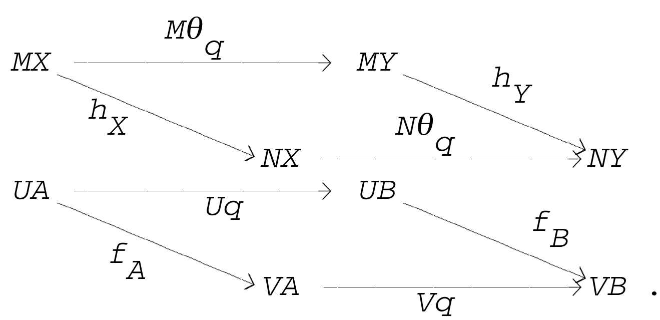
Appendix B: A fibrational theory of $\boldsymbol{L}$-equivalence

Consider fibrations $\begin{array}{c}\mathbf{E}_{\mathcal{C}}\\ \mathcal{C}\downarrow\\ \mathbf{B}_{\mathcal{C}}\end{array}$, $\begin{array}{c}\mathbf{E}_{\mathcal{D}}\\ \mathcal{D}\downarrow\\ \mathbf{B}_{\mathcal{D}}\end{array}$, and the category $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ of all maps

$$M=(M_1,M_2):\mathcal{C}\to\mathcal{D}\;::\;\begin{array}{ccc}\mathbf{E}_{\mathcal{C}} & \xrightarrow{M_2} & \mathbf{E}_{\mathcal{D}}\\ \mathcal{C}\downarrow & \circ & \downarrow\mathcal{D}\\ \mathbf{B}_{\mathcal{C}} & \xrightarrow[M_1]{} & \mathbf{B}_{\mathcal{D}}\end{array} \qquad (1)$$

of fibrations; $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ is a full subcategory of $[\mathcal{C},\mathcal{D}]$; see [M3]. $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ is the total category of a fibration denoted $\mathtt{Fib}\langle\mathcal{C},\mathcal{D}\rangle$; its base-category is the functor-category $[\mathbf{B}_{\mathcal{C}},\mathbf{B}_{\mathcal{D}}]$, and the fiber over $U:\mathbf{B}_{\mathcal{C}}\to\mathbf{B}_{\mathcal{D}}$ has objects all the M as in (1) with the fixed $U=M_1$, and arrows as in $\langle\mathcal{C},\mathcal{D}\rangle$ defined in [M3]; the fiber of $\mathtt{Fib}\langle\mathcal{C},\mathcal{D}\rangle$ over U is a full subcategory of the fiber of $\langle\mathcal{C},\mathcal{D}\rangle$ over U. Given $(f:U\to V)\in[\mathbf{B}_{\mathcal{C}},\mathbf{B}_{\mathcal{D}}]$, and $N:\mathcal{C}\to\mathcal{D}$ over V, the Cartesian arrow $M=f^{*}(N)\xrightarrow{h=\theta_f}N$ is obtained by the stipulation that for all $A\in\mathbf{B}_{\mathcal{C}}$, $X\in\mathcal{C}^{A}$, $M(X)\xrightarrow{h_X}X$ is a Cartesian arrow over $f_A:U(A)\to V(A)$; the definition of M on arrows is the obvious one; see also below. The fact that M so defined is a map of fibrations is shown by the diagram:



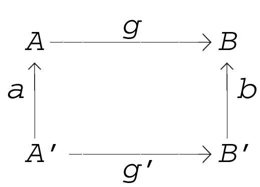
Here, $\theta_q:X\to Y$ is a Cartesian arrow over $q:A\to B$; the issue is to show that $M\theta_q$ is Cartesian (over Uq). The definition of M on arrows makes $M\theta_q$ an arrow over Uq making the upper quadrangle commute (unique such $M\theta_q$ exists by h_Y being Cartesian). As a composite of Cartesian arrows, $(N\theta_q)\circ h_X$ is Cartesian; as a left factor of the last, $M\theta_q$ is Cartesian.

In what follows, the base categories $\mathbf{B}_{\mathcal{C}}$, $\mathbf{B}_{\mathcal{D}}$ will have finite limits. $\mathtt{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ is the subfibration of $\mathrm{Fib}\langle\mathcal{C},\mathcal{D}\rangle$ with base-category $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, a full subcategory of $[\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}}]$, with fibers unchanged from $\mathrm{Fib}\langle\mathcal{C},\mathcal{D}\rangle$.

