§7. Equivalence of bicategories

For 2-categories and bicategories, see [M L], [Be], [S].

In this section, I discuss invariance of properties of bicategories, and of diagrams in bicategories, under biequivalence (however, I will call "biequivalence" "equivalence of bicategories"). To mention just two examples, the property of a bicategory having finite weighted (indexed) limits (see [S]) is a first-order property invariant under (bi)equivalence; but the property of a 2-category having finite 2-limits is not so invariant. The main result of this section (see the Corollary at the end) implies that the first-mentioned property can be expressed in FOLDS, although not quite in the language of the bicategory itself, but in a modification of it. In fact, the formulation of the said property in FOLDS can be done directly, quite easily.

One possible choice of a similarity type for 2-categories is the following graph $L_{2\text{-cat}}$:

$$
\begin{tikzcd}
 & T_1 \arrow[d, shift right=2, "t_{10}"'] \arrow[d, "t_{11}"] \arrow[d, shift left=2, "t_{12}"] & H \arrow[l, shift right, "h_0"'] \arrow[l, shift left, "h_1"] \arrow[d, shift right=2, "h_2"'] \arrow[d, "h_3"] \arrow[d, shift left=2, "h_4"] & \\
C_0 & C_1 \arrow[l, shift right, "c_{10}"'] \arrow[l, shift left, "c_{11}"] & C_2 \arrow[l, shift right, "c_{20}"'] \arrow[l, shift left, "c_{21}"] & T_2 \arrow[l, shift right=2, "t_{20}"'] \arrow[l, "t_{21}"] \arrow[l, shift left=2, "t_{22}"] \\
 & I_1 \arrow[u, "i_1"'] & I_2 \arrow[u, "i_2"'] &
\end{tikzcd}
$$

The following explains the meaning of these symbols in the case of a 2-category:

C_0 : (the set of all) objects (0-cells),
C_1 : arrows (1-cells),
C_2 : 2-cells;
c_{10} , c_{20} : domain,
c_{11} , c_{21} : codomain,
T_1 : commutative triangles

$$\tau \;=\; \begin{tikzcd} \cdot \arrow[r, "t_{10}\tau"] \arrow[rr, "t_{12}\tau"'] & \cdot \arrow[r, "t_{11}\tau"] & \cdot \end{tikzcd}$$

of 1-cells,

T_2 : commutative (for vertical composition) triangles

$$\theta \;=\; \begin{tikzcd}[column sep=huge] \cdot \arrow[r, bend left=60, ""{name=A, below}] \arrow[r, ""{name=B}] \arrow[r, bend right=60, ""{name=C, above}] & \cdot \arrow[Rightarrow, from=A, to=B, "t_{20}\theta"] \arrow[Rightarrow, from=B, to=C, "t_{21}\theta"] \end{tikzcd} \quad \begin{tikzcd}[column sep=huge] \cdot \arrow[r, bend left=60, ""{name=D, below}] \arrow[r, bend right=60, ""{name=E, above}] & \cdot \arrow[Rightarrow, from=D, to=E, "t_{22}\theta"] \end{tikzcd}$$

of 2-cells,

H : commutative (for horizontal composition) triangles

$$\eta \;=\; \begin{tikzcd}[column sep=huge] & c_{11}t_{10}h_0\eta \arrow[dr, bend left, "t_{11}h_0\eta", ""{name=D, below}] \arrow[dr, bend right, "t_{11}h_1\eta"', ""{name=E, above}] & \\ c_{10}t_{10}h_0\eta \arrow[ur, bend left, "t_{10}h_0\eta", ""{name=A, below}] \arrow[ur, bend right, "t_{10}h_1\eta"', ""{name=B, above}] \arrow[rr, bend left=20, "t_{12}h_1\eta", ""{name=F, below}] \arrow[rr, bend right=20, "t_{12}h_0\eta"', ""{name=G, above}] & & c_{11}t_{10}h_0\eta \arrow[Rightarrow, from=A, to=B, "h_2\eta"] \arrow[Rightarrow, from=D, to=E, "h_3\eta"'] \arrow[Rightarrow, from=G, to=F, "h_4\eta"'] \end{tikzcd}$$

of 2-cells;

I_1 : identity 1-cells,

I_2 : identity 2-cells.

A 2-category is the same as a structure for $L_{2\text{-cat}}$ satisfying certain axioms $\Sigma_{2\text{-cat}}$ in multisorted first order logic with equality(ies) over $L_{2\text{-cat}}$.

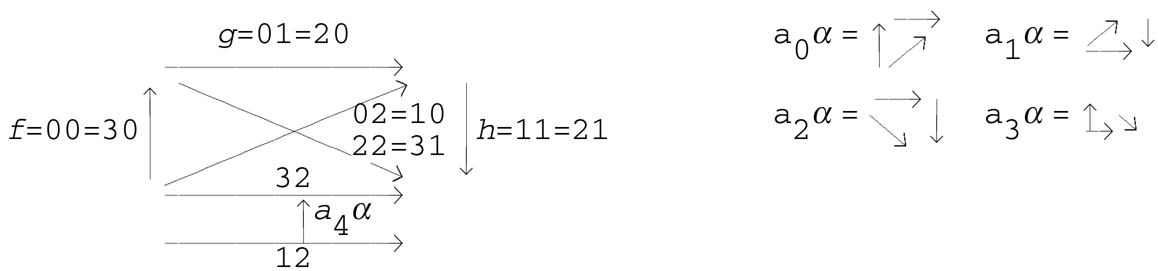
For the concept of bicategory we need, in addition, the sorts A , L and R , accommodating associativity isomorphisms, and left and right identity isomorphisms, respectively. More precisely, we introduce, besides these three new objects, the arrows

$$
\begin{tikzcd}
T_1 & A \arrow[l, shift right=3, "a_0"'] \arrow[l, shift right=1, "a_1"] \arrow[l, shift left=1, "a_2"] \arrow[l, shift left=3, "a_3"'] \arrow[r, "a_4"] & C_2
\end{tikzcd},
$$

