Appendix A: An alternative introduction of logic with dependent sorts.

The way we defined the basic concepts of FOLDS in §1 may look somewhat *ad hoc* because of the *a priori* role of the one-way (simple) categories as vocabularies. There is a more direct definition of FOLDS which does not start with assuming simple categories as vocabularies. The notion of "vocabulary" that arises naturally in the direct approach does, nevertheless, turn out to be equivalent to the one we started with in §1. More fully, the direct approach and the original approach turn out to be equivalent in all essential respects. This Appendix describes this state of affairs.

We first define the classes of entities called *kinds, sorts, variables*, *contexts* and *specializations*, and certain relation between these entities. Each kind, sort, variable, context and specialization has a certain *level*, which is a natural number; the definition of the said entities is by a simultaneous induction, proceeding by the level.

For the present purpose, we use the set-theoretic notion of *function* as a set of ordered pairs with the usual condition; the point is that we do not make the "categorical" specification of the codomain as part of the data for a function. Given functions s and t, $t\circ s$ is always defined and is a function; $\mathrm{dom}(t\circ s)=\{x\in\mathrm{dom}(s):\ s(x)\in\mathrm{dom}(t)\}$, and for $x\in\mathrm{dom}(t\circ s)$, $(t\circ s)(x)=t(s(x))$.

The *kinds of level* 0 are the entities of the form $\langle 0, \emptyset, a\rangle$, with a any set. We say that the kind $K=\langle 0, \emptyset, a\rangle$ *is of arity* $\emptyset$, and we write $K\overset{\circ}{\underset{\circ}{}}\emptyset$. The *sorts of level* 0 are the entities $\langle 1, K, \emptyset\rangle$, with K a kind of level 0; we put $\mathrm{Var}(K)=\emptyset$. A *variable of level* 0 is any entity of the form $\langle 2, X, a\rangle$ with X a sort of level 0, a any set; we say that the variable $x=\langle 2, X, a\rangle$ *is of sort* X, and we write $x{:}X$. (The definition ensures that every variable of level 0 has a unique sort of level 0.) A *context of level* 0 is a finite set of variables of level 0. A *specialization* of level 0 is a function s whose domain is a context of level 0, and for each $x\in\mathrm{dom}(s)$, $s(x)$ is a variable of the same sort as x.

Suppose n is a natural number, $n>0$, and we have defined what the *kinds, sorts, variables*, *contexts* and *specializations of level* k are, for each $k<n$, such that each context of level $<n$ is a finite set of variables of level $<n$, and each specialization of level $<n$ is a function whose domain and range are sets of variables of level $<n$. Suppose moreover that

we have defined the concept of a variable x being *of sort* X, for variables x and sorts X of level $<n$.

A *kind of level* n is an entity $\langle 0, \mathcal{Y}, a\rangle$, where $\mathcal{Y}$ is a context of level $n-1$, and a is an arbitrary set; we say that $\mathcal{Y}$ is *the arity* of $K=\langle 0, \mathcal{Y}, a\rangle$, and we write $K \overset{\circ}{\underset{\circ}{}} \mathcal{Y}$.

[Kinds are to form sorts (see below); kinds are incomplete sorts, with places for variables to fill; when these places are filled in a correct manner, then we have a sort. In our formulation, we did not introduce "places" as distinct from variables, although we could have done so; we used variables to denote "places"; this is the same as the "nameforms" in [K]. Our procedure may be compared to the one when, in ordinary first-order logic with several sorts, a relation symbol R is introduced in the form $R(x_0, x_1, \ldots x_{n-1})$, with distinct specific variables x_i of definite sorts; the arity of R then may be identified with the *set* $\mathcal{X}=\{x_0, x_1, \ldots x_{n-1}\}$; the atomic formula $R(y_0, y_1, \ldots y_{n-1})$ (y_i of the same sort as x_i) using R can then be identified with the pair (R, s) (= " $R(s)$ ") where s is the function with domain $\mathcal{X}$ for which $s(x_i)=y_i$.]

A *sort of level* n is any $X=\langle 1, K, s\rangle$, written more simply as $K(s)$, where K is a kind of level n, s is a specialization of level $n-1$, and $K \overset{\circ}{\underset{\circ}{}} \mathrm{dom}(s)$; $\mathrm{Var}(X) \underset{\mathrm{def}}{=} \mathrm{range}(s)$.

