§6. Equivalence of categories, and of diagrams of categories

The simplest application of the results of the last section is to invariance under equivalence of categories of first order properties of diagrams of objects and arrows in a category. In what follows, until further notice, $\mathbf{L}$ stands for $\mathbf{L}_{\mathrm{cat}}$, the DSV for categories introduced in §1; a category $\mathbf{A}$ may be regarded an $\mathbf{L}$-structure. A context of variables for $\mathbf{L}$ is essentially a functor $\mathbf{K}=\mathbf{L}_{\mathrm{graph}}\to\mathrm{Set}$, that is, a graph; we are mainly interested in finite contexts, although for the notions to be introduced next, there is no need to confine attention to finite contexts.

For a context $\mathcal{X}$, an *augmented* category of type $\mathcal{X}$ is a pair $(\mathbf{A}, \vec{a})$, with $\mathbf{A}$ a category, and $\vec{a}\in\mathbf{A}[\mathcal{X}]$ (that is, $\vec{a}$ a diagram of type the graph $\mathcal{X}$). Until further notice, notations such as $(\mathbf{A}, \vec{a})$, $(\mathbf{B}, \vec{b})$ denote augmented categories. Mere categories are considered special cases of augmented categories of type $\emptyset$; $\mathbf{A}$, $\mathbf{B}$ *etc.* denote categories.

For augmented categories $(\mathbf{A}, \vec{a})$, $(\mathbf{B}, \vec{b})$ of the same type, we write

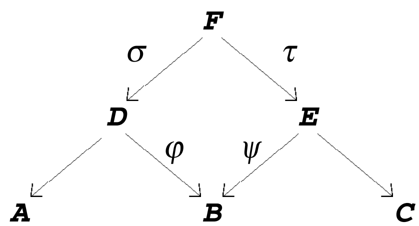
$$(\mathbf{A}, \vec{a}) \xrightarrow{\sim} (\mathbf{B}, \vec{b})$$

if there is an equivalence functor $F:\mathbf{A}\xrightarrow{\simeq}\mathbf{B}$ (F is full and faithful, and essentially surjective on objects) that maps $\vec{a}$ to $\vec{b}$; we may also write $(\mathbf{B}, \vec{b})\xleftarrow{\sim}(\mathbf{A}, \vec{a})$ for the same. Note that the relation $\xrightarrow{\sim}$ is reflexive and transitive but not symmetric (an equivalence functor $\mathbf{A}\xrightarrow{\simeq}\mathbf{B}$ may take two different objects $A\neq A'$ in $\mathbf{A}$ to the same B in $\mathbf{B}$; then $(\mathbf{A}, \langle A, A'\rangle)\xrightarrow{\sim}(\mathbf{B}, \langle B, B\rangle)$ but not *vice versa*). The special case when the type $\mathcal{X}$ is $\emptyset$, is, however, symmetric; $\mathbf{A}\xrightarrow{\sim}\mathbf{B}$ implies $\mathbf{A}\xleftarrow{\sim}\mathbf{B}$ since every equivalence functor has a quasi-inverse (by the Axiom of Choice); $\mathbf{A}\xrightarrow{\sim}\mathbf{B}$ is the same as equivalence of categories, $\mathbf{A}\simeq\mathbf{B}$.

The equivalence relation generated by the relation $\xrightarrow{\sim}$ is only "one step away" from $\xrightarrow{\sim}$; it is $\xleftrightarrow{\sim}$ defined as

$(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b}) \underset{\mathrm{def}}{\Longleftrightarrow}$ there is $(\boldsymbol{C}, \vec{c})$ such that $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftarrow} (\boldsymbol{C}, \vec{c}) \overset{\sim}{\longrightarrow} (\boldsymbol{B}, \vec{b})$.(1$^-$)

To see the transitivity of the relation $\overset{\sim}{\longleftrightarrow}$, assume $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftarrow} (\boldsymbol{D}, \vec{d}) \overset{\sim}{\longrightarrow} (\boldsymbol{B}, \vec{b})$ and $(\boldsymbol{B}, \vec{b}) \overset{\sim}{\longleftarrow} (\boldsymbol{E}, \vec{e}) \overset{\sim}{\longrightarrow} (\boldsymbol{C}, \vec{c})$, and consider the diagram



where the quadrangle has $\boldsymbol{F}$ the "isomorphism-comma" category, with objects $(D, E, \varphi D \overset{\cong}{\longrightarrow} \psi E)$, and arrows the usual commutative squares, with $\sigma : \boldsymbol{F} \to \boldsymbol{D}$, $\tau : \boldsymbol{F} \to \boldsymbol{E}$ the forgetful functors. Since φ , ψ are equivalence functors, so are σ , τ . Let $\vec{f} = \langle f_x \rangle_{x \in \mathcal{X}} \in \boldsymbol{F}[\mathcal{X}]$ be defined as follows. For $x \in \mathcal{X}$, $x : \mathrm{O}$, let $f_x = (d_x, e_x, \mathrm{id} : \varphi d_x \overset{\cong}{\longrightarrow} \psi e_x)$; note that $\varphi d_x = \psi e_x$ by assumption. For $x \in \mathcal{X}$, $x : \mathrm{A}(y, z)$, let $f_x = (d_x : d_y \to d_z, e_x : e_y \to e_z) : f_y \to f_z$; note that $\varphi d_x = \psi e_x$ by assumption. We have that $(\boldsymbol{F}, \vec{f}) \overset{\sim}{\longrightarrow} (\boldsymbol{D}, \vec{d})$, $(\boldsymbol{F}, \vec{f}) \overset{\sim}{\longrightarrow} (\boldsymbol{E}, \vec{e})$. Using the composites $\boldsymbol{F} \to \boldsymbol{A}$, $\boldsymbol{F} \to \boldsymbol{C}$, we obtain $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b})$ as desired.

Recall the relation $\approx_{\boldsymbol{L}}$ of the last section; $\approx_{\boldsymbol{L}}$ is, in particular, a relation between augmented categories. We have that $\approx_{\boldsymbol{L}}$ is the same as $\overset{\sim}{\longleftrightarrow}$.

(1) **(a)** $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b}) \iff (\boldsymbol{A}, \vec{a}) \approx_{\boldsymbol{L}} (\boldsymbol{B}, \vec{b})$;

(b) $\boldsymbol{A} \simeq \boldsymbol{B} \iff \boldsymbol{A} \approx_{\boldsymbol{L}} \boldsymbol{B}$.

Proof. Assume $(\boldsymbol{A}, \vec{a}) \approx_{\boldsymbol{L}} (\boldsymbol{B}, \vec{b})$. By §5, there is a normal $\boldsymbol{L}$,≈-equivalence $(W, u, v) : (\boldsymbol{A}, \vec{a}) \overset{\approx}{\underset{\boldsymbol{L}}{\longleftrightarrow}} (\boldsymbol{B}, \vec{b})$. Then, $\boldsymbol{C} = u^*(\boldsymbol{A}) = v^*(\boldsymbol{B})$ is a category, since, by 5.(1), as a standard $\boldsymbol{L}$-structure, $\boldsymbol{C}$ satisfies all the axioms of category which are formulated in FOLDS (see 5.(2)(a)). Furthermore, clearly, $\theta_u : \boldsymbol{C} \longrightarrow \boldsymbol{A}$, $\theta_v : \boldsymbol{C} \longrightarrow \boldsymbol{B}$ are surjective equivalence functors. This shows the right-to-left direction in (a). For the proof of the other direction, we prove the implication

$$(\boldsymbol{A}, \vec{a}) \xrightarrow{\sim} (\boldsymbol{B}, \vec{b}) \quad \Longrightarrow \quad (\boldsymbol{A}, \vec{a}) \sim_{\boldsymbol{L}} (\boldsymbol{B}, \vec{b}) \ ;$$

to this end, we "saturate" the given equivalence appropriately; we will do this proof in a more general situation below.

Knowing the transitivity of the relation $\sim_{\boldsymbol{L}}$, the transitivity of $\overset{\sim}{\longleftrightarrow}$ also follows from (1)(a).

(b) is a special case of (a).

Recall the translation $\varphi \mapsto \varphi^*$ in §1; this is just to say that any formula φ of FOLDS over $\boldsymbol{L}$ may be regarded a formula (φ^*) over $|\boldsymbol{L}|$ in ordinary multisorted logic.