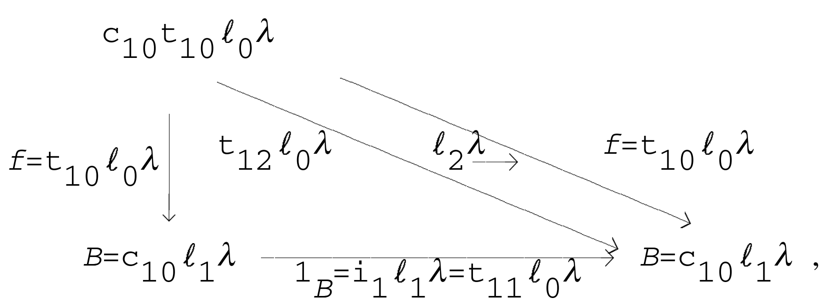
$$
\begin{tikzcd}
 & C_2 & \\
T_1 & L \arrow[l, "\ell_0"'] \arrow[u, "\ell_2"'] \arrow[r, "\ell_1"] & I_1
\end{tikzcd},
\quad
\begin{tikzcd}
 & C_2 & \\
T_1 & R \arrow[l, "r_0"'] \arrow[u, "r_2"'] \arrow[r, "r_1"] & I_1
\end{tikzcd};
$$

with these additions to $L_{2\text{-cat}}$, we obtain L_{bicat}.

In a bicategory, the symbols of $L_{2\text{-cat}}$ are interpreted as expected (as in a 2-category). A stands for the set of 5-tuples $\alpha = (a_0\alpha, a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha)$ where the $a_i\alpha$ ($i=0, 1, 2, 3$) are commutative triangles of 1-cells (elements of T_1), and $a_4\alpha$ is a 2-cell, fitting together as in



with ij standing for $t_{1j}(a_i\alpha)$, and $a_4\alpha$ is the associativity isomorphism $\alpha_{f,g,h}: h(gf) \xrightarrow{\cong} (hg)f$. L is the set of triples $\lambda = (\ell_0\lambda, \ell_1\lambda, \ell_2\lambda)$ as in



and $\ell_2\lambda$ is the identity isomorphism $\lambda_f: 1_B \circ f \xrightarrow{\cong} f$. R is similar, *mutatis mutandis.*

Bicategories are $\mathtt{L}_{\mathtt{bicat}}$-structures satisfying a set $\Sigma_{\mathtt{bicat}}$ of axioms, in multisorted first-order logic with equality over $\mathtt{L}_{\mathtt{bicat}}$. Of course, 2-categories are those bicategories for which each $\alpha_{f,g,h}$, λ_f, ρ_f are identity 2-cells. We write $\mathtt{T}_{\mathtt{bicat}}$ for the theory $(\mathtt{L}_{\mathtt{bicat}},\Sigma_{\mathtt{bicat}})$.

Now, we introduce the DSV $\mathbf{L}_{\mathtt{anabicat}}$. The underlying simple category is generated by the graph $\mathtt{L}_{\mathtt{bicat}}$, subject to the following equalities:

$$
\begin{aligned}
&c_{10}c_{20} = c_{10}c_{21},\ c_{11}c_{20} = c_{11}c_{21},\\
&c_{11}t_{10} = c_{10}t_{11},\ c_{10}t_{10} = c_{10}t_{12},\ c_{11}t_{11} = c_{11}t_{12},\\
&c_{10}i_1 = c_{11}i_1,\\
&c_{21}t_{20} = c_{20}t_{21},\ c_{20}t_{20} = c_{20}t_{22},\ c_{21}t_{21} = c_{21}t_{22},\\
&c_{20}i_2 = c_{21}i_2,\\
&t_{10}h_0 = c_{20}h_2,\ t_{10}h_1 = c_{21}h_2,\ t_{11}h_0 = c_{20}h_3,\ t_{11}h_1 = c_{21}h_3,\\
&t_{12}h_0 = c_{20}h_4,\ t_{12}h_1 = c_{21}h_4.
\end{aligned}
$$

$$
\begin{aligned}
&t_{10}a_0 = t_{10}a_3,\ t_{11}a_0 = t_{10}a_2,\ t_{12}a_2 = t_{11}a_3,\ t_{12}a_0 = t_{10}a_1,\\
&t_{11}a_1 = t_{11}a_2,\\
&c_{20}a_4 = t_{12}a_1,\ c_{21}a_4 = t_{12}a_3,\\
&i_1\ell_1 = t_{11}\ell_0,\ c_{20}\ell_2 = t_{12}\ell_0,\ c_{21}\ell_2 = t_{10}\ell_0,\\
&i_1r_1 = t_{10}r_0,\ c_{20}r_2 = t_{12}r_0,\ c_{21}r_2 = t_{11}r_0.
\end{aligned}
$$

The relations of $\mathbf{L}_{\mathtt{anabicat}}$ are exactly its maximal objects, that is, its level-3 objects, $\dot{\mathtt{I}}_2$, $\dot{\mathtt{T}}_2$, $\dot{\mathtt{H}}$, $\dot{\mathtt{A}}$, $\dot{\mathtt{L}}$ and $\dot{\mathtt{R}}$.

The equalities between composites arise naturally; they hold in a bicategory (as a $\mathtt{L}_{\mathtt{bicat}}$-structure); also, the relations of $\mathbf{L}_{\mathtt{anabicat}}$ are interpreted in a bicategory "relationally"; in brief, every bicategory is an $\mathbf{L}_{\mathtt{anabicat}}$-structure.

In [M2], the concepts of *anabicategory*, and *saturated anabicategory* were introduced. Although these concepts implicitly underlie all that follows, they will not be relied on explicitly.

An anabicategory is an $\mathbf{L}_{\mathtt{anabicat}}$-structure satisfying certain axioms $\Sigma_{\mathtt{anabicat}}$ in FOLDS (with restricted equality) over $\mathbf{L}_{\mathtt{anabicat}}$; a saturated anabicategory is one that satisfies a larger set $\Sigma_{\mathtt{sanabicat}}$ of axioms in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ (these facts will be seen upon inspecting the definitions in [M2]). An anabicategory is like a bicategory, with the composition functors replaced by composition anafunctors.

For the reader who has a copy of [M2], I now point out some details, which, however, are not needed later.

