Appendix C: More on $\boldsymbol{L}$-equivalence and equality.

Ordinary multisorted first-order logic without equality and without operation symbols (only relations are allowed) is a special case of FOLDS as follows. Let L be a multisorted, purely relational vocabulary. We associate a DSV $\boldsymbol{L}$ with L. The kinds of $\boldsymbol{L}$ are the sorts of L; the relations of $\boldsymbol{L}$ are the relation symbols of L. For R is sorted " $R\subset\prod_{i<n}X_i$ ", we have proper arrows $p_i^R: R\to X_i\ \ (i<n)$. This completes the description of $\boldsymbol{L}$. Clearly, the $\boldsymbol{L}$-structures are essentially the same as the L-structures.

$\boldsymbol{L}$ just constructed is a very simple DSV; its category of kinds has height 1.

Now, a natural notion of "isomorphism" for L-structures "without equality" is the ordinary notion of isomorphism modified by dropping single-valuedness and 1-1-ness. Let M, N be L-structures. By definition, $h: M\overset{\sim}{\underset{L}{\leftrightarrow}}N$ means a family of relations $h_X: MX\longrightarrow\!\!\!\!\!\!|\ \ NX$ ($X\in\mathrm{Sort}(L)$) such that $\mathrm{dom}(h_X)=MX$, $\mathrm{range}(h_X)=NX$, and for any " $R\subset\prod_{i<n}X_i$ " in L, $\vec{a}=\langle a_i\rangle_{i<n}\in\prod_{i<n}MX_i$, $\vec{b}=\langle b_i\rangle_{i<n}\in\prod_{i<n}NX_i$, we have that $a_i h_{X_i} b_i$ for all $i<n$ (briefly, $\vec{a}h\vec{b}$) implies that $\vec{a}\in MR\iff\vec{b}\in NR$. It is pretty clear that $h: M\overset{\sim}{\underset{L}{\leftrightarrow}}N$ preserves the meaning of L-formulas *without equality*: $\vec{a}h\vec{b}\Longrightarrow (M\models\varphi[\vec{a}]\iff N\models\varphi[\vec{a}])$; this would hold good for infinitary logic, and other extended notions of "formula". It is also clear that if for each sort X of L, there is a relation " $\mathrm{E}_X\subset X\times X$ " whose interpretation in both M and N is ordinary equality on X, then $h: M\overset{\sim}{\underset{L}{\leftrightarrow}}N$ is the same as an ordinary isomorphism $M\xrightarrow{\cong}N$.

The last-mentioned notion of "relational isomorphism" coincides with the relational version of $\boldsymbol{L}$-equivalence, for $\boldsymbol{L}$ the DSV constructed for L as above, defined as follows. For a general DSV $\boldsymbol{L}$, we call the $\boldsymbol{L}$-equivalence $(W, m, n): M\underset{\boldsymbol{L}}{\leftrightarrow}N$ *relational* if m and n are jointly monomorphic; we indicate the said quality by the letter r in $(W, m, n): M\overset{\mathrm{r}}{\underset{\boldsymbol{L}}{\leftrightarrow}}N$. This means that for every kind K in $\boldsymbol{L}$, the pair (m_K, n_K) of functions is jointly monomorphic, that is, the span $MK\xleftarrow{m_K}WK\xrightarrow{n_K}NK$ is a relation.

For simplicity, we deal with Set-valued structures in what follows. Suppose $(W, m, n): M \xleftrightarrow[\mathbf{L}]{r} N$. For each kind K, define the relation $\rho_K \subset MK \times NK$ by $a\rho_K b \iff \exists c \in WK.\, m_K c = a \wedge n_K c = b$. For $\mathcal{X}$ a finite context, $\vec{a} = \langle a_x \rangle_{x \in \mathcal{X}} \in M[\mathcal{X}]$, $\vec{b} = \langle b_x \rangle_{x \in \mathcal{X}} \in N[\mathcal{X}]$, we write $\vec{a}\rho_{\mathcal{X}}\vec{b} \iff \exists \vec{c} \in W[K].\, m\vec{c} = \vec{a} \wedge n\vec{c} = \vec{b}$. It turns out however that $\vec{a}\rho_{\mathcal{X}}\vec{b} \iff \forall x \in \mathcal{X}.\, a_x \rho_{K_x} b_x$. Indeed, the left-to-right direction is obvious. Conversely, let $c_x \in WK_x$ such that $m_{K_x} c_x = a_x \wedge n_{K_x} c_x = b_x$. I claim that $\vec{c} = \langle c_x \rangle_{x \in \mathcal{X}} \in W[K]$. For this, we need that if $y \in \mathcal{X}$, $p \in K_y | \mathbf{K}$, then

$$c_{x_{y,p}} = (Wp)(c_y) \ . \qquad (1)$$

But $m(Wp)(c_y) = (Mp)(mc_y) = (Mp)(a_y) = a_{x_{y,p}}$, and similarly $n(Wp)(c_y) = b_{x_{y,p}}$; since $c = c_{x_{y,p}} \in WK_{x_{y,p}}$ is uniquely determined by the property $m(c) = a_{x_{y,p}}$ & $n(c) = b_{x_{y,p}}$, (1) follows.

As a consequence, a relational equivalence can be described in terms of the relations ρ_K as follows. A *relational equivalence* $\rho: M \xleftrightarrow[\mathbf{L}]{r} N$ is a family $\rho = \langle \rho_K \rangle_{K \in \mathbf{K}}$ of relations $\rho_K \subset MK \times NK$ such that, with

$$\vec{a}\rho_{\mathcal{X}}\vec{b} \overset{\text{def}}{\iff} \forall x \in \mathcal{X}.\, a_x \rho_{K_x} b_x , \qquad (2)$$

the following hold:

(3) For any $p: K \to K_p$, $a \in MK$, $b \in NK$

$$a\rho_K b \implies (Mp)(a)\rho_{K_p}(Np)(b) \ .$$

(4) For any $K \in \mathbf{K}$, $\vec{a} \in M[K] = M[\mathcal{X}_K]$, $\vec{b} \in N[K] = N[\mathcal{X}_K]$,

$$\vec{a}\rho_{\mathcal{X}_K}\vec{b} \ \& \ a \in MK(\vec{a}) \implies \exists b \in NK(\vec{b}).\, \vec{a}a\rho_{\mathcal{X}_K^*}\vec{b}b .$$

$$\vec{a}\rho_{\mathcal{X}_K}\vec{b} \ \& \ b \in NK(\vec{b}) \implies \exists a \in MK(\vec{a}).\, \vec{a}a\rho_{\mathcal{X}_K^*}\vec{b}b .$$

(5) For any relation R in $\mathbf{L}$, and $\vec{a}\in M[R]=M[\mathcal{X}_R]$, $\vec{b}\in N[R]=N[\mathcal{X}_R]$,

$$\vec{a}\rho_{\mathcal{X}_R}\vec{b} \implies (\vec{a}\in MR \iff \vec{b}\in NR) .$$

(the notations $\mathcal{X}_K$, $\mathcal{X}_K^*$, $\mathcal{X}_R$ are from §4; $\vec{a}a$ denotes $\langle d_x\rangle_{x\in\mathcal{X}_K^*}\in M[\mathcal{X}_K^*]$ for which $d_x=a_x$ when $x\in\mathcal{X}_K$, and $d_{x_K}=a$).

