Let $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}] = (\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}], \Sigma_{\mathrm{diag}}[\boldsymbol{I}])$ be the theory of $\boldsymbol{I}$-diagrams of categories, functors, and natural transformations. $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$ is a theory in ordinary multisorted logic with equality. The models of $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$ are those $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$-structures that are isomorphic to some $D{:}\,\boldsymbol{I} \to \mathrm{Cat}$ as an $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$-structure (see above). Indeed, we can easily write down a set of axioms $\Sigma_{\mathrm{diag}}[\boldsymbol{I}]$ over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ whose models are, up to isomorphism, precisely the D's. Now, the construction $D \mapsto D^{\#}$ is related to an interpretation

$$\Phi{:}\,\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}] \longrightarrow [\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]] \qquad (19)$$

of the DSV $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ in the theory $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$; namely, $D^{\#} \cong \bar{D} \circ \Phi$; here, $\bar{D}{:}\,[\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]] \longrightarrow \mathrm{Set}$ is the coherent functor induced by $D{:}\,\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}] \to \mathrm{Set}$.

To describe Φ, I first introduce certain specific formulas over the language $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$. We refer to the (arbitrary) objects, arrows and 2-cell $I \underset{j}{\overset{i}{\rightrightarrows}} J$ in $\boldsymbol{I}$.

$$\bar{\mathrm{I}}_I(\kappa) \overset{=}{\mathrm{def}} \exists x \in \mathrm{I}_I . \mathrm{i}_I(x) = \kappa \qquad (\kappa{:}\mathrm{A}_I)$$

$$\dot{\mathrm{I}}_I(X, \kappa) \overset{=}{\mathrm{def}} \bar{\mathrm{I}}_I(\kappa) \wedge \mathrm{d}_I(\kappa) = X \qquad (X{:}\mathrm{O}_I, \kappa{:}\mathrm{A}_I)$$

$$\bar{\mathrm{T}}_I(f, g, h) \overset{=}{\mathrm{def}} \exists x \in \mathrm{T}_I . \mathrm{t}_{I0}(x) = f \wedge \mathrm{t}_{I1}(x) = g \wedge \mathrm{t}_{I0}(x) = h \qquad (f, g, h{:}\mathrm{A}_I)$$

$$\dot{\mathrm{T}}_I(X, Y, Z, f, g, h) \overset{=}{\mathrm{def}}$$
$$\mathrm{d}_I(f) = X \wedge \mathrm{c}_I(f) = Y \wedge \mathrm{d}_I(g) = Y \wedge \mathrm{c}_I(g) = Z \wedge \mathrm{d}_I(h) = X \wedge \mathrm{c}_I(h) = Z \wedge \bar{\mathrm{T}}_I(f, g, h)$$
$$(X, Y, Z{:}\mathrm{O}_I;\ f, g, h{:}\mathrm{A}_I)$$

$$\mathrm{Iso}_I(\mu) \overset{=}{\mathrm{def}} \exists \nu, \kappa, \lambda \in \mathrm{A}_I . \bar{\mathrm{I}}_I(\kappa) \wedge \bar{\mathrm{I}}_I(\lambda) \wedge \bar{\mathrm{T}}_I(\mu, \nu, \kappa) \wedge \bar{\mathrm{T}}_I(\nu, \mu, \lambda)$$
$$(\mu{:}\mathrm{A}_I)$$

$$\dot{\mathrm{O}}_i(X, A, \mu) \overset{=}{\mathrm{def}} \mathrm{Iso}_J(\mu) \wedge \mathrm{c}_J(x) = A \wedge \exists x \in \mathrm{O}_i . \mathrm{o}_{i0}(x) = X \wedge \mathrm{d}_J(\mu) = \mathrm{o}_{i1}(x).$$
$$(X{:}\mathrm{O}_I,\ A{:}\mathrm{O}_J,\ \mu{:}\mathrm{A}_J)$$

$$\mathrm{Comm}_J(\mu, g, h, \nu) \overset{=}{\mathrm{def}} \exists k \in \mathrm{A}_J . \bar{\mathrm{T}}_J(\mu, g, k) \wedge \bar{\mathrm{T}}_J(h, \nu, k)$$
$$(\mu, g, h, \nu{:}\mathrm{A}_J)$$

$$\dot{\mathrm{A}}_i(X, Y, A, B, \mu, \nu, f, g) \overset{=}{\mathrm{def}}$$
$$\dot{\mathrm{O}}_i(X, A, \mu) \wedge \dot{\mathrm{O}}_i(Y, B, \nu) \wedge \exists x \in \mathrm{A}_i . \mathrm{a}_{i0}(x) = f \wedge \mathrm{Comm}_J(\mu, g, \mathrm{a}_{i1}(x), \nu).$$
$$(X, Y{:}\mathrm{O}_I,\ A, B{:}\mathrm{O}_J,\ f{:}\mathrm{A}_I,\ \mu, \nu, g{:}\mathrm{A}_J)$$

$$\dot{\text{O}}_{\iota}(X, A, B, \mu, \nu, h) \;\underset{\text{def}}{\equiv}$$

$$\dot{\text{O}}_{i}(X, A, \mu)\wedge\dot{\text{O}}_{j}(X, B, \nu)\wedge\exists x\in\text{O}_{\iota}.\text{o}_{i0}\text{o}_{\iota 0}(x)=X\wedge\text{Comm}_{J}(\mu, h, \text{o}_{\iota 2}(x), \nu).$$
$$(X{:}\text{O}_{I},\ A, B{:}\text{O}_{J},\ \mu, \nu, h{:}\text{A}_{J})$$

