§6. Equivalence of categories, and of diagrams of categories

The simplest application of the results of the last section is to invariance under equivalence of categories of first order properties of diagrams of objects and arrows in a category. In what follows, until further notice, \( L \) stands for \( L_{\text{cat}} \), the DSV for categories introduced in §1; a category \( A \) may be regarded an \( L \)-structure. A context of variables for \( L \) is essentially a functor \( K=L_{\text{graph}} \to \text{Set} \), that is, a graph; we are mainly interested in finite contexts, although for the notions to be introduced next, there is no need to confine attention to finite contexts.

For a context \( \mathcal{X} \), an augmented category of type \( \mathcal{X} \) is a pair \( (A, \tilde{a}) \), with \( A \) a category, and \( \tilde{a} \in A[\mathcal{X}] \) (that is, \( \tilde{a} \) a diagram of type the graph \( \mathcal{X} \)). Until further notice, notations such as \( (A, \tilde{a}) \), \( (B, \tilde{b}) \) denote augmented categories. Mere categories are considered special cases of augmented categories of type \( \emptyset \); \( A \), \( B \) etc. denote categories.

For augmented categories \( (A, \tilde{a}) , (B, \tilde{b}) \) of the same type, we write

\[
(A, \tilde{a}) \longrightarrow (B, \tilde{b})
\]

if there is an equivalence functor \( F: A \cong B \) ( \( F \) is full and faithful, and essentially surjective on objects) that maps \( \tilde{a} \) to \( \tilde{b} \); we may also write \( (B, \tilde{b}) \leftarrow (A, \tilde{a}) \) for the same. Note that the relation \( \longrightarrow \) is reflexive and transitive but not symmetric (an equivalence functor \( A \cong B \) may take two different objects \( A \not\cong A' \) in \( A \) to the same \( B \) in \( B \); then \( (A, \langle A, A' \rangle) \longrightarrow (B, \langle B, B \rangle) \) but not vice versa). The special case when the type \( \mathcal{X} \) is \( \emptyset \), is, however, symmetric; \( A \longrightarrow B \) implies \( A \leftarrow B \) since every equivalence functor has a quasi-inverse (by the Axiom of Choice); \( A \longrightarrow B \) is the same as equivalence of categories, \( A \cong B \).

The equivalence relation generated by the relation \( \longrightarrow \) is only "one step away" from \( \longrightarrow \); it is \( \overset{\sim}{\longrightarrow} \) defined as

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\((\mathbf{A}, \tilde{a}) \xleftarrow{} (\mathbf{B}, \tilde{b})\) \xrightarrow{\text{def}} there is \((\mathbf{C}, \tilde{c})\) such that \((\mathbf{A}, \tilde{a}) \xleftarrow{} (\mathbf{C}, \tilde{c}) \xrightarrow{} (\mathbf{B}, \tilde{b})\). (1)

To see the transitivity of the relation \(\sim\), assume \((\mathbf{A}, \tilde{a}) \xleftarrow{} (\mathbf{D}, \tilde{d}) \xrightarrow{} (\mathbf{B}, \tilde{b})\) and \((\mathbf{B}, \tilde{b}) \xleftarrow{} (\mathbf{E}, \tilde{e}) \xrightarrow{} (\mathbf{C}, \tilde{c})\), and consider the diagram

\[
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\phi} & \mathbf{B} \\
\downarrow{\sigma} & \ & \downarrow{\psi} \\
\mathbf{D} & \xrightarrow{\phi} & \mathbf{E} \\
\downarrow{\phi} & \ & \downarrow{\psi} \\
\mathbf{C}
\end{array}
\]

where the quadrangle has \(\mathbf{F}\) the "isomorphism-comma" category, with objects

\((\mathbf{D}, \mathbf{E}, \phi \mathbf{D} \xrightarrow{\equiv} \psi \mathbf{E})\), and arrows the usual commutative squares, with \(\sigma: \mathbf{F} \to \mathbf{D}\), \(\tau: \mathbf{F} \to \mathbf{E}\) the forgetful functors. Since \(\phi, \psi\) are equivalence functors, so are \(\sigma, \tau\). Let

\[\hat{f} = \langle f_x \rangle_{x \in \mathbf{A}} \in \mathbf{F}[\mathbf{\bar{A}}]\]

be defined as follows. For \(x \in \mathbf{\bar{A}}\), \(x:0\), let

\[f_x = (d_x, e_x, \iota: \phi d_x \equiv \psi e_x)\]; note that \(\phi d_x = \psi e_x\) by assumption. For \(x \in \mathbf{\bar{A}}\), \(x: \mathbf{A}(y, z)\), let \(f_x = (d_x: d_y \to d_z, e_x: e_y \to e_z): f_y \to f_z\); note that \(\phi d_x = \psi e_x\) by assumption. We have that \((\mathbf{F}, \hat{f}) \sim (\mathbf{D}, \tilde{d})\), \((\mathbf{F}, \hat{f}) \sim (\mathbf{E}, \tilde{e})\). Using the composites \(\mathbf{F} \to \mathbf{A}, \mathbf{F} \to \mathbf{C}\), we obtain \((\mathbf{A}, \tilde{a}) \xleftarrow{} (\mathbf{B}, \tilde{b})\) as desired.

Recall the relation \(\approx\) of the last section; \(\approx\) is, in particular, a relation between augmented categories. We have that \(\approx\) is the same as \(\xleftarrow{}\).

\[(1)\]

(a) \((\mathbf{A}, \tilde{a}) \xleftarrow{} (\mathbf{B}, \tilde{b})\) \iff \((\mathbf{A}, \tilde{a}) \approx (\mathbf{B}, \tilde{b})\);

(b) \(\mathbf{A} \approx \mathbf{B} \iff \mathbf{A} \approx_{\mathbf{L}} \mathbf{B}\).

**Proof.** Assume \((\mathbf{A}, \tilde{a}) \approx (\mathbf{B}, \tilde{b})\). By §5, there is a normal \(\mathbf{L}, \approx\)-equivalence

\((\tilde{w}, u, v): (\mathbf{A}, \tilde{a}) \xleftarrow{\mathbf{L}} (\mathbf{B}, \tilde{b})\). Then, \(\mathbf{C} = u^* (\mathbf{A}) = v^* (\mathbf{B})\) is a category, since, by 5.(1), as a standard \(\mathbf{L}\)-structure, \(\mathbf{C}\) satisfies all the axioms of category which are formulated in FOLDS.
(see 5.(2)(a)). Furthermore, clearly, \( \theta_u: C \rightarrow A \), \( \theta_v: C \rightarrow B \) are surjective equivalence functors. This shows the right-to-left direction in (a). For the proof of the other direction, we prove the implication

\[ (A, \tilde{a}) \sim_r (B, \tilde{b}) \implies (A, \tilde{a}) \sim_L (B, \tilde{b}); \]

to this end, we "saturate" the given equivalence appropriately; we will do this proof in a more general situation below.