By what we said above, every $(W, m, n):M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$ gives rise to a $\rho:M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$ ((3) is naturality, (4) is the very surjective condition, (5) is the preservation of relations). Conversely, given $\rho:M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$, putting $WK=\{\langle K, a, b\rangle: a\rho_K b\}$, $m_K(\langle K, a, b\rangle)=a$, $n_K(\langle K, a, b\rangle)=b$ gives $(W, m, n):M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$.

We can make some steps towards Infinitary First Order Logic with Dependent Types. (We refer to [Ba] as a basic reference on infinitary logic and back-and-forth systems.) Let us fix the DSV $\mathbf{L}$ as before. The syntax of the logic $\mathbf{L}_{\infty,\omega}$ of FOLDS over $\mathbf{L}$ with arbitrary (set) size conjunction and disjunction, and finite quantification should be obvious; as usual, we only allow formulas that have finitely many free variables. To fix ideas, we consider logic without equality. $M\equiv_{\mathbf{L}_{\infty,\omega}}N$ means that M and N satisfy the same $\mathbf{L}_{\infty,\omega}$-sentences without equality. We have the following "back-and-forth" characterization of the relation $\equiv_{\mathbf{L}_{\infty,\omega}}$. A *weak relational $\mathbf{L}$-equivalence* $\rho:M\overset{r}{\underset{\mathbf{L}_{\infty,\omega}}{\longleftrightarrow}}N$ is a system $\rho=\langle\rho_{\mathcal{X}}\rangle_{\mathcal{X}}$ of relations $\rho_{\mathcal{X}}\subset M[\mathcal{X}]\times N[\mathcal{X}]$, indexed by all finite contexts, satisfying the following conditions (6)-(9):

(6) for any specialization $s:\mathcal{X}\longrightarrow\mathcal{Y}$, $\vec{a}\in M[\mathcal{Y}]$, $\vec{b}\in N[\mathcal{Y}]$,

$$\vec{a}\rho_{\mathcal{Y}}\vec{b} \implies (\vec{a}\circ s)\rho_{\mathcal{X}}(\vec{b}\circ s) ;$$

here, if $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}$, then $\vec{a}\circ s=\langle a_{s(x)}\rangle_{x\in\mathcal{X}}$.

(7) $\emptyset\rho_{\emptyset}\emptyset$ holds.

(8) For any finite contexts $\mathcal{X}$, $\mathcal{X}\dot{\cup}\{x\}$, $\vec{a}\in M[\mathcal{X}]$, $\vec{b}\in N[\mathcal{X}]$,

$$\vec{a}\rho_{\mathcal{X}}\vec{b} \ \& \ a \in MK(\vec{a}) \implies \exists b \in NK(\vec{b}) . \vec{a}a\rho_{\mathcal{X}\dot{\cup}\{x\}}\vec{b}b \ ,$$
$$\vec{a}\rho_{\mathcal{X}}\vec{b} \ \& \ b \in NK(\vec{b}) \implies \exists a \in MK(\vec{a}) . \vec{a}a\rho_{\mathcal{X}\dot{\cup}\{x\}}\vec{b}b \ .$$

(9) = (5)

We say that M and N are *weakly* $\boldsymbol{L}$*-equivalent*, $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$, if there is $\rho : M \xleftarrow[\boldsymbol{L}_{\infty,\,\omega}]{\mathrm{r}} N$.

Given $\rho : M \xleftarrow[\boldsymbol{L}]{\mathrm{r}} N$, then, with making the definitions as in (2), we also have $\rho : M \xleftarrow[\boldsymbol{L}_{\infty,\,\omega}]{\mathrm{r}} N$. The reader will see that in the case of ordinary multisorted logic, the definition of weak relational $\boldsymbol{L}$-equivalence reduces to the well-known concept of "back-and-forth system" that figures in the characterization of ∞,ω-equivalence. Thus, the following generalizes that characterization.

(10)(a) For $\boldsymbol{L}$-structures M and N, $M \equiv_{\boldsymbol{L}_{\infty,\,\omega}} N$ iff $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$.

(b) For countable $\boldsymbol{L}$-structures M and N, $M \equiv_{\boldsymbol{L}_{\infty,\,\omega}} N$ iff $M \sim_{\boldsymbol{L}} N$.

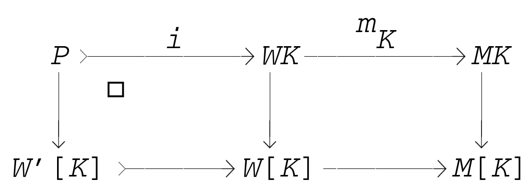
(c) For any countable $\boldsymbol{L}$, and countable $\boldsymbol{L}$-structure M, there is a ("Scott"-)sentence σ_M of $\boldsymbol{L}_{\omega_1,\,\omega}$ such that $N \equiv_{\boldsymbol{L}_{\infty,\,\omega}} M$ iff $N \models \sigma_M$.

The proofs are routine variants of those of the classical cases.

