Then, by (18), we have the rest such that, in addition, $(U',m',n):M'\underset{\#}{\leftrightarrow}N'$ and h' is pure. Taking the vertical composites, in particular $h=h'\circ h^*$, and $x=(h')_A(x^*)$, noting the purity of h^*, we have what we want.

Having (25), the proof of the conservativeness of $\mathcal{J}$ is as in *loc.cit.*

This completes the proof of (5) and (1).

The results proved may be applied to characterizations of formulas invariant under equivalence of categories, of diagrams of categories, and of bicategories, in category theory done in intuitionistic set-theory. However, the condition of being invariant under equivalence cannot, in most cases, be stated by using the traditional concept of equivalence. Note that in the proofs of 6.(5), 6.(23), 7.(5), one direction (passing from an $\mathbf{L}$-equivalence to a categorical equivalence) uses the Axiom of Choice, not available in intuitionistic set-theory. [M2] introduces "ana"-versions of certain concepts, among others functors of categories and functors of bicategories, that can be used in this context. The condition of invariance under categorical equivalence has to be strengthened, in general, to invariance under categorical anaequivalence, to have the characterizations analogous to the ones we proved for classical logic.

Let us note that statement (5), being in essence of a syntactical (arithmetical) nature, can be proved constructively, in intuitionistic set theory, by a general transfer result of H. Friedman [Fr]; thus, (5) is available when doing category theory intuitionistically. However, to be able to apply (5), the assumption of invariance under equivalence has to be available in the "provable" sense.

In the case of equivalence of categories, essentially because now there is no need to pass to a notion of "anacategory", we do have the direct analog of 6.(3) for intuitionistic logic. In particular:

(25') Let $\varphi(\mathcal{X})$ be a first-order formula on a finite diagram $\mathcal{X}$ of objects and arrows in the language of categories. Suppose that it is provable in intuitionistic set-theory that the property of $\varphi(\mathcal{X})$ being true is preserved and reflected along equivalence functors. Then there is a formula $\theta(\mathcal{X})$ in FOLDS over $\mathbf{L}_{\mathtt{cat}}$ such that $\forall\mathcal{X}(\varphi\longleftrightarrow\theta^{*})$ is provable in intuitionistic predicate calculus from the axioms of category (here, θ^* is the usual translate of θ into ordinary multisorted logic, given in §1).

In the rest of this Appendix, we discuss (simple) Craig interpolation and Beth definability for FOLDS.

For specificity, we consider FOLDS in the sense of classical FOLDS with (restricted) equality; theories *etc.* below are to be understood accordingly.

First, let us put ourselves in the context of Appendix A. Suppose $\mathbf{L}_1$ is a vocabulary. A *subvocabulary* of $\mathbf{L}_1$ is a subset $\mathbf{L}$ of $\mathbf{L}_1$ which itself is a vocabulary. Note that the set-theoretical intersection and union of any number of vocabularies are always again vocabularies.

In terms of the terminology of §1, instead of the above notions, we would use the following. Let $\mathbf{L}$, $\mathbf{L}_1$ DSV's, $i:\mathbf{L}\to\mathbf{L}_1$ a functor. I call i an *inclusion of* DSV's if it is (a) 1-1 on objects, (b) for any object R of the category $\mathbf{L}$, $R\in\mathrm{Rel}(\mathbf{L})$ iff $iR\in\mathrm{Rel}(\mathbf{L}_1)$, and (c) for every $A\in\mathbf{L}$, i induces a bijection $A|\mathbf{L}\to iA|\mathbf{L}_1$. Obviously, i preserves levels. A *sub-DSV* $\mathbf{L}$ of $\mathbf{L}_1$ is given by an inclusion $i:\mathbf{L}\to\mathbf{L}_1$ of DSV's for which i acts as the identity (i is a "real" inclusion). If we have inclusions $i_1:\mathbf{L}\to\mathbf{L}_1$, $i_2:\mathbf{L}\to\mathbf{L}_2$, we may consider the pushout $\mathbf{L}_1+_{\mathbf{L}}\mathbf{L}_2$; as a category, it is a pushout in the ordinary sense; the relations of $\mathbf{L}_1+_{\mathbf{L}}\mathbf{L}_2$ are defined to be the images of those of $\mathbf{L}_1$ and $\mathbf{L}_2$; clearly, the coprojections $\mathbf{L}_1\to\mathbf{L}_1+_{\mathbf{L}}\mathbf{L}_2$, $\mathbf{L}_2\to\mathbf{L}_1+_{\mathbf{L}}\mathbf{L}_2$ are inclusions too.