Next, assume that $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\exists$-fibrations. We have the *pre*fibration $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$, with base category $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, and total category $\wedge\exists(\mathcal{C},\mathcal{D})$. The fiber over $U\in\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$ is the full subcategory of the fiber of $\mathrm{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ over U with objects the maps of $\wedge\exists$-fibrations $M:\mathcal{C}\to\mathcal{D}$. $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$ is not a fibration; however, for certain maps $f:U\to V$, $f^{*}(N)$ calculated in $\mathrm{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ does belong to $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$, as we proceed to point out (from which it will of course follow that over such f , Cartesian arrows do exist in $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$).

Assume that $\mathcal{D}$ is a $\wedge\exists$-fibration, with $\mathcal{Q}_{\mathcal{D}}=\mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$. Let us call $q\in\mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$ *surjective* if $\exists_{q}\mathbf{t}_{A}=\mathbf{t}_{B}$. If q is surjective, then for any $Y\in\mathcal{D}^{B}$, $\exists_{q}q^{*}Y=\exists_{q}(\mathbf{t}_{A}\wedge q^{*}Y)$ $=\exists_{q}\mathbf{t}_{A}\wedge Y=Y$ (where the second equality is Frobenius reciprocity). It is clear that a pullback of a surjective arrow is surjective, and the composite of two surjective arrows is surjective. It is also clear that if qr is surjective, then so is q .

Let us call a commutative square in $\boldsymbol{B}_{\mathcal{D}}$

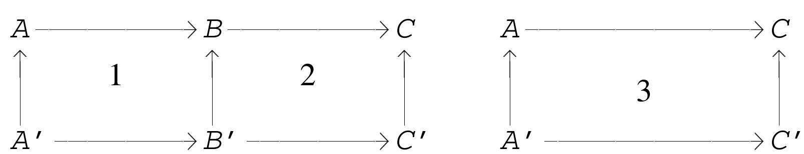


(1')

a *quasi-pullback* if the canonical arrow $p:A'\to A\times_{B}B'=P$ is surjective.

Using the stated properties of surjective maps, we easily see that if in the quasi-pullback (1'), g is surjective, then so is g' .

Consider two adjoining squares and their composite:



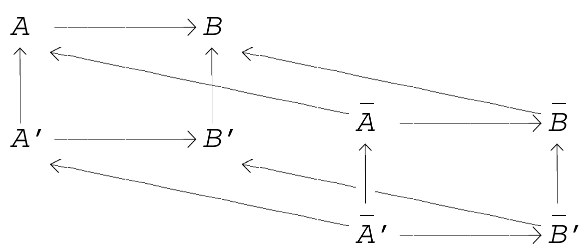
(1")

(2) The "composite" of two quasi-pullbacks is again a quasi-pullback: if both 1 and 2 are quasi-pullbacks, then so is 3 .

The verification uses both the pullback and composition properties of surjective arrows noted above.

(3) In (1"), if 3 is a quasi-pullback, 2 is a pullback, and 1 commmutes, then 1 is a quasi-pullback.

(3') If in the commutative diagram

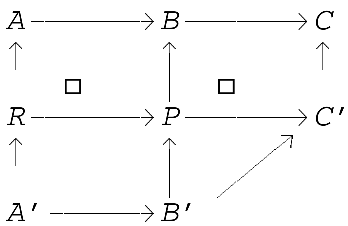


the two quadrangles $AA'\bar{A}\bar{A}'$ and $BB'\bar{B}\bar{B}'$ are pullbacks, and the square $AA'BB'$ is a quasi-pullback, then $\bar{A}\bar{A}'\bar{B}\bar{B}'$ is a quasi-pullback too.

This follows from (2) and (3).

(3") If in (1"), 3 is a quasi-pullback, and AB is surjective, then 2 is a quasi-pullback.

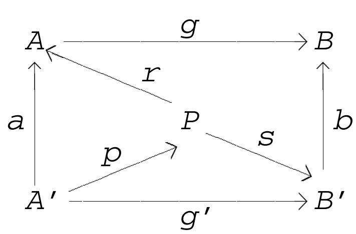
To see this, let $P{=}B\times_C C'$ for 2 , and $R{=}A\times_C C'$ for 3 . We have the commutative diagram



with two pullbacks as indicated. Since AB is surjective, so is RP. The assumption gives that $A'R$ is surjective. Now, the composite $A'P$ is surjective, and so is its left factor $B'P$, which is what we want.

