Appendix D: Calculations for §7.

D1. Define the *generalized* DS vocabulary $\boldsymbol{L}_{\texttt{2-cat}}$ as the full subcategory of $\boldsymbol{L}_{\texttt{anabicat}}$ on the objects of $\mathtt{L}_{\texttt{2-cat}}$, with relations $\dot{\mathtt{I}}_1$, $\dot{\mathtt{I}}_2$, $\dot{\mathtt{T}}_1$, $\dot{\mathtt{H}}$, $\dot{\mathtt{T}}_2$; it is *generalized* since a non-maximal object, $\mathtt{T}_1$, is also made into a relation. Accordingly, an $\boldsymbol{L}_{\texttt{2-cat}}$-structure is a functor from $\boldsymbol{L}_{\texttt{2-cat}}$ in which the listed relations (including $\mathtt{T}_1$) are interpreted relationally. This is the picture for $\boldsymbol{L}_{\texttt{2-cat}}$:

$$\begin{tikzcd}
 & \dot{\mathtt{T}}_1 \arrow[d, shift right=3, "t_{10}"'] \arrow[d, shift right=1, "t_{11}"] \arrow[d, shift left=2, "t_{12}"] & \dot{\mathtt{H}} \arrow[l, shift right, "h_0"'] \arrow[l, shift left, "h_1"] \arrow[d, shift right=3, "h_2"'] \arrow[d, shift right=1, "h_3"] \arrow[d, shift left=2, "h_4"] & \\
\mathtt{C}_0 & \mathtt{C}_1 \arrow[l, shift right, "c_{10}"'] \arrow[l, shift left, "c_{11}"] & \mathtt{C}_2 \arrow[l, shift right, "c_{20}"'] \arrow[l, shift left, "c_{21}"] & \dot{\mathtt{T}}_2 \arrow[l, shift right=2, "t_{20}"'] \arrow[l, "t_{21}"] \arrow[l, shift left=2, "t_{22}"] \\
 & \dot{\mathtt{I}}_1 \arrow[u] & \dot{\mathtt{I}}_2 \arrow[u] &
\end{tikzcd}$$

A *2-category-sketch* (2-cat-sketch) is, by definition, a structure of type $\boldsymbol{L}_{\texttt{2-cat}}$; maps of 2-cat-sketches are natural transformations of functors. For a 2-cat-sketch S , $|S|$ is its underlying 2-graph, its reduct to

$$\begin{tikzcd}
\mathtt{C}_0 & \mathtt{C}_1 \arrow[l, shift right, "c_{10}"'] \arrow[l, shift left, "c_{11}"] & \mathtt{C}_2 \arrow[l, shift right, "c_{20}"'] \arrow[l, shift left, "c_{21}"]
\end{tikzcd} .$$

Any bicategory has an underlying 2-cat-sketch. We will look at maps $S \longrightarrow \mathcal{A}$, $S \in \texttt{2-catSk}$, $\mathcal{A}$ a bicategory.

Let $S \underset{N}{\overset{M}{\rightrightarrows}} \mathcal{A}$. A *transformation* $\tau : M \to N$ is given by

(i) $\tau_X : MX \to NX$ for each $X \in S(\mathtt{C}_0)$;

(ii) for each $(f : X \to Y) \in S(\mathtt{C}_1)$, $\tau_f : Nf \circ \tau_X \xrightarrow{\cong} \tau_Y \circ Mf$ as in

$$\begin{tikzcd}
MX \arrow[r, "\tau_X"] \arrow[d, "Mf"'] \arrow[dr, phantom, "\overset{\tau_f}{\swarrow}\cong"] & NX \arrow[d, "Nf"] \\
MY \arrow[r, "\tau_Y"'] & NY
\end{tikzcd}$$

such that

(a) for any $X \overset{f}{\underset{g}{\underset{\varphi\Downarrow}{\rightrightarrows}}} Y$ in S,

$$\begin{tikzcd}
Nf\circ\tau_X \arrow[r, "\tau_f"] \arrow[d, "N\varphi\circ\tau_X"'] \arrow[dr, phantom, "\circ"] & \tau_Y\circ Mf \arrow[d, "\tau_Y\circ M\varphi"] \\
Ng\circ\tau_X \arrow[r, "\tau_g"'] & \tau_Y\circ Mg
\end{tikzcd};$$