Let $\mathtt{T}_{\mathtt{cat}} = (\ |\boldsymbol{L}|\ , \Sigma_{\mathtt{cat}})$ the theory of categories in ordinary multisorted logic ($\Sigma_{\mathtt{cat}}$ can be taken to be $\Sigma[\mathbf{L}_{\mathtt{cat}}] \cup \{\theta^* : \theta \in \Theta\}$; $\Sigma[\boldsymbol{L}]$ for any DSV $\boldsymbol{L}$ was defined in §1 ; Θ is the set of axioms in FOLDS for categories as given in §1.). When T is a theory extending $\mathtt{T}_{\mathtt{cat}}$ ($|\boldsymbol{L}| \subset \mathrm{L}_T$, $\Sigma_{\mathtt{cat}} \subset \Sigma_T$), and $M \models T$, we write $|M|$ for $M \restriction \boldsymbol{L}$, the underlying category of M.

(2)(a) Let T be a theory extending $\mathtt{T}_{\mathtt{cat}}$. Let $\mathcal{X}$ be a finite context over $\mathbf{L}_{\mathtt{cat}}$, σ an L_T-formula such that $\mathtt{Var}(\sigma) \subset \mathcal{X}$. If

for any $M, N \models T$ and diagrams $\vec{a} \in |M|\ [\mathcal{X}]$, $\vec{b} \in |N|\ [\mathcal{X}]$, $M \models \sigma[\vec{a}]$ and $(\ |M|\ , \vec{a}) \overset{\sim}{\longleftrightarrow} (\ |N|\ , \vec{b})$ imply $N \models \sigma[\vec{b}]$,

then

there is θ in FOLDS with restricted equality over $\mathbf{L}_{\mathtt{cat}}$ with $\mathtt{Var}(\theta) \subset \mathcal{X}$ such that for all $M \models T$ and diagrams $\vec{a} \in |M|\ [\mathcal{X}]$, we have $M \models \sigma[\vec{a}]$ iff $M \models \theta^*[\vec{a}]$.

(**b**) In particular, if σ is a sentence over L_T, and for any $M, N \models T$, $M \models \sigma$ and $|M| \simeq |N|$ imply $N \models \sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathtt{cat}}$ such that for any $M\models T$, $M\models\sigma$ iff $M\models\theta^*$.

Proof. We apply 5.(15) to $\mathbf{C}=[T]$, with $I:\mathbf{L}\to\mathbf{C}$ the composite of $I:\mathbf{L}\to[\mathrm{T}_{\mathrm{cat}}]$ defined in §5 before (7)(a) and the inclusion $[\mathrm{T}_{\mathrm{cat}}]\to[T]$; moreover, we take σ in 5.(15) to be $m^*([\mathcal{X}:\sigma])\rightarrowtail I[\mathcal{X}]$ ($m:I[\mathcal{X}]\rightarrowtail\{\mathcal{X}\}$; see §5 before (7)(a)). By (1)(a), the assumption implies that

$$M\models\sigma[\vec{a}]\ \ \&\ \ (|M|,\vec{a})\approx_{\mathbf{L}}(|N|,\vec{b})\quad\Longrightarrow\quad N\models\sigma[\vec{b}]\ .$$

The conclusion of 5.(15) is what we want. (b) is a special case of (a).

We say that a theory T extending $\mathrm{T}_{\mathrm{cat}}$ is *normal* if for any $M\models T$ and any category $\mathbf{A}$, if $\mathbf{A}\simeq|M|$, then there is a model $N\models T$ such that $\mathbf{A}=|N|$. In other words, normality of T says that the property of being the $\mathbf{L}_{\mathrm{cat}}$-reduct of a model of T is invariant under equivalence of categories. Most theories of categories (possibly) with additional structure are normal. E.g., so is the theory of monoidal categories, or the theory of categories with specified finite limits. Of course, $\mathrm{T}_{\mathrm{cat}}$ itself is normal.

Let $\mathcal{X}$ be a finite context, and σ be a formula over L_T with $\mathrm{Var}(\sigma)\subset\mathcal{X}$. Let us say that σ is *preserved along equivalence functors between models of* T if the following holds: whenever $M,N\models T$, $\vec{a}\in M[\mathcal{X}]$, $\vec{b}\in N[\mathcal{X}]$, then $M\models\sigma[\vec{a}]$ and $(|M|,\vec{a})\xrightarrow{\sim}(|N|,\vec{b})$ imply $N\models\sigma[\vec{b}]$. When in this definition, $(|M|,\vec{a})\xrightarrow{\sim}(|N|,\vec{b})$ is replaced by $(|M|,\vec{a})\xleftarrow{\sim}(|N|,\vec{b})$, we obtain the notion of being *reflected* along equivalence functors. Now, notice that for T a normal theory, the hypothesis of (2)(a) holds iff σ is preserved and reflected along equivalence functors of models of T (the point is that, in case T is normal, in (1^-), when $\mathbf{A}$ (and $\mathbf{B}$) are reducts of models of T, $\mathbf{C}$ can also be expanded to a model of T). Thus, we obtain the following variant of (2)(a):

(3) Let T be a normal theory of categories (possibly) with additional structure. Let $\mathcal{X}$ be a finite context over $\mathbf{L}_{\mathrm{cat}}$. Suppose that the first-order formula σ over L_T with free variables all in $\mathcal{X}$ is preserved and reflected along equivalence functors of models of T. Then there is a formula φ in FOLDS with restricted equality over $\mathbf{L}_{\mathrm{cat}}$ with $\mathrm{Var}(\varphi)\subset\mathcal{X}$ such that σ is equivalent to φ^{*} in models of T.

Freyd's and Blanc's characterization (see [F], [FS], [B]) of first order properties of finite diagrams invariant under equivalence is (3) for $T=\mathrm{T}_{\mathrm{cat}}$. In fact, the general result (3) can also be obtained by their methods, which is very different from the methods of this paper (we will comment on this in Appendix C). It seems however that the more general result (2), in particular, (2)(b), cannot be obtained by the Freyd's and Blanc's methods (although I should concede that the added generality in (2)(b) consisting in a reference to not-necessarily normal theories does not seem very important).

The results of §5 that are more general than 5.(15) (*e.g.*, the "interpolation-style" result (7)(b)) will also have consequences for equivalences of categories; we leave their formulation to the reader.

Extending the Freyd-Blanc result to more complex categorical structures will involve a new element. For instance, in the case of structures consisting of two categories and a functor between them (an example of which is a fibration), the first-order properties invariant under equivalence (in the appropriate standard sense; see also below) are not those expressible in FOLDS directly, but rather, those that are expressible in FOLDS in the language of the so-called saturated anafunctor associated with the given functor. Anafunctors are treated in [M2]; explanations will be given presently.

We now proceed to giving the framework for dealing with structures consisting of several (possibly infinitely many) categories, functors between them, and natural transformations between the latter. We will return to the simplest special case of two categories and a functor between them afterwards.

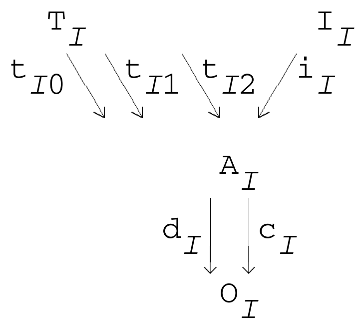
Let $\boldsymbol{I}$ be a small 2-graph; $\boldsymbol{I}:\mathbf{L}_{\text{2-graph}}\to\mathrm{Set}$. We associate the graph $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ with $\boldsymbol{I}$; $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ serves as a similarity type for diagrams $\boldsymbol{I}\to\mathrm{Cat}$ of (small) categories, functors and natural transformations. The objects of $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ are as follows:

O_I , A_I , I_I , T_I ($I \in Ob(\boldsymbol{I})$)

O_i , A_i ($i \in Arr(\boldsymbol{I})$)

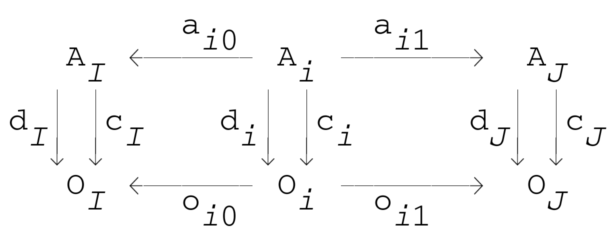
O_α ($\alpha \in 2\text{-}Cell(\boldsymbol{I})$)