(4) The Beck-Chevalley condition for $\exists$ holds (not just with pullback squares, but also) with quasi-pullback squares.

Indeed, consider the diagram



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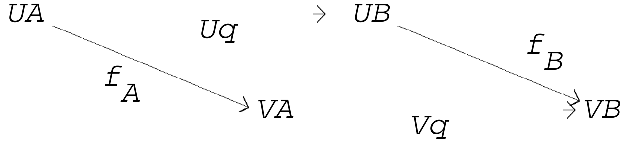
and calculate: $\exists_{g'} a^* X = \exists_s \exists_p a^* X = \exists_s \exists_p q^* r^* X = \exists_s r^* X = b^* \exists_g X$; the third equality is the "quasi-pullback" property, the last ordinary B-C .

Let us continue to assume that $\mathcal{D}$ is a "full" $\wedge\exists$-fibration ($\mathcal{Q}_{\mathcal{D}}$ contains all arrows), let $\mathcal{C}$ be an arbitrary $\wedge\exists$-fibration, $(q: A \to B) \in \mathbf{B}_{\mathcal{C}}$. We call a map $(f: U \to V) \in \mathrm{Lex}(\mathbf{B}_{\mathcal{C}}, \mathbf{B}_{\mathcal{D}})$ *very surjective with respect to* q if the square



is a quasi-pullback. (The concept of "very surjective" is relative to the fibration $\mathcal{D}$, although it does not depend on the fibration $\mathcal{C}$ except for its base-category.)

(5) If f is very surjective with respect to an arrow q, then so it is with respect to any pullback of q; if f is very surjective with respect to a pair composable arrows, then so it is with respect to their composite.

This follows by (3) and (2).

We say that f is *very surjective* if it is very surjective with respect to every $q \in \mathcal{Q}_{\mathcal{C}}$; by (5), it is enough to require the condition for a "generating set" of q's .

(6) The composite of very surjective arrows (in $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}}, \boldsymbol{B}_{\mathcal{D}})$) is very surjective; the pullback of a very surjective arrow is very surjective.

This follows by using (2) and (3).

Let $\boldsymbol{K}$ be a simple category, $\boldsymbol{B} = \mathrm{Con}(\boldsymbol{K})^{\mathrm{op}}$; $\mathrm{Lex}(\boldsymbol{B}, \boldsymbol{B}_{\mathcal{D}})$ can be identified with $\mathrm{Fun}(\boldsymbol{K}, \boldsymbol{B}_{\mathcal{D}})$; this is the kind of base-category for the fibrations we are interested in. In §4, we made two different choices for the class $\mathcal{Q}$ of quantifiable arrows in $\boldsymbol{B}$. The choice for the purposes of the main body of §5 is $\mathcal{Q}^{\neq}$; this, in the version that is closed under composition, is simply the class of epimorphisms of $\boldsymbol{B}$. When we make the choice of $\mathcal{Q}^{=}$ for $\mathcal{Q}$, we get as the very surjective maps in the sense of this section the ones we called *normal* ones in §5; we leave it to the reader to verify this.

(6') Let $(f: U \to V) \in \mathrm{Fun}(\boldsymbol{K}, \boldsymbol{B}_{\mathcal{D}})$ be very surjective (with respect to $\mathcal{Q}^{\neq}$). For every finite context $\mathcal{X}$ over $\boldsymbol{K}$, $f_{[\mathcal{X}]}: U[\mathcal{X}] \to V[\mathcal{X}]$ is surjective. For any $K \in \boldsymbol{K}$, $f_K: U(K) \to V(K)$ is surjective.