Let us use the terminology of Appendix A. Suppose that T_1 is a theory in FOLDS over $\mathbf{L}_1$, and $\mathbf{L}\subset\mathbf{L}_1$. Then $T_1\restriction\mathbf{L}$ denotes the theory $(\mathbf{L},\mathrm{Cn}_{\mathbf{L}}(T_1))$, where $\mathrm{Cn}_{\mathbf{L}}(T_1)$ is the set of $\mathbf{L}$-consequences (in classical FOLDS) of T_1. (A small point to make here is that an $\mathbf{L}$-formula is not necessarily an $\mathbf{L}_1$-formula, despite the fact that $\mathbf{L}\subset\mathbf{L}_1$. The reason is that a kind K in $\mathbf{L}$ may be maximal in $\mathbf{L}$, but not maximal in $\mathbf{L}_1$, in which case equality on K is allowed in FOLDS over $\mathbf{L}$, but not in FOLDS over $\mathbf{L}_1$. The definition of $\mathrm{Cn}_{\mathbf{L}}(T_1)$ is that it is the set of all $\mathbf{L}$-sentences *which are also* $\mathbf{L}_1$*-sentences*, and which are consequences of T_1.) If T_i is a theory over $\mathbf{L}_i$ (i=1,2), then $T_1\cup T_2$ is the theory over $\mathbf{L}_1\cup\mathbf{L}_2$ for which $\Sigma_{T_1\cup T_2}=\Sigma_{T_1}\cup\Sigma_{T_2}$. When two theories S_1 and S_2 are over the same language $\mathbf{L}$, then $S_1\cup S_2$ is also over $\mathbf{L}$.

In the §1 terminology, when T_i is a theory over $\mathbf{L}_i$ (i=1,2), we can define the "pushout" theory $T_1 +_{\mathbf{L}} T_2$ in the obvious way.

We will revert to the Appendix-A terminology.

Craig Interpolation for classical FOLDS. Suppose $\mathbf{L}_1$, $\mathbf{L}_2$ are vocabularies (for FOLDS), $\mathbf{L} = \mathbf{L}_1 \cap \mathbf{L}_2$, T_i is a theory over $\mathbf{L}_i$ $(i=1, 2)$. Then $T_1 \cup T_2$ is consistent if and only if $(T_1 \upharpoonright \mathbf{L}) \cup (T_2 \upharpoonright \mathbf{L})$ is consistent.

Of course, only the "if" part requires proof.

Let us illustrate the meaning of the above statement of the Craig interpolation theorem for FOLDS.

Suppose σ_i is a sentence over $\mathbf{L}_i$ $(i=1, 2)$, and $\sigma_1 \vDash \sigma_2$. Consider T_1 over $\mathbf{L}_1$ whose single axiom is σ_1, and T_2 over $\mathbf{L}_2$ whose single axiom is $\neg\sigma_2$. Then, $T_1 \cup T_2$ is inconsistent; hence, so is $(T_1 \upharpoonright \mathbf{L}) \cup (T_2 \upharpoonright \mathbf{L})$. This means that there are sentences θ_1, θ_2 over $\mathbf{L}$ such that $\sigma_1 \vDash \theta_1$, $\neg\sigma_2 \vDash \theta_2$ and $\{\theta_1, \theta_2\}$ is inconsistent; but then $\sigma_1 \vDash \theta_1$ and $\theta_1 \vDash \sigma_2$; we have the usual form of interpolation.