There is a simple categorical restatement of the notion of weak $\boldsymbol{L}$-equivalence. Consider $\boldsymbol{B} = (\mathrm{Set}^{\boldsymbol{K}})^{\mathrm{op}}_{\mathrm{fin}}$ as before. An $\boldsymbol{L}$*-pseudo-structure* P is a functor $\boldsymbol{B} \to \mathrm{Set}$, together with a subset $P(R) \subset P([\mathcal{X}])$ for each relation R of $\boldsymbol{L}$. A morphism of $\boldsymbol{L}$-pseudo-structures is a natural transformation of functors $\boldsymbol{B} \to \mathrm{Set}$ preserving each R in the obvious sense. Each $\boldsymbol{L}$-structure M can be regarded as a pseudo-structure, since any functor $\boldsymbol{K} \to \mathrm{Set}$ has a canonical extension $\boldsymbol{B} \to \mathrm{Set}$ which is in fact finite-limit preserving. Let $\mathrm{PStr}(\boldsymbol{L})$ be the category of pseudo-structures. We have a forgetful functor $\mathcal{E}' : \mathrm{PStr}(\boldsymbol{L}) \to \mathrm{Set}^{\boldsymbol{B}}$; $\mathcal{E}'$ can be seen to be a fibration. Now, a (not-necessarily-relational) *weak* $\boldsymbol{L}$*-equivalence* $(W, m, n) : M \xleftarrow[\boldsymbol{L},\,\mathrm{w}]{} N$ is, by definition, a functor $W \in \mathrm{Set}^{\boldsymbol{B}}$, together with arrows $m : W \to \mathcal{E}'M$, $n : W \to \mathcal{E}'N$ such that m, n are very surjective with respect to all epimorphisms in $\boldsymbol{B}$ (according to the definition before B.(5), with $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}}, \boldsymbol{B}_{\mathcal{D}})$ replaced by $\mathtt{Set}^{\boldsymbol{B}}$), and there is a pseudo-structure P, with Cartesian arrows $\theta_m : P \to M$, $\theta_n : P \to N$ over m and n, respectively. We write $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$ for: there exists $(W, m, n) : M \xleftarrow[\boldsymbol{L},\,\mathrm{w}]{} \!\!\to N$.

It is not hard to show that $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$ iff there is a weak relational $\boldsymbol{L}$-equivalence $\rho : M \xleftarrow[\boldsymbol{L}_{\infty,\,\omega}]{\mathrm{r}} \!\!\to N$; the proof is similar to the proof below concerning non-weak relational equivalences.

We return to ordinary (non-weak) equivalences. When M and N are $\mathtt{Set}$-valued $\boldsymbol{L}$-structures, with any $(W, m, n) : M \xleftarrow[\boldsymbol{L}]{} \!\!\to N$, there is a relational $(W', m', n') : M \xleftarrow[\boldsymbol{L}]{\mathrm{r}} \!\!\to N$; in fact, W' can be chosen as a subfunctor of W, with m' and n' being restrictions of m and n, respectively. To define $W'K \subset WK$, we use recursion on the level of K. Fix K. The induction hypothesis gives us the inclusion $W'[K] \rightarrowtail W[K]$. Consider the pullback $P = WK \times_{W[K]} W'[K]$ as in



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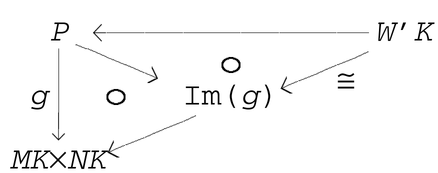
with i an inclusion; look at $g = \langle m_K i, n_K i \rangle : P \longrightarrow MK \times NK$, and, using the Axiom of Choice, split $h : P \twoheadrightarrow \mathrm{Im}(g)$ by an inclusion $k : W'K \rightarrowtail P$ as in



;

we have defined $W'K$. Inspection shows that W' is appropriate.

For not necessarily $\mathtt{Set}$-valued $\boldsymbol{L}$-structures M, N, let us write $M \sim_{\boldsymbol{L},\,\mathrm{r}} N$ for: there exists $(W, m, n) : M \xleftarrow[\boldsymbol{L}]{\mathrm{r}} \!\!\to N$.

What we saw says that the concept $M \sim_{\boldsymbol{L}} N$ remains unchanged, at least for $\mathtt{Set}$-valued models, if we ignore all but the relational $\boldsymbol{L}$-equivalences:

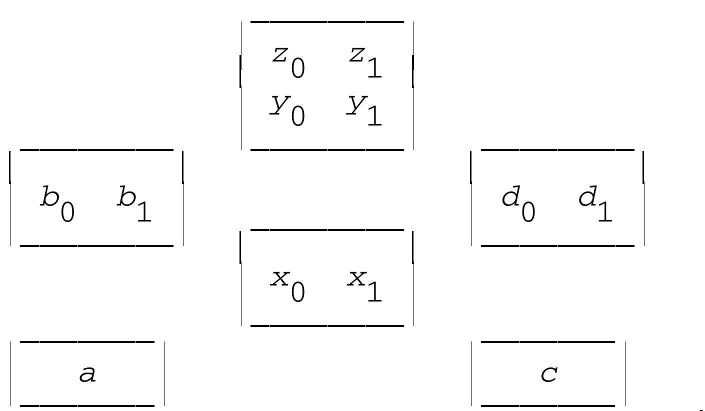
$$M \sim_{\boldsymbol{L}} N \iff M \sim_{\boldsymbol{L},\,\mathrm{r}} N. \tag{11}$$

However, the more general notion $(M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{b})$ goes wrong under the same alteration. For one thing, the need for not-necessarily-relational $\boldsymbol{L}$-equivalences is natural if we look at the proof of 5.(4). Given $\mathcal{X}$ and the tuples $\vec{a} \in M[\mathcal{X}]$, $\vec{b} \in N[\mathcal{X}]$ as there, the desired $\boldsymbol{L}$-equivalence $(W, m, n) : (M, \vec{a}) \underset{\boldsymbol{L}}{\leftrightarrow} (N, \vec{b})$ is constructed so as to continue the mappings $x \mapsto a_x$, $x \mapsto b_x$; if the latter two mappings are not jointly monomorphic, the resulting $\boldsymbol{L}$-equivalence will not be relational. On the other hand, the entry of non-relational $\boldsymbol{L}$-equivalences is not just a characteristic of the proof of 5.(4); it is in fact unavoidable.