The first assertion is shown by induction on the cardinality of $\mathcal{X}$. If $\mathcal{X}$ is of positive size, we can write $\mathcal{X}$ as $\mathcal{Y} \dot{\cup} \{x\}$ such that $\mathcal{Y}$ is a context too. By the paragraph after (4) in §4, for $K = \mathrm{K}_x$, we have a pushout diagram

$$\begin{array}{ccc} \mathcal{X}_K & \longrightarrow & \mathcal{X}_K^* \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

in $\mathtt{Con}(\boldsymbol{K})$, which, with $\mathcal{V}=\mathcal{X}_K$, $\mathcal{U}=\mathcal{X}_K^*$, gives rise to

$$\begin{array}{ccccccc} U[\mathcal{V}] & \longrightarrow & V[\mathcal{V}] & & & & \\ \uparrow & 1 & \uparrow & \nwarrow & U[\mathcal{Y}] & \longrightarrow & V[\mathcal{Y}] \\ U[\mathcal{U}] & \longrightarrow & V[\mathcal{U}] & & \uparrow & 2 & \uparrow \\ & \nwarrow & & \nwarrow & U[\mathcal{X}] & \longrightarrow & V[\mathcal{X}] \end{array}$$

to which (3') is applicable. The square 1 is a quasi-pullback (by f being very surjective), hence, so is 2 . Since by the induction hypothesis, $U[\mathcal{Y}] \to V[\mathcal{Y}]$ is surjective, so is $U[\mathcal{X}] \to V[\mathcal{X}]$.

The second assertion follows immediately from the first by the quasi-pullback

$$\begin{array}{ccc} U(K) & \xrightarrow{\pi_K^U} & U[K] \\ f_K \downarrow & & \downarrow f_{[K]} \\ V(K) & \xrightarrow[\pi_K^U]{} & V[K] \end{array} \quad ;$$

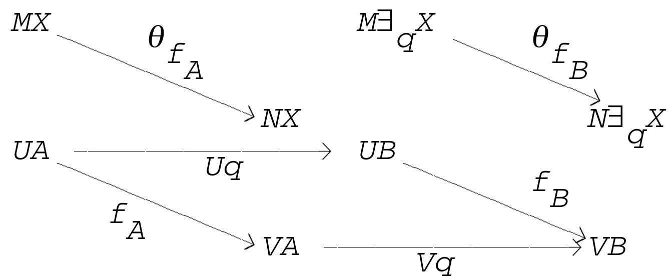
note that $U[K] = U[\mathcal{X}_K]$, *etc.*

Assume now that $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\vee\exists$-fibrations, $\mathcal{D}$ a "full" one.

(7) If $f:U\to V$ is very surjective, and $N\in\wedge\vee\exists(\mathcal{C},\mathcal{D})$, the $M=f^*(N)$ calculated in $\mathtt{Fib}(\mathcal{C},\mathcal{D})$ is in fact in $\wedge\vee\exists(\mathcal{C},\mathcal{D})$.

First of all, using that for each $g \in \mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$, g^* is a morphism of lattices, we immediately see that M preserves the fiberwise operations.

Consider



$$M\exists_q X = f_B^* N\exists_q X = f_B^* \exists_{Nq} NX = \exists_{Mq} f_A^* NX = \exists_{Mq} MX \ ;$$

here, the first equality is the definition of M; the second the quality of N being a morphism of $\exists$-fibrations; the third f being very surjective; and the last again the definition of M.

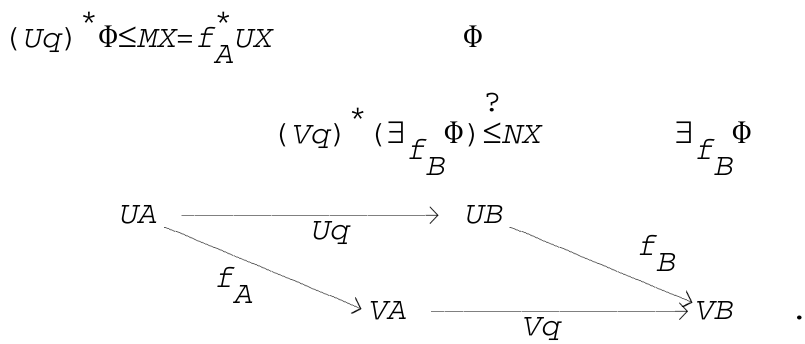
Now, assume in addition that both $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\vee\!\to\!\exists\forall$-fibrations, again with $\mathcal{Q}_{\mathcal{D}} = \mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$. I claim that

(8) If $f: U \to V$ is very surjective, then $N \in \wedge\vee\!\to\!\exists\forall(\mathcal{C}, \mathcal{D})$ implies that $M = f^*(N) \in \wedge\vee\!\to\!\exists\forall(\mathcal{C}, \mathcal{D})$.