There is a generalization of the above statement of interpolation, obtained by allowing individual constants in the theories. A *vocabulary* $\mathbf{L}$ *with individual constants* is a set of the form $\mathbf{L} = \mathbf{L}_0 \cup \mathcal{C}$, where $\mathbf{L}_0$ is a vocabulary, and $\mathcal{C}$ is a (not necessarily finite) context of variables (individual constants) such that for $c \in \mathcal{C}$, $\mathrm{K}_c \in \mathbf{L}_0$. Intersection and union of vocabularies with individual constants is again such. An $\mathbf{L}$*-sentence* is an $\mathbf{L}_0$-formula with all free variables in $\mathcal{C}$. A *structure* M for $\mathbf{L}$ is one, say M_0, for $\mathbf{L}_0$, together with an interpretation of the $\mathcal{C}$-symbols: some $\langle a_c \rangle_{c \in \mathcal{C}} \in M_0[\mathcal{C}]$. For an $\mathbf{L}$-sentence φ, $M \vDash \varphi \underset{\mathrm{def}}{\Longleftrightarrow} M_0 \vDash \varphi[\langle a_c \rangle_{c \in \mathcal{C}}]$. A *theory over* $\mathbf{L}$ is given by any set of $\mathbf{L}$-sentences; a model of the theory is an $\mathbf{L}$-structure satisfying all the axioms. Now, all the terms in the above statement of the Craig interpolation theorem have natural meanings when $\mathbf{L}_1$, $\mathbf{L}_2$ are vocabularies with individual constants; the theorem remains correct in the generalized form.

In the well-known manner, the Beth definability theorem can be deduced from Craig interpolation, by using individual constants. We obtain

Beth definability theorem for FOLDS. Suppose T is a theory in FOLDS, $\mathbf{L}\subset\mathbf{L}_T$, $\mathcal{X}$ is a finite context for $\mathbf{L}$, and φ is an $\mathbf{L}_T$-formula with $\mathrm{Var}(\varphi)\subset\mathcal{X}$. Suppose that for any two models M_1, M_2 of T, if $M_1\restriction\mathbf{L} = M_2\restriction\mathbf{L}$, then $M_1[\mathcal{X}:\varphi]=M_2[\mathcal{X}:\varphi]$. Then there is an $\mathbf{L}$-formula θ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that $M[\mathcal{X}:\varphi] = M[\mathcal{X}:\theta]$ for all models M of T.

For the proof, make two copies $\mathbf{L}_1$, $\mathbf{L}_2$ of the vocabulary $\mathbf{L}_T$, by renaming all kinds and relations $A\in\mathbf{L}_T-\mathbf{L}$ in two distinct ways as A_1 and A_2, and by putting $\mathbf{L}_i=\mathbf{L}\cup\{A_i:A\in\mathbf{L}_T-\mathbf{L}\}$; $\mathbf{L}_1\cap\mathbf{L}_2=\mathbf{L}$. For any $\mathbf{L}\cup\{\mathcal{X}\}$-sentence ψ, we have the $\mathbf{L}_i\cup\{\mathcal{X}\}$-sentence ψ_i, with the same free variables (in $\mathcal{X}$), obtained by the appropriate renaming. Applied to all members of Σ_T, this gives Σ_i, a set of $\mathbf{L}_i$-sentences. Consider the theories $T_1=(\mathbf{L}_1\cup\mathcal{X}, \Sigma_1\cup\{\varphi_1\})$, $T_2=(\mathbf{L}_2\cup\mathcal{X}, \Sigma_2\cup\{\neg\varphi_2\})$ over vocabularies $\mathbf{L}_1\cup\mathcal{X}$, $\mathbf{L}_2\cup\mathcal{X}$ with individual constants. Craig interpolation applied for T_1 and T_2 gives the desired conclusion.

We make some preparations for the proof of the Craig interpolation theorem.

Recall our definition of saturation in §5. We make some modifications on it.

Let us fix the DSV $\mathbf{L}$; $\mathbf{K}$ is its category of kinds. First of all, in contrast to §5, we now want to deal with logic with equality; formulas now may have equality. The definitions up to " $\mathcal{Y}$-$\mathbf{L}$-saturated " remain the same, except for the change in what counts as a formula. Consider a context $\mathcal{Y}$, and a $\mathcal{Y}$-set Φ of formulas; all formulas in Φ have variables in the context $\mathcal{Y}\dot{\cup}\{x\}$. Let us say that Φ is *low* if K_x is *low*, that is, it is not a maximal element of $\mathbf{K}$. This is the same as to say that no equality predicate is allowed on K_x.

