§4. The syntax of first-order logic with dependent sorts as a fibration

Let $\mathbf{L}$ be a DSV; let $\mathbf{K}$ be the full subcategory of the kinds. Consider the category $\mathbf{B}=\mathbf{B}_{\mathbf{K}}$ which is the free finite-limit completion of $\mathbf{K}$: $i:\mathbf{K}\to\mathbf{B}$, and for any category $\mathbf{S}$ with finite limits, $i^*:\mathrm{Lex}(\mathbf{B},\mathbf{S})\longrightarrow\mathrm{Fun}(\mathbf{K},\mathbf{S})$ is an equivalence of categories ($\mathrm{Lex}(\mathbf{B},\mathbf{S})$ is the category of left exact functors $\mathbf{B}\to\mathbf{S}$, $\mathrm{Fun}(\mathbf{K},\mathbf{S})=\mathbf{S}^{\mathbf{K}}$ the category of all functors $\mathbf{K}\to\mathbf{S}$, i^* is defined as composition with i).

It is well-known that for any (small) category $\mathbf{K}$, $\mathbf{B}_{\mathbf{K}}$ can be given as $(\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}}))^{\mathrm{op}}$ ($\mathrm{Fp}(\mathbf{M})$ is the full subcategory of finitely presentable objects of $\mathbf{M}$), with $i:\mathbf{K}\subset\mathbf{B}$ the functor $i:\mathbf{K}\to(\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}}))^{\mathrm{op}}$ induced by Yoneda. (The small-colimit completion of $\mathbf{A}$ is $Y:\mathbf{A}\to\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})}$; the finite-colimit completion of $\mathbf{A}$ is $Y:\mathbf{A}\to\mathrm{Fp}(\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})})$; therefore, the finite limit completion of $\mathbf{A}^{\mathrm{op}}$ is $Y:\mathbf{A}^{\mathrm{op}}\to(\mathrm{Fp}(\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})}))^{\mathrm{op}}$).

Now, for any simple category $\mathbf{K}$, $\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}})$ is the category of *finite* functors $\mathbf{K}\to\mathrm{Set}$; a functor $F:\mathbf{K}\to\mathrm{Set}$ is finite if $\mathrm{El}(F)=\{(K,a):K\in\mathrm{Ob}(\mathbf{K}),\ a\in FK\}$ is a finite set. Namely, each finite functor is finitely presentable, the finite functors are closed under finite colimits in $\mathrm{Set}^{\mathbf{K}}$, and every functor is the filtered colimit of the collection of its finite subfunctors (the latter uses that $\mathbf{K}$ has finite fan-out); this suffices.

Thus, $\mathbf{B}$ can be taken to be the opposite of the category $\mathrm{Fin}(\mathrm{Set}^{\mathbf{K}})$ of finite functors $\mathbf{K}\to\mathrm{Set}$; the canonical functor $i:\mathbf{K}\to\mathbf{B}$ is (induced by) Yoneda.

Let $\mathrm{Con}[\mathbf{K}]$ be the category whose objects are the contexts (of variables over $\mathbf{K}$), and whose arrows are the specializations. I **claim** that

$$\mathrm{Con}[\mathbf{K}]\simeq\mathrm{Fin}(\mathrm{Set}^{\mathbf{K}})\ .$$

Let $F:\mathbf{K}\longrightarrow\mathrm{Set}$ be a finite functor. I define a mapping

$$(K, a) \longmapsto y^F_{K, a} : \mathrm{El}(F) \longrightarrow \mathrm{VAR}$$

into the class VAR of variables as follows:

$$y^F_{K, a} \stackrel{=}{\mathrm{def}} \langle 2, Y^F_{K, a}, a\rangle$$

where

$$Y^F_{K, a} = K(\langle y^F_{K_p}, (Fp)(a)\rangle_{p\in K|\boldsymbol{K}}) \ .$$

This is a legitimate definition by recursion on the level of K. $Y^F_{K, a}$ is a type; this requires that

$$K_p(\langle y^F_{K_{qp}}, (F(qp))(a)\rangle_{q\in K_p|\boldsymbol{K}}) = Y^F_{K_p}, (Fp)(a) ,$$

which is true since $(F(qp))(a) = (Fq)((Fp)(a))$. Hence, $y^F_{K, a}$ is a variable.