Let $\mathcal{A}$ be an anabicategory as in [M2]. In explaining in what way $\mathcal{A}$ is an $\mathbf{L}_{\mathtt{anabicat}}$-structure, we will write T_1 for $\mathcal{A}\mathrm{T}_1$, *etc*. For a diagram

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd} \quad , \qquad (1)$$

$\mathrm{T}_1(f, g, h)$ (short for $\mathrm{T}_1(A, B, C, f, g, h)$) is the set $|\circ_{A, B, C}|((f, g), h)$, the set of specifications s for h being the composite of f and g , $h = g \circ_s f$ (see 3.1.(iv) in [M2]). For $f: A \to A \in \mathrm{C}_1$, $\mathrm{I}_1(A, f)$ is $|1_A|(*, f)$, the set of specifications i for f being the identity 1-cell on A , $f = 1_{A, i}$ (see 3.1.(iii) in [M2]). For

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "k" near end] & C \arrow[d, "h"] \\ A \arrow[u, "f"] \arrow[ur, "i"' near end] \arrow[r, shift left=2, "\ell"] \arrow[r, shift right=2, "j"'] \arrow[r, phantom, "\Uparrow\alpha"] & D \end{tikzcd} \qquad (2)$$

in $\mathcal{A}$, and

$$a \in \mathrm{T}_1(f, g, i)\ ,\ \ b \in \mathrm{T}_1(i, h, j)\ ,\ \ c \in \mathrm{T}_1(g, h, k)\ ,\ \ d \in \mathrm{T}_1(f, j, \ell)\ , \qquad (3)$$

and $\alpha: j \to \ell$,we have

$$\mathrm{A}(a, b, c, d; \alpha) \iff \alpha = \alpha_{a, b, c, d}$$

(see 3.1.(vi) in [M2]). (According to our conventions in logic with dependent sorts, $\mathrm{A}(a, b, c, d; \alpha)$ is short for $\mathrm{A}(A, B, C, D; f, g, h, i, j, k, \ell; a, b, c, d; \alpha)$) .

Every bicategory (as an $\mathbf{L}_{\mathtt{anabicat}}$-structure) is an anabicategory, although not necessarily saturated.

Whereas the interpretation of $\mathtt{T}_1$ in a bicategory, the notion of "commutative triangle of 1-cells", is a *relation* on triangles of 1-cells (where a triangle of 1-cells is three objects and three arrows (1-cells) appropriately related via the domain/codomain functions), in an anabicategory, we have a sort of entity that may be called "specification for a commutative triangle of 1-cells". Such a specification does specify a unique triangle (via the maps $\mathtt{t}_{1i}$); however, the property "commutative" does not figure separately. You may say that a triangle is commutative if *there is* a specification for it to be commutative, but in the concept of anabicategory, we do not work with this notion, we only work with the specifications. In an anabicategory, the expression "commutative triangle (of 1-cells)" should always be interpreted as "specification for a commutative triangle".

Next, we define a translation of the language $\mathbf{L}_{\mathtt{anabicat}}$ into the theory $\mathtt{T}_{\mathtt{bicat}}$; that is, a $[\mathtt{T}_{\mathtt{bicat}}]$-bicat structure $I:\mathbf{L}_{\mathtt{anabicat}} \longrightarrow [\mathtt{T}_{\mathtt{bicat}}]$. Via this translation, every bicategory $\mathcal{A}$ gives rise to $\mathcal{A}^{\#}=\mathcal{A}\circ I$, an $\mathbf{L}_{\mathtt{anabicat}}$-structure. $\mathcal{A}^{\#}$ is in fact a saturated anabicategory; however, for the main result, we will not need this fact; we will use the actual construction of $\mathcal{A}^{\#}$ as an $\mathbf{L}_{\mathtt{anabicat}}$-structure only. (In [M2], $\mathcal{A}^{\#}$ was defined for the special case of a monoidal category (one-object bicategory) $\mathcal{A}$ only.) We define the passage $\mathcal{A}\mapsto\mathcal{A}^{\#}$; this will describe the said interpretation as well.

In $\mathcal{A}^{\#}$, the interpretation of the part

$$\begin{tikzcd}
\mathtt{C}_0 & \mathtt{C}_1 \arrow[l, shift right, "\mathtt{c}_{10}"'] \arrow[l, shift left, "\mathtt{c}_{11}"] & \mathtt{C}_2 \arrow[l, shift right, "\mathtt{c}_{20}"'] \arrow[l, shift left, "\mathtt{c}_{21}"] & \mathtt{T}_2 \arrow[l, shift right=2, "\mathtt{t}_{20}"'] \arrow[l, "\mathtt{t}_{21}" description] \arrow[l, shift left=2, "\mathtt{t}_{22}"] \\
 & & \mathtt{I}_2 \arrow[u] &
\end{tikzcd}$$

of $\mathbf{L}_{\mathtt{anabicat}}$ is the same as in $\mathcal{A}$.

Under (1) (0-cells and 1-cells in $\mathcal{A}$ as well as in $\mathcal{A}^{\#}$),

$$\mathcal{A}^{\#}\mathrm{T}_1(f, g, h) \underset{\text{def}}{\equiv} \mathrm{Iso}^{\mathcal{A}}(gf, h)$$

= the set of all isomorphism 2-cells $gf \xrightarrow{\cong} h$.