(b) $(f\colon X\to Y)\in S(I_1) \Longrightarrow \tau_f = 1_{\tau_X}$; and

(c) for every $\begin{tikzcd}[column sep=small] & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd} \in S(\mathrm{T}_1)$ (note that $MgMf=Mh$, $NgNf=Nh$),

$$\begin{tikzcd}
MA \arrow[r, "Mf"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\overset{\tau_f}{\swarrow}"] & MB \arrow[r, "Mg"] \arrow[d, "\tau_B"'] \arrow[dr, phantom, "\overset{\tau_g}{\swarrow}"] & MC \arrow[d, "\tau_C"'] \\
NA \arrow[r, "Nf"'] & NB \arrow[r, "Ng"'] & NC
\end{tikzcd}
=
\begin{tikzcd}
MA \arrow[r, "Mh"] \arrow[d, "\tau_A"'] \arrow[dr, phantom, "\overset{\tau_h}{\swarrow}"] & MC \arrow[d, "\tau_C"'] \\
NA \arrow[r, "Nh"'] & NC
\end{tikzcd},$$

that is,

$$\begin{tikzcd}
(\tau_C Mg)Mf \arrow[r, "\tau_g Mf"] & (Ng\tau_B)Mf \arrow[dr, phantom, "\circ"] & Ng(\tau_B Mf) \arrow[l, "\alpha"'] \arrow[d, "\tau_f Ng"] \\
\tau_C(MgMf) \arrow[u, "\alpha"] \arrow[r, "\tau_h"'] & (NgNf)\tau_A & Ng(Nf\tau_A) \arrow[l, "\alpha"]
\end{tikzcd}.$$

Given $S \overset{M}{\underset{N}{\underset{\Downarrow\tau}{\rightrightarrows}}} \mathcal{X}$ and $\Phi\colon T\longrightarrow S$, we have $T \overset{M\Phi}{\underset{N\Phi}{\underset{\Downarrow\tau\Phi}{\rightrightarrows}}} \mathcal{X}$ for which $(\tau\Phi)_f = \tau_{\Phi f}$ for $f\in T(\mathrm{C}_1)$.

D2. Going back to the definition of $\mathcal{R}\mathrm{T}_1$ in part (B) of the proof of 7.5, and using the notation there, that definition can be put as follows. Consider the 2-cat-sketch S_0:

$$\begin{tikzcd} B \arrow[r, "g"] & C \\ A \arrow[u, "f"] \arrow[ur, "gf"] \arrow[ur, shift right=2, "i"'] & \end{tikzcd} \qquad a\colon gf \xrightarrow{\cong} i \qquad \left(S_0(T_1)=\{(f,g,gf)\}\ ,\ S_0(C_2)=\{a,a^{-1},1_i,1_{gf}\} \right)$$

,

and the two diagrams $S_0 \overset{\Phi}{\underset{\Psi}{\rightrightarrows}} \mathcal{A}$ defined as

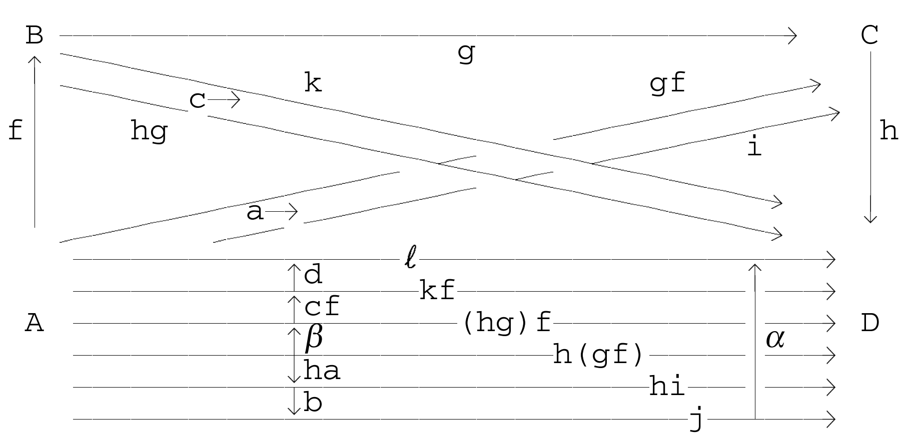
$$\Phi = \begin{tikzcd} FY \arrow[r, "Fg"] & FZ \\ FX \arrow[u, "Ff"] \arrow[ur, "FgFf"] \arrow[ur, shift right=2, "Fi"'] & \end{tikzcd} \quad ; \qquad \Psi = \begin{tikzcd} B \arrow[r, "\bar{g}"] & C \\ A \arrow[u, "\bar{f}"] \arrow[ur, "\bar{g}\bar{f}"] \arrow[ur, shift right=2, "\bar{i}"'] & \end{tikzcd} \quad .$$

$$\Phi a = FaF_{f,\,g} : FgFf \xrightarrow{\cong} Fi \qquad\qquad \Psi a = \bar{a} : \bar{g}\bar{f} \xrightarrow{\cong} \bar{i}$$

Then $\mathcal{R}\mathrm{T}_1(\varphi, \gamma, \eta)[a, \bar{a}]$ iff $x, y, z, \varphi, \gamma, \iota$ are the components of a map $\Phi \to \Psi$.