The arrows of $L_{diag}[\boldsymbol{I}]$ are shown in the following three diagrams:

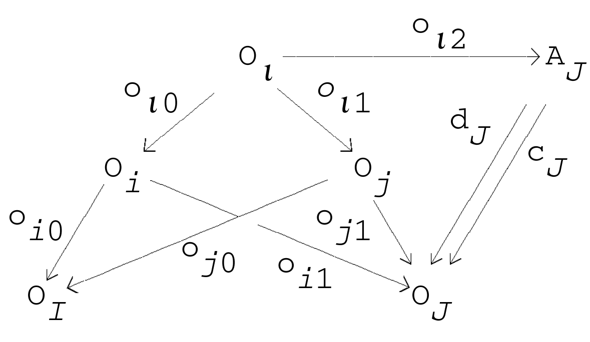


(3)

displaying the arrows associated with an object I ;



which displays the arrows associated with an arrow $i : I \to J$ in $\boldsymbol{I}$; and

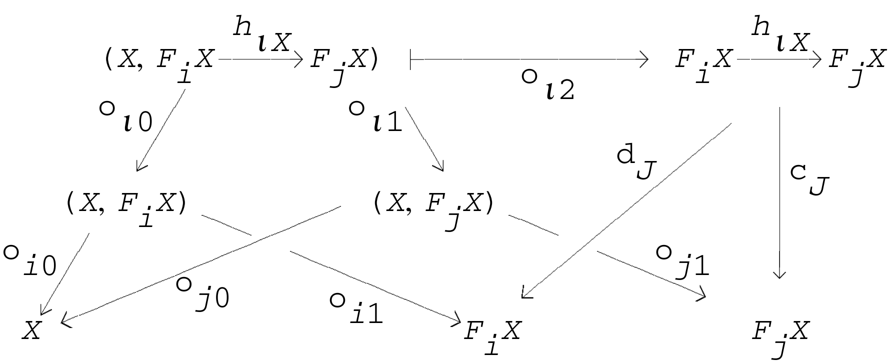


which displays the ones associated with the 2-cell $\iota : i \to j$ ($I \underset{j}{\overset{i}{\rightrightarrows}} J$, $\downarrow \iota$) .

Given a $\boldsymbol{I}$-diagram

$$D:\boldsymbol{I}\to\mathrm{Cat} : (\langle \boldsymbol{C}_I\rangle_{I\in\boldsymbol{I}}, \langle F_i:\boldsymbol{C}_I\to\boldsymbol{C}_J\rangle_{i:I\to J}, \langle h_\iota : F_i\to F_j\rangle_{I \overset{i}{\underset{j}{\Downarrow\iota}} J}) \qquad (4)$$

of categories, functors and natural transformations, we construe D as an $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$-structure as follows. (3) is interpreted as the category $\boldsymbol{C}_I$. When $i:I\to J$, O_i is the set of pairs (X, F_iX) with $X\in\mathrm{Ob}(\boldsymbol{C}_I)$, with $(X, F_iX)\overset{\mathrm{o}_{i0}}{\longmapsto}X$, $(X, F_iX)\overset{\mathrm{o}_{i1}}{\longmapsto}F_iX$. A_i is the set of pairs $(f, F_if)=(X\xrightarrow{f}Y, F_iX\xrightarrow{F_if}F_iY)$, with $(f, F_if)\overset{\mathrm{d}_i}{\longmapsto}(X, F_iX)$, $(f, F_if)\overset{\mathrm{c}_i}{\longmapsto}(Y, F_iY)$, $(f, F_if)\overset{\mathrm{a}_{i0}}{\longmapsto}f$, $(f, F_if)\overset{\mathrm{a}_{i1}}{\longmapsto}F_if$. For $I \overset{i}{\underset{j}{\Downarrow\iota}} J$, O_ι is the set of pairs $(X, F_iX\xrightarrow{h_{\iota X}}F_jX)$. The effect of the remaining arrows, as well as the corresponding commutativities, are shown by the following picture:



Let $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ be the DSV defined as follows. The underlying simple category of $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ is generated by the graph $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$, subject to the following equalities between arrows:

the ones ensuring that (3) generates a copy of $\mathbf{L}_{\mathrm{cat}}$ (see §1);

$\mathrm{o}_{i0}\mathrm{d}_i=\mathrm{d}_I\mathrm{a}_{i0}$, $\mathrm{o}_{i1}\mathrm{d}_i=\mathrm{d}_J\mathrm{a}_{i1}$, $\mathrm{o}_{i0}\mathrm{c}_i=\mathrm{c}_I\mathrm{a}_{i0}$, $\mathrm{o}_{i1}\mathrm{c}_i=\mathrm{c}_J\mathrm{a}_{i1}$,

$$\mathrm{o}_{i0}\mathrm{o}_{\iota 0}=\mathrm{o}_{j0}\mathrm{o}_{\iota 1},\quad \mathrm{d}_J\mathrm{o}_{\iota 2}=\mathrm{o}_{i1}0_{\iota 0},\quad \mathrm{c}_J\mathrm{o}_{\iota 2}=\mathrm{o}_{j1}0_{\iota 1}. \qquad (5)$$

The relations of $\boldsymbol{L}$ are exactly its top-level objects; that is, T_I, I_I, A_i, O_ι, for I, i and ι ranging over the 0-cells, 1-cells and 2-cells of $\boldsymbol{I}$, respectively.

The equalities on arrows are suggested by what is true for $\boldsymbol{I}$-diagrams as structures. In fact, every $\boldsymbol{I}$-diagram is a functor $D:\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]\longrightarrow\mathrm{Set}$, that is, the equalities listed are true in it (as identities). Also, the relations of $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ are interpreted in D relationally (the corresponding family of functions is monomorphic). In summary, every $\boldsymbol{I}$-diagram is an $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structure.

$\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ is the similarity type of what we call the "anadiagrams" of type $\boldsymbol{I}$. An *anadiagram* $M:\boldsymbol{I}\xrightarrow{\mathrm{a}}\mathrm{Cat}$ is an $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structure satisfying the following axioms (A0) to (A6) in FOLDS with equality ($I \underset{j}{\overset{i}{\rightrightarrows}} J$ with $\iota: i \Downarrow j$ range over objects, arrows and 2-cells in $\boldsymbol{I}$ as shown; the unique existential quantifiers in (A2) and (A5) are abbreviations in the usual way, and they refer to equality on the sorts $\mathrm{A}_J(\cdot,\cdot)$).

(A0): axioms expressing that for each $I\in\mathrm{Ob}(\boldsymbol{I})$, the part of M referring to I is a category .

(A1) $\forall X:\mathrm{O}_I.\exists A:\mathrm{O}_J.\exists s:\mathrm{O}_i(X,A).\,\mathbf{t}$.

(A2) $\forall X,Y:\mathrm{O}_I.\forall A,B:\mathrm{O}_J\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_i(Y,B).\forall f:\mathrm{A}_I(X,Y).$
$\exists!g:\mathrm{A}_J(A,B).\mathrm{A}_i(s_{\mathrm{d}_i},t_{\mathrm{c}_i},f_{\mathrm{a}_{i0}},g_{\mathrm{a}_{i1}})$.

(A3) $\forall X:\mathrm{O}_I.\forall A:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall\alpha:\mathrm{A}_I(X,X).\forall\bar{\alpha}:\mathrm{A}_J(A,A)$
$[\mathrm{A}_i(s,s,\alpha,\bar{\alpha})\longrightarrow(\mathrm{I}_I(X_{\mathrm{d}_I\mathrm{i}_I},\alpha_{\mathrm{i}_I})\longrightarrow\mathrm{I}_J(\mathrm{A},\bar{\alpha}))]$.

(A4) $\forall X,Y,Z:\mathrm{O}_I.\forall A,B,C:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_i(Y,B).\forall u:\mathrm{O}_i(Z,C)$
$\forall f:\mathrm{A}_I(X,Y).\forall g:\mathrm{A}_I(Y,Z).\forall h:\mathrm{A}_I(X,Z)$
$\forall\bar{f}:\mathrm{A}_I(A,B).\forall\bar{g}:\mathrm{A}_I(B,C).\forall\bar{h}:\mathrm{A}_I(A,C)$
$[((\mathrm{A}_i(s,t,f,\bar{f})\wedge\mathrm{A}_i(t,u,g,\bar{g})\wedge\mathrm{A}_i(s,u,h,\bar{h}))\longrightarrow$
$(\mathrm{T}_I(f,g,h)\longrightarrow\mathrm{T}_J(\bar{f},\bar{g},\bar{h}))]$.