Knowing the transitivity of the relation \( \sim_L \), the transitivity of \( \sim_r \) also follows from (1)(a).

(b) is a special case of (a).

Recall the translation \( \varphi \mapsto \varphi^* \) in §1; this is just to say that any formula \( \varphi \) of FOLDS over \( L \) may be regarded a formula \( (\varphi^*) \) over \( |L| \) in ordinary multisorted logic.

Let \( T_{\text{cat}} = (|L|, \Sigma_{\text{cat}}) \) the theory of categories in ordinary multisorted logic (\( \Sigma_{\text{cat}} \) can be taken to be \( \Sigma[\text{cat}] \cup \{ \theta^* : \theta \in \Theta \} \); \( \Sigma[|L|] \) for any DSV \( L \) was defined in §1; \( \Theta \) is the set of axioms in FOLDS for categories as given in §1.). When \( T \) is a theory extending \( T_{\text{cat}} \) \( (|L| \subseteq L_T, \Sigma_{\text{cat}} \subseteq \Sigma_T) \), and \( M \models T \), we write \( |M| \) for \( M_L \), the underlying category of \( M \).

(2)(a) Let \( T \) be a theory extending \( T_{\text{cat}} \). Let \( \mathcal{X} \) be a finite context over \( L_{\text{cat}} \), \( \sigma \) an \( L_{\text{T}} \)-formula such that \( \text{Var}(\sigma) \subseteq \mathcal{X} \). If

for any \( M, N \models T \) and diagrams \( \tilde{a} \in |M| \ [\mathcal{X}] \), \( \tilde{b} \in |N| \ [\mathcal{X}] \), \( M \models \sigma[\tilde{a}] \) and

\( (|M|, \tilde{a}) \leftrightarrow (|N|, \tilde{b}) \) imply \( M \models \sigma[\tilde{b}] \),

then there is \( \theta \) in FOLDS with restricted equality over \( L_{\text{cat}} \) with \( \text{Var}(\theta) \subseteq \mathcal{X} \) such that for all \( M \models T \) and diagrams \( \tilde{a} \in |M| \ [\mathcal{X}] \), we have \( M \models \sigma[\tilde{a}] \) iff \( M \models \theta^*[\tilde{a}] \).

(b) In particular, if \( \sigma \) is a sentence over \( L_T \), and for any \( M, N \models T \), \( M \models \sigma \) and \( |M| \simeq |N| \) imply \( M \models \sigma \), then there is a sentence \( \theta \) of FOLDS over \( L_{\text{cat}} \) such that for
any $\mathcal{M} = T$, $\mathcal{M} = \sigma$ iff $\mathcal{M} = \theta^*$. 

Proof. We apply 5.(15) to $\mathcal{C} = [T]$, with $I: \mathcal{L} \rightarrow \mathcal{C}$ the composite of $I: \mathcal{L} \rightarrow [\mathcal{T}_{\text{cat}}]$ defined in §5 before (7)(a) and the inclusion $[\mathcal{T}_{\text{cat}}] \rightarrow [T]$; moreover, we take $\sigma$ in 5.(15) to be $m^* (\{\mathcal{X} : \sigma\}) \rightarrow I[\mathcal{X}]$ ($m: I[\mathcal{X}] \rightarrow \{\mathcal{X}\}$; see §5 before (7)(a)). By (1)(a), the assumption implies that

$$
\mathcal{M} = \sigma[\bar{a}] \quad \text{and} \quad (|\mathcal{M}|, \bar{a}) \cong L (|\mathcal{N}|, \bar{b}) \implies \mathcal{M} = \sigma[\bar{b}].
$$

The conclusion of 5.(15) is what we want. (b) is a special case of (a).

We say that a theory $T$ extending $\mathcal{T}_{\text{cat}}$ is normal if for any $\mathcal{M} = T$ and any category $\mathcal{A}$, if $\mathcal{A} \cong |\mathcal{M}|$, then there is a model $\mathcal{M} = T$ such that $\mathcal{A} = |\mathcal{N}|$. In other words, normality of $T$ says that the property of being the $\mathcal{L}_{\text{cat}}$-reduct of a model of $T$ is invariant under equivalence of categories. Most theories of categories (possibly) with additional structure are normal. E.g., so is the theory of monoidal categories, or the theory of categories with specified finite limits. Of course, $\mathcal{T}_{\text{cat}}$ itself is normal.

Let $\mathcal{X}$ be a finite context, and $\sigma$ be a formula over $\mathcal{L}_{\mathcal{T}}$ with $\text{Var}(\sigma) \subseteq \mathcal{X}$. Let us say that $\sigma$ is preserved along equivalence functors between models of $T$ if the following holds: whenever $\mathcal{M}, \mathcal{N} = T$, $\bar{a} \in M[\mathcal{X}]$, $\bar{b} \in N[\mathcal{X}]$, then $\mathcal{M} = \sigma[\bar{a}]$ and $(|\mathcal{M}|, \bar{a}) \rightarrow (|\mathcal{N}|, \bar{b})$ imply $\mathcal{M} = \sigma[\bar{b}]$. When in this definition, $(|\mathcal{M}|, \bar{a}) \rightarrow (|\mathcal{N}|, \bar{b})$ is replaced by

$$(|\mathcal{M}|, \bar{a}) \leftarrow (|\mathcal{N}|, \bar{b}),$$

we obtain the notion of being reflected along equivalence functors. Now, notice that for $T$ a normal theory, the hypothesis of (2)(a) holds iff $\sigma$ is preserved and reflected along equivalence functors of models of $T$ (the point is that, in case $T$ is normal, in (1'), when $\mathcal{A}$ (and $\mathcal{B}$) are reducts of models of $T$, $\mathcal{C}$ can also be expanded to a model of $T$). Thus, we obtain the following variant of (2)(a):

(3) Let $T$ be a normal theory of categories (possibly) with additional structure. Let $\mathcal{X}$ be a finite context over $\mathcal{L}_{\text{cat}}$. Suppose that the first-order formula $\sigma$ over $\mathcal{L}_T$ with free
variables all in $\mathcal{X}$ is preserved and reflected along equivalence functors of models of $T$. Then there is a formula $\varphi$ in FOLDS with restricted equality over $L_{\text{cat}}$ with $\text{Var}(\varphi) \subseteq \mathcal{X}$ such that $\sigma$ is equivalent to $\varphi^*$ in models of $T$.