Consider the following example of a DSV, called $\boldsymbol{L}$:

$$\begin{array}{ccc} E_1 & \overset{e_{10}}{\underset{e_{11}}{\rightrightarrows}} & K_1 \\ & & \big\downarrow p \\ E_0 & \overset{e_{00}}{\underset{e_{01}}{\rightrightarrows}} & K_0 \end{array} \qquad pe_{10}=pe_{11} \qquad .$$

A *standard* structure M for $\boldsymbol{L}$ is one for which, for $b_0, b_1 \in M[K_1]$, that is, $(Mp)b_0 = (Mp)b_1$, we have $b_0 (ME_1) b_1 \iff b_0 = b_1$, and also, ME_0 is ordinary equality on MK_0. Consider the following example for an $\boldsymbol{L}$-equivalence $(W, m, n) : M \underset{\boldsymbol{L}}{\leftrightarrow} N$, for certain M and N:



Here, $MK_0 = \{a\}$, $MK_1 = \{b_0, b_1\}$, $NK_0 = \{c\}$, $MK_1 = \{d_0, d_1\}$, $WK_0 = \{x_0, x_1\}$,

$WK_1=\{y_0, y_1, z_0, z_1\}$, $y_0, z_0 \overset{Wp}{\longmapsto} x_0$, $y_1, z_1 \overset{Wp}{\longmapsto} x_1$, and

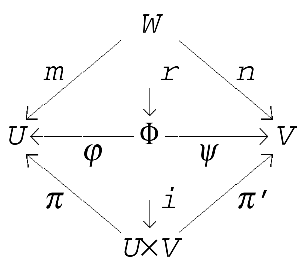
$$y_0 \overset{m}{\longmapsto} b_0 \ , \ y_1 \overset{m}{\longmapsto} b_1 \ , \ z_0 \overset{m}{\longmapsto} b_1 \ , \ z_1 \overset{m}{\longmapsto} b_0 \ ,$$

$$y_0 \overset{n}{\longmapsto} d_0 \ , \ y_1 \overset{n}{\longmapsto} d_0 \ , \ z_0 \overset{n}{\longmapsto} d_1 \ , \ z_1 \overset{n}{\longmapsto} d_1 \ ;$$

E_0 and E_1 are interpreted in M and N as equality.

This shows that, for the context $\mathcal{X}=\{x_0, x_1:K_0; y_0:K_1(x_0); y_1:K_1(x_1)\}$, and for $\vec{a}=\langle a/x_0, a/x_1, b_0/y_0, b_1/y_1\rangle$, $\vec{c}=\langle c/x_0, c/x_1, d_0/y_0, d_0/y_1\rangle$, we have $(M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{c})$. On the other hand, there is no relational equivalence $(W', m', n'):(M, \vec{a}) \underset{\boldsymbol{L}}{\longleftrightarrow} (N, \vec{c})$. In any such, $W'K_0$ is a singleton $\{x\}$; $x \overset{m'}{\longmapsto} a$, $x \overset{n'}{\longmapsto} c$; we have some $u_0, u_1 \in W'K_1(x)$ such that $u_0 \overset{m'}{\longmapsto} b_0$, $u_1 \overset{m'}{\longmapsto} b_1$; and the preservation of E_1 implies $n'(u_0)\neq n'(u_1)$, contradiction.

This example also dispels the possible belief that an $\boldsymbol{L}$-equivalence $(W, m, n):M \underset{\boldsymbol{L}}{\longleftrightarrow} N$ can always be reduced to a relational one by taking the image of (W, m, n) . Let $U=M\restriction\boldsymbol{K}$, $V=N\restriction\boldsymbol{K}$, and consider



(12)

where r and i form the surjective/injective factorization of $\langle m, n\rangle:W\to U\times V$. In other words, when $i:\Phi\to U\times V$ is an inclusion, for any $K\in\boldsymbol{K}$, the relation $\Phi K\subset MK\times NK$ is given by $a(\Phi K)b \iff \exists c\in WK.mc=a\&nc=b$. When applied in our example, (Φ, φ, ψ) so defined does not preserve E_1 .

I now turn to some remarks on equality.

Let $\boldsymbol{L}$ be an arbitrary DSV. Let us augment $\boldsymbol{L}$ to $\boldsymbol{L}^{\mathrm{G}}$, another DSV, by adding a relation

$\dot{\mathrm{G}}_K$ to $\boldsymbol{L}$ for every $K\in\mathrm{Kind}(\boldsymbol{L})$, with proper arrows $g_{K0}:\dot{G}_K\to K$, $g_{K1}:\dot{G}_K\to K$, together with all composites $pg_{Ki}:\dot{G}_K\to K_p$, $p\in K|\boldsymbol{L}$ $(i=0,1)$. We do *not* identify pg_{K0} with pg_{K1}. For an $\boldsymbol{L}^{\mathrm{G}}$-structure M,

$$M[\dot{\mathrm{G}}_K] = \{(\vec{a}, a, \vec{b}, b) : \vec{a}, \vec{b}\in M[K], a\in MK(\vec{a}), b\in MK(\vec{b})\} .$$

The letter G is used because we are dealing with *global* equality as opposed to fiberwise equality (see below). A *standard* $\boldsymbol{L}^{\mathrm{G}}$-structure M is one in which, for $\vec{a}, \vec{b}\in M[K]$, $a\in MK(\vec{a})$, $b\in MK(\vec{b})$, $(\vec{a}, a, \vec{b}, b)\in M(\dot{\mathrm{G}}_K)$ iff $a=b$; more briefly, $M(\dot{\mathrm{G}}_K)$ as a subset of $MK\times MK$ is $\{(a,a):M(K)\}$. Any $\boldsymbol{L}$-structure can be made into a standard $\boldsymbol{L}^{\mathrm{G}}$-structure in exactly one way. When an $\boldsymbol{L}$-structure is used as an $\boldsymbol{L}^{\mathrm{G}}$-structure, we mean the corresponding standard $\boldsymbol{L}^{\mathrm{G}}$-structure.

The effect of adding global equalities is that all $\boldsymbol{L}$-equivalences can be canonically replaced by relational ones, by taking the image of the given one. If $(W, m, n) : M\xleftrightarrow[\boldsymbol{L}^{\mathrm{G}}]{} N$, then for (Φ, φ, ψ) defined above, we have $(\Phi, \varphi, \psi) : M\xleftrightarrow[\boldsymbol{L}^{\mathrm{G}}]{r} N$.

To see this, first we show that the arrow r in (12) is very surjective; that is, for any $K\in\boldsymbol{K}$, the diagram

$$\begin{array}{ccc} W(K) & \xrightarrow{\;r_K\;} & \Phi(K) \\ \downarrow & & \downarrow \\ W[K] & \xrightarrow[\;r_{[K]}\;]{} & \Phi(K) \end{array} \qquad (13)$$

is a quasi-pullback. Assume $\vec{a}\in M[K]$, $\vec{b}\in N[K]$, $a\in MK(\vec{a})$, $b\in NK(\vec{b})$ such that $(\vec{a}, \vec{b})\in\Phi[K]$, $(a, b)\in\Phi K(\vec{a}, \vec{b})$, and $\vec{c}\in W[K]$ with $m\vec{c}=\vec{a}$, $n\vec{c}=\vec{b}$ (that is, $r_{[K]}(\vec{c})=(\vec{a}, \vec{b})$); we want $c\in WK(\vec{c})$ such that $mc=a$ and $nc=b$. By the definition of Φ, there is $d\in WK$ with $md=a$, $nd=b$. By the very surjectivity of n, there is $c\in WK(\vec{c})$

such that $nc{=}b$. But by the presence of the relation G_K, $md(M\mathrm{G}_K)mc$ iff $nd(N\mathrm{G}_K)nc$; that is, $md{=}mc$ iff $nd{=}nc$; which says that $mc{=}a$ as desired.