D3. In what follows, we will consider the following 2-cat-sketch S and various of its parts (subsketches):

S =



$S(\mathrm{T}_1)$ has six elements, (f, g, gf), $(gf, h, h(gf))$, (i, h, hi), (g, h, hg), $(f, hg, h(gf))$, (f, k, kf); the notations showing composition are purely symbolic. The horizontal compositions cf and ha signify the presence of elements "1_f" and "1_h" of $S(\mathrm{I}_2)$, and two corresponding elements of $S(\mathrm{H})$. $S(\mathrm{I}_1)=\emptyset$. There are further 2-cells and elements of $S(\mathrm{I}_2)$ and $S(\mathrm{T}_2)$ to the effect that a, b, c, d, α and β are isomorphisms, and α is the composite $d(cf)\beta(ha)^{-1}b^{-1}$.

In case of a general 2-cat-sketch S, for a sketch-map $M:S\to\mathcal{X}$ and a functor $F:\mathcal{X}\to\mathcal{A}$ of bicategories, the composite FM cannot be defined (think of a sketch in which a 1-cell is a composite in two different ways); in the case of our S however, since S is sufficiently "free", a useful sense can be ascribed to FM. First of all, for S_0 from D2, for $M:S_0\to\mathcal{X}$, $F:\mathcal{X}\to\mathcal{A}$, FM is defined as Φ was above: for

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ X \arrow[u, "f"] \arrow[ur, "gf"] \arrow[ur, shift right=2, "i"'] & \end{tikzcd}$$

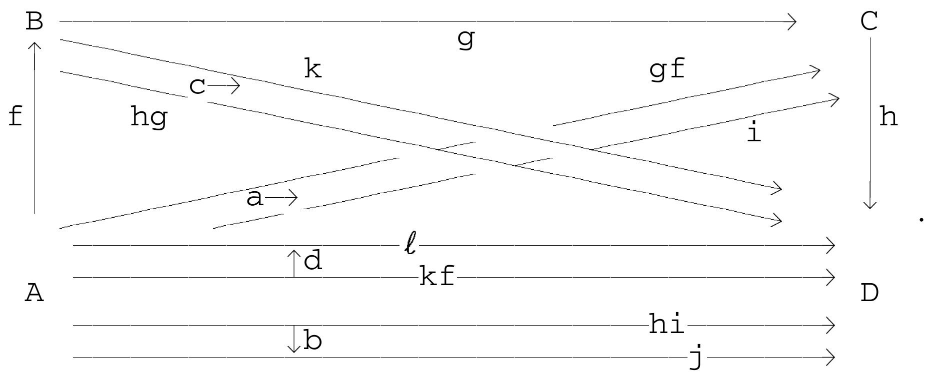
$$a:gf\xrightarrow{\cong}i$$

as M, we put FM to be

$$FM = \begin{tikzcd} FY \arrow[r, "Fg"] & FZ \\ FX \arrow[u, "Ff"] \arrow[ur, "FgFf"] \arrow[ur, shift right=2, "Fi"'] & \end{tikzcd} .$$

$$Fa=FaF_{f,\,g}:FgFf\xrightarrow{\cong}Fi$$

Now, there are four mappings of the form $S_0\to S$, corresponding to the four items $a:gf\xrightarrow{\cong}i$, $b:hi\xrightarrow{\cong}j$, $c:hg\xrightarrow{\cong}k$, $d:kf\xrightarrow{\cong}\ell$. We define, for any $M:S\to\mathcal{X}$ and $F:\mathcal{X}\to\mathcal{A}$, $FM:S\to\mathcal{A}$ as follows. First, we make sure that for any of the four maps $\sigma:S_0\to S$, $(FM)\sigma = F(M\sigma)$. This requirement determines FM as far as its restriction to the subsketch



is concerned. But then the effect of FM is uniquely determined on the items h(gf), (hg)f, cf, ha . Next, we define $(FM)(\beta)$ so that the following diagram commutes; we wrote f, g, h for Mf , Mg , Mh :

$$
\begin{tikzcd}
(FhFg)Ff \arrow[r, "F_{h,g}Ff"] & F(hg)Ff \arrow[r, "F_{f,hg}"] & F((hg)f) \\
Fh(FgFf) \arrow[u, "(FM)(\beta)"] \arrow[r, "FhF_{f,g}"'] & FhF(gf) \arrow[u, phantom, "\circ"] \arrow[r, "F_{gf,h}"'] & F(h(gf)) \arrow[u, "F(M\beta)"']
\end{tikzcd}
\quad .
$$

Finally, the effect of FM on α in S is now uniquely determined. It is worth noting that if $M\beta = \alpha_{f,\,g,\,h}$, then $(FM)(\beta) = \alpha_{Ff,\,Fg,\,Fh}$ ($f{=}M$f , etc.); the reason is that F "preserves" α (see above).