This is the description of the effect of Φ on objects:

$$\Phi(\text{O}_{I}) = [X\in\text{O}_{I}\text{: }\mathbf{t}]$$
$$\Phi(\text{A}_{I}) = [X\in\text{A}_{I}\text{: }\mathbf{t}]$$
$$\Phi(\text{I}_{I}) = [X\in\text{O}_{I},\ \kappa\in\text{A}_{I}\text{: }\dot{\text{I}}_{I}(X, \kappa)]$$
$$\Phi(\text{T}_{I}) = [X, Y, Z\in\text{O}_{I};f, g, h\in\text{A}_{I}\text{: }\dot{\text{T}}_{I}(X, Y, Z, f, g, h)]$$
$$\Phi(\text{O}_{i}) = [X\in\text{O}_{I},\ A\in\text{O}_{J},\ \mu\in\text{A}_{J}\text{: }\dot{\text{O}}_{i}(X, A, \mu)]$$
$$\Phi(\text{A}_{i}) = [X, Y\in\text{O}_{I};\ A, B\in\text{O}_{J};\ f\in\text{A}_{I};\ \mu, \nu, g\in\text{A}_{J}\text{: }\dot{\text{A}}_{i}(X, Y, A, B, \mu, \nu, f, g)]$$
$$\Phi(\text{O}_{\iota}) = [X\in\text{O}_{I};\ A, B\in\text{O}_{J};\ \mu, \nu, h\in\text{A}_{J}\text{: }\dot{\text{O}}_{\iota}(X, A, B, \mu, \nu, h)]$$

To complete the definition of Φ as in (19), we should also specify the effect of Φ on arrows; this is done in the way straightforwardly suggested by our intentions with Φ.

The fact mentioned above that $D^{\#}\cong\bar{D}\circ\Phi$ holds will be seen directly. In fact, if we define $\bar{D}$ in the standard way (among the possibilities that differ by isomorphisms only), we obtain an equality $D^{\#}=\bar{D}\circ\Phi$.

Next, we explain a translation of formulas to formulas induced by Φ. Temporarily, let us call a FOLDS variable μ *special* if $\mu{:}\text{O}_{i}(X, A)$ for (unique) suitable $i{:}I\to J\in\text{Arr}(\boldsymbol{I})$ and $X{:}\text{O}_{I},\ A{:}\text{O}_{J}$. Let us fix a 1-1 mapping $\mu\mapsto\mu^{*}$ of special variables μ to variables μ^{*} in ordinary multisorted logic over $\text{L}_{\text{diag}}[\boldsymbol{I}]$ such that, when μ is as above, $\mu^{*}{:}\text{A}_{J}$. The non-special variables over $\mathbf{L}_{\text{anadiag}}[\boldsymbol{I}]$ are considered variables over $\text{L}_{\text{diag}}[\boldsymbol{I}]$; if $x{:}\text{O}_{I}$, $x{:}\text{O}_{I}$ in the sense of multisorted logic, and if $x{:}\text{A}_{I}(y, z)$, then $x{:}\text{A}_{I}$ in the sense of multisorted logic.

For a special variable μ as above, we have the formula $\varphi_{[\mu]}\underset{\text{def}}{\equiv}\dot{\text{O}}_{i}(X, A, \mu^{*})$, with the latter formula introduced above. For a finite context $\mathcal{X}$, we let $\mathcal{X}^*=\mathcal{X}-\{\mu\in\mathcal{X}:\ \mu$ special$\}\cup\{\mu^*\in\mathcal{X}:\ \mu$ special$\}$ (exchange every special variable μ for μ^*), and consider the formula $\varphi_{[\mathcal{X}]}\ \overset{=}{\mathrm{def}}\ \bigwedge\{\varphi_{[\mu]}:\mu\in\mathcal{X}$ special$\}$; $\mathrm{Var}(\varphi_{[\mathcal{X}]})=\mathcal{X}^*$. For a finite set $\mathcal{Y}$ of variables over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$, we write $\{\mathcal{Y}\}$ for the product-object $[\mathcal{Y}{:}\mathbf{t}]=\prod_{y\in\mathcal{Y}}[y\in\mathrm{K}_Y:\ \mathbf{t}]$ in $[\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]]$, where $y{:}\mathrm{K}_Y$.