Freyd's and Blanc's characterization (see [F], [FS], [B]) of first order properties of finite diagrams invariant under equivalence is (3) for $T=T_{\text{cat}}$. In fact, the general result (3) can also be obtained by their methods, which is very different from the methods of this paper (we will comment on this in Appendix C). It seems however that the more general result (2), in particular, (2)(b), cannot be obtained by the Freyd's and Blanc's methods (although I should concede that the added generality in (2)(b) consisting in a reference to not-necessarily normal theories does not seem very important).

The results of §5 that are more general than 5.(15) (e.g., the "interpolation-style" result (7)(b)) will also have consequences for equivalences of categories; we leave their formulation to the reader.

Extending the Freyd-Blanc result to more complex categorical structures will involve a new element. For instance, in the case of structures consisting of two categories and a functor between them (an example of which is a fibration), the first-order properties invariant under equivalence (in the appropriate standard sense; see also below) are not those expressible in FOLDS directly, but rather, those that are expressible in FOLDS in the language of the so-called saturated anafunctor associated with the given functor. Anafunctors are treated in [M2]; explanations will be given presently.

We now proceed to giving the framework for dealing with structures consisting of several (possibly infinitely many) categories, functors between them, and natural transformations between the latter. We will return to the simplest special case of two categories and a functor between them afterwards.

Let $\mathcal{I}$ be a small 2-graph; $\mathcal{I}:L_{2\text{-graph}} \to \text{Set}$. We associate the graph $L_{\text{diag}}[\mathcal{I}]$ with $\mathcal{I}$; $L_{\text{diag}}[\mathcal{I}]$ serves as a similarity type for diagrams $\mathcal{I} \to \text{Cat}$ of (small) categories, functors and natural transformations. The objects of $L_{\text{diag}}[\mathcal{I}]$ are as follows:
\[ O_\alpha A_\alpha, I_\alpha, T_\alpha \quad (I \in \text{Ob}(I)) \]
\[ O_i A_i \quad (i \in \text{Arr}(I)) \]
\[ O^\alpha \quad (\alpha \in 2\text{-Cell}(I)) \]

The arrows of \( L_{\text{diag}}[I] \) are shown in the following three diagrams:

![Diagram 1](described_in_text)

Displaying the arrows associated with an object \( I \):

![Diagram 2](described_in_text)

Which displays the arrows associated with an arrow \( i : I \to J \) in \( I \); and

![Diagram 3](described_in_text)

Which displays the ones associated with the 2-cell \( t : i \to j \). 

Given a \( I \)-diagram

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of categories, functors and natural transformations, we construe $D$ as an $L_{\text{diag}}[\mathcal{I}]$-structure as follows. (3) is interpreted as the category $\mathcal{C}_I$. When $i: I \to J$, $O_i$ is the set of pairs $(X, F_i X)$ with $X \in \text{Ob}(\mathcal{C}_I)$, with $(X, F_i X) \overset{O_i}{\overset{0}{\to}} X$, $A_i$ is the set of pairs $(f, F_i f) = (X \overset{f}{\to} Y, F_i X \overset{F_i f}{\to} F_i Y)$, with $(f, F_i f) \overset{A_i}{\overset{0}{\to}} f$, $(f, F_i f) \overset{A_i}{\overset{1}{\to}} F_i f$. For $I \overset{i}{\to} J$, $O_i$ is the set of pairs $(X, F_i X \overset{h_{i X}}{\to} F_j X)$. The effect of the remaining arrows, as well as the corresponding commutativities, are shown by the following picture:

Let $L_{\text{anadiag}}[\mathcal{I}]$ be the DSV defined as follows. The underlying simple category of $L_{\text{anadiag}}[\mathcal{I}]$ is generated by the graph $L_{\text{diag}}[\mathcal{I}]$, subject to the following equalities between arrows:

the ones ensuring that (3) generates a copy of $L_{\text{cat}}$ (see §1);

$$O_i 0 0 0 d_i = d_i a_{i 0}, \quad O_i 1 1 d_i = d_i a_{i 1}, \quad O_i 0 0 c_i = c_i a_{i 0}, \quad O_i 1 1 c_i = c_i a_{i 1},$$

$$O_i 0 0 0 t_0 = o_{j 0} o_{i 0}, \quad d_i o_{i 2} = o_{i 1 0} t_0, \quad c_i o_{i 2} = o_{j 1 0} t_1.$$

The relations of $L$ are exactly its top-level objects; that is, $T_i$, $I_i$, $A_i$, $O_i$, for $i$, $i$ and $i$ ranging over the 0-cells, 1-cells and 2-cells of $\mathcal{I}$, respectively.
The equalities on arrows are suggested by what is true for \( \mathcal{I} \)-diagrams as structures. In fact, every \( \mathcal{I} \)-diagram is a functor \( D: \mathbf{L}_{\text{anadiag}}[\mathcal{I}] \to \mathbf{Set} \), that is, the equalities listed are true in it (as identities). Also, the relations of \( \mathbf{L}_{\text{anadiag}}[\mathcal{I}] \) are interpreted in \( D \) relationally (the corresponding family of functions is monomorphic). In summary, every \( \mathcal{I} \)-diagram is an \( \mathbf{L}_{\text{anadiag}}[\mathcal{I}] \)-structure.

\( \mathbf{L}_{\text{anadiag}}[\mathcal{I}] \) is the similarity type of what we call the "anadiagrams" of type \( \mathcal{I} \). An \emph{anadiagram} \( M: \mathcal{I} \to \mathbf{Cat} \) is an \( \mathbf{L}_{\text{anadiag}}[\mathcal{I}] \)-structure satisfying the following axioms (A0) to (A6) in FOLDS with equality (\( \mathcal{I}, \mathcal{J} \) range over objects, arrows and 2-cells in \( \mathcal{I} \) as shown; the unique existential quantifiers in (A2) and (A5) are abbreviations in the usual way, and they refer to equality on the sorts \( \mathbf{A}_{\mathcal{J}}(\cdot, \cdot) \).

(A0): axioms expressing that for each \( \mathcal{I} \in \mathbf{Ob}(\mathcal{I}) \), the part of \( M \) referring to \( \mathcal{I} \) is a category.