The additional fiber-wise operation, Heyting implication, is dealt with as before. Let $(q: A \to B) \in \mathcal{Q}_{\mathcal{C}}$, $X \in \mathcal{C}^A$; we want to show that $M\forall_q X = \forall_{Mq} MX$; that is, for any $\Phi \in \mathcal{D}^{UB}$, $\Phi \leq_{UB} M\forall_q X \iff (Uq)^* \Phi \leq_{UA} MX$. The left-to-right implication is automatic. Assume

$$(Uq)^* \Phi \leq_{UA} MX \ , \qquad (9)$$

and consider



As indicated, we consider the object $\exists_{f_B}\Phi$ over VB, and claim that the inequality marked ? is true.

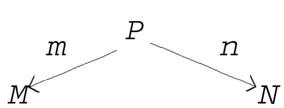
$$(Vq)^*(\exists_{f_B}\Phi) = \exists_{f_A}(Uq)^*\Phi \tag{10}$$

by the (generalized) B-C property for $\exists$ with quasi-pullbacks. (9) implies that

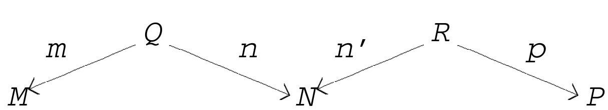
$$\exists_{f_A}(Uq)^*\Phi \le_{UA} \exists_{f_A} MX = \exists_{f_A} f_A^* NX \le NX \ . \tag{11}$$

(10) and (11) imply what we wanted. Now, from this, $\exists_{f_B}\Phi \le \forall_{Vq} NX = N(\forall_q X)$, and $\Phi \le f_B^* \exists_{f_B}\Phi \le f_B^* N(\forall_q X) = M(\forall_q X)$ as desired.

$M, N \in \wedge\vee\to\exists\forall(\mathcal{C}, \mathcal{D})$ are said to be *equivalent*, $M \sim N$, if there is a diagram



such that m, n are Cartesian in $\mathtt{Fiblex}(\mathcal{C}, \mathcal{D})$, and $m_1 : P_1 \to M_1$, $n_1 : P_1 \to N_1$ are very surjective. Equivalence is clearly reflexive and symmetric; it is transitive too; given



with the relevant properties, one forms the pullback

$$\begin{array}{ccccc} & & S_1 & & \\ & \overset{q_1}{\swarrow} & & \overset{r_1}{\searrow} & \\ Q_1 & & & & R_1 \\ & \underset{n_1}{\searrow} & & \underset{n'_1}{\swarrow} & \\ & & N_1 & & \end{array}$$

in $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, and defines S as $(n''_1)^*(N_1)$, for $n''_1=n_1q_1=n'_1r_1$; let $n'':S\to N$ be the Cartesian arrow over n''_1. Then n being Cartesian implies that there is a (unique) q over q_1 such that $nq=n''$; similarly for r over r_1. Since n'' is Cartesian, so are q and r. Since q_1, r_1 are pullbacks of very surjective arrows, they are very surjective. We conclude that mq and pr are Cartesian arrows over very surjective ones, which proves what we want.

Let us take $T=(\boldsymbol{L},\emptyset)$, the "empty theory" over the DSV $\boldsymbol{L}$, and let $\mathcal{C}=[T]$, a $\wedge\vee\to\exists\forall$-fibration with base-category $\boldsymbol{B}=(\mathrm{Con}[\boldsymbol{K}])^{\mathrm{op}}$ and class of quantifiable arrows $\mathcal{Q}=\mathcal{Q}^{\neq}$. Recall the canonical $i:\boldsymbol{K}\to\boldsymbol{B}$ induced by Yoneda. $\mathrm{Mod}_{\boldsymbol{C}}(\mathrm{T}) = \mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$, and we have the fibration $\mathcal{E}:\mathrm{Mod}_{\boldsymbol{C}}(T)\to\boldsymbol{C}^{\boldsymbol{K}}$ as explained in §5. We also have the fibration

$$\mathcal{D}=\mathrm{Fiblex}\langle\mathcal{C},\mathcal{P}(\boldsymbol{C})\rangle : \mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]\longrightarrow\mathrm{Lex}(\boldsymbol{B},\boldsymbol{C}) .$$

We have a "forgetful" morphism $()^{-}:\mathcal{D}\to\mathcal{E}$; $()^{-}_{1}$ is the equivalence

$$U\mapsto U\circ i : \mathrm{Lex}(\boldsymbol{B},\boldsymbol{C})\xrightarrow{\simeq}\boldsymbol{C}^{\boldsymbol{K}};$$

and $()^{-}_{2}$ is defined as $P\mapsto P^{-}$ was defined in §4 (see (5)) for the special case when $P\in\mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C})\subset\mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]$. It is easy to verify that $()^{-}$ is a morphism of fibrations.