(A5) $\forall X:\mathrm{O}_I.\forall A:\mathrm{O}_J.\forall B:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_j(X,B)$

$$\exists! f{:}A_J(A, B).O_\iota(s_{\iota 0}, t_{\iota 1}, f_{\iota 2}).$$

(A6) $$\forall X, Y{:}O_I.\forall A, B, C, G{:}O_J$$
$$\forall s{:}O_i(X, A)\forall t{:}O_i(Y, B).\forall u{:}O_j(X, C).\forall v{:}O_j(Y, G)$$
$$\forall f{:}A_I(X, Y).\forall g{:}A_J(A, B).\forall k{:}A_J(C.G)$$
$$\forall \ell{:}A_J(A, C).\forall m{:}A_J(B, G)$$
$$[(O_i(s, t, f, g)\wedge O_j(u, v, f, k)\wedge O_\iota(s, u, \ell)\wedge O_\iota(t, v, m))\longrightarrow$$
$$\exists n{:}A_J(B, C).(T_J(g, m, n)\wedge T_J(\ell, k, n))].$$

For a less formal explanation of the notion of anadiagram, I refer to [M2]. In that paper, I introduce the notion of *anafunctor* between categories, a generalization of the notion of functor. An anafunctor defines its values on objects only up to isomorphism. Formally, the definition of anafunctor is obtained by specializing the definition of "anadiagram" to the case when $\boldsymbol{I}$ is the (2-)graph $0\xrightarrow{\langle 0, 1\rangle}1$ (without 2-cells). Anadiagrams have anafunctors instead of functors as 1-cells, and natural transformations of anafunctors as 2-cells.

Note that any $\boldsymbol{I}$-diagram $D{:}\boldsymbol{I}\to\mathrm{Cat}$ is an anadiagram; all the axioms for "anadiagram" are satisfied in D (as an $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structure). In fact, the diagrams are essentially the same as those anadiagrams M in which the sorts O_i ($i\in\mathrm{Arr}(\boldsymbol{I})$) are interpreted *relationally*, that is, the family $\langle Mp\rangle_{p:O_i\to K_p}$ is jointly monomorphic.

On the other hand, any anadiagram gives rise to a diagram, obtained by making some non-canonical choices. Let M be an anadiagram $M{:}\boldsymbol{I}\xrightarrow{a}\mathrm{Cat}$; we construct $D{:}\boldsymbol{I}\longrightarrow\mathrm{Cat}$; we use the notation (4) for the ingredients of D. For $I\in\mathrm{Ob}(\boldsymbol{I})$, the category $\boldsymbol{C}_I$ is given directly by the data in M corresponding to I (see (A0)). By (A1), for any $i{:}I\to J$ in $\boldsymbol{I}$ and $X\in\mathrm{Ob}(\boldsymbol{C}_I)=MO_I$, we make a choice of $A^i_X=A_X\in MO_J$ and $s^i_X=s_X\in MO_i(X, A)$; we put $F_iX=A_X$. Starting with $f{:}X\to Y$, and using (A2) with $A=A_X$, $B=A_Y$, $s=s_X$, $t=s_Y$, we let $F_if=g$ whose unique existence (A2) states. (A3) and (A4) assure that F_i so defined is a functor $F_i{:}\boldsymbol{C}_I\to\boldsymbol{C}_J$. Using (A5) with $A=A^i_X$, $B=A^j_X$, $s=s^i_X$, $t=s^j_X$, we put $h_{\iota X}=f$ for the f whose existence (5) asserts. (A6) ensures that h_ι is a natural transformation $h_\iota{:}F_i\to F_j$. Let us refer D as the diagram obtained from M by *cleavage* (in analogy to the terminology used with fibration); of course, it is not uniquely determined.

Next, we describe the *saturation* $D^{\#}$ of a diagram $D:\mathbf{I}\to\mathrm{Cat}$, an anadiagram canonically associated with D. (As a matter of fact, the components corresponding to the 1-cells $i:I\to J$ will be the "saturated anafunctors" $F_i^{\#}$ associated with the given functors F_i, in the sense of [M2].)

In $D^{\#}$, the interpretation of each part of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ as in (3) is the same as in D.

For $i:I\to J$, a 1-cell in $\mathbf{I}$, $D^{\#}\mathrm{O}_i$ is the set of triples $\mu=(X, A, F_iX\xrightarrow[\mu]{\cong}A)$ with $X\in\mathbf{C}_I$, $A\in\mathbf{C}_J$ and μ an isomorphism as shown; $\mu\overset{\mathrm{O}_{i0}}{\longmapsto}X$, $\mu\overset{\mathrm{O}_{i1}}{\longmapsto}A$. $D^{\#}A_i$ is the set of all entities

$$\left(\begin{array}{c}X\\ \downarrow f\\ Y\end{array},\quad \begin{array}{ccc}F_iX & \xrightarrow[\mu]{\cong} & A\\ \downarrow F_if & \circ & \downarrow g\\ F_iY & \xrightarrow[\nu]{\cong} & B\end{array}\right)$$

with the displayed entity mapped to (X, A, μ) by d_i, to (Y, B, ν) by c_i, to f by a_{i0}, and to g by a_{i1}. For $I\underset{j}{\overset{i}{\rightrightarrows}}J$, $D^{\#}\mathrm{O}_{\iota}$ consists of all

$$\left(X,\quad \begin{array}{ccc}F_iX & \xrightarrow[\cong]{\mu} & A\\ h_{\iota X}\downarrow & \circ & \downarrow g\\ F_jX & \xrightarrow[\rho]{\cong} & B\end{array}\right),$$

and the displayed item is mapped to (X, A, μ) by $\mathrm{o}_{\iota 0}$, to (X, B, ρ) by $\mathrm{o}_{\iota 1}$, and to g by $\mathrm{o}_{\iota 2}$.

We leave it to the reader to verify that $D^{\#}$ so defined is an anadiagram.

$D^{\#}$ satisfies a property that distinguishes it from diagrams; it is *saturated*, by which we mean that it satisfies, for each $i:I\to J$ in $\mathbf{I}$, the FOLDS sentence

(A7) $$\forall X{:}\mathrm{O}_I.\forall A, B{:}\mathrm{O}_J.\forall s{:}\mathrm{O}_i(X, A).\forall f{:}\mathrm{A}_J(A, B)$$
$$(\mathrm{Iso}(f) \longrightarrow \exists! t{:}\mathrm{O}_i(X, B).\exists g{:}\mathrm{A}_I(X, X).(\mathrm{I}_I(g) \wedge \mathrm{A}_i(s, t, g, f)))\ ;$$

here, $\mathrm{Iso}(f)$ abbreviates

$$\exists h{:}\mathrm{A}_J(B, A)\exists k{:}\mathrm{A}_J(A, A)\exists \ell{:}\mathrm{A}_J(B, B).(\mathrm{I}_J(k) \wedge \mathrm{I}_J(\ell) \wedge \mathrm{T}_J(f, h, k) \wedge \mathrm{T}_J(h, f, \ell))$$
.

In fact, it can be proved (although we will not need this result) that, up to *isomorphism* as $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structures, the saturated $\boldsymbol{I}$-anadiagrams are precisely the ones of the form $D^{\#}$, for some diagram D.