If

$$\begin{tikzcd}
 & B \arrow[dr, bend left, "h"] \arrow[dr, bend right, "i"'] \arrow[dr, phantom, "\overset{\delta}{\Leftarrow}"] & \\
A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\underset{\beta}{\Rightarrow}"] \arrow[rr, bend left=15, "k"] \arrow[rr, bend right=15, "j"'] \arrow[rr, phantom, "\Uparrow\varepsilon"] & & C
\end{tikzcd}$$

and $s \in \mathcal{A}^{\#}\mathrm{T}_1(f, h, j)$, $t \in \mathcal{A}^{\#}\mathrm{T}_1(g, i, k)$, then

$$\mathcal{A}^{\#}\dot{\mathrm{H}}(s, t; \beta, \delta, \varepsilon) \underset{\text{def}}{\Longleftrightarrow} \begin{tikzcd}
hf \arrow[r, "s"] \arrow[d, "\delta\cdot\beta"'] \arrow[dr, phantom, "\circ"] & j \arrow[d, "\varepsilon"] \\
ig \arrow[r, "t"'] & k
\end{tikzcd} .$$

Under (2) and (3) in $\mathcal{A}^{\#}$,

$$\mathcal{A}^{\#}\dot{\mathrm{A}}(a, b, c, d; \alpha) \underset{\text{def}}{\Longleftrightarrow} \begin{tikzcd}
h(gf) \arrow[r, "\alpha_{f,g,h}"] \arrow[d, "ha"] & (hg)f \arrow[d, "cf"'] \\
hi \arrow[d, "b"] \arrow[r, phantom, "\circ"] & kf \arrow[d, "d"'] \\
j \arrow[r, "\alpha"] & \ell
\end{tikzcd} ; \qquad (4)$$

here a reference is made to the associativity isomorphism $\alpha_{f,g,h}$ given with $\mathcal{A}$.

For a 1-cell $f: A \to A$, $\mathcal{A}^{\#}\mathrm{I}_1(A; f) = \mathrm{Iso}(1_A, f)$.

For

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "h"] & \\ B \arrow[r, "g"'] & B \end{tikzcd} ,$$

$a\in\mathcal{A}^{\#}\mathrm{T}_1(A, B, B; f, g, h)$, $i\in\mathrm{I}_1(B; g)$, $\lambda:\mathrm{C}_2(h, f)$,

$$\mathcal{A}^{\#}\dot{\mathrm{L}}(a, i, \lambda) \underset{\mathrm{def}}{\Longleftrightarrow} \begin{tikzcd} 1_B f \arrow[r, "if"] \arrow[d, "\lambda_f"'] \arrow[dr, phantom, "\circ"] & gf \arrow[d, "a"] \\ f & h \arrow[l, "\lambda"] \end{tikzcd} ,$$

where a reference is made to the identity isomorphism λ_f given with $\mathcal{A}$. The definition of $\mathcal{A}^{\#}\dot{\mathrm{R}}$ is a straightforward variant.

In a bicategory $\mathcal{A}$, a 1-cell $f:B\to A$ is an *equivalence* if there is $f':A\to B$ such that $f\circ f'\cong 1_A$, $f'\circ f\cong 1_B$; this is equivalent to saying that for any $C\in\mathcal{A}$, the induced functor $f^*:\mathcal{A}(C, B)\longrightarrow\mathcal{A}(C, A)$ is an equivalence of categories.

We have the notion of *functor of bicategories*; this is just a different expression for "homomorphism of bicategories" (see [Be], [S]). A functor $F:\mathcal{X}\longrightarrow\mathcal{A}$ of bicategories is an *equivalence* (*of bicategories*) [instead of "biequivalence"], in notation $F:\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$, if

(i) for every $A\in\mathcal{A}$, there is $X\in\mathcal{X}$ and an equivalence $f:FX\xrightarrow{\simeq}A$;

and

(ii) for $X, Y\in\mathcal{X}$, F induces an equivalence of categories $\mathcal{X}(X, Y)\longrightarrow\mathcal{A}(FX, FY)$.

See [S].

We say that the bicategories $\mathcal{X}$, $\mathcal{A}$ are *equivalent* [instead of "biequivalent"] if there is an equivalence $\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$. Equivalence of bicategories is an equivalence relation (this requires the Axiom of Choice; the fact is well-known, but it also follows from (5) below).

Let $\boldsymbol{L} = \mathbf{L}_{\mathtt{anabicat}}$.

(5) For any bicategories $\mathcal{X}$, $\mathcal{A}$, $\mathcal{X} \simeq \mathcal{A}$ iff $\mathcal{X} \approx_{\boldsymbol{L}} \mathcal{A}$.

Proof. (A) ("if") Let $(\mathcal{R}, r_0, r_1) : \mathcal{X}^{\#} \overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}} \mathcal{A}^{\#}$. We construct $F:\mathcal{X} \xrightarrow{\simeq} \mathcal{A}$.

We write $\langle\varepsilon\rangle$ for $r_0(\varepsilon)$, and $[\varepsilon]$ for $r_1(\varepsilon)$. We will write $\mathcal{R}$ for $r_0^*\mathcal{X}^{\#}=r_1^*\mathcal{A}^{\#}$ too.

Given any $X\in\mathcal{X}C_0$, we pick (by Choice) $\bar{X}\in\mathcal{R}C_0$ such that $\langle\bar{X}\rangle=X$. We put $FX \underset{\text{def}}{=} [\bar{X}]$. For any $f:X\to Y$ in $\mathcal{X}$, pick (by Choice) $\bar{f}\in\mathcal{R}C_1(\bar{X}, \bar{Y})$ such that $\langle\bar{f}\rangle=f$, and for $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $(\Downarrow\beta)$, $\bar{\beta}\in C_2(\bar{f}, \bar{g})$ with $\langle\bar{\beta}\rangle=\beta$ ($\bar{\beta}$ is uniquely determined); define $Ff = [\bar{f}]$, $F\beta = [\bar{\beta}]$.

For $X\xrightarrow{f}Y\xrightarrow{g}Z$ in $\mathcal{X}$, $a \underset{\text{def}}{=} 1_{gf}\in\mathcal{X}^{\#}T_1(f, g, gf)$; let $\bar{a}\in\mathcal{R}T_1(\bar{f}, \bar{g}, \overline{gf})$ such that $\langle\bar{a}\rangle=a$; then $[\bar{a}]\in\mathcal{A}T_1(Ff, Fg, F(gf))$, that is, $[\bar{a}]:Fg\circ Ff\xrightarrow{\cong}F(gf)$. Therefore, we may define $F_{f,g} \underset{\text{def}}{=} [\bar{a}]$.