By B.(6'), the induced map $r_{[\mathcal{X}]}: W[\mathcal{X}] \longrightarrow \Phi[\mathcal{X}]$ is surjective.

Now, looking at

$$\begin{array}{ccccc} W[K] & \xrightarrow{r_{[K]}} & \Phi[K] & \xrightarrow{\varphi_{[K]}} & M[K] \\ \uparrow & & \uparrow & & \uparrow \\ W(K) & \xrightarrow[r_K]{} & \Phi(K) & \xrightarrow[\varphi_K]{} & M(K) \end{array} \qquad \begin{array}{ccc} W[K] & \xrightarrow{m_{[K]}} & M[K] \\ \uparrow & & \uparrow \\ W(K) & \xrightarrow[m_K]{} & M(K) \end{array}$$

we see that B.(3") is applicable to yield that φ is very surjective.

Given a relation $R \in \mathrm{Rel}(\boldsymbol{L}^{\mathrm{G}})$, if $(\vec{a}, \vec{b}) \in \Phi[R]$, then by $r_{[R]}: W[R] \longrightarrow \Phi[R]$ being surjective, there is $\vec{c} \in W[R]$ with $r_{[R]}(\vec{c}) = (\vec{a}, \vec{b})$, that is, $m\vec{c}{=}\vec{a}$, $n\vec{c}{=}\vec{b}$, and thus $\vec{a} \in MR$ iff $\vec{b} \in NR$. This completes showing that $(\Phi, \varphi, \psi): M \overset{r}{\underset{\boldsymbol{L}^{\mathrm{G}}}{\longleftrightarrow}} N$.

We have shown something more general (and more technical), which is independent of equality. This is that

(14) If $(W, m, n): M \underset{\boldsymbol{L}}{\longleftrightarrow} N$ and we have

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{m} & \downarrow r & {}^{n}\searrow & \\ U & \xleftarrow[\varphi]{} & \Phi & \xrightarrow[\psi]{} & V \end{array}$$

such that r is very surjective, then $(\Phi, \varphi, \psi): M \underset{\boldsymbol{L}}{\longleftrightarrow} N$;

the relational quality of (φ, ψ) is not relevant to this.

Clearly, a relational equivalence preserving global equalities on all kinds is nothing but an isomorphism. We have shown that $M\sim_{\mathbf{L}^{\mathrm{G}}}N$ implies that $M\cong N$, and $(M,\vec{a})\sim_{\mathbf{L}^{\mathrm{G}}}(N,\vec{b})$ implies $(M,\vec{a})\cong(N,\vec{b})$. But, all formulas in multisorted logic over $|\mathbf{L}|$ are preserved by isomorphism. By the invariance theorem 5.(12), we conclude the following.

(15) For any context $\mathcal{X}$ over $\mathbf{L}$, and any formula σ of multisorted logic over $|\mathbf{L}|$ with $\mathrm{Var}(\sigma)\subset\mathcal{X}$ [remember, a variable $x{:}X$ of FOLDS counts as a variable of sort K_X in multisorted logic], there is a FOLDS formula θ over $\mathbf{L}^{\mathrm{G}}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that σ and θ^* are logically equivalent (over $\mathcal{X}$):

$\models\forall\mathcal{X}(\sigma\longleftrightarrow\theta^*)$; or in other words, $M[\mathcal{X}{:}\sigma]=M[\mathcal{X}{:}\theta^*]$ for any $\mathbf{L}$-structure M. (We apply 5.(12) to $I{:}\mathbf{L}^{\mathrm{G}}\to[(|\mathbf{L}|,\Sigma[\mathbf{L}])]$; for $\Sigma[\mathbf{L}]$, see §1. I is essentially the identity except that all the G_K's are interpreted as equality. In $M[\mathcal{X}{:}\theta^*]$, M is understood as a standard $\mathbf{L}^{\mathrm{G}}$-structure.)

Notice the small point that in the statement of (15), we are not allowed to start with a $|\mathbf{L}|$-formula σ with arbitrary free variables; the free variables have to form a context. E.g., in the case of the language of categories, a formula with a single arrow-variable cannot (of course) have an equivalent in FOLDS *with the same free variables*; we have to add the "domain and the codomain of the arrow-variable" as free variables.

Let us hasten to add that it is possible to show (15) directly, by a rather simple structural induction on the formula σ.

We have an instance of what we may call *expressive completeness of* FOLDS: full first-order logic over $|\mathbf{L}|$ can be expressed in $\mathbf{L}^{\mathrm{G}}$. This is accompanied by a mode of deductive completeness. We will give a deductive system for entailments over $\mathbf{L}^{\mathrm{G}}$, extending the standard system for $\mathbf{L}^{\mathrm{G}}$ for logic without equality by specific rules related to the G-predicates, which is complete for semantics restricted to standard $\mathbf{L}^{\mathrm{G}}$-structures, that is, semantics of true equality.

The set $\mathrm{G}_K|\boldsymbol{L}$, the arity of the relation G_K, is the set

$$\{p\mathrm{g}_{K0}: p\in K|\boldsymbol{L}\}\cup\{\mathrm{g}_{K0}\}\cup\{p\mathrm{g}_{K1}: p\in K|\boldsymbol{L}\}\cup\{\mathrm{g}_{K1}\}$$

Accordingly, we will write atomic formulas $\mathrm{G}_K(\vec{z})$, $\vec{z}$ indexed by $\mathrm{G}_K|\boldsymbol{L}$, in the form $\mathrm{G}_K(\vec{x}, x, \vec{y}, y)$; here, $\vec{x}=\langle x_{p\mathrm{g}_{K0}}\rangle_{p\in K|\boldsymbol{L}}$, $x:K(\vec{x})$, $\vec{y}=\langle y_{p\mathrm{g}_{K0}}\rangle_{p\in K|\boldsymbol{L}}$, $y:K(\vec{y})$.