(A1) \( \forall X: O_\mathcal{I}. \exists A: O_\mathcal{J}. \exists s: O_\mathcal{I}(X, A). \mathbf{t} . \)

(A2) \( \forall X, Y: O_\mathcal{I}. \forall A, B: O_\mathcal{J}. \forall s: O_\mathcal{I}(X, A). \forall t: O_\mathcal{I}(Y, B). \forall f: A_\mathcal{I}(X, Y). \\
\exists ! g: A_\mathcal{J}(A, B). A_\mathcal{I}(s, t, f, g). \\
\ \ \ \ \ \ \ \ d_i c_i a_i o a_i i_i \)

(A3) \( \forall X: O_\mathcal{I}. \forall A: O_\mathcal{J}. \forall s: O_\mathcal{I}(X, A). \forall \alpha: A_\mathcal{I}(X, X). \forall \tilde{\alpha}: A_\mathcal{J}(A, A). \\
[ A_\mathcal{I}(s, s, \alpha, \tilde{\alpha}) \to (I_\mathcal{I}(X, \alpha) \to I_\mathcal{J}(A, \tilde{\alpha})) ] . \\
\ \ \ \ \ \ \ \ d_i i_i I_\mathcal{I} i_\mathcal{I} \)

(A4) \( \forall X, Y, Z: O_\mathcal{I}. \forall A, B, C: O_\mathcal{J}. \forall s: O_\mathcal{I}(X, A). \forall t: O_\mathcal{I}(Y, B). \forall u: O_\mathcal{I}(Z, C). \\
\forall f: A_\mathcal{I}(X, Y). \forall g: A_\mathcal{I}(Y, Z). \forall h: A_\mathcal{I}(X, Z). \\
\forall \bar{f}: A_\mathcal{I}(A, B). \forall \bar{g}: A_\mathcal{I}(B, C). \forall \bar{h}: A_\mathcal{I}(A, C). \\
\left[ \left( (A_\mathcal{I}(s, t, f, \bar{f}) \land A_\mathcal{I}(t, u, g, \bar{g}) \land A_\mathcal{I}(s, u, h, \bar{h})) \to (T_\mathcal{I}(f, g, h) \to T_\mathcal{J}(\bar{f}, \bar{g}, \bar{h})) \right) \right] . \\
\ \ \ \ \ \ \ \ d_i i_i I_\mathcal{I} i_\mathcal{I} \)

(A5) \( \forall X: O_\mathcal{I}. \forall A: O_\mathcal{J}. \forall B: O_\mathcal{J}. \forall s: O_\mathcal{I}(X, A). \forall t: O_\mathcal{J}(X, B) \\
\ \ \ \ \ \ \ \ d_i i_i I_\mathcal{I} i_\mathcal{I} \)

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\exists f : A \rightarrow B \cdot O \cdot (s \cdot t \cdot f).

(A6) \quad \forall X, Y : O \cdot \forall A, B, C, G : O \cdot \\
\forall s : O \cdot \forall t : O \cdot \forall u : O \cdot \forall v : O \cdot (s, t, f, g, u, v, k, m, n)
\Rightarrow \exists n : A \rightarrow B \cdot (T(g, m, n) \cdot T(k, n)).

For a less formal explanation of the notion of anadiagram, I refer to [M2]. In that paper, I introduce the notion of anafunctor between categories, a generalization of the notion of functor. An anafunctor defines its values on objects only up to isomorphism. Formally, the definition of anafunctor is obtained by specializing the definition of "anadiagram" to the case when \( \mathbf{I} \) is the (2-)graph \( \langle 0, 1 \rangle \) (without 2-cells). Anadiagrams have anafunctors instead of functors as 1-cells, and natural transformations of anafunctors as 2-cells.

Note that any \( \mathbf{I} \)-diagram \( D : \mathbf{I} \rightarrow \mathbf{Cat} \) is an anadiagram; all the axioms for "anadiagram" are satisfied in \( D \) (as an \( \mathbf{L}_{\text{anadiag}}[\mathbf{I}] \)-structure). In fact, the diagrams are essentially the same as those anadiagrams \( M \) in which the sorts \( O_i \) (\( i \in \text{Arr}(\mathbf{I}) \)) are interpreted relationally, that is, the family \( \langle \mathcal{M}_p \rangle_{p : O_i \rightarrow K_p} \) is jointly monomorphic.

On the other hand, any anadiagram gives rise to a diagram, obtained by making some non-canonical choices. Let \( M \) be an anadiagram \( M : \mathbf{I} \rightarrow \mathbf{Cat} \); we construct \( D : \mathbf{I} \rightarrow \mathbf{Cat} \); we use the notation (4) for the ingredients of \( D \). For \( \mathbf{I} \in \text{Ob}(\mathbf{I}) \), the category \( \mathbf{C}_i \) is given directly by the data in \( M \) corresponding to \( \mathbf{I} \) (see (A0)). By (A1), for any \( i : \mathbf{I} \rightarrow \mathbf{J} \) in \( \mathbf{I} \) and \( X \in \text{Ob}(\mathbf{C}_i) = \text{MO}_i \), we make a choice of \( A^i_X = A_X \in \text{MO}_J \) and \( s^i_X = s_X \in \text{MO}_i \); we put \( F_iX = A_X \). Starting with \( f : X \rightarrow Y \), and using (A2) with \( A = A_X, B = A_Y, s = s_X, t = s_Y \), we let \( F_if = g \) whose unique existence (A2) states. (A3) and (A4) assure that \( F_i \) so defined is a functor \( F_i : \mathbf{C}_i \rightarrow \mathbf{C}_j \). Using (A5) with \( A = A^i_X, B = A^j_Y, s = s^i_X, t = s^j_Y \), we put \( h_{iX} = f \) for the \( f \) whose existence (5) asserts. (A6) ensures that \( h_{iX} \) is a natural transformation \( h_{iX} : F_i \rightarrow F_j \). Let us refer \( D \) as the diagram obtained from \( M \) by cleavage (in analogy to the terminology used with fibration); of course, it is not uniquely determined.
Next, we describe the saturation $D^\#$ of a diagram $D : \mathcal{I} \to \text{Cat}$, an anadiagram canonically associated with $D$. (As a matter of fact, the components corresponding to the 1-cells $i : \mathcal{I} \to \mathcal{J}$ will be the "saturated anafunctors" $F_i^\#$ associated with the given functors $F_i$, in the sense of [M2].)

In $D^\#$, the interpretation of each part of $\text{anadiag}[\mathcal{I}]$ as in (3) is the same as in $D$.

For $i : \mathcal{I} \to \mathcal{J}$, a 1-cell in $\mathcal{I}$, $D^\#_i$ is the set of triples $\langle X, A, F_i^\# X \cong \mu \to A \rangle$ with $X \in \mathcal{C}_{\mathcal{I}}, A \in \mathcal{C}_{\mathcal{J}}$ and $\mu$ an isomorphism as shown; $\mu \cong_i 0 \to X$, $\mu \cong_i 1 \to A$. $D^\# A_i$ is the set of all entities

$$
\begin{align*}
X & \xrightarrow{\mu} A \\
\downarrow f & \quad \downarrow F_i f \circ \downarrow g \\
Y & \quad \downarrow F_i Y \cong_i 1 \\
\equiv & \\
\end{align*}
$$

with the displayed entity mapped to $(X, A, \mu)$ by $\cong_i$, to $(Y, B, \nu)$ by $\cong_i$, to $f$ by $\cong_{i0}$, and to $g$ by $\cong_{i1}$. For $\mathcal{I} \xrightarrow{i} \mathcal{J}$, $D^\#_i$ consists of all

$$
\begin{align*}
F_i^\# X & \equiv \mu \to A \\
\downarrow h_{1X} & \quad \circ \downarrow g \\
F_j^\# X & \equiv \rho \to B \\
\end{align*}
$$

and the displayed item is mapped to $(X, A, \mu)$ by $\circ_{i0}$, to $(X, B, \rho)$ by $\circ_{i1}$, and to $g$ by $\circ_{i2}$.