Given D as in (5), and another $\boldsymbol{I}$-type diagram

$$\hat{D}{:}\boldsymbol{I}\to\mathrm{Cat} : (\langle\hat{\boldsymbol{C}}_I\rangle_{I\in\boldsymbol{I}}, \langle\hat{F}_i{:}\hat{\boldsymbol{C}}_I\to\hat{\boldsymbol{C}}_J\rangle_{i:I\to J}, \langle\hat{h}_\iota{:}\hat{F}_i\to\hat{F}_j\rangle_{I\underset{j}{\overset{i}{\rightrightarrows}}J\ (\iota:i\to j)})\ , \tag{6}$$

we say that D and $\hat{D}$ are *equivalent*, and write $D\simeq\hat{D}$, if there exist a family $\langle E_I{:}\boldsymbol{C}_I\xrightarrow{\simeq}\hat{\boldsymbol{C}}_I\rangle_{I\in\boldsymbol{I}}$ of equivalence functors, and a family $\langle e_i\rangle_{i:I\to J}$ of natural isomorphisms as in

$$\begin{array}{ccc} \boldsymbol{C}_I & \xrightarrow{E_I} & \hat{\boldsymbol{C}}_I \\ {\scriptstyle F_i}\downarrow & \cong \swarrow e_i & \downarrow{\scriptstyle \hat{F}_i} \\ \boldsymbol{C}_J & \xrightarrow[E_J]{} & \hat{\boldsymbol{C}}_J \end{array} \qquad e_i{:}\hat{F}_i\circ E_I \xrightarrow{\cong} E_J\circ F_i\ ,$$

satisfying the additional naturality condition:

$$\begin{array}{ccc} E_J\circ F_i & \xleftarrow{e_i} & \hat{F}_i\circ E_I \\ {\scriptstyle E_J\circ h_\iota}\downarrow & \circ & \downarrow{\scriptstyle \hat{h}_\iota\circ E_I} \\ E_J\circ F_j & \xleftarrow[e_j]{} & \hat{F}_j\circ E_I \end{array}$$

for every $I \overset{\xrightarrow{\;i\;}}{\underset{\xrightarrow[\;j\;]{}}{\Downarrow\iota}} J$ in $\boldsymbol{I}$. The data $E=(\langle E_I\rangle_{I\in\boldsymbol{I}},\langle e_i\rangle_{i\in\mathrm{Arr}(\boldsymbol{I})})$ form an *equivalence* of D and $\hat{D}$, in notation,

$$E=(\langle E_I\rangle_{I\in\boldsymbol{I}},\langle e_i\rangle_{i\in Arr(\boldsymbol{I})}):D\xrightarrow{\simeq}\hat{D}. \tag{7}$$

This notion of equivalence is a "bicategorical" notion; it is the equivalence in the internal sense of the bicategory (actually, 2-category) $\mathrm{Hom}(\langle\boldsymbol{I}\rangle,\mathrm{Cat})$ of homomorphisms of bicategories, pseudo-natural transformations and modifications, with $\langle\boldsymbol{I}\rangle$ the 2-category generated by the 2-graph $\boldsymbol{I}$. (The main part of the fact that the "one-way" formulation of equivalence given above as the definition, and the "internal" concept just mentioned coincide, is the symmetry of the relation $\simeq$; an outline of the proof of the symmetry of $\simeq$ is given below.) It is the "good" notion of equivalence, the one that comes up in practice. For instance, in Chapter 4 of [MP], diagrams of sketches, and diagrams of accessible categories are dealt with, and the present notion of equivalence is the one which is operative. Specifically, the Uniform Sketchability Theorem, one of the main results of [MP] (4.4.1 in [MP]) says that a small diagram of accessible categories is *equivalent* to one obtained from a diagram of sketches by taking the categories of models of the sketches involved.

Although the fact is well-known, I outline the proof that the relation $D\simeq\hat{D}$ is symmetric. Since it is easily seen to be transitive and reflexive, $\simeq$ is an equivalence relation.

Assume data as in (7); see also (4) and (6). We define $\hat{E}:\hat{D}\xrightarrow{\simeq}D$. With $I\in\mathrm{Ob}(\boldsymbol{I})$, $A\in\mathrm{Ob}(\hat{\boldsymbol{C}}_I)$, choose $X_A^I=X_A\in\mathrm{Ob}(\boldsymbol{C}_I)$ and $\varepsilon_A^I=\varepsilon_A:E_IX_A\xrightarrow{\cong}A\in\mathrm{Arr}(\hat{\boldsymbol{C}}_I)$. Put $\hat{E}_IA=X_A$. For $f:A\to B\in\mathrm{Arr}(\hat{\boldsymbol{C}}_I)$, $\hat{E}_If$ is the arrow that makes the square

$$\begin{array}{ccc} E_IX_A & \xrightarrow[\cong]{\varepsilon_A} & A \\ {\scriptstyle \hat{E}_If}\downarrow & \circ & \downarrow{\scriptstyle f} \\ E_IX_B & \xrightarrow[\varepsilon_B]{\cong} & B \end{array}$$

commute. $\hat{E}_I$ so defined is a functor $\hat{E}_I:\hat{\boldsymbol{C}}_I\longrightarrow\boldsymbol{C}_I$; it is an equivalence functor; it is a quasi-inverse of E_I : we have $\varepsilon_I : E_I\hat{E}_I \xrightarrow{\cong} 1_{\hat{\mathbf{C}}_I}$ with components the ε^I_A , and $\eta_I : 1_{\mathbf{C}_I} \xrightarrow{\cong} \hat{E}_I E_I$ with components $\eta_{I,X}$ for which $E_I(\eta_{I,X}) = (\varepsilon^I_{E_I X})^{-1}$. For $i: I \to J$ in $\mathbf{I}$, we define $\hat{e}_i : F_i\hat{E}_J \xrightarrow{\cong} \hat{E}_J\hat{F}_i$ as the composite

$$F_i\hat{E}_J \xrightarrow[\cong]{\eta_J F_i \hat{E}_J} \hat{E}_J E_J F_i \hat{E}_J \xrightarrow[\cong]{\hat{E}_J e_i^{-1} \hat{E}_I} \hat{E}_J \hat{F}_i E_I \hat{E}_J \xrightarrow[\cong]{\hat{E}_J F_i \varepsilon_I} \hat{E}_J \hat{F}_i \quad .$$

$\langle \hat{e}_i \rangle_{i \in \mathrm{Arr}(\mathbf{I})}$ will be compatible with the h_ι , and give $\hat{E}: \hat{D} \xrightarrow{\simeq} D$ as desired.

Let $\mathbf{K}_0$ be the full subcategory of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ consisting of the objects O_I and A_I for all $I \in \mathrm{Ob}(\mathbf{I})$. A *restricted* context is a context over $\mathbf{K}_0$. We have $D{\restriction}\mathbf{K}_0 = D^{\#}{\restriction}\mathbf{K}_0$, and hence, for a restricted context $\mathcal{X}$, $D[\mathcal{X}] = D^{\#}[\mathcal{X}]$.

With $\mathcal{X}$ a restricted context, an *augmented* $\mathbf{I}$*-diagram of type* $\mathcal{X}$ is a pair $(D, \vec{a})$ where $D: \mathbf{I} \to \mathrm{Cat}$, and $\vec{a} \in D[\mathcal{X}]$; notations such as $(D, \vec{a})$, $(\hat{D}, \vec{b})$ denote augmented $\mathbf{I}$-diagrams. We write $E: (D, \vec{a}) \xrightarrow{\simeq} (\hat{D}, \vec{b})$ for the following: $E: D \xrightarrow{\simeq} \hat{D}$ with E as in (7) such that $E(\vec{a}) = \vec{b}$ in the obvious sense that $E_I(a_X) = b_X$. The relation $\xrightarrow{\sim}$ between augmented diagrams is defined thus:

$$(D, \vec{a}) \xrightarrow{\sim} (\hat{D}, \vec{b}) \iff \text{there exists } E: (D, \vec{a}) \xrightarrow{\simeq} (\hat{D}, \vec{b}) \quad .$$

We write $(D, \vec{a}) \xleftrightarrow{\sim} (\hat{D}, \vec{b})$ for: there exists $(\tilde{D}, \vec{c})$ such that $(D, \vec{a}) \xleftarrow{\sim} (\tilde{D}, \vec{c}) \xrightarrow{\sim} (\hat{D}, \vec{b})$. The relation $\xleftrightarrow{\sim}$ is the equivalence relation generated by $\xrightarrow{\sim}$; this can be seen directly, but it also follows from (8) below. In particular, when $\mathcal{X} = \emptyset$, the relation $\xleftrightarrow{\sim}$ coincides with $\simeq$ for $\mathbf{I}$-diagrams (since $\simeq$ is an equivalence relation).

(8) For augmented $\mathbf{I}$-diagrams $(D, \vec{a})$, $(\hat{D}, \vec{b})$ of the same type, we have

$$(D, \vec{a}) \overset{\sim}{\longleftrightarrow} (\hat{D}, \vec{b}) \quad \Longleftrightarrow \quad (D^{\#}, \vec{a}) \approx_{\boldsymbol{L}} (\hat{D}^{\#}, \vec{b}) \;;$$

here, $\boldsymbol{L} = \mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$.