I **claim** that, for $FM\colon S \to \mathcal{A}$ so defined, $(FM)(\alpha) = F(M(\alpha))$. This is demonstrated by the following commutative diagram:

$$
\begin{tikzcd}
 & FkFf \arrow[r, "F_{f,k}"] \arrow[rr, bend left, "(FM)(\alpha)"] & F(kf) \arrow[r, "Fd"'] \arrow[ur, phantom, "4"] & F\ell \\
(FhFg)Ff \arrow[ur, "(FM)(c)"] \arrow[r, "F_{h,g}Ff"] \arrow[u, phantom, "2"] & F(hg)Ff \arrow[u, "FcFf"'] \arrow[r, "F_{f,hg}"'] \arrow[ur, phantom, "6"] & F((hg)f) \arrow[u, "F(cf)"] & \\
Fh(FgFf) \arrow[u, "(FM)(\beta)"] \arrow[r, "FhF_{f,g}"'] & FhF(gf) \arrow[u, phantom, "1"] \arrow[r, "F_{gf,h}"'] & F(h(gf)) \arrow[u, "F(M\beta)"'] \arrow[uur, "F(M(\alpha))"'] \arrow[ur, phantom, "8"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[drr, "(FM)(a)"'] & \arrow[r, phantom, "3"] & {} \arrow[d, "FhFa"] \arrow[r, phantom, "7"] & {} \arrow[d, "F(ha)"'] & {} \arrow[d, dash] \\
 & & FhFi \arrow[r, "F_{i,\, h}"'] \arrow[rr, bend right, "(FM)(b)"'] & F(hi) \arrow[r, "Fb"'] \arrow[d, phantom, "5"] & Fj \\
 & & & {} &
\end{tikzcd}
$$

Here, the cell 1 commutes by the definition of $(FM)(\beta)$; $2, 3, 4, 5$ commute by the definition of FM on the 2-cells a, b, c, d; 6 and 7 by the naturality of $F_{\_,\,\_}$; and 8 by the fact that $M\alpha$ is the appropriate composite. The assertion is the commutativity of the outside perimeter of the diagram.

D4. Let S_1 be the following subsketch of S :

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "hg"' near end] & C \arrow[d, "h"] \\
A \arrow[u, "f"] \arrow[ur, "gf" near start] \arrow[r, shift left, "(hg)f"'] \arrow[r, shift right=2, "h(gf)"'] & D
\end{tikzcd}
$$

($S_1(t)=\emptyset$ for all $t\in L_{2\text{-cat}}$, except for $t=C_0, C_1, T_1$) , and let S_2 be the sketch (subsketch of S) obtained by adding the 2-cell $\alpha\colon h(gf) \longrightarrow (hg)f$ to S_1 . Suppose we have $M, N\colon S_2 \to \mathcal{A}$ such that $M\alpha = \alpha_{Mf,\, Mg,\, Mh}$ and $N\alpha = \alpha_{Nf,\, Ng,\, Nh}$ (associativity isomorphisms), and, also writing M for $M\restriction S_1$, we have

$$
\begin{tikzcd}
S_1 \arrow[r, shift left=2, "M"] \arrow[r, shift right=2, "N"'] \arrow[r, phantom, "\downarrow\tau"] & \mathcal{A}
\end{tikzcd}
\tag{1}
$$

Then τ is a map with respect to S_2 , that is,

$$
\begin{tikzcd}
S_2 \arrow[r, shift left=2, "M"] \arrow[r, shift right=2, "N"'] \arrow[r, phantom, "\downarrow\tau"] & \mathcal{A}
\end{tikzcd}
\ .
$$

This fact expresses the naturality of the associativity isomorphism in a sense that is considerably stronger than the one required in the definition of bicategory. The proof of the assertion is contained in the diagram