The $\mathbf{L}$-structure M is said to be *strictly* $\mathcal{Y}$-$\mathbf{L}$-*saturated* if for every $\vec{a}\in M[\mathcal{Y}]$ and every $\mathcal{Y}$-set Φ, if Φ is finitely satisfiable in $(M, \vec{a})$, then (1) Φ is satisfiable in $(M, \vec{a})$, and (2) if Φ is a low set, then Φ is satisfiable by an element a for which $a\neq a_y$ for all $y\in\mathcal{Y}$; here, $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}$. We say that M is *strictly* κ-$\mathbf{L}$-*saturated* if it is strictly $\mathcal{Y}$-saturated for all $\mathcal{Y}$ of cardinality $<\kappa$.

There are two issues: existence and uniqueness; let's deal with existence first. To that end, we give a simple general construction.

Let M, N be $\boldsymbol{L}$-structures. We write $M\prec_{\boldsymbol{L}}N$ if M is a subfunctor of N (note that both M and N are functors $\boldsymbol{L}\to\mathrm{Set}$), and for any $\mathcal{X}$, $\vec{a}\in M[\mathcal{X}]$ $(\subset N[\mathcal{X}])$, $M\models\varphi[\vec{a}]$ iff $N\models\varphi[\vec{a}]$.

(26) Let M be any $\boldsymbol{L}$-structure, K a low kind, $\vec{a}\in M[K]$, and $MK(\vec{a})\neq\emptyset$. We can construct another structure N such that $M\prec_{\boldsymbol{L}}N$ and $MK(\vec{a})\subsetneqq NK(\vec{a})$.

For simplicity, we assume that M is separated (the MK are pairwise disjoint). Let $b\in MK(\vec{a})$.

Let $U=M\restriction\boldsymbol{K}$. Construct $V:\boldsymbol{K}\to\mathrm{Set}$ as follows. Say of $x\in|U|$ that it is *above* b if there is $f:K'\to K$ (possibly the identity) such that $(Uf)x=b$. Note that

(27) if $g:K_1\to K_2$, $x_1\in UK_1$ and $x_2=(Ug)(x_1)$, then if x_2 is above b, so is x_1.

Introduce a new element $\bar{x}$ for every x above b, distinct from each other and from the elements of U. Put $VK'=UK'\dot{\cup}\{\bar{x}:x\in UK'\text{ above }b\}$. The effect of V on arrows is defined so that U is a subfunctor of V, and by the following determinations. For $g:K_1\to K_2$, $x_1\in UK_1$ above b, let $x_2=(Ug)(x_1)$; $\bar{x}_1\in VK_1\xmapsto[\mathrm{def}]{Vg}\bar{x}_2$ if x_2 is above b, $\bar{x}_1\in VK_1\xmapsto[\mathrm{def}]{Vg}x_2$ otherwise. It is easy to see, using (27), that V is a functor, we have the inclusion $i:U\to V$, and we have the retraction $r:V\to U$ for which $\bar{x}\xmapsto{r}x$; $ri=1_U$. I claim that r is very surjective. If $\vec{y}=\langle y_p\rangle_{p\in K|\boldsymbol{K}}\in V[K]$, $\vec{y}\xmapsto{r}\vec{x}$, $x\in UK(\vec{x})$, then if x is not above b, then no y_p is above b and $x\in VK(\vec{y})$, and of course $x\xmapsto{r}x$; but if x is above b, then $\bar{x}\in VK(\vec{y})$, and of course $\bar{x}\xmapsto{r}x$.

Returning to M, using the very surjective $r:V\to U$, define $N=r^{*}M$ (see §5). When we regard M and N as structures for $\boldsymbol{L}^{\mathrm{eq}}$, with standard equality for the equality predicates, then still $N=r^{*}M$. This amounts to the following: if K' is a maximal kind, $\vec{y}\in V[K']$, $y_1,y_2\in VK'(\vec{y})$, $\vec{y}\xmapsto{r}\vec{x}$, $y_1\xmapsto{r}x_1$, $y_2\xmapsto{r}x_2$, then $x_1=x_2$ implies $y_1=y_2$. If

$x_1=x_2$, the only way $y_1 \neq y_2$ could be the case is that x_1 is above b, $y_1=x_1$ and $y_2=\bar{x}_1$ (or the other way around). However, if so, then since $K' \neq K$ (K is low), we have $p:K' \to K$ proper such that $(Up)x_1=b$, hence, $(Up)\bar{x}_1=\bar{b}$, and, since $b \neq \bar{b}$, $y_1=x_1$, $y_2=\bar{x}_1$ cannot both be in $VK'(\vec{y})$ for the same $\vec{y}$, contradiction.