For a sort X, a *variable of sort* X is any $x = \langle 2, X, a\rangle$; we write $x{:}X$.

A *context of level* n is any set of the form $\mathcal{Y}\cup\mathcal{X}$ where $\mathcal{Y}$ is a context of level $n-1$, $\mathcal{X}$ is a (non-empty, for having level exactly $=n$) finite set, and each $x\in\mathcal{X}$ is a variable of level n such that if $x{:}X$, then $\mathrm{Var}(X)\subset\mathcal{Y}$.

If $X=K(s)(=\langle 1, K, s\rangle)$, then $X|t$ denotes $K(t\circ s)(=\langle 1, K, t\circ s\rangle)$. [$X|t$ is the sort obtained "by substituting $t(x)$ simultaneously for each $x\in\mathrm{Var}(X)$ in X ".] t is *a specialization* (of level n) if t is a function whose domain is a context, and for every $x\in\mathrm{dom}(t)$, if $x{:}X$, then $X|t$ is a sort (of level $\le n$), and $t(x)$ is a variable (of level $\le n$) of sort $X|t$ (and there is at least one $x\in\mathrm{dom}(t)$ of level n).

The above may be put in a more compact manner, without talking about levels, as follows. We define classes

$\mathtt{KIND}$, $\mathtt{CONTEXT}$, $\mathtt{SORT}$, $\mathtt{SPEC}$, $\mathtt{VARIABLE}$

such that

$\mathcal{X}\in\mathtt{CONTEXT} \implies \mathcal{X}$ is a finite subset of $\mathtt{VARIABLE}$,
$s\in\mathtt{SPEC} \implies s$ is a function, $\mathtt{dom}(s)$ and $\mathtt{range}(s) \subset \mathtt{CONTEXT}$;

predicates

$\overset{\circ}{\circ} \subset \mathtt{KIND}\times\mathtt{CONTEXT}$ (read $K\overset{\circ}{\circ}\mathcal{Y}$ as " K is a kind of arity $\mathcal{Y}$ ")
$: \subset \mathtt{VARIABLE}\times\mathtt{SORT}$ (read $x{:}X$ as " x is a variable of sort X ")

and the function

$\mathtt{Var} : \mathtt{SORT} \longrightarrow \mathcal{P}_{\mathtt{fin}}(\mathtt{VARIABLE})$,

by the closure conditions:

1 $\mathcal{X}\in\mathtt{CONTEXT} \implies \langle 0,\mathcal{X},\mathtt{a}\rangle \in \mathtt{KIND}$ and $\langle 0, \mathcal{X}, a\rangle \overset{\circ}{\circ} \mathcal{X}$;
2 $\varnothing \in \mathtt{CONTEXT}$;
3 $\mathcal{X}\in\mathtt{CONTEXT}$, $X\in\mathtt{SORT}$, $x{:}X$, $\mathtt{Var}(X) \subset \mathcal{X} \implies \mathcal{X}\cup\{x\} \in \mathtt{CONTEXT}$;
4 $s\in\mathtt{SPEC}$, $K\in\mathtt{KIND}$, $K\overset{\circ}{\circ}\mathtt{dom}(s) \implies$
$\langle 1, K, s\rangle \in \mathtt{SORT}$ and $\mathtt{Var}(\langle 1, K, s\rangle) = \mathtt{range}(s)$;
5 $\varnothing \in \mathtt{SPEC}$;
6 $\mathcal{X}\in\mathtt{CONTEXT}$, $s\in\mathtt{SPEC}$, $X\in\mathtt{SORT}$, $X|s\in\mathtt{SORT}$,
$x{:}X$, $x\notin\mathtt{dom}(s)$, $\mathtt{Var}(X)\subset\mathcal{X}$, $y{:}X|s \implies s\cup\{(x,y)\} \in \mathtt{SPEC}$;
($\langle 1, K, s\rangle | t \underset{\mathrm{def}}{=} \langle 1, K, t\circ s\rangle$)
7 $X\in\mathtt{SORT} \implies \langle 2, X, a\rangle \in \mathtt{VARIABLE}$ and $\langle 2, X, a\rangle{:}X$.