We let $\mathcal{Y}_F \stackrel{=}{\mathrm{def}} \{y^F_{K, a} : (K, a)\in \mathrm{El}(F)\}$. It is immediate that $\mathcal{Y}_F$ is a context. We have a bijection

$$(K, a) \longmapsto y^F_{K, a} : \mathrm{El}(F) \xrightarrow{\cong} \mathcal{Y}_F \ .$$

If $h:F\longrightarrow G$ is a natural transformation, we let $s=s_h:\mathcal{Y}_F\rightarrow\mathcal{Y}_G$ be defined by $s(y^F_{K, a})=y^G_{K, h_K(a)}$. s is a specialization: this requires that $Y^F_{K, a}|s = Y^G_{K, h_K(a)}$, which is the same as $h_{K_p}((Fp)(a)) = (Gp)(h_K a)$ $(p:K\rightarrow K_p)$, which holds by the naturality of h. It is immediate that we have a bijection

$$h \longmapsto s_h : \mathrm{Nat}(F, G) \xrightarrow{\cong} \mathrm{Spec}(\mathcal{Y}_F, \mathcal{Y}_G) \ .$$

Also, if $F\xrightarrow{h}G\xrightarrow{k}H$, then $s_{kh} = s_k \circ s_h$, and $s_{1_F}=1_{\mathcal{Y}_F}$.

Thus far, we have seen that we have the full and faithful functor

$$\begin{array}{c}F\\ \downarrow h\\ G\end{array} \longmapsto s_{h}\begin{array}{c}\mathcal{Y}_F\\ \downarrow\\ \mathcal{Y}_G\end{array} : \mathrm{Fin}(\mathrm{Set}^{\boldsymbol{K}}) \longrightarrow \mathrm{Con}[\boldsymbol{K}] \tag{1}$$

Now, given a context $\mathcal{Z}$, define $F=F_{\mathcal{Z}}:\boldsymbol{K}\to\mathrm{Set}$ by $FK=\{z\in\mathcal{Z}:\mathrm{K}_z=K\}$, and $Fp:FK\to F\mathrm{K}_p$ by $(Fp)(z)=\mathrm{x}_{z,p}$; F is a finite functor. Moreover, we have the map

$$s : z \longmapsto \mathrm{y}^F_{\mathrm{K}_z, z} : \mathcal{Z}\longrightarrow\mathcal{Y}_F \;;$$

s is a specialization since

$$z : \mathrm{K}_z(\langle \mathrm{x}_{z,p}\rangle_{p\in K|\boldsymbol{K}}) , \qquad\qquad \mathrm{y}^F_{\mathrm{K}_z, z} : \mathrm{K}_z\{\langle \mathrm{y}^F_{\mathrm{K}_p, (Fp)(z)}\rangle_{p\in \mathrm{K}_z|\boldsymbol{K}} ,$$

and $s(\mathrm{x}_{z,p}) = \mathrm{y}^F_{\mathrm{K}_z, \mathrm{x}_{z,p}} = \mathrm{y}^F_{\mathrm{K}_p, (Fp)(z)}$, by the definition of F.

It is clear that s is a bijection, i.e., an isomorphism in $\mathrm{Con}[\boldsymbol{K}]$.

We have verified that (1) is an equivalence of categories, thus our **claim**.

It is easy to see that the image of (1) consists of those contexts $\mathcal{Z}$ for which $z\in\mathcal{Z} \mapsto (\mathrm{K}_z, \mathrm{a}(z))$ is a 1-1 function.