$$
\begin{tikzcd}
(\bar h(\bar g\bar f))\tau_A \arrow[rr] & & ((\bar h\bar g)\bar f)\tau_A & & \\
\bar h((\bar g\bar f)\tau_A) \arrow[u] \arrow[r] & \bar h(\bar g(\bar f\tau_A)) \arrow[r] & (\bar h\bar g)(\bar f\tau_A) \arrow[u] & & \mathit{IV} \\
\mathit{I} & \bar h(\bar g(\tau_B f)) \arrow[u, "\bar h(\bar g\tau_f)"] \arrow[d] \arrow[r] & (\bar h\bar g)(\tau_B f) \arrow[u, "(\bar h\bar g)\tau_f"'] \arrow[r] & ((\bar h\bar g)\tau_B)f & \\
& \bar h((\bar g\tau_B)f) \arrow[rr] & & (\bar h(\bar g\tau_B))f \arrow[u] & \\
& h((\tau_C g)f) \arrow[u, "\bar h(\tau_g f)"] \arrow[rr] & & (\bar h(\tau_C g))f \arrow[u, "(\bar h\tau_g)f"'] & \mathit{II} \\
& \bar h(\tau_C(gf)) \arrow[u] \arrow[r] \arrow[uuuul, "\bar h(\tau_{gf})"] & (\bar h\tau_C)(gf) \arrow[r] & \bar h(\tau_C(gf)) \arrow[u] & \\
\mathit{III} & & (\tau_D h)(gf) \arrow[u, "\tau_h(gf)"] \arrow[r] & ((\tau_D h)g)f \arrow[u, "(\tau_h g)f"'] & (\tau_D(hg))f \arrow[l] \arrow[uuuul, "\tau_{hg}f"'] \\
& & \tau_D(h(gf)) \arrow[u] \arrow[uuuuuuull, bend left=60, "\tau_{h(gf)}"] & & \tau_D((hg)f) \arrow[ll] \arrow[u] \arrow[uuuuuuull, bend right=60, "\tau_{(hg)f}"']
\end{tikzcd}
$$

in which t is written for Mt, $\bar t$ for Nt, for all relevant values of t, and all unmarked arrows are instances of associativity isomorphisms, possibly horizontally composed with a 1-cell. The issue is the commutativity of the outside quadrangle. The four cells marked I, II, III and IV commute by the definition of τ being a map as in (1). The commutativity of the pentagons are the associativity coherence axioms for bicategory; the commutativity of the small quadrangles are instances of the (ordinary) naturality of the associativity isomorphism. Since all cells commute, the outside commutes as a consequence, and this is what we want.

D5. Now, start with the part (subsketch) S_3

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "k" near start] & C \arrow[d, "h"] \\
A \arrow[u, "f"] \arrow[ur, "i"' near start] \arrow[r, shift left, "\ell"] \arrow[r, shift right, "j"'] & D
\end{tikzcd}
$$

of S ($S_3(t)=\emptyset$ for all $t\in L_{2\text{-cat}}$, except for $t=C_0, C_1$), and a map

$$S_3 \underset{N}{\overset{M}{\underset{\downarrow\sigma}{\rightrightarrows}}} \mathcal{A} \ . \qquad (2)$$

It is clear that if we have any $T \underset{Q}{\overset{P}{\underset{\downarrow\theta}{\rightrightarrows}}} \mathcal{A}$, and T' is the sketch obtained by adding a new element " $gf=h$ " to $T(\mathrm{T}_1)$, where f and g are already in T, but h is new, then P, Q and θ uniquely extend to $T' \underset{Q}{\overset{P}{\underset{\downarrow\theta}{\rightrightarrows}}} \mathcal{A}$. Now, let S_4 be the part of S which is S without the 2-cells ($S_4(t)=S(t)$ for $t=\mathrm{C}_0,\mathrm{C}_1,\mathrm{T}_1$ and $S_4(t)=\emptyset$ otherwise). Applying the above remark four times, we have, a unique extension

$$S_4 \underset{N}{\overset{M}{\underset{\downarrow\sigma}{\rightrightarrows}}} \mathcal{A}$$

of (2).

D6. Suppose T is a sketch, T' is a subsketch of T missing only some 2-cells and T_2-elements of T, and that T is generated by T' in the sense that T is the least subsketch T'' of T such that T'' contains T' and every time when $(\rho,\ \sigma.\ \theta)\in T(\mathrm{T}_2)$, $\rho,\ \sigma\in T''(\mathrm{C}_2)$, then $\theta\in T''(\mathrm{C}_2)$, and every time when $(\rho,\ \sigma.\ \theta)\in T(\mathrm{H})$, $\rho,\ \sigma\in T''(\mathrm{C}_2)$, then $\theta\in T''(\mathrm{C}_2)$. Then every transformation $T' \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}$ is also one as in $T \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}$. This is immediate.