By definition, the intended system $(\mathtt{KIND}, \ldots)$ is the *minimal* one satisfying the given closure conditions.

Let us give some examples. Let $\underline{\mathtt{O}}$, $\underline{\mathtt{A}}$, $\underline{\mathtt{A}}_1$, $\underline{U}$, $\underline{V}$, $\underline{u}$, $\underline{v}$ be arbitrary entities, $\underline{U}\neq\underline{V}$, $\underline{u}\neq\underline{v}$. Here are specific kinds, variables, sorts and contexts, introduced by the above rules; at the start of the line, the number of the clause used is shown:

2	$\emptyset\in\mathtt{CONTEXT}$,
1	$\bar{\mathtt{O}} \underset{\text{def}}{=} \langle\emptyset,\emptyset,\underline{\mathtt{O}}\rangle \in \mathtt{KIND}$, $\bar{\mathtt{O}} \overset{\circ}{\underset{\circ}{}} \emptyset$.
5	$s_0 \underset{\text{def}}{=} (\emptyset:\emptyset\to\emptyset) \in \mathtt{SPEC}$
4	$\mathtt{O} \underset{\text{def}}{=} \langle 1,\bar{\mathtt{O}},s_0\rangle \in \mathtt{SORT}$, $\mathtt{Var(O)} = \emptyset$
7	$U \underset{\text{def}}{=} \langle 2,\mathtt{O},\underline{U}\rangle \in \mathtt{VARIABLE}$, $U{:}\mathtt{O}$
	$V \underset{\text{def}}{=} \langle 2,\mathtt{O},\underline{V}\rangle \in \mathtt{VARIABLE}$, $V{:}\mathtt{O}$
3 twice	$\{U, V\} \in \mathtt{CONTEXT}$
1	$\mathtt{A} \underset{\text{def}}{=} \langle 0,\{U, V\},\underline{\mathtt{A}}\rangle \in \mathtt{KIND}$, $\mathtt{A} \overset{\circ}{\underset{\circ}{}} \{U, V\}$
6 twice	$s_1 \underset{\text{def}}{=} \mathrm{id}_{\{U, V\}}:\{U, V\}\to\{U, V\} \in \mathtt{SPEC}$
4	$\mathtt{A}(U, V) \underset{\text{def}}{=} \langle 1,\mathtt{A},s_1\rangle \in \mathtt{SORT}$
7	$u \underset{\text{def}}{=} \langle 2,\mathtt{A}(U, V),\underline{u}\rangle \in \mathtt{VARIABLE}$, $u{:}\mathtt{A}(U, V)$
	$v \underset{\text{def}}{=} \langle 2,\mathtt{A}(U, V),\underline{v}\rangle \in \mathtt{VARIABLE}$, $v{:}\mathtt{A}(U, V)$
3...	$\{U, V, u, v\} \in \mathtt{CONTEXT}$
1	$\mathtt{A}_1 \underset{\text{def}}{=} \langle 0,\{U, V, u, v\},\underline{\mathtt{A}}_1\rangle \in \mathtt{KIND}$

For a variable x , we have a unique sort $\mathtt{X}_x$ for which $x{:}\mathtt{X}_x$; $\mathtt{X}_x=\mathtt{K}_x(\mathtt{s}_x)$ for a uniquely determined kind $\mathtt{K}_x$ and specialization $\mathtt{s}_x$. For a kind K , $\mathcal{X}_K$ is the context for which $K \overset{\circ}{\underset{\circ}{}} \mathcal{X}_K$.

A *pre-vocabulary* is a set $\boldsymbol{K}$ of kinds such that $K\in\boldsymbol{K}$, $x\in\mathcal{X}_K$ imply that $\mathtt{K}_x\in\boldsymbol{K}$. (I am talking about *pre*-vocabularies because relations are not yet contemplated.)