Recall that, with Φ as in (19), for any finite context $\mathcal{X}$, we have the object $\Phi[\mathcal{X}]$ defined as a certain pullback. Inspection shows that $\Phi[\mathcal{X}]$ can be taken to be $|[\mathcal{X}^*:\varphi_{[\mathcal{X}]}]|$, the domain of the subobject $[\mathcal{X}^*:\varphi_{[\mathcal{X}]}]$ of $\{\mathcal{X}^*\}$; we have a canonical monomorphism $m:\Phi[\mathcal{X}]\rightarrowtail\{\mathcal{X}^*\}$. Thus, for any θ in FOLDS with restricted equality, with $\mathrm{Var}(\theta)\subset\mathcal{X}$, $\Phi[\mathcal{X}:\theta]\rightarrowtail\Phi[\mathcal{X}]$ may be regarded a subobject $\Phi[\mathcal{X}:\theta]\rightarrowtail\Phi[\mathcal{X}]\overset{m}{\rightarrowtail}\{\mathcal{X}^*\}$ of $\{\mathcal{X}^*\}$. We can produce a formula θ^* such that $\mathrm{Var}(\theta^*)=\mathrm{Var}(\theta)^*$ and

$$\Phi[\mathcal{X}:\theta]\ =_{\{\mathcal{X}^*\}}\ [\mathcal{X}^*:\theta^*]$$

(equality of subobjects of $\{\mathcal{X}^*\}$) as follows. We have, for atomic formulas

$$(\mathrm{I}_I(X,\kappa))^*\ \equiv\ \dot{\mathrm{I}}_I(X,\kappa)$$
$$(X{:}\mathrm{O}_I,\ \kappa{:}\mathrm{A}_I)$$
$$(\mathrm{T}_I(X,Y,Z,f,g,h))^*\ \equiv\ \dot{\mathrm{T}}_I(X,Y,Z,f,g,h)$$
$$(X,Y,Z{:}\mathrm{O}_I;\ f{:}\mathrm{A}_I(X,Y);\ g{:}\mathrm{A}_I(Y,Z);\ h{:}\mathrm{A}_I(X,Z))$$
$$(\mathrm{A}_i(X,Y,A,B,\mu,\nu,f,g))^*\ \equiv\ \dot{\mathrm{A}}_i(X,Y,A,B,\mu^*,\nu^*,f,g)$$
$$(X,Y{:}\mathrm{O}_I;\ A,B{:}\mathrm{O}_J;\ \mu{:}\mathrm{O}_i(X,A),\ \nu{:}\mathrm{O}_i(Y,B),\ f{:}\mathrm{A}_I(X,Y),\ g{:}\mathrm{A}_J(A,B))$$
$$(\mathrm{O}_\iota(X,A,B,\mu,\nu,h))^*\ \equiv\ \dot{\mathrm{O}}_\iota(X,A,B,\mu^*,\nu^*,h)$$
$$(X{:}\mathrm{O}_I,\ A,B{:}\mathrm{O}_J;\ \mu{:}\mathrm{O}_i(X,A),\ \nu{:}\mathrm{O}_i(Y,B),\ h{:}\mathrm{A}_J(A,B))$$
$$(f=_{\mathrm{A}_I(X,Y)}g)^*\ \equiv\ \mathrm{d}_I(f)=\mathrm{d}_I(g)=X\wedge\mathrm{c}_I(f)=\mathrm{c}_I(g)=Y\wedge f=_{\mathrm{A}_I}g$$
$$(X,Y{:}\mathrm{O}_I;\ f{:}\mathrm{A}_I(X,Y),\ g{:}\mathrm{A}_I(X,Y))$$
$$(\mu=_{\mathrm{O}_i(X,A)}\nu)^*\ \equiv\ \dot{\mathrm{O}}_i(X,A,\mu)\wedge\dot{\mathrm{O}}_i(X,A,\nu)\wedge\mu^*=_{\mathrm{A}_J}\nu^*$$
$$(X{:}\mathrm{O}_I;\ A{:}\mathrm{O}_J;\ \mu,\nu{:}\mathrm{O}_i(X,A));$$

for connectives

$$\mathbf{t}^* \equiv \mathbf{t}$$

$$\mathbf{f}^* \equiv \mathbf{f}$$

$$(\theta\wedge\rho)^* \equiv \theta^*\wedge\rho^*$$

$$(\theta\vee\rho)^* \equiv \theta^*\vee\rho^*$$

$$(\theta\to\rho)^* \equiv \varphi_{[\mathcal{X}]}\wedge(\theta^*\to\rho^*) \qquad (\mathcal{X}=\mathrm{Var}(\theta\to\rho))$$

and for quantifiers

$$(\forall x\theta)^* \equiv \forall x\in \mathrm{O}_I.\theta^* \qquad (x:\mathrm{O}_I)$$

$$(\forall x\theta)^* \equiv \forall x\in \mathrm{A}_I.((\mathrm{d}_I(x)=y\wedge \mathrm{c}_I(x)=z)\longrightarrow\theta^*) \qquad (x:\mathrm{A}_I(y,z))$$

$$(\forall x\theta)^* \equiv \forall x^*\in \mathrm{A}_J.(\dot{\mathrm{O}}_i(y,z,x^*)\longrightarrow\theta^*) \qquad (i:I\to J,\ x:\mathrm{O}_i(y,z));$$

the existential quantifier is dealt with correspondingly.