As a special case, for (mere) $\boldsymbol{I}$-diagrams D and $\hat{D}$,

$$D \simeq \hat{D} \quad \Longleftrightarrow \quad D^{\#} \approx_{\boldsymbol{L}} \hat{D}^{\#} \quad .$$

Proof. (A)($\Longleftarrow$:) Let $(\mathcal{R}, r_0, r_1) : (D^{\#}, \vec{a}) \underset{\boldsymbol{L}}{\longleftrightarrow} (\hat{D}^{\#}, \vec{b})$ be a normal $\boldsymbol{L}$, $\approx$-equivalence (see 5.(2")). Let $\vec{c} \in \mathcal{R}[\mathcal{X}]$ ($\mathcal{X}$ the type of $(D^{\#}, \vec{a})$, $(\hat{D}^{\#}, \vec{b})$) for which $r_0(\vec{c}) = \vec{a}$, $r_1(\vec{c}) = \vec{b}$.

Let $M = r_0^{*}(D^{\#}) = r_1^{*}(\hat{D}^{\#})$, a standard $\boldsymbol{L}$-structure. Since $D^{\#}$ is an anadiagram, and the concept of "anadiagram" is elementary in FOLDS over $\boldsymbol{L}$, by 5.(1)(a), M is an anadiagram. Let $\tilde{D}$ be obtained from M by cleavage. We show that there is an equivalence $E : \tilde{D} \xrightarrow{\simeq} D$ which extends

$$m \restriction \boldsymbol{K}_0 = \theta_m \restriction \boldsymbol{K}_0 : M \restriction \boldsymbol{K}_0 = \tilde{D} \restriction \boldsymbol{K}_0 \to D^{\#} \restriction \boldsymbol{K}_0 = D \restriction \boldsymbol{K}_0$$

(that is, $E_I = (\theta_m)_I$ for all $I \in \mathrm{Ob}(\boldsymbol{I})$; here, we used the notation (7) for E), and similarly, there is $\hat{E} : \tilde{D} \xrightarrow{\simeq} \hat{D}$ extending $n \restriction \boldsymbol{K}_0$. In particular, it will follow that $E(\vec{c}) = \vec{a}$, $\hat{E}(\vec{c}) = \vec{b}$ and $(D, \vec{a}) \overset{\sim}{\longleftarrow} (\tilde{D}, \vec{c}) \overset{\sim}{\longrightarrow} (\hat{D}, \vec{b})$ as desired.

We use a notation for $\tilde{D}$ that is analogous to (6). The functor $E_I : \tilde{\boldsymbol{C}}_I \longrightarrow \boldsymbol{C}_I$ is defined by the effect m_{O_I} and m_{A_I} ; since $\theta_m : M \to D$ preserves the relations I_I and T_I , E_I is a functor. By the normality of r_0 , E_I induces bijections on hom-sets, and by the surjectivity of r_0 on O_I , E_I is a surjective equivalence.

Let $i : I \to J$. Looking back at how the cleavage $\tilde{D}$ was defined, we see that $\tilde{F}_i X = A_X$, with $s_X \in M\mathrm{O}_i(X, A_X)$. Then $m(s_X) \in D^{\#}\mathrm{O}_i(mX, mA_X) = D^{\#}\mathrm{O}_i(E_I X, E_J \tilde{F}_i X)$. By the definition of $D^{\#}$, this means that $ms_X : F_i E_I X \xrightarrow{\cong} E_J \tilde{F}_i X$. We put $e_{iX} = ms_X$. To see that $e_i = \langle e_{iX} \rangle_{X \in \mathrm{Ob}(\mathbf{C}_I)}$ is a natural transformation $e_i : F_i E_I \xrightarrow{\cong} E_J \tilde{F}_i$, let $f: X \to Y \in \tilde{\mathbf{C}}_I$. We see that $\tilde{F}_i f$ is defined by the property that $M(\mathrm{A}_i)(s_X, s_Y, f, \tilde{F}_i f)$ should hold. But θ_m preserves A_i ; hence, $D^{\#}(A_i)(e_{iX}, e_{iY}, E_I f, E_J \tilde{F}_i f)$, which, by the definition of $D^{\#}$, means

$$\begin{array}{ccc} F_i E_I X & \xrightarrow{\quad e_{iX} \quad} & E_J \tilde{F}_i X \\ {\scriptstyle F_i E_I f} \Big\downarrow & \bigcirc & \Big\downarrow {\scriptstyle E_J \tilde{F}_i f} \\ F_i E_I Y & \xrightarrow[\quad e_{iY} \quad]{} & E_J \tilde{F}_i Y \end{array} ,$$

which is the naturality of e_i .

Let $I \overset{i}{\underset{j}{\rightrightarrows}} J$ (with ι : $i \Downarrow j$) be given. The naturality condition on (e_i, e_j) with respect to $\iota : i \to j$ is seen as follows. Let $X \in \mathrm{Ob}(\tilde{\mathbf{C}}_I)$. The definition of the component $\tilde{h}_{\iota X} : \tilde{F}_i X \to \tilde{F}_j X$ is defined (in the process of cleavage) by the condition $M\mathrm{O}_\iota(s^i_X, s^j_X, \tilde{h}_{\iota X})$. The map $\theta_m : M \to D^{\#}$ preserves the relation O_ι . It follows that $D^{\#}\mathrm{O}_\iota(ms^i_X, ms^j_X, m\tilde{h}_{\iota X})$ holds; that is, $D^{\#}\mathrm{O}_\iota(e_{iX}, e_{jX}, E_J \tilde{h}_{\iota X})$ holds. Considering the definition of $D^{\#}\mathrm{O}_\iota$, this says that

$$\begin{array}{ccc} E_J \tilde{F}_i X & \xleftarrow{\quad e_{iX} \quad} & F_i E_I X \\ {\scriptstyle E_J \tilde{h}_{\iota X}} \Big\downarrow & \bigcirc & \Big\downarrow {\scriptstyle h_{\iota E_I X}} \\ E_J \tilde{F}_j X & \xleftarrow[\quad e_{jX} \quad]{} & F_j E_I X \end{array}$$

which is what we wanted.

(B)("only if") We show that $(D, \vec{a}) \xrightarrow{\sim} (\hat{D}, \vec{b})$ implies $(D, \vec{a}) \approx_{\boldsymbol{L}} (\hat{D}, \vec{b})$. Since $\approx_{\boldsymbol{L}}$ is an equivalence relation, the desired assertion will follow.

Suppose that $E:(D,\vec{a})\xrightarrow{\simeq}(\hat{D},\vec{b})$; E is taken in the notation in (7) ; we construct $(\mathcal{R},r_0,r_1):(D^{\#},\vec{a})\underset{\boldsymbol{L}}{\overset{\approx}{\longleftrightarrow}}(\hat{D}^{\#},\vec{b})$. The kinds of $\boldsymbol{L}$ are as in

$$\begin{array}{ccccc} \mathrm{A}_I & & & & \mathrm{A}_J \\ {\scriptstyle \mathrm{d}_I}\downarrow\ \downarrow{\scriptstyle \mathrm{c}_I} & & & & {\scriptstyle \mathrm{d}_J}\downarrow\ \downarrow{\scriptstyle \mathrm{c}_J} \\ \mathrm{O}_I & \xleftarrow[\mathrm{o}_{i0}]{} & \mathrm{O}_i & \xrightarrow[\mathrm{o}_{i1}]{} & \mathrm{O}_J \end{array}$$

with $i:I\to J$ in $\boldsymbol{I}$; we have to define $\mathcal{R}$ on these kinds.