We leave it to the reader to verify that $D^\#$ so defined is an anadiagram.

$D^\#$ satisfies a property that distinguishes it from diagrams; it is saturated, by which we mean that it satisfies, for each $i : \mathcal{I} \to \mathcal{J}$ in $\mathcal{I}$, the FOLDS sentence
(Iso(f)→∃t:O_j(X,B).∃g:A_I(X,X).(I_I(g)∧A_1(s,t,g)));

here, Iso(f) abbreviates

∃h:A_J(B,A)∃k:A_J(A,A)∃ℓ:A_J(B,B). (I_J(k)∧I_J(ℓ)∧T_J(f,h,k)∧T_J(h,f,ℓ)) .

In fact, it can be proved (although we will not need this result) that, up to isomorphism as L_{anadiag}[I]-structures, the saturated I-anadiagrams are precisely the ones of the form D^#, for some diagram D.

Given D as in (5), and another I-type diagram

\[ \hat{D}:I→\text{Cat} : (\langle C_I \rangle_{I∈I}, \langle \hat{F}_I:C_I→C_J \rangle_{I→J}, \langle \hat{h}_I:F_I→F_J \rangle_{I→J} ) , \] (6)

we say that D and \( \hat{D} \) are equivalent, and write \( D≃\hat{D} \), if there exist a family

\( \langle E_I:C_I→C_I \rangle_{I∈I} \) of equivalence functors, and a family \( \langle e_I : \hat{F}_I→F_I \rangle_{I→J} \) of natural isomorphisms as in

\[
\begin{array}{ccc}
C_I & \xrightarrow{E_I} & \hat{C}_I \\
\hat{F}_I & \equiv & F_I \\
C_J & \xrightarrow{E_J} & \hat{C}_J \\
\end{array}
\]

satisfying the additional naturality condition:

\[
\begin{array}{ccc}
E_J^*F_I & \xleftarrow{e_I} & \hat{F}_I^*E_I \\
E_J^*h_I & \circ & \hat{h}_I^*E_I \\
E_J^*F_J & \xleftarrow{e_I} & \hat{F}_J^*E_I \\
\end{array}
\]
for every $\xymatrix{I \ar[r]^{i} \ar[d]_{j} & J}$ in $\mathcal{I}$. The data $E = (\langle E_i \rangle_{i \in \mathcal{I}}, \langle e_i \rangle_{i \in \text{Arr}(\mathcal{I})})$ form an equivalence of $D$ and $\hat{D}$, in notation,

$$E = (\langle E_i \rangle_{i \in \mathcal{I}}, \langle e_i \rangle_{i \in \text{Arr}(\mathcal{I})}) : D \sim \hat{D}. \quad (7)$$

This notion of equivalence is a "bicategorical" notion; it is the equivalence in the internal sense of the bicategory (actually, 2-category) $\text{Hom}(\langle \mathcal{I} \rangle, \text{Cat})$ of homomorphisms of bicategories, pseudo-natural transformations and modifications, with $\langle \mathcal{I} \rangle$ the 2-category generated by the 2-graph $\mathcal{I}$. (The main part of the fact that the "one-way" formulation of equivalence given above as the definition, and the "internal" concept just mentioned coincide, is the symmetry of the relation $\sim$; an outline of the proof of the symmetry of $\sim$ is given below.) It is the "good" notion of equivalence, the one that comes up in practice. For instance, in Chapter 4 of [MP], diagrams of sketches, and diagrams of accessible categories are dealt with, and the present notion of equivalence is the one which is operative. Specifically, the Uniform Sketchability Theorem, one of the main results of [MP] (4.4.1 in [MP]) says that a small diagram of accessible categories is equivalent to one obtained from a diagram of sketches by taking the categories of models of the sketches involved.

Although the fact is well-known, I outline the proof that the relation $D \sim \hat{D}$ is symmetric. Since it is easily seen to be transitive and reflexive, $\sim$ is an equivalence relation.

Assume data as in (7); see also (4) and (6). We define $\hat{E} : D \sim \hat{D}$. With $I \in \text{Ob}(\mathcal{I})$, $A \in \text{Ob}(\hat{\mathcal{C}}_{I})$, choose $X_A = X_A \in \text{Ob}(\mathcal{C}_{I})$ and $\varepsilon_A \circ \varepsilon_A : E_I \cdot X_A \equiv A \in \text{Arr}(\hat{\mathcal{C}}_{I})$. Put $\hat{E}_I A = X_A$. For $f : A \rightarrow B \in \text{Arr}(\hat{\mathcal{C}}_{I})$, $\hat{E}_I f$ is the arrow that makes the square

$$
\begin{array}{ccc}
E_I X_A & \xrightarrow{\varepsilon_A} & A \\
\hat{E}_I f \downarrow & \equiv & \downarrow f \\
\hat{E}_I X_B & \xrightarrow{\varepsilon_B} & B
\end{array}
$$

commute. $\hat{E}_I$ so defined is a functor $\hat{E}_I : \hat{\mathcal{C}}_I \to \hat{\mathcal{C}}_I$; it is an equivalence functor; it is a
quasi-inverse of \( E_I \); we have \( \epsilon_I: E_I \xrightarrow{\Xi} \check{1} \) with components the \( \epsilon^I_A \), and

\[ \eta_I: \check{1} \xrightarrow{\Xi} E_I E_I \]

with components \( \eta_I, X \) for which \( E_I(\eta_I, X) = (\epsilon^I_A X)^{-1} \). For

\( i:I \to J \) in \( I \), we define \( \hat{e}_{i}: F_i \xrightarrow{\Xi} \check{E}_{J} \) as the composite

\[ \check{E}_{J} \xrightarrow{\hat{\eta}_I F_i} \check{E}_{J} \xrightarrow{\hat{E}_{J} \hat{e}_{i}^{-1} F_i} \check{E}_{J} \xrightarrow{\hat{E}_{J} \hat{F}_i} \check{E}_{J} \xrightarrow{\hat{\eta}_I} \check{E}_{J} F_i. \]  

\( \hat{e}_{i} \) will be compatible with the \( h_{i} \), and give \( \check{E}: \check{D} \xrightarrow{\sim} D \) as desired.