Notice that if $\mathrm{Var}(\theta)$ is a restricted context, then $\mathrm{Var}(\theta^*)=\mathrm{Var}(\theta)$.

The upshot of all this as follows. For an $\boldsymbol{I}$-diagram $D:\boldsymbol{I}\to\mathrm{Cat}$, and its saturation $D^\#$, if $\mathcal{X}$ is a finite restricted context over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$, θ is a FOLDS formula with $\mathrm{Var}(\theta)\subset\mathcal{X}$, and $\vec{a}\in D[\mathcal{X}]$, then

$$D^\#\models\theta[\vec{a}] \iff D\models\theta^*[\vec{a}] \quad .$$

For a structure M over a language extending $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$, $|M|$ denotes its reduct to $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$; $|M|$ is the underlying $\boldsymbol{I}$-diagram of M.

(20)(a) Let T be a theory extending $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$, σ an L_T-formula such that $\mathrm{Var}(\sigma)\subset\mathcal{X}$. The following two conditions (i), (ii) are equivalent.

(i) For any $M, N \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, $\vec{b} \in |N|[\mathcal{X}]$, $M \models \sigma[\vec{a}]$ and $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$ imply $N \models \sigma[\vec{b}]$.

(ii) There is θ in FOLDS over $\mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$ with $\mathtt{Var}(\theta) \subset \mathcal{X}$ such that for all $M \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, we have $M \models \sigma[\vec{a}]$ iff $M \models \theta^{*}[\vec{a}]$.

(**b**) In particular, if σ is a sentence over L_T, and for any $M, N \models T$, $M \models \sigma$ and $|M| \simeq |N|$ imply $N \models \sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$ such that for any $M \models T$, $M \models \sigma$ iff $M \models \theta^{*}$.

Proof. ((ii)$\longrightarrow$(i)) Given θ as (ii), we have

$$M \models \sigma[\vec{a}] \iff M \models \theta^{*}[\vec{a}] \iff |M| \models \theta^{*}[\vec{a}] \iff |M|^{\#} \models \theta[\vec{a}]$$

and similarly,

$$N \models \sigma[\vec{b}] \iff |N|^{\#} \models \theta[\vec{b}] \quad .$$

Assume the hypotheses of (i), in particular, $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$. By (8), for $\boldsymbol{L} = \mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$, $(|M|^{\#}, \vec{a}) \approx_{\boldsymbol{L}} (|N|^{\#}, \vec{b})$, hence, by 5.(2)(b), $|M|^{\#} \models \theta[\vec{a}] \iff |N|^{\#} \models \theta[\vec{b}]$. By what we saw above, this means $M \models \sigma[\vec{a}] \iff N \models \sigma[\vec{b}]$ as desired.

((i)$\longrightarrow$(ii)) Assume (i). We apply 5.(15) with $\bar{\sigma} = m^{*}([\mathcal{X}:\sigma] \in \mathrm{S}(\Phi[\mathcal{X}])$ in place of σ; $m:\Phi[\mathcal{X}] \rightarrowtail \{\mathcal{X}\}$ as above. The condition $M \models \sigma[\vec{a}]$ translates into $\langle \vec{a} \rangle \in \bar{M}[\bar{\sigma}]$; now, $\langle \vec{a} \rangle = \vec{a}$. Recall that $\bar{M} \restriction \boldsymbol{L} = \bar{M} \circ \Phi = |M|^{\#}$. Thus, also using (8), we have

for all $M, N \models T$, $\vec{a} \in (\bar{M} \restriction \boldsymbol{L})[\mathcal{X}]$, $\vec{b} \in (\bar{N} \restriction \boldsymbol{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in \bar{M}[\bar{\sigma}] \ , \ (\bar{M} \restriction \boldsymbol{L}, \vec{a}) \approx (\bar{N} \restriction \boldsymbol{L}, \vec{b}) \implies \langle \vec{b} \rangle \in \bar{N}[\bar{\sigma}] \quad .$$

Since every $P \models \mathbf{C}$ is isomorphic to one of the form $\bar{M}$, with $M \models T$, we have the hypothesis of 5.(12). The conclusion gives θ in FOLDS over $\mathbf{L}$ such that $\bar{\sigma} =_{\Phi[\mathcal{X}]} \Phi[\mathcal{X}:\theta]$, which suffices.

The result of (20) can be paraphrased by saying that a first-order property of a diagram of categories, functors and natural transformations is invariant under equivalence iff the property is expressible in FOLDS with restricted equality as a property of the saturated anadiagram canonically associated with the diagram.