Here are some other pieces of notation. For any object A of $\boldsymbol{L}$ (kind or relation), and tuples $\vec{x}=\langle x_p\rangle_{p\in A|\boldsymbol{L}}$, $\vec{y}=\langle y_p\rangle_{p\in A|\boldsymbol{L}}$ for which $A(\vec{x})$, $A(\vec{y})$ (types or atomic formulas) are well-formed, $\vec{x}\mathrm{G}_{[A]}\vec{y}$ denotes the formula

$$\bigwedge_{p\in A|\boldsymbol{L}}\mathrm{G}_{\mathrm{K}_p}(\langle x_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}, x_p, \langle y_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}, y_p) .$$

When $\vec{x}=\langle x_p\rangle_{p\in K|\boldsymbol{L}}$, $\vec{x}^p \overset{=}{\mathrm{def}} \langle x_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}$.

V. Global-equality axioms.

(G_1)
$$\frac{}{\mathbf{t} \underset{\mathcal{X}}{\Longrightarrow} \mathrm{G}_K(\vec{x}, x, \vec{x}, x)}$$

(G_2)
$$\frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \underset{\mathcal{X}}{\Longrightarrow} \mathrm{G}_K(\vec{y}, y, \vec{x}, x)}$$

(G_3)
$$\frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \wedge \mathrm{G}_K(\vec{y}, y, \vec{z}, z) \underset{\mathcal{X}}{\Longrightarrow} \mathrm{G}_K(\vec{x}, x, \vec{z}, z)}$$

(G_4)
$$\frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \underset{\mathcal{X}}{\Longrightarrow} \mathrm{G}_K(\vec{y}^p, y_p, \vec{x}^p, x_p)} \qquad (p\in K|\boldsymbol{L})$$

(G_5)
$$\frac{}{\vec{x}\mathrm{G}_{[K]}\vec{y} \underset{\mathcal{X}}{\Longrightarrow} \exists y:K(\vec{y}).\mathrm{G}_K(\vec{x}, x, \vec{y}, y)} \qquad (x:K(\vec{x}))$$

$$(\mathrm{G}_6) \qquad \frac{}{\vec{x}\mathrm{G}_{[R]}\vec{y} \underset{\mathcal{X}}{\Longrightarrow} R(\vec{x}) \longleftrightarrow R(\vec{y})}$$

The proof of the said completeness is done in the traditional manner; we use completeness for logic without equality over $\boldsymbol{L}^{\mathrm{G}}$ for the theory whose axioms are the (conclusion-)entailments in the equality rules. Given any structure M for $\boldsymbol{L}^{\mathrm{G}}$ satisfying the equality axioms, we construct a standard $\boldsymbol{L}^{\mathrm{G}}$-structure $M/\sim$ which is $\boldsymbol{L}^{\mathrm{G}}$-elementary equivalent to M. For a kind K, let $\sim_K$ be the relation on the set MK defined by $a\sim_K b \Longleftrightarrow M\mathrm{G}_K([a], a, [b], b)$ holds; here $[a]=\langle (Mp)(a)\rangle_{p\in K|\boldsymbol{L}}$, and similarly for $[b]$. By (G_1), (G_2) and (G_3), each $\sim_K$ is an equivalence relation; let us write $a/\sim$ for the equivalence class containing a. (G_4) implies that if $f:K\to K'$, $a_i\in MK$, $a'_i=(Mf)(a_i)\in MK'$, then $a_1\sim_K a_2 \Rightarrow a'_1\sim_{K'} a'_2$. Let $U=M\restriction\boldsymbol{K}$. We define $U/\sim:\boldsymbol{K}\to\mathrm{Set}$ by $(U/\sim)(K)=(UK)/\sim$ $(\underset{\mathrm{def}}{=}\{a/\sim : a\in UK\})$, and $((U/\sim)(f))(a/\sim)=((Uf)(a))/\sim$, which is well-defined.

For $\vec{a}=\langle a_p\rangle_{p\in R|\boldsymbol{K}}\in M[R]$, we put $\vec{a}/\sim\ =\langle a_p/\sim\rangle_{p\in R|\boldsymbol{K}}\in (M/\sim)R$.

We define $M/\sim$ by $(M/\sim)\restriction\boldsymbol{K}=U/\sim$, and

$$(M/\sim)R(\vec{a}/\sim) \underset{\mathrm{def}}{\Longleftrightarrow} MR(\vec{a}) ;$$

by (G_6), this is well-defined; we have completed the definition of $M/\sim$.

For any finite context $\mathcal{X}$, we have $(M/\sim)[\mathcal{X}] = (M[\mathcal{X}])/\sim$ $(\underset{\mathrm{def}}{=}\{\vec{a}/\sim : \vec{a}\in M[\mathcal{X}]\})$. Moreover, when $\vec{a}\in M[K]$, then $(M/\sim)K(\vec{a}/\sim) = MK(\vec{a})/\sim$ $(\underset{\mathrm{def}}{=}\{a/\sim : a\in MK(\vec{a})\}$. This is not automatic; it requires (G_5). Finally, we show, by structural induction, that for any θ over $\boldsymbol{L}^{\mathrm{G}}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$, and $\vec{a}\in M[\mathcal{X}]$,

$$M/\sim\ \vDash \theta[\vec{a}/\sim] \iff M\vDash \theta[\vec{a}] \quad .$$

Having the construction $M\mapsto M/\sim$ with the properties shown, the proof of the standard completeness for $\boldsymbol{L}^{\mathrm{G}}$ can be completed in the expected manner.

In place of global equality, it seems natural to consider *fiberwise equality* for FOLDS. Let, for any DSV $\boldsymbol{L}$, $\boldsymbol{L}^{\mathrm{E}}$ denote the DSV obtained by adding to $\boldsymbol{L}$ a new relation E_K for every kind K, with $\mathrm{E}_K \underset{\mathrm{e}_{K1}}{\overset{\mathrm{e}_{K0}}{\rightrightarrows}} K$ and $p\mathrm{e}_{K0}=p\mathrm{e}_{K1}$ $(p\in K|\boldsymbol{K})$ as for maximal kinds in $\boldsymbol{L}^{\mathrm{eq}}$. A *standard* $\boldsymbol{L}^{\mathrm{E}}$-structure is one in which each E_K is interpreted as equality; to give a standard $\boldsymbol{L}^{\mathrm{E}}$-structure is the same as to give an $\boldsymbol{L}$-structure. In what follows, M and N are $\boldsymbol{L}$-structures; when they figure as $\boldsymbol{L}^{\mathrm{E}}$-structures, they mean the corresponding standard ones.