It is clear that although the categories $\mathrm{Fin}(\mathrm{Set}^{\boldsymbol{K}})$, $\mathrm{Con}[\boldsymbol{K}]$ are large, they are essentially small.

Thus, $\boldsymbol{B}$, the free finite-limit completion of $\boldsymbol{K}$, can be taken to be the opposite of the category $\mathrm{Con}[\boldsymbol{K}]$ of contexts with specializations as arrows. To describe the canonical embedding $i:\boldsymbol{K}\to\boldsymbol{B}$ under the latest construal of the completion $\boldsymbol{B}$, let us define, for any $K\in\boldsymbol{K}$, the context

$$\mathcal{X}_K \stackrel{=}{\mathrm{def}} \{x^K_p : p\in K|\boldsymbol{K}\} \tag{2}$$

for which $X = \mathrm{X}_K \overset{=}{\mathrm{def}} K(\langle \mathrm{x}_p^K \rangle_{p\in K|\boldsymbol{K}})$ is a sort, and $\mathrm{a}(X)=\langle p \rangle_{p\in K|\boldsymbol{K}}$. In the definition of $\mathcal{X}_K$, the only essential points are that $K(\langle \mathrm{x}_p^K \rangle_{p\in K|\boldsymbol{K}})$ is a sort, and that the mapping $p \mapsto \mathrm{x}_p^K$ is 1-1. X_K is "the most general sort" of the kind K; every other such sort is of the form $\mathrm{X}_K|s$ for some specialization s with domain $\mathcal{X}_K$. Further, let

$$\mathcal{X}^*_K \overset{=}{\mathrm{def}} \mathcal{X}_K \dot{\cup} \{\mathrm{x}_K\} \ ,$$

where $\mathrm{x}_K : \mathrm{X}_K$, and, for the sake of definiteness, x_K is taken to be the specific variable for which $\mathrm{a}(\mathrm{x}_K)=1_K$. Note that under the equivalence $F \mapsto \mathcal{Y}_F$ between finite functors and contexts, $\mathcal{X}^*_K$ is the context that corresponds to the representable functor $\boldsymbol{K}(K,-):\boldsymbol{K}\to \mathrm{Set}$.

When a context $\mathcal{X}$ is considered an object of $\boldsymbol{B} = (\mathrm{Con}[\boldsymbol{K}])^{\mathrm{op}}$, it is written as $[\mathcal{X}]$. Arrows $s:\mathcal{X}\to\mathcal{Y}$ of $\mathrm{Con}[\boldsymbol{K}]$ correspond to arrows $[s]:[\mathcal{Y}]\to[\mathcal{X}]$.

The canonical embedding $i:\boldsymbol{K}\to\boldsymbol{B}$ (having the universal property of the finite limit completion) has $i(K) = [\mathcal{X}^*_K]$.

The morphism $p:K\to K_p$ is taken by i to the arrow

$$[s_p]:[\mathcal{X}^*_K] \longrightarrow [\mathcal{X}^*_{\mathrm{K}_p}]$$

for the specialization

$$s_p:\mathcal{X}^*_{\mathrm{K}_p} \to \mathcal{X}^*_K : \ \mathrm{x}_q^{\mathrm{K}_p} \overset{s}{\longmapsto} \mathrm{x}_{qp}^K \ \ (q:\mathrm{K}_p\to\mathrm{K}_q) \ , \ \ \mathrm{x}_{\mathrm{K}_p} \overset{s}{\longmapsto} \mathrm{x}_p^K \ . \qquad (3)$$

Note that in the category $\boldsymbol{B}$, the object $[\mathcal{X}_K]$ is the same as $i[K]$ for the "$\boldsymbol{B}$-valued $\boldsymbol{K}$-structure $i:\boldsymbol{K}\to\boldsymbol{B}$", that is, the limit of the composite $K{\downarrow}(\boldsymbol{K}-\{K\}) \xrightarrow{\Phi} \boldsymbol{K} \xrightarrow{i} \boldsymbol{B}$.