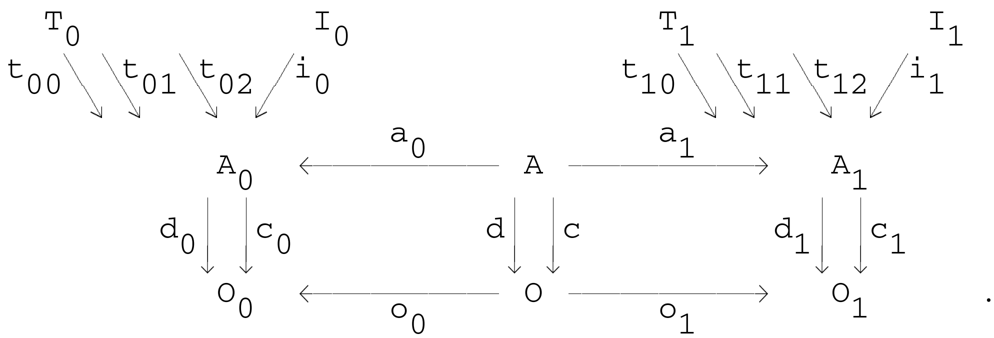
It is left to the reader to formulate stronger versions of (20), based on results of §5.

A *normal* theory for $\mathbf{I}$-diagrams is a theory T extending $\mathrm{T}_{\mathrm{diag}}[\mathbf{I}]$ such that if $M \models T$ and $D \simeq |M|$, then there is $N \models T$ such that $|N| = D$. For a restricted context $\mathcal{X}$, and formula σ of $\mathrm{L}_{\mathrm{diag}}[\mathbf{I}]$ with $\mathrm{Var}(\sigma) \subset \mathcal{X}$, we define the concepts "σ is preserved/reflected along equivalences of models of T" in the obvious way, in analogy to the case of a single category (see above). We have the following analog of (3).

(20') Let T be a normal theory of $\mathbf{I}$-diagrams of categories, functors and natural transformations. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$. Suppose that the first-order formula σ over $\mathrm{L}_{\mathrm{diag}}[\mathbf{I}]$ with free variables all in $\mathcal{X}$ is preserved and reflected along equivalences of models of T. Then there is a formula φ in FOLDS over $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ such that σ is equivalent to φ^* (defined above) in models of T.

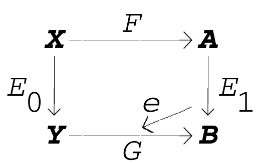
Let us discuss the special case of $\mathbf{I} = (\,0 \xrightarrow{\langle 0,1\rangle} 1\,)$ consisting of two objects and an arrow between them; there are no 2-cells. The intended structures for $\mathrm{L}_{\mathrm{fun}} = \mathrm{L}_{\mathrm{diag}}[(0 \xrightarrow{\langle 0,1\rangle} 1)]$ are *functors*; more precisely, structures consisting of two categories connected by a functor. *Fibrations* are such structures. There are many first-order conditions on fibrations and on objects and morphisms in fibrations that are of interest. On the other hand, in [MR2], several elementary (first-order definable) classes of $\mathrm{L}_{\mathrm{fun}}$-structures were introduced as categorical formulations of modal logic; these "modal categories" are not in general fibrations.

Let me restate the basic concepts for $\mathtt{L}_{\mathtt{fun}}$. $\mathtt{L}_{\mathtt{fun}}$ is the following graph:



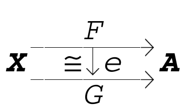
$\mathbf{L}_{\mathtt{anafun}} = \mathbf{L}_{\mathtt{anadiag}}[(0 \xrightarrow{\langle 0,1\rangle} 1)]$ is generated by $\mathtt{L}_{\mathtt{fun}}$, subject to appropriate equalities of composite arrows. A functor $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ is regarded an $\mathtt{L}_{\mathtt{fun}}$-structure in such a way that the interpretation of the relations $\mathtt{O}$ and $\mathtt{A}$ are the graphs of the object-function and of the arrow-function of F , respectively.

Given functors $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ and $G:\boldsymbol{Y}\longrightarrow\boldsymbol{B}$, an *equivalence* between them is a triple (E_0, E_1, e) as in



$: \quad e:E_1F \xrightarrow{\cong} GE_0$

in which E_0 and E_1 are equivalence functors. This notion of equivalence of functors can be motivated by saying that it is the combination of two simpler notions: one is the isomorphism of two parallel functors



,

and the other is the relation between $\boldsymbol{X}\xrightarrow{F}\boldsymbol{A}$ and the composites $\boldsymbol{Y}\xrightarrow{E_0}\boldsymbol{X}\xrightarrow{F}\boldsymbol{A}$, $\boldsymbol{X}\xrightarrow{F}\boldsymbol{A}\xrightarrow{E_1}\boldsymbol{B}$ where E_0 , E_1 are equivalence functors. Since the second notion only involves changing a category to an equivalent one, the change affected on the functor should be an "inessential" one; the resulting composites should be "equivalent" to F ; they are, according to our definition. It is clear that the equivalence relation generated by the two special cases of "equivalence" is the full notion of "equivalence".