D7. Let us turn to the proof that $\mathcal{R}$ preserves A . What we need to show is this. Assume that we have

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "k"] & Z \arrow[d, "h"] \\ X \arrow[u, "f"] \arrow[ur, "i"'] \arrow[r, shift left, "\ell"] \arrow[r, shift right, "j"'] & W \end{tikzcd} \text{ in } \mathcal{X}, \qquad \begin{tikzcd} B \arrow[r, "\bar{g}"] \arrow[dr, "\bar{k}"] & C \arrow[d, "\bar{h}"] \\ A \arrow[u, "\bar{f}"] \arrow[ur, "\bar{i}"'] \arrow[r, shift left, "\bar{\ell}"] \arrow[r, shift right, "\bar{j}"'] & D \end{tikzcd} \text{ in } \mathcal{A} ,$$

the items listed under (*) in §7, the further items

$$(w\colon FW \xrightarrow{\simeq} D) \in \mathcal{R}\mathrm{C}_0[W, D] \ ,$$
$$(b\colon hi \xrightarrow{\cong} j) \in \mathcal{X}^{\#}\mathrm{T}_1(i, h, j) \ , \quad (\bar{b}\colon \bar{h}\bar{i} \xrightarrow{\cong} \bar{j}) \in \mathcal{A}^{\#}\mathrm{T}_1(\bar{i}, \bar{h}, \bar{j}) \ ,$$
$$(c\colon hg \xrightarrow{\cong} k) \in \mathcal{X}^{\#}\mathrm{T}_1(g, h, k) \ , \quad (\bar{c}\colon \bar{h}\bar{g} \xrightarrow{\cong} \bar{k}) \in \mathcal{X}^{\#}\mathrm{T}_1(\bar{g}, \bar{h}, \bar{k}) \ ,$$
$$(d\colon kf \xrightarrow{\cong} \ell) \in \mathcal{X}^{\#}\mathrm{T}_1(f, k, \ell) \ , \quad (\bar{d}\colon \bar{k}\bar{f} \xrightarrow{\cong} \bar{\ell}) \in \mathcal{X}^{\#}\mathrm{T}_1(\bar{f}, \bar{k}, \bar{\ell}) \ ,$$
$$\eta \in \mathcal{R}\mathrm{C}_1(z, w)[h, \bar{h}] \ , \quad \psi \in \mathcal{R}\mathrm{C}_1(x, w)[j, \bar{j}] \ , \quad \kappa \in \mathcal{R}\mathrm{C}_1(y, w)[k, \bar{k}] \ ,$$
$$\lambda \in \mathcal{R}\mathrm{C}_1(x, w)[\ell, \bar{\ell}] \ ;$$

and assume that

$$\mathcal{R}\mathrm{T}_1(\varphi, \gamma, \iota)[a, \bar{a}] \ , \quad \mathcal{R}\mathrm{T}_1(\iota, \eta, \psi)[b, \bar{b}] \ ,$$
$$\mathcal{R}\mathrm{T}_1(\gamma, \eta, \kappa)[c, \bar{c}] \ , \quad \mathcal{R}\mathrm{T}_1(\varphi, \psi, \lambda)[d, \bar{d}]$$

hold. Under these conditions, we want that if $\mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}]$, then

$$\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha) \iff \mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha}) \ .$$

I claim that it suffices to show that

$$\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha) \ \text{ and } \ \mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha}) \ \text{ imply } \ \mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}] \ .$$

We use that for the given a, b, c, d , there is a unique α such that $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha)$ (see (4) in §7), and similarly for $\bar{a}, \bar{b}, \bar{c}, \bar{d}$; and we use that for the given $x, w; \psi, \lambda$, the relation $\mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}]$ of the variables $\alpha, \bar{\alpha}$ establishes a bijection $\alpha \mapsto \bar{\alpha} : \mathcal{X}\mathrm{C}_2(j, \ell) \xrightarrow{\cong} \mathcal{A}\mathrm{C}_2(\bar{j}, \bar{\ell})$. The claim now is easily seen.

Thus, we assume $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha)$ and $\mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha})$.

Recall the sketch S . The data give us diagrams $M_0 : S \longrightarrow \mathcal{X}$, $N : S \longrightarrow \mathcal{A}$; the effect of

M_0 , N are given by the notation, except that $M_0\beta = \alpha_{f,\ g,\ i}$ (associativity iso in $\mathcal{X}$) and $N\beta = \alpha_{\bar{f},\ \bar{g},\ \bar{i}}$ (associativity iso in $\mathcal{A}$). Composing M_0 with F, we get $M{=}FM_0 : S \longrightarrow \mathcal{A}$ (see D3). Consider the restrictions $M{:}\, S_3 \to \mathcal{A}$, $N{:}\, S_3 \to \mathcal{A}$. The data $x,\ y,\ z,\ w,\ \varphi,\gamma,\iota,\eta,\psi,\kappa,\lambda$ supply the components of a map

$$S_3 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}\ .$$

By D5, we have a unique extension of τ, also denoted by τ, as in

$$S_4 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}\ .$$

Let S_5 be the subsketch of S that consists of S_4, and the 2-cells $a,\ b,\ c,\ d$. The assumptions and D2 (applied to the four maps $S_0 \to S$) tell us that we have