We have the quasi-inverse

$$U\mapsto[U] : \boldsymbol{C}^{\boldsymbol{K}}\xrightarrow{\simeq}\mathrm{Lex}(\boldsymbol{B},\boldsymbol{C}) \qquad (12)$$

specified so that $[U]([\mathcal{X}]) = U[\mathcal{X}]$; we have the canonical isomorphism $j_U\colon [U]^- \cong U$ natural in U. $()^-\colon\mathcal{D}\to\mathcal{E}$ restricts to an equivalence

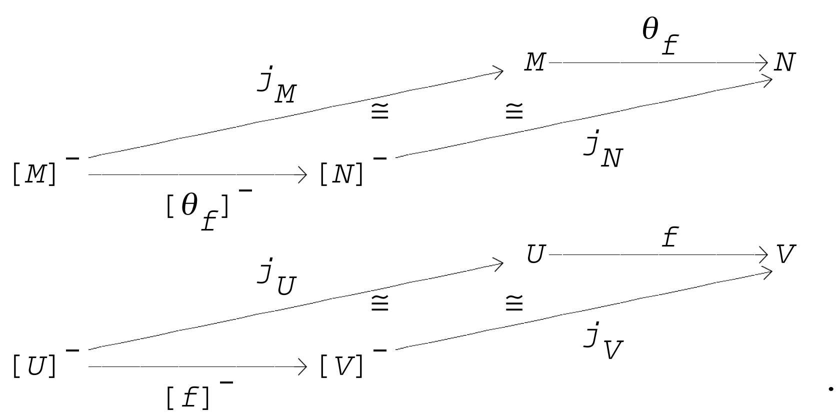
$$()^- : \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C}) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \quad , \tag{13}$$

whose quasi-inverse is

$$M \mapsto [M] : \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C}) \subset \mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]$$

constructed in §4 , with the canonical isomorphism $j_M\colon [M]^- \cong M$ natural in M. These are connected to (12) by $[M]_1 = [M{\restriction}\boldsymbol{K}]$, $(j_M)_1 = j_{M{\restriction}\boldsymbol{K}}$.

Let us deduce (1)(b) of §5 from (8); let's use the notation and hypotheses of 5.(1)(b). Consider the following diagram in the fibration $\mathcal{E}$:



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The two quadrangles commute, by the naturality of j. It follows that

$[\theta_f]^-\colon [M]^- \longrightarrow [N]^-$ is Cartesian over $[f]^-\colon [U]^- \longrightarrow [V]^-$. Consider the Cartesian arrow $\theta_{[f]}\colon [f]^*[N] \longrightarrow [N]$ over $[f]\colon [U] \longrightarrow [V]$ in $\mathcal{D}$. Since $()^-$ is a morphism of fibrations,

$$(\theta_{[f]})^- : ([f]^*[N])^- \longrightarrow [N]^-$$

is Cartesian over the same $[f]^-\colon [U]^- \longrightarrow [V]^-$. It follows that there is an isomorphism

$([f]^*[N])^- \xrightarrow{\cong} M$ over $1_{[U]^-}$. But then, since (13) is full and faithful, it follows that $[f]^*[N] = M$. Hence,

$$M[\mathcal{X}:\varphi] = ([f]^*[N])[\mathcal{X}:\varphi] = f^*_{\mathcal{X}}([N][\mathcal{X}:\varphi]) = f^*_{\mathcal{X}}(N[\mathcal{X}:\varphi]) ,$$

where the second equality is the description of Cartesian arrows in $\mathcal{D}$, the last is the definition of $[N]$; and this is what was to be proved.

Continuing in this manner, we see that, for $M, N \in \mathrm{Mod}_{\boldsymbol{C}}(T)$, $M \sim_{\boldsymbol{L}} N$ in the sense of §4 iff $[M] \sim [N]$ in the sense of this Appendix.