For $F:\mathbf{X}\to\mathbf{A}$ as an $\mathrm{L}_{\mathrm{fun}}$-structure, $F^{\#}$, the saturation of F, an $\mathbf{L}_{\mathrm{anafun}}$-structure, has, among others,

$$F^{\#}\mathrm{O} = \{ (X, A, \mu) : X\in\mathbf{X}, A\in\mathbf{A}, \mu: FX\xrightarrow{\cong}A\} ,$$

and

$$F^{\#}\mathrm{A} = \{ (X, A, \mu, Y, B, \nu, f, g) :$$

$$(X\xrightarrow{f}Y)\in\mathbf{X},\ (A\xrightarrow{g}B)\in\mathbf{A},\ \mu: FX\xrightarrow{\cong}A,\ \nu: FY\xrightarrow{\cong}B \text{ such that } \begin{array}{ccc} FX & \xrightarrow{\mu} & A \\ Ff\downarrow & \circ & \downarrow g \\ FY & \xrightarrow[\nu]{} & B \end{array} \} .$$

In the spirit of [M2] , within the notation for $F^{\#}\mathrm{A}$, the object A is also written as $F^{\#}_{\mu}(X)$, and $g = F^{\#}_{\mu,\ \nu}(f)$.

The various kinds of modal categories of [MR2] are each defined by a finite set of first-order axioms, and each kind of modal category is invariant under equivalence: if $F:\mathbf{X}\to\mathbf{A}$ belongs to the given kind, and $G:\mathbf{Y}\to\mathbf{B}$ is equivalent to $F:\mathbf{X}\to\mathbf{A}$, then so does $G:\mathbf{Y}\to\mathbf{B}$. It follows by our invariance theorem (15) that the axioms can be formulated in FOLDS, although *not as statements about the functor itself, but as statements about its saturation.* However, it is not necessary to use the invariance theorem (which is anyway proved in a non-constructive way) to obtain the individual FOLDS-statements; in each case, one can find them directly, rather easily. I will give an example of an axiom thus reformulated in FOLDS.

Suppose the functor $F:\mathbf{X}\to\mathbf{A}$ preserves monomorphisms, and consider the following condition on F:

(21) For any $X\in\mathbf{X}$, the induced map $F_X: \mathrm{S}_{\mathbf{X}}(X) \longrightarrow \mathrm{S}_{\mathbf{A}}(FX)$ of posets has a right adjoint (denoted $Y\mapsto\Box Y$, the necessity operator).

I want to show that the (21) can be equivalently written as a statement about $F^{\#}$. The simple observation is that if (21) holds, and $\mu : FX \xrightarrow{\cong} A$, then the map $\varphi = \varphi[\mu] : \mathrm{S}_{\mathbf{X}}(X) \longrightarrow \mathrm{S}_{\mathbf{A}}(A)$ defined by $\varphi([Z \overset{r}{\rightarrowtail} X]) = [FZ \overset{\mu\circ Fr}{\rightarrowtail} A]$ also has a right adjoint ($[Z \overset{r}{\rightarrowtail} X]$ is the subobject of X given by r); it is this latter, more general, statement that we can (almost) directly formulate in FOLDS about $F^{\#}$.

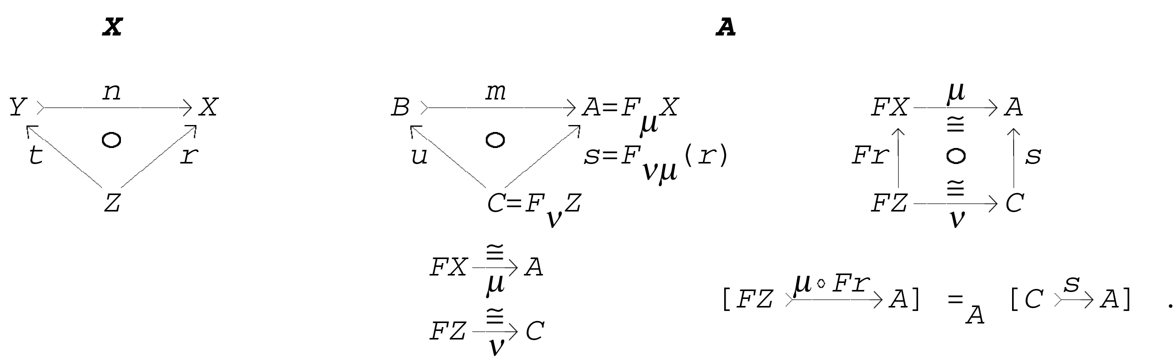
For variables $U, V{:}\mathrm{O}_0$, $u{:}\mathrm{A}_0(U, V)$, let $\mathrm{M}_0(U, V, u)$, abbreviated as $\mathrm{M}_0(u)$, and intended to say that u is a monomorphism, be the $\mathbf{L}_{\mathrm{anafun}}$-formula

$$\forall W{:}\mathrm{O}_0 . \forall v, w{:}\mathrm{A}_0(W, U)(\exists z{\in}\mathrm{A}_0(W, V) . \mathrm{T}_0(v, u, z) \wedge \mathrm{T}_0(w, u, z) . \longrightarrow v =_{\mathrm{A}_0(W, U)} w) .$$