The coherence condition that the $F_{f,g}$ have to satisfy (the sense in which F preserves the associativity isomorphisms) reads as follows: given

$$X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}W,$$

we have

$$
\begin{tikzcd}
Fh(FgFf) \arrow[r, "\alpha_{Ff,\, Fg,\, Fh}"] \arrow[d, "FhF_{f,\, g}"'] & (FhFg)Ff \arrow[d, "F_{g,\, h}Ff"] \\
FhF(gf) \arrow[d, "F_{gf,\, h}"'] \arrow[r, phantom, "\circ?"] & F(hg)Ff \arrow[d, "F_{f,\, hg}"] \\
F(h(gf)) \arrow[r, "F(\alpha_{f,\, g,\, h})"'] & F((hg)f)
\end{tikzcd} .
$$

Writing $a=1_{gf}$, $b=1_{h(gf)}$, $c=1_{hg}$, $d=1_{(hg)f}$, this amounts to the same as

$$
\begin{tikzcd}
[\bar{h}]([\bar{g}][\bar{f}]) \arrow[r, "\alpha_{[\bar{f}],\, [\bar{g}],\, [\bar{h}]}"] \arrow[d, "{[\bar{h}][\bar{a}]}"'] & ([\bar{h}][\bar{g}])[\bar{f}] \arrow[d, "{[\bar{c}][\bar{f}]}"] \\
[\bar{h}][\overline{gf}] \arrow[d, "{[\bar{b}]}"'] \arrow[r, phantom, "\circ?"] & [\overline{hg}][\bar{f}] \arrow[d, "{[\bar{d}]}"] \\
[\overline{h(gf)}] \arrow[r, "{[\overline{\alpha_{f,\, g,\, h}}]}"'] & [\overline{hg}][\bar{f}]
\end{tikzcd} .
$$

But by (4), the last commutativity is equivalent to saying that $\mathcal{A}^{\#}\mathrm{A}([\bar{a}], [\bar{b}], [\bar{c}], [\bar{d}]; [\overline{\alpha_{f,\, g,\, h}}])$ holds. The latter is a consequence of $\mathcal{R}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \overline{\alpha_{f,\, g,\, h}})$, which in turn follows from $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha_{f,\, g,\, h})$, which, finally, holds by (4) since a, b, c and d are identities.

The preservation by F of identity isomorphisms, and that of horizontal composition (see [MP], §4.1, (2)(v) and (2)(iv)) are similar, and use L , R and H , respectively.

The facts that F preserves identity 2-cells and vertical composition of 2-cells are immediate.

We **claim** that for any $A\in\mathcal{A}\mathrm{C}_0$, there is $X\in\mathcal{X}\mathrm{C}_0$ such that $FX\simeq A$. Given A , pick $\hat{X}\in\mathcal{R}\mathrm{C}_0$ with $[\hat{X}]=A$, and let $X=\langle\hat{X}\rangle$. (Picture:

$$\begin{tikzcd}
 & \bar{X}^{\,4} \arrow[ddl] \arrow[ddrr] & & \\
 & \hat{X}\ 2 \arrow[dl] \arrow[dr] & & \\
\underset{3}{X} & & \underset{1}{A} & \underset{5}{FX}
\end{tikzcd} \quad .)$$

Consider $1_X \in \mathcal{X}\mathrm{C}_1(X, X)$, and let $i \in \mathcal{R}\mathrm{C}_1(\bar{X}, \hat{X})$, $j \in \mathcal{R}\mathrm{C}_1(\hat{X}, \bar{X})$ such that $\langle i \rangle = \langle j \rangle = 1_X$. We have

$$[i]: FX \longrightarrow A\,, \quad [j]: A \longrightarrow FX$$

in $\mathcal{A}$. Let $f = 1_X \circ 1_X \in \mathcal{X}\mathrm{C}_1(X, X)$, and $\bar{f} \in \mathcal{R}\mathrm{C}_1(\bar{X}, \bar{X})$, $\hat{f} \in \mathcal{R}\mathrm{C}_1(\hat{X}, \hat{X})$ such that $\langle \bar{f} \rangle = \langle \hat{f} \rangle = f$. Consider $1_f \in \mathcal{X}\mathrm{C}_2(1_X \circ 1_X, f)$; then $1_f \in \mathcal{X}^{\#}\mathrm{T}_1(X, X, X; 1_X, 1_X, f)$. Let $\iota \in \mathrm{T}_1(\bar{X}, \hat{X}, \bar{X}; i, j, \bar{f})$, $\iota' \in \mathrm{T}_1(\hat{X}, \bar{X}, \hat{X}; j, i, \hat{f})$ such that $\langle \iota \rangle = \langle \iota' \rangle = 1_f$. Then $[\iota] \in \mathcal{A}^{\#}\mathrm{T}_1(FX, A, FX; [i], [j], [\bar{f}])$, and thus

$$[\iota]: [j] \circ [i] \xrightarrow{\ \cong\ } [\bar{f}]\ ; \qquad (6)$$

similarly,

$$[\iota']: [i] \circ [j] \xrightarrow{\ \cong\ } [\hat{f}]\ .$$

But, $\varphi \overset{=}{\mathrm{def}} \lambda_{1_X}: f \xrightarrow{\cong} 1_X$, that is, $\varphi \in \mathcal{X}^{\#}\mathrm{I}_1(X, f)$. Thus, there is $\bar{\varphi} \in \mathcal{R}\mathrm{I}_1(\bar{X}, \bar{f})$ (such that $\langle \bar{\varphi} \rangle = \varphi$). Then, $[\bar{\varphi}] \in \mathcal{A}^{\#}\mathrm{I}_1(FX, [\bar{f}])$, *i.e.*, $[\bar{\varphi}]: 1_{FX} \xrightarrow{\cong} [\bar{f}]$. Combined with (6), we get $[j] \circ [i] \cong 1_{FX}$. Similarly, $[i] \circ [j] \cong 1_A$. The data $[i], [j]$ provide an equivalence of FX and A as **claimed**.