$$S_5 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}\ .$$

Now, add also β back to S_5, getting S_6. Since by D3,

$$M\beta = (FM_0)(\beta) = \alpha_{Ff,\ Fg,\ Fh} = \alpha_{Mf,\ Mg,\ Mh}\ ,$$

D4 says that we have

$$S_6 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}\ ,$$

and finally D6 says that

$$S \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}\ .$$

The fact that τ is natural with respect to α is the desired fact $\mathcal{R}C_2(x,\ w;\ \psi,\ \lambda)[\alpha,\ \bar{\alpha}]$, since, by D3, $M\alpha = F(M_0\alpha)$.

D8. The proof that $(\mathcal{R}, r_0, r_1)$ preserves H is similar, and simpler. Now, the situation is this. We have

$$\begin{tikzcd}
 & Y \arrow[dr, shift left=2, "h"] \arrow[dr, shift right=2, "i"'] \arrow[dr, phantom, "\overset{\delta}{\Leftarrow}"] & \\
X \arrow[ur, shift left=2, "f"] \arrow[ur, shift right=2, "g"'] \arrow[ur, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[rr, shift left=2, "k"] \arrow[rr, shift right=2, "j"'] \arrow[rr, phantom, "\Uparrow\varepsilon"] & & Z
\end{tikzcd}$$

in $\mathcal{X}$, and

$$\begin{tikzcd}
 & B \arrow[dr, shift left=2, "\bar{h}"] \arrow[dr, shift right=2, "\bar{i}"'] \arrow[dr, phantom, "\overset{\bar{\delta}}{\Leftarrow}"] & \\
A \arrow[ur, shift left=2, "\bar{f}"] \arrow[ur, shift right=2, "\bar{g}"'] \arrow[ur, phantom, "\overset{\bar{\beta}}{\Rightarrow}"] \arrow[rr, shift left=2, "\bar{k}"] \arrow[rr, shift right=2, "\bar{j}"'] \arrow[rr, phantom, "\Uparrow\bar{\varepsilon}"] & & C
\end{tikzcd}$$

in $\mathcal{A}$; we have

$(x{:}FX\xrightarrow{\simeq}A)\in\mathcal{R}\mathrm{C}_0[X, A]$, $(y{:}FY\xrightarrow{\simeq}B)\in\mathcal{R}\mathrm{C}_0[Y, B]$, $(z{:}FZ\xrightarrow{\simeq}C)\in\mathcal{R}\mathrm{C}_0[Z, C]$,
$\varphi\in\mathcal{R}\mathrm{C}_1(x, y)[f, \bar{f}]$, $\eta\in\mathcal{R}\mathrm{C}_1(y, z)[h, \bar{h}]$, $\psi\in\mathcal{R}\mathrm{C}_1(x, z)[j, \bar{j}]$,
$\gamma\in\mathcal{R}\mathrm{C}_1(x, y)[g, \bar{g}]$, $\iota\in\mathcal{R}\mathrm{C}_1(y, z)[i, \bar{i}]$, $\kappa\in\mathcal{R}\mathrm{C}_1(x, z)[k, \bar{k}]$,
$s\in\mathcal{X}^{\#}\mathrm{T}_1(f, h, j)$, $t\in\mathcal{X}^{\#}\mathrm{T}_1(g, i, k)$, $\bar{s}\in\mathcal{A}^{\#}\mathrm{T}_1(\bar{f}, \bar{h}, \bar{j})$, $\bar{t}\in\mathcal{A}^{\#}\mathrm{T}_1(\bar{g}, \bar{i}, \bar{k})$

such that

$$\mathcal{R}\mathrm{C}_2(x, y; \varphi, \gamma)[\beta, \bar{\beta}] \text{ , } \mathcal{R}\mathrm{C}_2(y, z; \eta, \iota)[\delta, \bar{\delta}] \text{ ,} \tag{3}$$

$$\mathcal{R}\mathrm{T}_1(\varphi, \eta, \psi)[s, \bar{s}] \text{ and } \mathcal{R}\mathrm{T}_1(\gamma, \iota, \kappa)[t, \bar{t}] \text{ .} \tag{4}$$

Under these conditions, we want that

$$\mathcal{R}C_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}] \Longrightarrow (\mathcal{X}^{\#}H(s, t; \beta, \delta, \varepsilon) \Longleftrightarrow \mathcal{A}^{\#}H(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}))\ .$$