Changing all subscripts 0 to 1 , we get the formula $\mathrm{M}_1(u)$. Here is the sentence θ for which $F^{\#} \vDash \theta$ is equivalent to (21):

$$\begin{aligned}
&\forall X{:}\mathrm{O}_0 \forall A{:}\mathrm{O}_1 \forall \mu{:}\mathrm{O}(X, A) \forall B{:}\mathrm{O}_1 \forall m{:}\mathrm{A}_1(B, A) \{\mathrm{M}_1(m) \longrightarrow \\
&\exists Y{:}\mathrm{O}_0 \exists n{:}\mathrm{A}_0(Y, X)[\mathrm{M}_0(n) \wedge \forall Z{:}\mathrm{O}_0 \forall C{:}\mathrm{O}_1 \forall \nu{:}\mathrm{O}(Z, C) \forall r{:}\mathrm{A}_0(Z, X) \forall s{:}\mathrm{A}_1(C, A) \\
&(\mathrm{M}_0(r) \wedge \mathrm{M}_0(s) \wedge \mathrm{A}(\underset{d}{\nu}, \underset{c}{\mu}, \underset{a_0}{r}, \underset{a_1}{s}) \longrightarrow \\
&\exists t{:}\mathrm{A}_0(Z, Y) . \mathrm{T}_0(Z, Y, X, t, n, r) \longleftrightarrow \exists u{:}\mathrm{A}_1(C, B) . \mathrm{T}_1(C, B, A, u, m, s))]\} \quad .
\end{aligned}$$

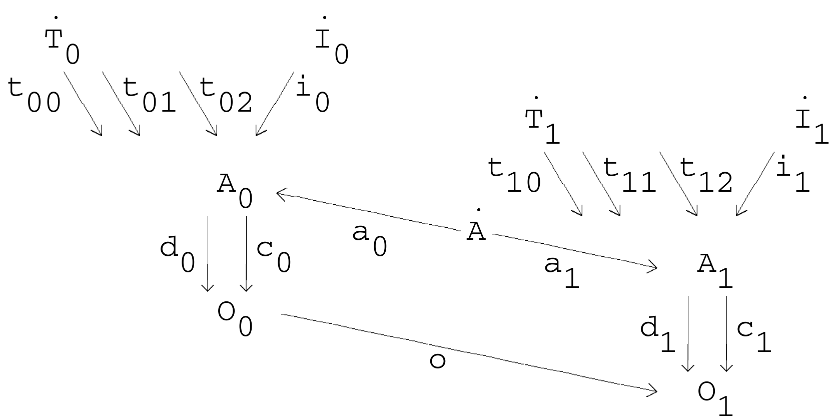
To help reading the sentence interpreted in $F^{\#}$, here is a display of the data involved:



Let us discuss fibrations. The first thing to say is that the concept of fibration is *not* invariant under equivalence of functors. An equivalence functor is, clearly, not necessarily a fibration; an identity functor is one, however; it follows that the concept of fibration is not invariant under equivalences of the form (E_0, Id, id) .

On the other hand, once we know that $F:\mathbf{X}\to\mathbf{A}$ and $G:\mathbf{Y}\to\mathbf{B}$ *are* fibrations, then the usually considered additional properties of F , and of diagrams in the fibration F , *are* inherited along arbitrary equivalences $F\xrightarrow{\simeq}G$. The reason is that any equivalence $F\xrightarrow{\simeq}G$ gives rise to a "strong" equivalence from F to G ; and the usually considered properties are invariant under the strong equivalences. In fact, the notion of strong equivalence is related to looking at a fibration as a structure for a new DS vocabulary $\mathbf{L}_{\mathrm{fib}}$. Let me explain.

Consider the following DSV $\mathbf{L}_{\mathrm{fib}}$:



;

here, besides the two obvious copies of $\mathbf{L}_{\mathrm{cat}}$, we have the equalities

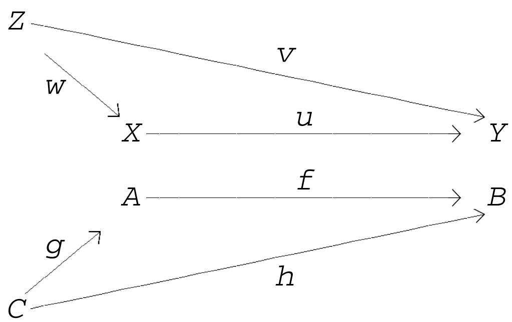
$$od_0a_0 = d_1a_1 \ , \ oc_0a_0 = c_1a_1 \ .$$

(The simpler version that has an arrow $A_0\to A_1$ in place of $A_0\longleftarrow A\longrightarrow A_1$ is not suitable; we need equality on A_1 to express fully the properties of the arrows of the base category; with the version indicated, A_1 would not be a top kind, therefore would not be eligible for carrying an equality predicate in the language.)