Suppose $\rho: M \overset{\mathrm{r}}{\underset{\boldsymbol{L}^{\mathrm{E}}}{\leftrightarrow}} N$. I claim that each $\rho_K \subset MK\times NK$ is the graph of a bijection $MK\to NK$. By (6'), $\mathrm{dom}(\rho_K)=MK$, $\mathrm{codom}(\rho_K)=NK$. Thus, it remains to show that

$$a_i\in MK,\ b_i\in NK,\ a_i\rho_K b_i\ (i=1,2) \implies a_1=a_2 \iff b_1=b_2 \tag{16}$$

We show this by induction on the level of K. Assume the hypotheses of (16). Let $a_i\in MK(\vec{a}^i)$, $b_i\in NK(\vec{b}^i)$. Then, if $\vec{a}^i=\langle a_p^i\rangle_{p\in K|\boldsymbol{K}}$, $\vec{b}^i=\langle b_p^i\rangle_{p\in K|\boldsymbol{K}}$, then $a_p^i\rho_{K_p} b_p^i$ (by (3)).

Assume (*e.g.*) $a_1=a_2$. Then $\vec{a}^1=\vec{a}^2 \underset{\mathrm{def}}{=} \vec{a}$, that is, $a_p^1=a_p^2$ for all $p\in K|\boldsymbol{K}$. By the induction hypothesis, (16) applied to K_p, we have $b_p^1=b_p^2$, that is, $\vec{b}^1=\vec{b}^2 \underset{\mathrm{def}}{=} \vec{b}$. We have $a_1, a_2\in MK(\vec{a})$, $b_1, b_2\in NK(\vec{b})$, and $\vec{a}a_i\rho_{\mathcal{X}_K^*}\vec{b}b_i$. Therefore, by (6), $M\mathrm{E}_K(\vec{a}, a_1, a_2) \iff N\mathrm{E}_K(\vec{b}, b_1, b_2)$; that is, $a_1=a_2 \iff b_1=b_2$ as desired.

Given that each ρ_K is a bijection, clearly, ρ is an isomorphism $\rho: M\xrightarrow{\cong} N$ (of $\boldsymbol{L}$-structures). We conclude

$$M \sim_{\boldsymbol{L}^{\mathrm{E}},\mathrm{r}} N \implies M\cong N \tag{17}$$

(the above argument did not depend essentially on the fact that we dealt with Set-valued structures)

Applying 5.(12), we obtain

(18) For every *sentence* σ in multisorted logic (with equality) over $|\mathbf{L}|$ there is a sentence $\bar{\sigma}$ of FOLDS over $\mathbf{L}^{\mathrm{E}}$ such that for every $\mathbf{L}$-structure M, $M\models\sigma \iff M\models\bar{\sigma}$ (here, in the first instance, M figures as an $|\mathbf{L}|$-structure; in the second instance as a standard $\mathbf{L}^{\mathrm{E}}$-structure).

Proof. Consider the interpretation $I:\mathbf{L}^{\mathrm{E}}\to[T]$, where $T=(|\mathbf{L}|, \Sigma_{\mathbf{L}})$, extending the "identity" interpretation $\mathbf{L}\to[T]$, and interpreting each E_K as equality. We apply 5.(12) to I, $\mathcal{X}=\emptyset$ and σ. Suppose $M, N\models T$ are Set-valued models (!),

$$M\restriction\mathbf{L}^{\mathrm{E}} \sim_{\mathbf{L}^{\mathrm{E}}} N\restriction\mathbf{L}^{\mathrm{E}} \tag{19}$$

and $M\models\sigma$. M and N are $\mathbf{L}$-structures, and $M\restriction\mathbf{L}^{\mathrm{E}}$, $N\restriction\mathbf{L}^{\mathrm{E}}$ are the corresponding standard $\mathbf{L}^{\mathrm{E}}$-structures. By (19) and (11), it follows that $M\cong N$. Since "everything" is invariant under isomorphism, $N\models\sigma$. Thus, the hypothesis of 5.(12) holds. The conclusion is exactly what we want.

Note that the result of (18) cannot be generalized to formulas with free variables in place of sentences. That is, the statement of (15), with $\mathbf{L}^{\mathrm{E}}$ replacing $\mathbf{L}^{\mathrm{G}}$ is not true. This is shown by the example that we gave above; in that example, $\mathbf{L}=\mathbf{L}_0^{\mathrm{E}}$ for $\mathbf{L}_0$ consisting of K_0, K_1 and p (and no relations). With $\mathcal{X}=\{x_0, x_1, y_0, y_1\}$ as in the example, *if* for the formula $\sigma\equiv y_0=y_1$ (whose free variables are in $\mathcal{X}$) there were θ in FOLDS over $\mathbf{L}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that, for every $\mathbf{L}_0$-structure M (also counted as a standard $\mathbf{L}$-structure) and

$$\vec{a}=\langle a_0, a_1\in MK_0;\ b_0\in MK_1(a_0);\ b_1\in MK_1(a_1)\rangle,$$

$$M\models\sigma(\vec{a}) \iff b_0=b_1 \overset{?}{\iff} M\models\theta(\vec{a})$$

then for every equivalence $(W, m, n):(M, \vec{a})\overset{\sim}{\underset{\mathbf{L}}{\longleftrightarrow}}(N, \vec{c})$, where

$$\vec{c}=\langle c_0,\ c_1\in NK_0;\ d_0\in NK_1(c_0);\ d_1\in NK_1(c_1)\rangle\ ,$$

since it would preserve θ, we would have

$$b_0=b_1 \iff d_0=d_1\ ;$$

but the example shows that this conclusion is false.

(18) can be used to give another proof of 6.(3), the Freyd-Blanc characterization result, at least for $\mathcal{X}=\emptyset$; this proof is a variant of what is contained in [FS].

Let T be a normal theory of categories with additional structure. Assume σ is an L_T-sentence such that for $M, N\models T$, $|M|\simeq|N|$ implies that $M\models\sigma$ iff $N\models\sigma$. In particular,

for $M, N\models T$, $|M|\cong|N|$ implies that $M\models\sigma$ iff $N\models\sigma$.