Let \( K_0 \) be the full subcategory of \( \mathbf{L}_{\text{anadiag}}[I] \) consisting of the objects \( O_I \) and \( A_I \) for all \( I \in \mathbf{Ob}(I) \). A restricted context is a context over \( K_0 \). We have \( D \uparrow K_0 = D^\# \uparrow K_0 \), and hence, for a restricted context \( \mathbf{X}, D[\mathbf{X}] = D^\# [\mathbf{X}] \).

With \( \mathbf{X} \) a restricted context, an augmented \( I \)-diagram of type \( \mathbf{X} \) is a pair \( (D, \check{a}) \) where

\( D: I \to \mathbf{Cat}, \) and \( \check{a} \in D[\mathbf{X}] \); notations such as \( (D, \check{a}), (\hat{D}, \check{b}) \) denote augmented \( I \)-diagrams. We write \( E: (D, \check{a}) \xrightarrow{\sim} (\hat{D}, \check{b}) \) for the following: \( E: \check{D} \xrightarrow{\sim} \hat{D} \) with \( E \) as in (7) such that \( E(\check{a}) = \check{b} \) in the obvious sense that \( E_I(a_{\mathbf{X}}) = b_{\mathbf{X}} \). The relation \( \sim \) between augmented diagrams is defined thus:

\( (D, \check{a}) \xrightarrow{\sim} (\hat{D}, \check{b}) \iff \) there exists \( E: (D, \check{a}) \xrightarrow{\sim} (\hat{D}, \check{b}) \).

We write \( (D, \check{a}) \xleftarrow{\sim} (\hat{D}, \check{b}) \) for: there exists \( (\hat{D}, \check{c}) \) such that

\( (D, \check{a}) \xleftarrow{\sim} (\hat{D}, \check{c}) \xrightarrow{\sim} (\hat{D}, \check{b}) \). The relation \( \xleftarrow{\sim} \) is the equivalence relation generated by \( \sim \); this can be seen directly, but it also follows from (8) below. In particular, when \( \mathbf{X} = \emptyset \), the relation \( \xleftarrow{\sim} \) coincides with \( \simeq \) for \( I \)-diagrams (since \( \simeq \) is an equivalence relation).

(8) For augmented \( I \)-diagrams \( (D, \check{a}), (\hat{D}, \check{b}) \) of the same type, we have
Here, \( L = L_{\text{anadiag}}[I] \).

As a special case, for (mere) \( I \)-diagrams \( D \) and \( \hat{D} \),

\[
D \simeq \hat{D} \iff D^\# = L_{\hat{D}}^\#.
\]

**Proof.** (A)(\(\iff:\)) Let \((\mathcal{R}, x_0, x_1) : (D^\#, \hat{a}) \leftrightarrow (\hat{D}^\#, \hat{b})\) be a normal \( L, \simeq \)-equivalence (see 5.(2\(\ast\))). Let \( \hat{c} \in \mathcal{R}[\mathcal{X}] \) (\( \mathcal{X} \) the type of \((D^\#, \hat{a}), (\hat{D}^\#, \hat{b})\)) for which \( x_0(\hat{c}) = \hat{a}, \quad x_1(\hat{c}) = \hat{b} \).

Let \( M = x_0^*(D^\#) = x_1^*(\hat{D}^\#) \), a standard \( L \)-structure. Since \( D^\# \) is an anadiagram, and the concept of "anadiagram" is elementary in FOLDS over \( L \), by 5.(1)(a), \( M \) is an anadiagram.

Let \( \hat{D} \) be obtained from \( M \) by cleavage. We show that there is an equivalence \( E : \hat{D} \simeq D \) which extends

\[
m_k^K_0 = \theta_m^K_0 : M_k^K_0 \simeq \hat{D} k^K_0 \to D^\# k^K_0 = D k^K_0
\]

(that is, \( E_I = (\theta_m)_{M_I} \) for all \( I \in \text{Ob}(I) \); here, we used the notation (7) for \( E \)), and similarly, there is \( \hat{E} : \hat{D} \simeq D \) extending \( n k^K_0 \). In particular, it will follow that \( E(\hat{c}) = \hat{a}, \quad E(\hat{c}) = \hat{b} \) and \((D, \hat{a}) \leftrightarrow (\hat{D}, \hat{c}) \rightarrow (D, \hat{b})\) as desired.

We use a notation for \( \hat{D} \) that is analogous to (6). The functor \( E_I : C_I \rightarrow C_I \) is defined by the effect \( m^I_{O_I} \) and \( m^I_{A_I} \); since \( \theta_m : M \rightarrow D \) preserves the relations \( I_I \) and \( T_I \), \( E_I \) is a functor. By the normality of \( x_0 \), \( E_I \) induces bijections on hom-sets, and by the surjectivity of \( x_0 \) on \( O_I \), \( E_I \) is a surjective equivalence.

Let \( i : I \rightarrow J \). Looking back at how the cleavage \( \hat{D} \) was defined, we see that \( F_i X = A_X \),
with \( s, x \in M_0(X, A_X) \). Then \( m(s, x) = D^\# \circ \Omega \circ (m, mA_X) = D^\# \circ \Omega \circ (E_{iX}, E_{jY}) \). By the definition of \( D^\# \), this means that \( ms_X: F_{iX} \to E_{jY} \). We put \( e_{iX} = m s_X \). To see that

\[
e_i = \langle e_{iX} \rangle \in \text{Ob}(C_i)
\]
is a natural transformation \( e_i: F_{iX} \to E_{jY} \), let \( f: X \to Y \in C \).

We see that \( \tilde{F}_i f \) is defined by the property that \( M(A_i) (s_X, s_Y, f, \tilde{F}_i f) \) should hold. But \( \theta_m \) preserves \( A_i \); hence, \( D^\# (A_i) (e_{iX}, e_{iY}, E_{iX}, E_{jY}) \), which, by the definition of \( D^\# \), means

\[
\begin{array}{ccc}
F_{iX} & \xrightarrow{e_{iX}} & E_{jY} \\
\downarrow F_{iX} & \circ & \downarrow E_{jY} \\
F_{jY} & \xrightarrow{e_{jY}} & E_{jY}
\end{array}
\]

which is the naturality of \( e_i \).