Among the $\mathbf{L}_{\mathrm{fib}}$-structures, we find the functors; given $F:\mathbf{X}\to\mathbf{A}$, it is understood as an $\mathbf{L}_{\mathrm{fib}}$-structure in the natural way in which the 0-copy of $\mathbf{L}_{\mathrm{cat}}$ is $\mathbf{X}$, the 1-copy $\mathbf{A}$, o is the object-function of F , and the relation A is the graph of the arrow-function of F . Note that whereas $\mathbf{L}_{\mathrm{fib}}$ is a simplification of L_{fun} , its height is 4 , and that of L_{fun} is 3 . Here is an axiom in FOLDS over $\mathbf{L}_{\mathrm{fib}}$ that formulates the existence of (strongly) Cartesian arrows:

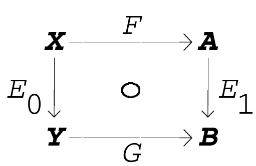
$$\forall A{:}\mathrm{O}_1\forall B{:}\mathrm{O}_1\forall f{:}\mathrm{A}_1(A,B)\forall Y{:}\mathrm{O}_0(B)\exists u{:}\mathrm{A}_0(A,B,X,Y)\{\dot{\mathrm{A}}(u,f)\wedge$$
$$\forall C{:}\mathrm{O}_1\forall g{:}\mathrm{A}_1(C,A)\exists h{:}\mathrm{A}_1(C,B)[\dot{\mathrm{T}}_1(g,f,h)\wedge\forall Z{:}\mathrm{O}_0\forall v{:}\mathrm{A}_0(C,A,Z,Y)(\dot{\mathrm{A}}(v,h)\rightarrow$$
$$\exists!w{:}\mathrm{A}_0(C,A,Z,X)(\dot{\mathrm{A}}(w,g)\wedge\dot{\mathrm{T}}_0(w,u,v)))]\}\ .$$

Here is a diagram to accompany the sentence:



We have employed the usual abbreviations in writing the atomic formulas; the unique existential quantifier $\exists!$ may be expanded in the expected way. Adding further axioms that are easily obtained, we get a sentence in FOLDS over $\mathbf{L}_{\mathrm{fib}}$ that axiomatizes the notion of fibration. This would not be possible to do over $\mathbf{L}_{\mathrm{anafun}}$.

Let us call functors $F{:}\boldsymbol{X}\rightarrow\boldsymbol{A}$ and $G{:}\boldsymbol{Y}\rightarrow\boldsymbol{B}$ *strongly equivalent*, $F\simeq_{\mathrm{s}}G$, if there is an equivalence $(E_0, E_1, id):F\simeq G$ (in the previous sense), with an identity in the third component;



(22)

(23) For functors F and G, $F\simeq_{\mathrm{s}}G$ iff $F\approx_{\mathbf{L}_{\mathrm{fib}}}G$. As a consequence, a first order property of objects and arrows in a prefibration (functor), in particular, in a fibration, is invariant under strong equivalence iff the property is expressible in FOLDS over $\mathbf{L}_{\mathrm{fib}}$.

I only outline the proof. Of course, the second statement is obtained *as a consequence of the first by* §5. Given $(\mathcal{R}, r, s) : F \xleftarrow[\mathbf{L}_{\mathrm{fib}}]{} \to G$, for any $A \in F\mathrm{O}_1 = \mathrm{Ob}(\boldsymbol{A})$, let us pick $\bar{A} \in \mathcal{R}\mathrm{O}_1$ by the Axiom of Choice such that $r(\bar{A}) = A$, and put $E_1(A) = s(\bar{A})$. For $X \in \mathrm{Ob}(\boldsymbol{X})$, let $A = F(X)$; thus, $X \in F\mathrm{O}_0(A)$. By the very surjectivity of r, there is $\bar{X} \in \mathcal{R}\mathrm{O}_0(\bar{A})$ such that $r(\bar{X}) = X$; we let $E_0(X) = s(\bar{X})$. We have defined the object-functions of equivalence functors $E_1 : \boldsymbol{A} \to \boldsymbol{B}$, $E_0 : \boldsymbol{X} \to \boldsymbol{Y}$, and note (the main point) that, at least as far as the effect on objects is concerned, the diagram (22) commutes (and not just up to an isomorphism). The rest of the verification is left to the reader.

Note that the treatment of fibrations did not require a passage to an "anafunctor". The usually considered properties of fibrations are invariant under strong equivalence. On the other hand, there is a simple, and well-known, "transfer property" for morphisms of fibrations which ensures that for *fibrations* F and G, $F \simeq G$ iff $F \simeq_s G$; in fact if $(E_0, E_1, e) : F \simeq G$, there is $E'_0 : \boldsymbol{X} \to \boldsymbol{Y}$ such that $E'_0 \cong E_0$ and $(E'_0, E_1, id) : F \simeq G$.