Let us see that $F_{X,Y}: \mathcal{X}(X, Y) \longrightarrow \mathcal{A}(FX, FY)$ is an equivalence of categories. That it is a bijection on hom-sets is a consequence of the fact that $(\mathcal{R}, r_0, r_1)$ respects the equalities on C_2-sorts. To see essential surjectivity on objects, let $g: FX \to FY$, that is, $g \in \mathcal{A}^{\#}\mathrm{C}_1([\bar{X}], [\bar{Y}])$. There is $\hat{f} \in \mathcal{R}\mathrm{C}_1(\bar{X}, \bar{Y})$ such that $[\hat{f}] = g$; let $f = \langle \hat{f} \rangle$. We now have $\bar{f}, \hat{f}$ both in $\mathcal{R}\mathrm{C}_1(\bar{X}, \bar{Y})$, and both "over" f. There are $i\in\mathcal{R}\mathrm{C}_2(\bar{f}, \hat{f})$, $j\in\mathcal{R}\mathrm{C}_2(\hat{f}, \bar{f})$, $\bar{\ell}\in\mathcal{R}\mathrm{C}_2(\bar{f}, \bar{f})$, $\hat{\ell}\in\mathcal{R}\mathrm{C}_2(\hat{f}, \hat{f})$ such that $\langle i\rangle=\langle j\rangle=\langle \bar{\ell}\rangle=\langle \hat{\ell}\rangle=1_f$. We have $\mathcal{X}^{\#}\mathrm{I}_2(f; 1_f)$, hence, $\mathcal{R}\mathrm{I}_2(\bar{f}; \bar{\ell})$ and $\mathcal{A}^{\#}\mathrm{I}_2(Ff; [\bar{\ell}])$; that is, $[\bar{\ell}]=1_{Ff}$. Similarly, $[\hat{\ell}]=1_g$. Since $\mathcal{X}^{\#}\mathrm{T}_2(f, f, f; 1_f, 1_f, 1_f)$, we have $\mathcal{R}\mathrm{T}_2(\bar{f}, \hat{f}, \bar{f}; i, j, \bar{\ell})$ and $\mathcal{R}\mathrm{T}_2(\hat{f}, \bar{f}, \hat{f}; j, i, \hat{\ell})$, and as a consequence, $\mathcal{A}^{\#}\mathrm{T}_2(Ff, g, Ff; [i], [j], 1_{Ff})$ and $\mathcal{A}^{\#}\mathrm{T}_2(g, Ff, g; [j], [i], 1_g)$; that is, $[j][i]=1_{Ff}$, $[i][j]=1_g$. This shows that $g\cong Ff$ as desired.

(B) ("only if") Let $F:\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$, we construct $(\mathcal{R}, r_0, r_1):\mathcal{X}^{\#}\xleftarrow[\mathbf{L}]{}\!\!\rightarrow\mathcal{A}^{\#}$. We will again write $\langle\varepsilon\rangle$ for $r_0(\varepsilon)$, $[\varepsilon]$ for $r_1(\varepsilon)$.

We put $\mathcal{R}\mathrm{C}_0 \overset{=}{\mathrm{def}} \{(X, A, x): X\in\mathcal{X}\mathrm{C}_0,\ A\in\mathcal{A}\mathrm{C}_0,\ x \text{ is an equivalence } x:FX\xrightarrow{\simeq}A\}$; $\langle(X, A, x)\rangle \overset{=}{\mathrm{def}} X$, $[(X, A, x)] \overset{=}{\mathrm{def}} A$.

Let us introduce a helpful notation. For any object D of $\mathbf{L}_{\mathrm{anabicat}}$, any $d_1\in\mathcal{X}D$ and $d_2\in\mathcal{A}D$, $\mathcal{R}D[d_1, d_2]$ stands for $\{d\in\mathcal{R}D: \langle d\rangle=d_1, [d]=d_2\}$, "the fiber of $\mathcal{R}D$ over (d_1, d_2)". We extend this definition to any sort $\mathcal{R}D(e, e', \ldots)$ in $\mathcal{R}$, in place of $\mathcal{R}D$;

$$\mathcal{R}D(e, e', \ldots)[d_1, d_2] = \{d\in\mathcal{R}D(e, e', \ldots): \langle d\rangle=d_1, [d]=d_2\} ;$$

here, it is assumed that $d_1\in\mathcal{X}^{\#}D(\langle e\rangle, \langle e'\rangle, \ldots)$, $d_2\in\mathcal{A}^{\#}D([e], [e'], \ldots)$.

The definition of $\mathcal{R}\mathrm{C}_0$ together with effect of r_1, r_2 on it, can be put, more succinctly, as

$$\mathcal{R}\mathrm{C}_0[X, A] = \mathrm{Equiv}(FX, A) = \{x: x:FX\xrightarrow{\simeq}A\} .$$

Continuing, we define, for $f:X\to Y$, $\bar{f}:A\to B$, $x=(X, A, x)$, $y=(Y, B, y)\in\mathcal{R}\mathrm{C}_0$,

$$\mathcal{R}\mathrm{C}_1(x, y)[f, \bar{f}] = \mathrm{Iso}(y\circ Ff, \bar{f}\circ x) ,$$

the set of all 2-cell-isomorphisms φ as in

$$\begin{tikzcd} FX \arrow[r, "x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\cong}{\nearrow}\varphi"] & A \arrow[d, "\bar{f}"] \\ FY \arrow[r, "y"'] & B \end{tikzcd} .$$

$\mathcal{R}C_2$ is relational, meaning that its fibers are either $\{*\}$, or $\emptyset$. Instead of "$* \in \mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$", we just write "$\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$".

For $X \underset{g}{\overset{f}{\underset{\downarrow\mu}{\rightrightarrows}}} Y$ in $\mathcal{X}$, $A \underset{\bar{g}}{\overset{\bar{f}}{\underset{\downarrow\nu}{\rightrightarrows}}} B$ in $\mathcal{A}$, x, y and φ as before, and

$\gamma \in \mathcal{R}C_1(x, y)[g, \bar{g}]$,

$$\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu] \quad \underset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} FX \arrow[rr, "x"] \arrow[d, "Ff"', shift right=4] \arrow[d, "Fg", shift left=4] \arrow[d, phantom, "\xrightarrow{F\mu}"] & \text{"}\circ\text{"} & A \arrow[d, "\bar{f}"', shift right=2] \arrow[d, "\bar{g}", shift left=2] \arrow[d, phantom, "\overset{\nu}{\to}"] \\ FY \arrow[rr, "y"'] & \varphi \nearrow \quad \gamma \nearrow & B \end{tikzcd}$$

$$\underset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} y \circ Ff \arrow[r, "y \circ F\mu"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & y \circ Fg \arrow[d, "\gamma"] \\ \bar{f} \circ x \arrow[r, "\nu \circ x"'] & \bar{g} \circ x \end{tikzcd} .$$