Let \( i \to j \) be given. The naturality condition on \( (e_i, e_j) \) with respect to \( t: i \to j \) is seen as follows. Let \( x \in \text{Ob}(C_i) \). The definition of the component \( \tilde{h}_tX: F_{iX} \to F_jX \) is defined (in the process of cleavage) by the condition \( M_{\Omega_t} (s_i, s_j, \tilde{h}_tX) \). The map \( \theta_m: M \to D^\# \) preserves the relation \( \circ \). It follows that \( D^\# \circ \Omega_t (ms_i, ms_j, \tilde{h}_tX) \) holds; that is, \( D^\# \circ \Omega_t (e_{iX}, e_{jY}, E_{iX}, E_{jY}) \) holds. Considering the definition of \( D^\# \circ \Omega_t \), this says that

\[
\begin{array}{ccc}
E_{jY} & \xleftarrow{e_{iX}} & F_{iX} \\
\downarrow E_{jY} & \circ & \downarrow F_{iX} \\
E_{jY} & \xleftarrow{e_{jY}} & F_{jY}
\end{array}
\]

which is what we wanted.

\textbf{(B)("only if") } We show that \( (D, \tilde{a}) \to (D, \tilde{b}) \) implies \( (D, \tilde{a}) \approx (D, \tilde{b}) \). Since \( \approx \) is an equivalence relation, the desired assertion will follow.
Suppose that $E : (D, \vec{a}) \xrightarrow{\sim} (\hat{D}, \vec{b})$; $E$ is taken in the notation in (7); we construct

$(\mathcal{R}, r_0, r_1) : (D^#, \vec{a}) \xleftarrow{\sim} (D^#, \vec{b})$. The kinds of $L$ are as in

$$
\begin{array}{c}
\begin{array}{ccc}
\mathcal{A}_I & \xrightarrow{\mathcal{A}_J} \\
\downarrow d_I & & \downarrow d_J \\
O_I & \xleftarrow{o_0} & O_J
\end{array}
\end{array}
$$

with $i : I \to J$ in $I$; we have to define $\mathcal{R}$ on these kinds.

We put $\mathcal{R}_I = \{(X, \hat{X}, \sigma) : X \in DO_I, \hat{X} \in DO_I, \sigma : E_I X \xrightarrow{\sim} \hat{X}\}$, with $(X, \hat{X}, \sigma) \xrightarrow{r_0} X$,

$(X, \hat{X}, \sigma) \xrightarrow{\hat{X}} \hat{X}$. The "very surjective" condition on $r_0$, $r_1$ at $O_I$ holds by the essential surjectivity of $E_I$.

$$
\begin{array}{c}
\begin{array}{ccc}
\mathcal{R}_I & \xrightarrow{\mathcal{R}_I} \\
\downarrow X & & \downarrow X \\
Y & \xleftarrow{\sigma} & Y
\end{array}
\end{array}
$$

with the displayed item being mapped to $(X, \hat{X}, \sigma)$ by $\mathcal{R}_{d_I}$, to $(Y, \hat{Y}, \tau)$ by $\mathcal{R}_{c_I}$, to $f$ by $r_0$, and to $\hat{f}$ by $r_1$. The mapping

$$
f \longmapsto \hat{f} : D^# A(X, Y) \longrightarrow D^# A(\hat{X}, \hat{Y})$$

so defined, with fixed $(X, \hat{X}, \sigma)$, $(Y, \hat{Y}, \tau) \in \mathcal{R}_I$, is a bijection; this holds since $E_I$ is an equivalence of categories. This shows the "very surjective" condition for $r_0$, $r_1$ at $A_I$, as well as the preservation of $E_{A_I}$.

$$
\begin{array}{c}
\begin{array}{ccc}
\mathcal{R}_I & \xrightarrow{\mathcal{R}_I} \\
\downarrow X & & \downarrow X \\
\mathcal{R}_I & \xleftarrow{\mathcal{R}_I} & \mathcal{R}_J
\end{array}
\end{array}
$$
with the displayed item being mapped to \((X, \hat{X}, \sigma)\) by \(\mathcal{R}_O \hat{1}_0\), to \((A, \hat{A}, \alpha)\) by \(\mathcal{R}_O \hat{1}_1\), to \((X, A, \mu) \in DO\hat{1}\) by \(x_0\), and to \((\hat{X}, \hat{A}, \hat{\mu}) \in \hat{DO}\hat{1}\) by \(x_1\) where \(\hat{\mu}\) is determined by the following commutativity:

\[
\begin{array}{c}
\begin{array}{ccc}
\hat{F}_I F_X & \xrightarrow{\sigma} & \hat{F}_I X \\
\downarrow & & \downarrow \\
\hat{E}_I X & & \hat{A} \\
\end{array}
\end{array}
\]

Note that since all given arrows are isomorphisms, \(\hat{\mu}\) is uniquely determined, and it is an isomorphism. Moreover, since \(E_J\) is an equivalence of categories, the mapping

\[
\mu \mapsto \hat{\mu} : D^\# O\hat{1}(X, A) \longrightarrow D^\# O\hat{1}(\hat{X}, \hat{A})
\]

so defined (with the rest of the data fixed) is a bijection, which shows the "very surjective" condition at \(O\hat{1}\), and the preservation of \(E_O\hat{1}\).

This completes the data for \((\mathcal{R}, x_0, x_1)\); it remains to verify the necessary properties.

Let us consider the preservation of the relation \(\hat{A}\) by \((\mathcal{R}, x_0, x_1)\). What we have to do is this. We take four items

\[
\begin{array}{c}
x_{d_i} \in \mathcal{R}_O \hat{1}, x_{c_i} \in \mathcal{R}_O \hat{1}, x_{a_{i0}} \in \mathcal{R}_A \hat{1}, x_{a_{i1}} \in \mathcal{R}_A \hat{J}
\end{array}
\]

such that \((x_{d_i}, x_{c_i}, x_{a_{i0}}, x_{a_{i1}}) \in \mathcal{R}[\hat{A}]\), that is,

\[
\begin{array}{c}
\mathcal{R}_O \hat{1}_0(x_{d_i}) = \mathcal{R}_d \hat{I}(x_{a_{i0}}), \mathcal{R}_O \hat{1}_1(x_{d_i}) = \mathcal{R}_d \hat{J}(x_{a_{i1}}), \\
\mathcal{R}_O \hat{1}_0(x_{c_i}) = \mathcal{R}_c \hat{I}(x_{a_{i0}}), \mathcal{R}_O \hat{1}_1(x_{c_i}) = \mathcal{R}_c \hat{J}(x_{a_{i1}});
\end{array}
\]

(10)
we consider their \( r_0 \) and \( r_\perp \)-projections; and we have to show that

\[
(r_0 x_{a_1}, r_0 x_{c_1}, r_0 x_{a_0}, r_0 x_{a_1}) \in D^A_i
\]  

(11)

if and only if

\[
(r_1 x_{a_1}, r_1 x_{c_1}, r_1 x_{a_0}, r_1 x_{a_1}) \in D^\#A_i.
\]  

(12)

Let

\[
x_{a_i} = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu)
\]

with \( \sigma: E_i X \rightarrow \hat{X} \), \( \alpha: E_j A \rightarrow \hat{A} \), \( \mu: F_i X \rightarrow A \);

\[
x_{c_i} = ((Y, \hat{Y}, \tau), (B, \hat{B}, \beta), \nu)
\]

with \( \tau: E_i Y \rightarrow \hat{Y} \), \( \beta: E_j B \rightarrow \hat{B} \), \( \nu: F_i Y \rightarrow B \);

note that \( x_{a_i} \vdash o_{i0} \sigma \), \( x_{a_i} \vdash o_{i1} \alpha \), \( x_{c_i} \vdash o_{i0} \tau \), \( x_{c_i} \vdash o_{i1} \beta \).