We put $\mathcal{R}\mathrm{O}_I=\{(X,\hat{X},\sigma):X\in D\mathrm{O}_I,\ \hat{X}\in\hat{D}\mathrm{O}_I,\ \sigma:E_IX\xrightarrow{\cong}\hat{X}\}$, with $(X,\hat{X},\sigma)\overset{r_0}{\longmapsto}X$, $(X,\hat{X},\sigma)\overset{r_1}{\longmapsto}\hat{X}$. The "very surjective" condition on r_0 , r_1 at O_I holds by the essential surjectivity of E_I .

$$\mathcal{R}\mathrm{A}_I \overset{=}{\mathrm{def}} \{((X,\hat{X},\sigma),(Y,\hat{Y},\tau),\ \begin{array}{c}X\\ f\downarrow\\ Y\end{array},\ \begin{array}{c}\hat{X}\\ \hat{f}\downarrow\\ \hat{Y}\end{array}):$$

$$(X,\hat{X},\sigma),\ (Y,\hat{Y},\tau)\in\mathcal{R}\mathrm{O}_I,\ \begin{array}{ccc} E_IX & \xrightarrow{\sigma} & \hat{X} \\ {\scriptstyle E_If}\downarrow & \circ & \downarrow{\scriptstyle \hat{f}} \\ E_IY & \xrightarrow[\tau]{} & \hat{Y} \end{array}\} ,$$

with the displayed item being mapped to $(X,\hat{X},\sigma)$ by $\mathcal{R}\mathrm{d}_I$, to $(Y,\hat{Y},\tau)$ by $\mathcal{R}\mathrm{c}_I$, to f by r_0 , and to $\hat{f}$ by r_1 . The mapping

$$f\longmapsto\hat{f}:D^{\#}\mathrm{A}(X,Y)\longrightarrow\hat{D}^{\#}\mathrm{A}(\hat{X},\hat{Y})$$

so defined, with fixed $(X,\hat{X},\sigma)$, $(Y,\hat{Y},\tau)\in\mathcal{R}\mathrm{O}_I$, is a bijection; this holds since E_I is an equivalence of categories. This shows the "very surjective" condition for r_0 , r_1 at A_I , as well as the preservation of $\dot{\mathrm{E}}_{\mathrm{A}_I}$.

$$\mathcal{R}\mathrm{O}_i \overset{=}{\mathrm{def}} \{((X,\hat{X},\sigma),(A,\hat{A},\alpha),\mu):(X,\hat{X},\sigma)\in\mathcal{R}\mathrm{O}_I,\ (A,\hat{A},\alpha)\in\mathcal{R}\mathrm{O}_J,\ \mu:F_iX\xrightarrow{\cong}A\} ,$$

with the displayed item being mapped to $(X, \hat{X}, \sigma)$ by $\mathcal{R}\mathrm{o}_{i0}$, to $(A, \hat{A}, \alpha)$ by $\mathcal{R}\mathrm{o}_{i1}$, to $(X, A, \mu) \in D\mathrm{O}_i$ by r_0, and to $(\hat{X}, \hat{A}, \hat{\mu}) \in \hat{D}\mathrm{O}_i$ by r_1 where $\hat{\mu}$ is determined by the following commutativity:

$$\begin{array}{ccc}
\hat{F}_i E_I X & \xrightarrow[\cong]{\sigma} & \hat{F}_i \hat{X} \\
{\scriptstyle e_{iX}}\downarrow{\scriptstyle \cong} & & \Big| \\
E_J F_i X & \circ & \Big| \; \hat{\mu} \\
{\scriptstyle E_J \mu}\downarrow{\scriptstyle \cong} & & \downarrow \\
E_J A & \xrightarrow[\alpha]{\cong} & \hat{A}
\end{array} \quad . \qquad (9)$$

Note that since all given arrows are isomorphisms, $\hat{\mu}$ is uniquely determined, and it is an isomorphism. Moreover, since E_J is an equivalence of categories, the mapping

$$\mu \mapsto \hat{\mu} : D^{\#}\mathrm{O}_i(X, A) \longrightarrow \hat{D}^{\#}\mathrm{O}_i(\hat{X}, \hat{A})$$

so defined (with the rest of the data fixed) is a bijection, which shows the "very surjective" condition at O_i, and the preservation of $\dot{\mathrm{E}}_{\mathrm{O}_i}$.

This completes the data for $(\mathcal{R}, r_0, r_1)$; it remains to verify the necessary properties.

Let us consider the preservation of the relation A_i by $(\mathcal{R}, r_0, r_1)$. What we have to do is this. We take four items

$$x_{\mathrm{d}_i} \in \mathcal{R}\mathrm{O}_i\,,\; x_{\mathrm{c}_i} \in \mathcal{R}\mathrm{O}_i\,,\; x_{\mathrm{a}_{i0}} \in \mathcal{R}\mathrm{A}_I\,,\; x_{\mathrm{a}_{i1}} \in \mathcal{R}\mathrm{A}_J$$

such that $(x_{\mathrm{d}_i}, x_{\mathrm{c}_i}, x_{\mathrm{a}_{i0}}, x_{\mathrm{a}_{i1}}) \in \mathcal{R}[\mathrm{A}_i]$, that is,

(10) $$\begin{array}{l}
\mathcal{R}\mathrm{o}_{i0}(x_{\mathrm{d}_i}) = \mathcal{R}\mathrm{d}_I(x_{\mathrm{a}_{i0}})\,,\;\; \mathcal{R}\mathrm{o}_{i1}(x_{\mathrm{d}_i}) = \mathcal{R}\mathrm{d}_J(x_{\mathrm{a}_{i1}})\,, \\
\mathcal{R}\mathrm{o}_{i0}(x_{\mathrm{c}_i}) = \mathcal{R}\mathrm{c}_I(x_{\mathrm{a}_{i0}})\,,\;\; \mathcal{R}\mathrm{o}_{i1}(x_{\mathrm{c}_i}) = \mathcal{R}\mathrm{c}_J(x_{\mathrm{a}_{i1}})\,;
\end{array}$$

we consider their r_0 and r_1-projections; and we have to show that

$$(r_0 x_{\mathrm{d}_i}, r_0 x_{\mathrm{c}_i}\rangle, r_0 x_{\mathrm{a}_{i0}}, r_0 x_{\mathrm{a}_{i1}}) \in D^{\#}\mathrm{A}_i \tag{11}$$

if and only if

$$(r_1 x_{\mathrm{d}_i}, r_1 x_{\mathrm{c}_i}, r_1 x_{\mathrm{a}_{i0}}, r_1 x_{\mathrm{a}_{i1}}) \in \hat{D}^{\#}\mathrm{A}_i\ . \tag{12}$$

Let $x_{\mathrm{d}_i} = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu)$

with $\sigma: E_I X \xrightarrow{\cong} \hat{X}$, $\alpha: E_J A \xrightarrow{\cong} \hat{A}$, $\mu: F_i X \xrightarrow{\cong} A$;

$x_{\mathrm{c}_i} = ((Y, \hat{Y}, \tau), (B, \hat{B}, \beta), \nu)$

with $\tau: E_I Y \xrightarrow{\cong} \hat{Y}$, $\beta: E_J B \xrightarrow{\cong} \hat{B}$, $\nu: F_i Y \xrightarrow{\cong} B$;

note that $x_{\mathrm{d}_i} \xmapsto{\circ_{i0}} \sigma$, $x_{\mathrm{d}_i} \xmapsto{\circ_{i1}} \alpha$, $x_{\mathrm{c}_i} \xmapsto{\circ_{i0}} \tau$, $x_{\mathrm{c}_i} \xmapsto{\circ_{i1}} \beta$.

The first and third of the above conditions (10) force the first two components of $x_{\mathrm{a}_{i0}}$ to be $(X, \hat{X}, \sigma)$ and $(Y, \hat{Y}, \tau)$, respectively. Let

$$x_{\mathrm{a}_{i0}} = ((X, \hat{X}, \sigma), (Y, \hat{Y}, \tau), f: X \to Y, \hat{f}: \hat{X} \to \hat{Y}) \ ;$$

we have

$$\begin{array}{ccc} E_I X & \xrightarrow{\ \sigma\ } & \hat{X} \\ {\scriptstyle E_I f}\downarrow & \circ & \downarrow{\scriptstyle \hat{f}} \\ E_I Y & \xrightarrow[\ \tau\ ]{} & \hat{Y} \end{array} \ . \tag{13}$$

Similarly,

$$x_{\mathrm{a}_{i1}} = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g: A \to B, \hat{g}: \hat{A} \to \hat{B})$$

with

$$
\begin{array}{ccc}
E_J A & \xrightarrow{\alpha} & \hat{A} \\
{\scriptstyle E_J g}\downarrow & \circ & \downarrow{\scriptstyle \hat{g}} \\
E_J B & \xrightarrow[\beta]{} & \hat{B}
\end{array} . \tag{14}
$$