We compare the present approach to the one in §1. Let $\boldsymbol{K}$ be a pre-vocabulary. We make $\boldsymbol{K}$ into a category with objects the elements of $\boldsymbol{K}$. Arrows of $\boldsymbol{K}$ are the identity arrows, and the $\mathtt{p}^K_x{:}K\to\mathtt{K}_x$, one for each pair $K\in\boldsymbol{K}$, $x\in\mathcal{X}_K$. Composition is defined thus. Given

$$K \xrightarrow{\mathtt{p}^K_x} \mathtt{K}_x \xrightarrow{\mathtt{p}^{\mathtt{K}_x}_y} \mathtt{K}_y \qquad (x\in\mathcal{X}_K ,\; y\in\mathcal{X}_{\mathtt{K}_x}) ,$$

$\mathrm{X}_x = \mathrm{K}_x(\mathrm{s}_x)$, with $\mathrm{s}_x{:}\mathcal{X}_{\mathrm{K}_x} \twoheadrightarrow \mathrm{Var}(\mathrm{X}_x)$. $z \overset{=}{\mathrm{def}} \mathrm{s}_x(y) \in \mathrm{Var}(\mathrm{X}_x) \subset \mathcal{X}_K$; also, $\mathrm{K}_z = \mathrm{K}_y$; therefore, $K \xrightarrow{\mathrm{p}_z^K} \mathrm{K}_y$. We define $\mathrm{p}_y^{\mathrm{K}_x} \circ \mathrm{p}_x^K = \mathrm{p}_z^K$.

This composition is associative as is seen by using the equality $s(\mathrm{s}_y(u)) = \mathrm{s}_{s(y)}(u)$, which in turn is part of the definition of s being a specialization.

The category $\boldsymbol{K}$ so defined is clearly a simple category; the levels of kinds as given in the definition above are the same as their levels in $\boldsymbol{K}$.

Let us use $\boldsymbol{K}$ as a category of kinds in the way done in §1. I claim that the resulting notions of variable$_1$, sort$_1$, and context$_1$ are essentially the same as those of variable$_{\boldsymbol{K}}$, sort$_{\boldsymbol{K}}$ and context$_{\boldsymbol{K}}$ in the sense of the present Appendix, with the only kinds allowed the ones in $\boldsymbol{K}$. More precisely, we define, by a simultaneous recursion, functions

$$X \longmapsto \bar{X} : \mathrm{Sort}_{\boldsymbol{K}} \longrightarrow \mathrm{Sort}_1 \qquad (1)$$

$$x \longmapsto \bar{x} : \mathrm{Variable}_{\boldsymbol{K}} \longrightarrow \mathrm{Variable}_1 \; ; \qquad (2)$$

by putting $\overline{\langle 2, X, a\rangle} = \langle 2, \bar{X}, a\rangle$, and $\overline{\langle 1, K, s\rangle} = \langle 1, K, \langle x_p\rangle_{p \in K|\boldsymbol{K}}\rangle$, where $x_p = \overline{s(y)}$ for the unique y for which $p = \mathrm{p}_y^K$. I leave it to the reader to check that (1) and (2) are bijections, and $\bar{x}{:}\bar{X} \iff x{:}X$. Moreover, we have that the bijection (2) induces a bijection between $\mathrm{Context}_1$ and $\mathrm{Context}_{\boldsymbol{K}}$.

Let us return to the development started in this Appendix. A *relation-symbol* is an entity of the form $\langle 3, \mathcal{X}, a\rangle$ where $\mathcal{X}$ is a context; $\mathcal{X}$ is the *arity* of the relation-symbol $R = \langle 3, \mathcal{X}, a\rangle$; $R \overset{\circ}{\underset{\circ}{}} \mathcal{X}$. A *vocabulary* is a set $\boldsymbol{L}$ of kinds and relation-symbols such that the set $\boldsymbol{K}$ of kinds in $\boldsymbol{L}$ is a pre-vocabulary, and if R is a relation-symbol in $\boldsymbol{L}$, $R \overset{\circ}{\underset{\circ}{}} \mathcal{X}$, $x \in \mathcal{X}$, then $\mathrm{K}_x \in \boldsymbol{K}$.

Our comparison above of pre-vocabularies and simple categories of §1 clearly extends to an essential bijection between vocabularies as defined here, and DSV's of §1.

An *atomic formula* (in logic without equality) is any $\langle 4, R, s\rangle$ where R is a relation-symbol, s is a specialization, and $R\overset{\diamond}{\underset{\diamond}{}}\mathrm{dom}(s)$.

I leave the rest of the development of FOLDS in the style of this Appendix, and its comparison to the main body of the paper, to the reader.