The first and third of the above conditions (10) force the first two components of \( x_{a_{i0}} \) to be

\( (X, \hat{X}, \sigma) \) and \( (Y, \hat{Y}, \tau) \), respectively. Let

\[
x_{a_{i0}} = ((X, \hat{X}, \sigma), (Y, \hat{Y}, \tau), \hat{f}: X \rightarrow \hat{Y}, \hat{f}: X \rightarrow \hat{Y})
\]

we have

\[
\begin{array}{ccc}
E_i X & \overset{\sigma}{\longrightarrow} & \hat{X} \\
\downarrow E_i \hat{f} & \circ & \downarrow \hat{f} \\
E_i Y & \overset{\tau}{\longrightarrow} & \hat{Y}
\end{array}
\]  

(13)

Similarly,

\[
x_{a_{i1}} = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g: A \rightarrow B, g: \hat{A} \rightarrow \hat{B})
\]

with
(11) means

\[
\begin{array}{c}
F\downarrow X & \xrightarrow{\mu} & A \\
\downarrow F\downarrow f & \circ & \downarrow g \\
\downarrow F\downarrow Y & \xrightarrow{\nu} & B
\end{array}
\]

(15)

whereas (12) means

\[
\begin{array}{c}
\hat{F}\downarrow \hat{X} & \xrightarrow{\hat{\mu}} & \hat{A} \\
\downarrow \hat{F}\downarrow \hat{f} & \circ & \downarrow \hat{g} \\
\downarrow \hat{F}\downarrow \hat{Y} & \xrightarrow{\hat{\nu}} & B
\end{array}
\]

(16)

where \( \hat{\mu} \) and \( \hat{\nu} \) are defined as \( \hat{\mu} \) is in (9); we want to see that (15) iff (16). Consider the following diagram:

The cells 1 and 6 commute, by the definitions of \( \hat{\mu} \) and \( \hat{\nu} \) (see (9)). 2 commutes by (13), 3 by the naturality of \( e\downarrow \), and 5 by (14). Note that all arrows except the vertical ones are isomorphisms. If (15) commutes, then so does 4; the resulting commutativity of the outside square is (16) as desired. Conversely, if (16) commutes, then so does 4 (using the isomorphisms in the diagram), and since \( E_J \) is faithful, so does (15).
Let us look at the similar verification of preservation of $O_t : I \xrightarrow{\jmath} J$. We take

$$(x_{t0}^O, x_{t1}^O, x_{t2}^O) \in \mathcal{R}[O_t]$$

that is,

$$x_{t0}^O = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu) \in \mathcal{R}_i$$

with $\sigma : F_\jmath X \cong \hat{X}$, $\alpha : E_\jmath A \cong \hat{A}$, $\mu : F_\jmath X \cong A$;

$$x_{t1}^O = ((X, \hat{X}, \sigma), (B, \hat{B}, \beta), \rho) \in \mathcal{R}_j$$

with the same $\sigma$ as above, and $\beta : E_\jmath B \cong \hat{B}$, $\rho : F_\jmath X \cong B$

(since we must have $\mathcal{R}_i (x_{t0}^O) = \mathcal{R}_j (x_{t1}^O)$ (see the first equation in (4)), the first components of $x_{t0}^O$ and $x_{t1}^O$ have to agree);

$$x_{t2}^O = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g : A \rightarrow B, \hat{g} : A \rightarrow B)$$

with (9) holding (see the other two equations in (4)). Looking at the definition of $\mathcal{D}^\# O_t$, $\hat{\mathcal{D}}^\# O_t$, what we have to see is that

$$\begin{array}{ccc}
F_\jmath X & \xrightarrow{\mu} & A \\
\downarrow h_{\jmath X} & \circ & \downarrow g \\
F_\jmath X & \xrightarrow{\rho} & B
\end{array} \quad \text{iff} \quad \begin{array}{ccc}
\hat{F}_\jmath X & \xrightarrow{\hat{\mu}} & \hat{A} \\
\downarrow \hat{h}_{\jmath X} & \circ & \downarrow \hat{g} \\
\hat{F}_\jmath X & \xrightarrow{\hat{\rho}} & \hat{B}
\end{array}  \quad (17)$$

Consider
The cells 1 and 6 commute for reasons as before. 2 commutes because of the naturality of $\hat{h}_I$, 3 because of the naturality of $(e_i, e_j)$ with respect to $i : i \to j$, 5 because of (14). 4 is the antecedent of (17) with $E_J$ applied to it, the outer square is the succedent of (17). The assertion in (17) follows.

The remaining properties are the preservation of the $T_I, \mathcal{I}_I$, and of the equalities on the $A_I, O_I$. These are immediately seen.

We need that $(\mathcal{R}, r_0, r_1)$ "relates $\hat{a}$ to $\hat{b}$". For $\mathcal{X}$ the restricted context involved, $\hat{a} = \langle a_x \rangle_{x \in \mathcal{X}}, \hat{b} = \langle b_x \rangle_{x \in \mathcal{X}}$; we want $\hat{c} = \langle c_x \rangle_{x \in \mathcal{X}} \in \hat{D}[\mathcal{X}]$ such that $r_0(\hat{c}) = \hat{a}$, $r_1(\hat{c}) = \hat{b}$. For $x \in \mathcal{X}, K = O_I$, define $c_x = 1_{E_I a x} : E_I a x \Rightarrow b x \in \mathcal{R} O_I$; we have $r_0 c_x = a_x, r_1 c_x = b_x$. For $x \in \mathcal{X}, x : A_I(y, z)$, define $c_x = (c_y, c_z, a_x, b_x) \in \mathcal{R} A_I$; $c_x \in \mathcal{R} A_I$ indeed holds since this means

$$\begin{align*}
E_I(a_y) & \xrightarrow{c_y} b_y \\
E_I(a_x) & \xrightarrow{c_x} b_x \\
E_I(a_z) & \xrightarrow{c_z} b_z
\end{align*}$$

and this holds since $E_I(a_x) = b_x$; also, $r_0(c_x) = a_x, r_1(c_x) = b_x$.

This completes the proof of (8).