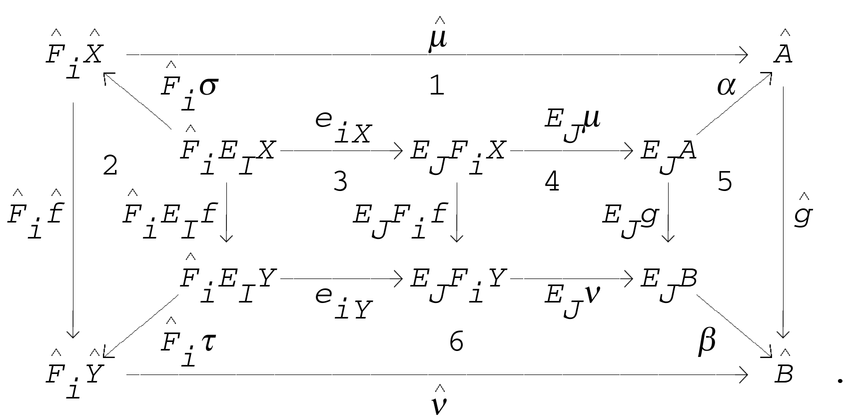
(11) means

$$
\begin{array}{ccc}
F_i X & \xrightarrow{\mu} & A \\
\downarrow{\scriptstyle F_i f} & \circ & \downarrow{\scriptstyle g} \\
F_i Y & \xrightarrow[\nu]{} & B
\end{array} , \tag{15}
$$

whereas (12) means

$$
\begin{array}{ccc}
\hat{F}_i \hat{X} & \xrightarrow{\hat{\mu}} & \hat{A} \\
\downarrow{\scriptstyle \hat{F}_i \hat{f}} & \circ & \downarrow{\scriptstyle \hat{g}} \\
\hat{F}_i \hat{Y} & \xrightarrow[\hat{\nu}]{} & \hat{B}
\end{array} \tag{16}
$$

where $\hat{\mu}$ and $\hat{\nu}$ are defined as $\hat{\mu}$ is in (9); we want to see that (15) iff (16). Consider the following diagram:



The cells 1 and 6 commute, by the definitions of $\hat{\mu}$ and $\hat{\nu}$ (see (9)). 2 commutes by (13), 3 by the naturality of e_i, and 5 by (14). Note that all arrows except the vertical ones are isomorphisms. If (15) commutes, then so does 4 ; the resulting commutativity of the outside square is (16) as desired. Conversely, if (16) commutes, then so does 4 (using the isomorphisms in the diagram), and since E_J is faithful, so does (15).

Let us look at the similar verification of preservation of $\bigcirc_\iota$; $I \underset{j}{\overset{i}{\rightrightarrows}} J$. We take $(x_{\bigcirc_{\iota 0}}, x_{\bigcirc_{\iota 1}}, x_{\bigcirc_{\iota 2}}) \in \mathcal{R}[\bigcirc_\iota]$, that is,

$$x_{\bigcirc_{\iota 0}} = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu) \in \mathcal{R}\bigcirc_i$$

with $\sigma : E_I X \xrightarrow{\cong} \hat{X}$, $\alpha : E_J A \xrightarrow{\cong} \hat{A}$, $\mu : F_i X \xrightarrow{\cong} A$;

$$x_{\bigcirc_{\iota 1}} = ((X, \hat{X}, \sigma), (B, \hat{B}, \beta), \rho) \in \mathcal{R}\bigcirc_j$$

with the same σ as above, and $\beta : E_J B \xrightarrow{\cong} \hat{B}$, $\rho : F_i X \xrightarrow{\cong} B$

(since we must have $\mathcal{R}\bigcirc_{i0}(x_{\bigcirc_{\iota 0}}) = \mathcal{R}\bigcirc_{j0}(x_{\bigcirc_{\iota 1}})$ (see the first equation in (4)), the first components of $x_{\bigcirc_{\iota 0}}$ and $x_{\bigcirc_{\iota 1}}$ have to agree);

$$x_{\bigcirc_{\iota 2}} = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g : A \to B, \hat{g} : \hat{A} \to \hat{B})$$

with (9) holding (see the other two equations in (4)). Looking at the definition of $D^{\#}\bigcirc_\iota$, $\hat{D}^{\#}\bigcirc_\iota$, what we have to see is that

$$\begin{array}{ccc} F_i X & \xrightarrow[\cong]{\mu} & A \\ {\scriptstyle h_{\iota X}}\downarrow & \bigcirc & \downarrow{\scriptstyle g} \\ F_j X & \xrightarrow[\rho]{\cong} & B \end{array} \qquad \text{iff} \qquad \begin{array}{ccc} \hat{F}_i \hat{X} & \xrightarrow[\cong]{\hat{\mu}} & \hat{A} \\ {\scriptstyle \hat{h}_{\iota X}}\downarrow & \bigcirc & \downarrow{\scriptstyle \hat{g}} \\ \hat{F}_j \hat{X} & \xrightarrow[\rho]{\cong} & \hat{B} \end{array} \quad . \tag{17}$$

Consider

$$
\begin{array}{ccccccc}
\hat{F}_i\hat{X} & & & \xrightarrow{\quad\hat{\mu}\quad} & & & \hat{A} \\
 & \nwarrow{\scriptstyle \hat{F}_i\sigma} & & 1 & & \nearrow{\scriptstyle \alpha} & \\
 & 2 & \hat{F}_iE_IX & \xrightarrow{e_{iX}} E_JF_iX \xrightarrow{E_J\mu} & E_JA & 5 & \\
{\scriptstyle \hat{h}_\iota\hat{X}}\Big\downarrow & & {\scriptstyle \hat{h}_\iota E_IX}\Big\downarrow & 3 \quad {\scriptstyle E_Jh_\iota X}\Big\downarrow \quad 4 & \Big\downarrow{\scriptstyle E_Jg} & & \Big\downarrow{\scriptstyle \hat{g}} \\
 & & \hat{F}_jE_IX & \xrightarrow{e_{jX}} E_JF_jX \xrightarrow{E_J\rho} & E_JB & & \\
 & \swarrow{\scriptstyle \hat{F}_j\tau} & & 6 & & \searrow{\scriptstyle \beta} & \\
\hat{F}_j\hat{X} & & & \xrightarrow[\quad\hat{\rho}\quad]{} & & & \hat{B}
\end{array}
\quad .
$$

The cells 1 and 6 commute for reasons as before. 2 commutes because of the naturality of $\hat{h}_\iota$, 3 because of the naturality of (e_i, e_j) with respect to $\iota : i \to j$, 5 because of (14). 4 is the antecedent of (17) with E_J applied to it, the outer square is the succedent of (17). The assertion in (17) follows.

The remaining properties are the preservation of the T_I, I_I, and of the equalities on the A_I, O_i. These are immediately seen.

We need that $(\mathcal{R}, r_0, r_1)$ "relates $\vec{a}$ to $\vec{b}$". For $\mathcal{X}$ the restricted context involved, $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}$, $\vec{b}=\langle b_x\rangle_{x\in\mathcal{X}}$; we want $\vec{c}=\langle c_x\rangle_{x\in\mathcal{X}}\in\tilde{D}[\mathcal{X}]$ such that $r_0(\vec{c})=\vec{a}$, $r_1(\vec{c})=\vec{b}$. For $x\in\mathcal{X}$, $\mathrm{K}_x=\mathrm{O}_I$, define $c_x=1_{E_Ia_x}: E_Ia_x \xrightarrow{\cong} b_x \in \mathcal{R}\mathrm{O}_I$; we have $r_0c_x=a_x$, $r_1c_x=b_x$. For $x\in\mathcal{X}$, $x{:}\mathrm{A}_I(y, z)$, define $c_x=(c_y, c_z, a_x, b_x)\in\mathcal{R}\mathrm{A}_I$; $c_x\in\mathcal{R}\mathrm{A}_I$ indeed holds since this means

$$
\begin{array}{ccc}
E_I(a_y) & \xrightarrow{c_y} & b_y \\
{\scriptstyle E_I(a_x)}\Big\downarrow & \circ & \Big\downarrow{\scriptstyle b_x} \\
E_I(a_z) & \xrightarrow[c_z]{} & b_z
\end{array}
$$

and this holds since $E_I(a_x)=b_x$; also, $r_0(c_x)=a_x$, $r_1(c_x)=b_x$.

This completes the proof of (8).