Using that x, y are equivalences, and that F is an equivalence of bicategories, we see that, for fixed x, y, φ, γ, the relation $\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$ of the variables μ, ν is a bijection

$$\mu \mapsto \nu : \mathcal{X}C_2(f, g) \xrightarrow{\cong} \mathcal{A}C_2(\bar{f}, \bar{g}) .$$

This implies that $(\mathcal{R}, r_0, r_1)$ preserves the equality relation $\dot{E}_{C_2}$. Also, with reference to

$X \underset{h}{\overset{f}{\underset{\downarrow\rho}{\overset{\downarrow\mu}{\underset{g}{\to}}}}} \big\downarrow \xi \to Y$, $A \underset{\bar{h}}{\overset{\bar{f}}{\underset{\downarrow\sigma}{\overset{\downarrow\nu}{\underset{\bar{g}}{\to}}}}} \big\downarrow \zeta \to B$, and $\eta \in \mathcal{R}C_1[h, \bar{h}]$, we easily see that

$\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$, $\mathcal{R}C_2(y, z; \gamma, \eta)[\rho, \sigma]$, $\rho\mu = \xi$, $\sigma\nu = \zeta \Longrightarrow$

$$\mathcal{R}\mathrm{C}_2(x, z; \gamma, \eta)[\xi, \zeta] \ ,$$

from which it follows (by the above bijection $\mu \mapsto \nu$) that

$$\mathcal{R}\mathrm{C}_2(x, y; \varphi, \gamma)[\mu, \nu] \ , \ \mathcal{R}\mathrm{C}_2(y, z; \gamma, \eta)[\rho, \sigma] \ , \mathcal{R}\mathrm{C}_2(x, z; \gamma, \eta)[\xi, \zeta] \implies \rho\mu = \xi \iff \sigma\nu = \zeta \ .$$

This means that $r_0^{-1}(\mathcal{X}^{\#}\mathrm{T}_2) = r_1^{-1}(\mathcal{A}^{\#}\mathrm{T}_2)$; that is, $(\mathcal{R}, r_0, r_1)$ preserves T_2 .

Given

$$(x\colon FX \xrightarrow{\simeq} A) \in \mathcal{R}\mathrm{C}_0[X, A] \ , \ f\colon X \to X \text{ in } \mathcal{X} \ , \ \bar{f}\colon A \to A \text{ in } \mathcal{A} \ , \ \varphi \in \mathcal{R}\mathrm{C}_1(x, x)[f, \bar{f}] \ ,$$

that is,

$$\begin{tikzcd}
FX \arrow[r, "x"] \arrow[d, "Ff"'] & A \arrow[d, "\bar{f}"] \\
FX \arrow[r, "x"'] \arrow[ur, phantom, "\overset{\cong}{\nearrow}\,\varphi"] & A
\end{tikzcd} \ ,$$

and $a\colon 1_X \xrightarrow{\cong} f$, $\bar{a}\colon 1_A \xrightarrow{\cong} \bar{f}$, we have

$$\mathcal{R}\mathrm{I}_1(\underset{c_{10}i_1}{x}, \varphi)[a, \bar{a}] \underset{\text{def}}{\iff} \begin{tikzcd}
xFf \arrow[r, "\varphi", "\cong"'] & \bar{f}x \\
xF(1_X) \arrow[u, "xFa", "\cong"'] \arrow[d, "xF_X"', "\cong"] \arrow[r, phantom, "\circ"] & 1_A x \arrow[u, "\cong", "\bar{a}x"'] \arrow[d, "\cong"', "\lambda_x"] \\
x1_{FX} \arrow[r, "\cong", "\rho_x"'] & x
\end{tikzcd} \ .$$

Given

(*) $\quad (x\colon FX \xrightarrow{\simeq} A) \in \mathcal{R}\mathrm{C}_0[X, A] \ , \ (y\colon FY \xrightarrow{\simeq} B) \in \mathcal{R}\mathrm{C}_0[Y, B] \ ,$

$\qquad (z\colon FZ \xrightarrow{\simeq} C) \in \mathcal{R}\mathrm{C}_0[Z, C] \ ,$

$$
\begin{tikzcd}[column sep=small]
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "i"'] & & Z
\end{tikzcd}
\text{ in } \mathcal{X},\quad
\begin{tikzcd}[column sep=small]
& B \arrow[dr, "\bar g"] & \\
A \arrow[ur, "\bar f"] \arrow[rr, "\bar i"'] & & C
\end{tikzcd}
\text{ in } \mathcal{A},
$$

$$(a: gf \xrightarrow{\cong} i) \in \mathcal{X}^{\#}\mathrm{T}_1(f, g, i)\ ,\quad (\bar a: \bar g\bar f \xrightarrow{\cong} \bar i) \in \mathcal{A}^{\#}\mathrm{T}_1(\bar f, \bar g, \bar i)\ ,$$

$$\varphi \in \mathcal{R}\mathrm{C}_1(x, y)[f, \bar f]\ ,\quad \gamma \in \mathcal{R}\mathrm{C}_1(y, z)[g, \bar g]\ ,\quad \iota \in \mathcal{R}\mathrm{C}_1(x, z)[i, \bar i]\ ,$$

we have

$$\mathcal{R}\mathrm{T}_1(\varphi, \gamma, \iota)[a, \bar a] \quad \underset{\mathrm{def}}{\Longleftrightarrow}$$

$$
\begin{tikzcd}
(zFg)Ff \arrow[d, "(right)\ \gamma Ff"'] & z(FgFf) \arrow[l, "\cong"', "\alpha"] \arrow[rr, "\overset{(front)}{zFaF_{f,\,g}}"] & & zFi \arrow[d, "\iota\ \ (bottom)"] \\
(\bar g y)Ff & \bar g(yFf) \arrow[l, "\cong"', "\alpha"] \arrow[dr, "(left)\ \ \bar g\varphi"'] \arrow[rr, phantom, "\circ"] & & \bar i x \\
& & \bar g(\bar f x) \arrow[r, "\cong", "\alpha"'] & (\bar g\bar f)x \arrow[u, "\bar a x\ \ (back)"']
\end{tikzcd}
$$

(we have referred to the following diagram of 1-cells, and its "faces":

$$
\begin{tikzcd}
& & & & B \arrow[dr, "\bar g"] & \\
& FY \arrow[urrr, "y"] \arrow[ddr, "Fg"] & & & & \\
& & & A \arrow[uur, "\bar f"] \arrow[rr, "\bar i"] & & C \\
FX \arrow[uur, "Ff"] \arrow[urrr, "x"] \arrow[rr, "Fi"'] & & FZ \arrow[urrr, "z"'] & & &
\end{tikzcd}
$$
).