By ordinary model theory (a version of Beth definability), it follows that there is a sentence τ in multisorted logic over $|\boldsymbol{L}_{\mathrm{cat}}|$ such that for models of T, σ and τ are equivalent. By (18), there is a sentence ψ in FOLDS over $\boldsymbol{L}^{\mathrm{E}}_{\mathrm{cat}}$ which is equivalent to σ in all $\boldsymbol{L}_{\mathrm{cat}}$-structures (also counted as standard $\boldsymbol{L}^{\mathrm{E}}_{\mathrm{cat}}$-structures). There are two E-predicates in ψ, E_{O} and E_{A}. Replace each occurrence $\mathrm{E}_{\mathrm{O}}(X, Y)$ of E_{O} by the formula

$$\text{"}X\cong Y\text{"} \equiv \exists f\in\mathrm{A}(X, Y).\exists g\in\mathrm{A}(Y, X).\exists h\in\mathrm{A}(X, X).\exists i\in\mathrm{A}(Y, Y)$$
$$(\mathrm{I}(h)\wedge\mathrm{I}(i)\wedge\mathrm{T}(f, g, h)\wedge\mathrm{T}(g, f, i))\ ;$$

call the result θ. Notice that θ is a FOLDS formula of $\mathrm{L}^{\mathrm{eq}}_{\mathrm{cat}}$ (it has only the allowable equality predicates in $\mathrm{L}^{\mathrm{eq}}_{\mathrm{cat}}$). I claim that for all $M\models T$,

$$M\models\sigma \iff M\models\theta\ .$$

Let $M\models T$. $|M|$ is a category; let $|M|_{\mathrm{s}}$ be its skeleton. Since $|M|\simeq|M|_{\mathrm{s}}$, by the normality of T, there is $N\models T$ such that $|N| = |M|_{\mathrm{s}}$. Now

$$\begin{aligned} M\models\sigma \quad &\Longleftrightarrow \quad N\models\sigma && \text{since } |M|\simeq|N|\text{ , and } M, N\models T \\ &\Longleftrightarrow \quad N\models\tau && \text{since } N\models T \\ &\Longleftrightarrow \quad |N|\models\tau \\ &\Longleftrightarrow \quad |N|\models\psi \\ &\Longleftrightarrow \quad |N|\models\theta && \text{since } |N| \text{ is skeletal (that is, for objects } X, Y, X=Y \text{ iff } X\cong Y) \\ &\Longleftrightarrow \quad |M|\models\theta && \text{since } |M|\simeq|N|\text{ , and } \theta \text{ is a FOLDS formula with equality over } \mathbf{L}_{\mathtt{cat}} \\ &\Longleftrightarrow \quad M\models\theta\ . \end{aligned}$$

This method of proof is also applicable to the "higher" cases. Let us consider the case of bicategories; let us show that if a sentence σ in multisorted logic over $\mathrm{L}_{\mathtt{bicat}}=|\mathbf{L}_{\mathtt{anabicat}}|$ is invariant under equivalence of bicategories, then σ is equivalent in bicategories to θ^{*} for a FOLDS sentence θ over $\mathbf{L}_{\mathtt{anabicat}}$; θ^{*} is the translate of θ such that $\mathcal{A}\models\theta^{*}\Longleftrightarrow\mathcal{A}^{\#}\models\theta$.

A bicategory $\mathcal{A}$ is *skeletal* if any two equivalent objects are equal, and any two isomorphic parallel 1-cells are equal. For any bicategory $\mathcal{A}$, there is a skeletal one, $\mathcal{A}_{\mathtt{s}}$, which is (bi)equivalent to $\mathcal{A}$.

The first step is to use Beth definability to the interpretation $\Phi:\mathbf{L}_{\mathtt{anabicat}}\to[\mathrm{T}_{\mathtt{bicat}}]$. Since $\mathcal{A}^{\#}\cong\mathcal{B}^{\#}$ implies that $\mathcal{A}\simeq\mathcal{B}$, it follows that there is a sentence τ in multisorted logic over $|\mathbf{L}_{\mathtt{anabicat}}|$ such that for every bicategory $\mathcal{A}$, $\mathcal{A}\models\sigma\Longleftrightarrow\mathcal{A}^{\#}\models\tau$. By (18), we can find a sentence ψ in FOLDS over $\mathbf{L}^{\mathrm{E}}_{\mathtt{anabicat}}$ such that, in particular, $\mathcal{A}^{\#}\models\tau\Longleftrightarrow\mathcal{A}^{\#}\models\psi$. Now, transform ψ in the following way. Each occurrence $\mathrm{E}_{C_0}(X, Y)$ of E_{C_0} is replaced by the formula

$$\text{" } X\simeq Y \text{ "} \equiv \ \ldots$$

and each occurrence $\mathrm{E}_{C_1}(X\overset{f}{\underset{g}{\rightrightarrows}}Y)$ of E_{C_1} is replaced by the formula

$$\text{" } f\cong g \text{ "} \equiv \ \ldots$$

The resulting sentence θ is in $\boldsymbol{L}^{\mathtt{eq}}_{\mathtt{anabicat}}$. I claim that for any bicategory $\mathcal{A}$, $\mathcal{A}\models\sigma \iff \mathcal{A}\models\theta^{*}$. Indeed,

$$\mathcal{A}\models\sigma \iff \mathcal{A}_{\mathtt{s}}\models\sigma \iff (\mathcal{A}_{\mathtt{s}})^{\#}\models\tau \iff (\mathcal{A}_{\mathtt{s}})^{\#}\models\psi \iff (\mathcal{A}_{\mathtt{s}})^{\#}\models\theta \iff \mathcal{A}^{\#}\models\theta \iff \mathcal{A}\models\theta^{*} \quad ;$$

the next-to-last last biconditional holds because $\mathcal{A}_{\mathtt{s}}\simeq\mathcal{A}$, of which $(\mathcal{A}_{\mathtt{s}})^{\#}\sim_{\boldsymbol{L}^{\mathtt{eq}}}\mathcal{A}^{\#}$ ($\boldsymbol{L}=\boldsymbol{L}_{\mathtt{anabicat}}$) is a consequence, and because θ is a FOLDS sentence over $\boldsymbol{L}^{\mathtt{eq}}$.

This proof replaces the general invariance theorem 5.(12) by Beth definability, and a special case of that invariance theorem, (18). It falls somewhat short of the results of §7, partly because we have confined the situation to an empty context $\mathcal{X}$. Also, this approach is not available in constructive category theory; the existence of the skeleton (already in the classical case of mere categories) depends on the Axiom of Choice. As we will see in Appendix E, the main theory of equivalence of §5 has a constructive version involving intuitionistic logic. Modifying the notions of equivalence to notions of "anaequivalence" (using, and building on, [M2]), we obtain versions of the results of sections 6 and 7 for constructive category theory.