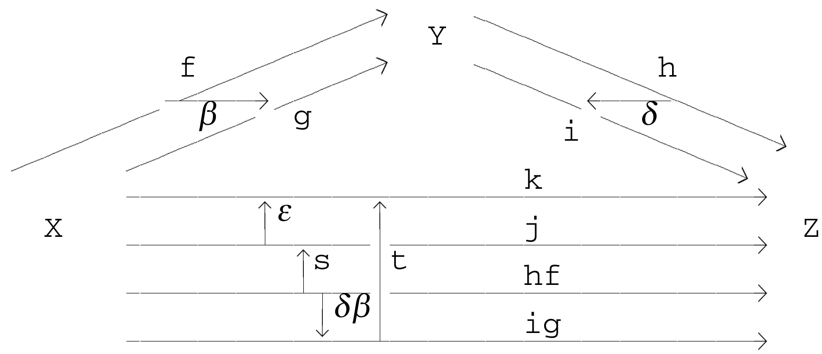
Again, it suffices to show that

$$\mathcal{X}^{\#}H(s, t; \beta, \delta, \varepsilon) \text{ and } \mathcal{A}^{\#}H(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}) \tag{5}$$

imply $\mathcal{R}C_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}]$. (6)

Assume (5). Consider the 2-cat-sketch

$T =$



We have (f, h, hf), $(g, i, ig) \in T(T_1)$, $(\beta, \delta, \delta\beta) \in T(H)$, and $\begin{array}{ccc} hf & \xrightarrow{s} & j \\ \delta\beta \downarrow & \circ & \downarrow \varepsilon \\ ig & \xrightarrow[t]{} & k \end{array}$ (the latter by an (unmarked) 2-cell σ , and (s, ε, σ), $(\delta\beta, t, \sigma) \in T(T_2)$) .

The conditions in (5) ensure that the data we have give rise to morphisms $M_0 : T \to \mathcal{X}$, $N: T \to \mathcal{A}$. As in the case of the sketch S , we can form the composite $M = FM_0 : T \to \mathcal{A}$; we have $M(s) = Fs \circ F_{f,\, h}$, $M(t) = Ft \circ F_{g,\, i}$; the commutativity of the diagram

$$\begin{tikzcd}
FhFf \arrow[r, "F\delta F\beta"] \arrow[d, "F_{f,\, h}"'] \arrow[dr, phantom, "\circ"] & FiFg \arrow[d, "F_{g,\, i}"] \\
F(hf) \arrow[r, "F(\delta\beta)"] \arrow[d, "Fs"'] \arrow[dr, phantom, "\circ"] & F(ig) \arrow[d, "Ft"] \\
F(j) \arrow[r, "F\varepsilon"'] & F(k)
\end{tikzcd}$$

ensures that M is indeed $M: T \to \mathcal{A}$. Consider the following subsketches of T:

$$
T_1 = \begin{tikzcd}[column sep=huge, row sep=large]
 & Y \arrow[dr, shift left=2, "h"] \arrow[dr, shift right=2, "i"'] \arrow[dr, phantom, "\overset{\delta}{\Leftarrow}"] & \\
X \arrow[ur, shift left=2, "f"] \arrow[ur, shift right=2, "g"'] \arrow[ur, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[rr, shift left=2, "k"] \arrow[rr, shift right=2, "j"'] & & Z
\end{tikzcd}
$$

$$
T_2 = \begin{tikzcd}[column sep=huge, row sep=large]
 & Y \arrow[dr, shift left=2, "h"] \arrow[dr, shift right=2, "i"'] \arrow[dr, phantom, "\overset{\delta}{\Leftarrow}"] & \\
X \arrow[ur, shift left=2, "f"] \arrow[ur, shift right=2, "g"'] \arrow[ur, phantom, "\overset{\beta}{\Rightarrow}"] \arrow[rr, shift left=6, "k"] \arrow[rr, shift left=2, "j"] \arrow[rr, shift right=2, "hf"] \arrow[rr, shift right=6, "ig"'] \arrow[rr, phantom, shift left=4, "\Uparrow s"] \arrow[rr, phantom, "\Uparrow t"] & & Z
\end{tikzcd}
$$

The data x, y, z, φ, γ, η, ι, ψ, κ give, via the relation (3), a map

$$
\begin{tikzcd}
T_1 \arrow[r, shift left=2, "M"] \arrow[r, shift right=2, "N"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{A}
\end{tikzcd},
$$

which, by (4) and D2, uniquely extends to

$$
\begin{tikzcd}
T_2 \arrow[r, shift left=2, "M"] \arrow[r, shift right=2, "N"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{A}
\end{tikzcd}.
$$

By D6, this extends to

$$
\begin{tikzcd}
T \arrow[r, shift left=2, "M"] \arrow[r, shift right=2, "N"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{A}
\end{tikzcd}.
$$

The naturality of τ with respect to ε is the desired relation (6).