The facts that $\dot{\mathrm{E}}_{\mathrm{I}_1}$, $\dot{\mathrm{E}}_{\mathrm{T}_1}$ are preserved are shown through the facts that the definitions of $\mathcal{R}\mathrm{I}_1$, $\mathcal{R}\mathrm{T}_1$ give bijections $a \mapsto \bar a$.

The proof that $(\mathcal{R}, r_0, r_1)$ so defined preserves $\dot{A}$ and $\dot{H}$ is put into Appendix D .

We have an "augmented" version of (5), similarly to §6. I will state this without proof; for the proof the details of the notion of anafunctor would be needed, together with a concept of cleavage; the proof is, in outline, quite similar to the proof of 5.(8).

Let $\boldsymbol{K}_0$ be the full subcategory of $\boldsymbol{L}=\mathbf{L}_{\mathtt{anabicat}}$ consisting of the objects C_0 , C_1 and C_2 . A *restricted context* is a context of $\boldsymbol{K}_0$. For a bicategory $\mathcal{A}$ and its saturation $\mathcal{A}^{\#}$, $\mathcal{A}\restriction\boldsymbol{K}_0=\mathcal{A}^{\#}\restriction\boldsymbol{K}_0$; $\mathcal{A}[\mathcal{X}]=\mathcal{A}^{\#}[\mathcal{X}]$ whenever $\mathcal{X}$ is restricted.

Let $\mathcal{X}$ be a restricted context. An *augmented bicategory* of type $\mathcal{X}$ is a pair $(\mathcal{A}, \vec{a})$ of a bicategory $\mathcal{A}$ and a tuple $\vec{a}\in\mathcal{A}[\mathcal{X}]$; symbols such as $(\mathcal{A}, \vec{a})$, $(\mathcal{X}, \vec{x})$ stand for augmented bicategories. The notation $E:(\mathcal{X}, \vec{x})\xrightarrow{\simeq}(\mathcal{A}, \vec{a})$ signifies that $E:\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$ and $E(\vec{x})=\vec{a}$. The relations $\xrightarrow{\sim}$ and $\xleftrightarrow{\sim}$ are now defined in the same way as for $\boldsymbol{I}$-diagrams in §6. For bicategories, that is type-$\emptyset$ augmented bicategories, the relations $\xrightarrow{\sim}$, $\xleftrightarrow{\sim}$ coincide with equivalence $\simeq$. Generalizing (5), we have

(7) For augmented bicategories $(\mathcal{X}, \vec{x})$, $(\mathcal{A}, \vec{a})$, $(\mathcal{X}^{\#}, \vec{x})\approx_{\boldsymbol{L}}(\mathcal{A}^{\#}, \vec{a})$ iff $(\mathcal{X}, \vec{x})\xleftrightarrow{\sim}(\mathcal{A}, \vec{a})$.

We can, analogously to §6, define a recursive translation $\theta\mapsto\theta^{*}$ from FOLDS formulas θ over $\boldsymbol{L}$ to formulas θ^{*} in ordinary multisorted logic over $\mathrm{L}_{\mathtt{bicat}}$ such that, if $\mathcal{X}=\mathrm{Var}(\theta)$ is a restricted context, then $\mathrm{Var}(\theta^{*})=\mathcal{X}$, and for any bicategory $\mathcal{A}$, $\vec{a}\in\mathcal{A}[\mathcal{X}]$, $\mathcal{A}^{\#}\models\theta[\vec{a}]$ iff $\mathcal{A}\models\theta^{*}[\vec{a}]$. We obtain the following analogs of 5.(20) and 5.(20').

(8)(a) Let T be a theory extending $\mathrm{T}_{\mathtt{bicat}}$. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathtt{anabicat}}$, σ an L_T-formula such that $\mathrm{Var}(\sigma)\subset\mathcal{X}$. The following two conditions (i), (ii) are equivalent.

(i) For any $M, N \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, $\vec{b} \in |N|[\mathcal{X}]$, $M \models \sigma[\vec{a}]$ and $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$ imply $N \models \sigma[\vec{b}]$.

(ii) There is θ in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ with $\mathtt{Var}(\theta) \subset \mathcal{X}$ such that for all $M \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, we have $M \models \sigma[\vec{a}]$ iff $M \models \theta^{*}[\vec{a}]$.

(**b**) In particular, if σ is a sentence over L_{T}, and for any $M, N \models T$, $M \models \sigma$ and $|M| \simeq |N|$ imply $N \models \sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ such that for any $M \models T$, $M \models \sigma$ iff $M \models \theta^{*}$.

(9) Let T be a normal theory of bicategories. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathtt{anabicat}}$. Suppose that the first-order formula σ over $\mathrm{L}_{\mathtt{bicat}}$ with free variables all in $\mathcal{X}$ is preserved and reflected along equivalences of models of T. Then there is a formula φ in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ such that σ is equivalent to φ^{*} in models of T.

(8)(b) follows from (5) (proved in detail above) and §5. As was mentioned, the proofs of (8)(a) and (9) require a more detailed look at anabicategories, similarly to what we did in §5 on anadiagrams in the proof of (20)(a); this work is omitted here.

A paraphrase of (8) can be stated as follows. A first-order property of a bicategory, or of a diagram of 0-cells, 1-cells and 2-cells in a bicategory, is invariant under (bi)equivalence of bicategories if and only if it can be expressed in FOLDS as a statement about the saturation of the bicategory.