We have, by 5.(1), that θ_r is elementary (with respect to logic over $\boldsymbol{L}^{\mathrm{eq}}$ *without* equality; i.e., with respect to logic over $\boldsymbol{L}$ *with* equality). Combining this with $ri=1_U$, we immediately obtain that $\theta_i:M \to N$ is elementary, that is, $M \prec_{\boldsymbol{L}} N$ as desired. This proves (26).

The usual proof of the existence of saturated models (see [CK]), using unions of elementary chains, is now easily supplemented by uses of (26) to provide

(28) For any infinite cardinal $\kappa \geq \#\boldsymbol{L}$ ($\boldsymbol{L}$ any vocabulary with individual constants), any consistent theory T over $\boldsymbol{L}$ has a strictly κ^+, $\boldsymbol{L}$–saturated model of cardinality $\leq 2^\kappa$.

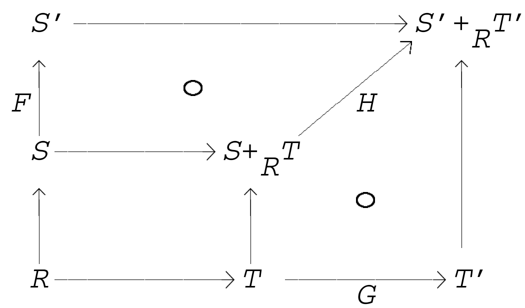
(29) If M, N are strictly κ, $\boldsymbol{L}$–saturated $\boldsymbol{L}$-structures, $M \equiv_{\boldsymbol{L}} N$, both of cardinality $\leq \kappa$, then they are isomorphic.

Proof. Inspecting the proof of 5.(4), we see that we can make both maps m and n bijective. This suffices.

Proof of Craig. Suppose $(T_1 \restriction \boldsymbol{L}) \cup (T_2 \restriction \boldsymbol{L})$ is consistent. Let M be a model of it; M is an $\boldsymbol{L}$-structure. Let Σ be the set of all sentences in FOLDS over $\boldsymbol{L}$ that are true in M; $T=(\boldsymbol{L}, \Sigma)$. Both $T_1 \cup T$ and $T_2 \cup T$ are consistent; if not, we would have (say) $\tau \in \Sigma$ such that $T_1 \models \neg\tau$; but then, by definition, $\neg\tau \in \Sigma_{T_1 \restriction \boldsymbol{L}}$, hence $M \models \neg\tau$; contradiction to $\tau \in \Sigma$.

Choose $\lambda \geq \#\boldsymbol{L}_1$, $\geq \#\boldsymbol{L}_2$ such that $\kappa=\lambda^+=2^\lambda$. By (28), let $M_i \models T_i \cup T$ $(i=1, 2)$ strictly κ, $\boldsymbol{L}_i$-saturated, of cardinality $\leq\kappa$. Then $M_i \restriction \boldsymbol{L}$ is also strictly κ, $\boldsymbol{L}_i$–saturated, of cardinality $\leq\kappa$. By (29), there is an isomorphism $f:M_1 \restriction \boldsymbol{L} \xrightarrow{\cong} M_2 \restriction \boldsymbol{L}$. There is M_2' and an isomorphism $g: M_2' \xrightarrow{\cong} M_2$ such that $M_2' \restriction \boldsymbol{L} = M_1 \restriction \boldsymbol{L}$ (and $g \restriction \boldsymbol{L} = f$). But then the $\boldsymbol{L}_1 \cup \boldsymbol{L}_2$-structure N for which $N \restriction \boldsymbol{L}_1 = M_1$, $N \restriction \boldsymbol{L}_2 = M_2'$, is a model of $T_1 \cup T_2$.

Finally, let us note that Craig interpolation and Beth definability hold for intuitionistic FOLDS. Looking at the above formulation for classical FOLDS, we are led to the following formulation:



F, G conservative $\Longrightarrow$ H conservative

This is to be understood in a suitable doctrine. Above we proved, in essence, this in the doctrine of $\wedge\vee\neg\exists$-fibrations (see §3) restricted to fibrations obtained from simple base-categories as described in §4 , with arrows restricted to inclusions as defined above. The claim is that the same holds when we switch to $\wedge\vee\rightarrow\exists\forall$-fibrations. The proof is along the lines we presented in the first part of this Appendix.