We single out four classes of arrows in $\mathrm{Con}[\boldsymbol{K}]$, $\mathcal{Q}_0\subset\mathcal{Q}_1\subset\mathcal{Q}_2\subset\mathcal{Q}_3$. $\mathcal{Q}_0$ consists of the inclusion-arrows $\mathrm{incl}:\mathcal{X}_K\to\mathcal{X}^*_K$, where K ranges over the kinds. $\mathcal{Q}_1$ consists of the inclusion-arrows of the form $\mathrm{incl}:\mathcal{X}\to\mathcal{X}\dot{\cup}\{x\}$, where $\mathcal{X}$ is any (finite) context, and

$\mathcal{X}\dot\cup\{x\}$ is also a context (for this, it is necessary and sufficient that $x\notin\mathcal{X}$ and $\mathtt{Dep}(x)\subset\mathcal{X}$). $\mathcal{Q}_2$ is the class of all 1-1 arrows $i:\mathcal{X}\to\mathcal{Y}$ where $\mathtt{card}(\mathcal{Y})=\mathtt{card}(\mathcal{X})+1$. Finally, $\mathcal{Q}_3$ is the class of all 1-1 arrows $\mathcal{X}\to\mathcal{Y}$.

Every time $s:\mathcal{Y}\to\mathcal{X}$ is a specialization, and $t:\mathcal{Y}\dot\cup\{y\}\to\mathcal{X}\dot\cup\{x\}$ extends s, with $t(y)=x$, we have the pushout diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\mathtt{incl}} & \mathcal{Y}\dot\cup\{y\} \\ s\downarrow & & \downarrow t \\ & \square & \\ \mathcal{X} & \xrightarrow[\mathtt{incl}]{} & \mathcal{X}\dot\cup\{x\} \end{array} \qquad (4)$$

in $\mathtt{Con}[\boldsymbol{K}]$. All arrows in $\mathcal{Q}_1$ are pushouts of ones in $\mathcal{Q}_0$. To see this, for a given $\mathcal{X}\xrightarrow{\mathtt{incl}}\mathcal{X}\dot\cup\{x\}$, apply (4) to $\mathcal{X}_{K_x}\xrightarrow{\mathtt{incl}}\mathcal{X}^{*}_{K_x}$ as $\mathcal{Y}\xrightarrow{\mathtt{incl}}\mathcal{Y}\dot\cup\{y\}$, and $s:\mathcal{X}_{K_x}\to\mathcal{X}$ given by $s(x_p^{K_x}) = x_{x,p}$.

It is clear that $\mathcal{Q}_2$ is the closure of $\mathcal{Q}_1$ under isomorphisms (meaning that $q:A\to B\in\mathcal{Q}_2$ iff there is $q':A'\to B'\in\mathcal{Q}_1$ with some commutative

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ \cong\downarrow & \circ & \downarrow\cong \\ A' & \xrightarrow[q']{} & B' \end{array} \text{).}$$

(4) shows that any arrow $q:A\to B$ in $\mathcal{Q}_1$ has a pushout along any $a:A\to A'$ that is again in $\mathcal{Q}_1$. Thus, $\mathcal{Q}_2$ is closed under pushout, and in fact it is the closure of $\mathcal{Q}_0$ under pushout.

$\mathcal{Q}_3$ is the closure of $\mathcal{Q}_2$ under composition. Indeed, given any inclusion $i:\mathcal{X}\to\mathcal{Y}$, there is a finite sequence $\mathcal{X}=\mathcal{X}_0\subset\mathcal{X}_1\subset\ldots\subset\mathcal{X}_{n-1}\subset\mathcal{X}_n=\mathcal{Y}$ of contexts such that $\mathtt{card}(\mathcal{X}_{i+1})=\mathtt{card}(\mathcal{X}_i)+1$; enumerate $\mathcal{Y}-\mathcal{X}$ as $\langle y_i\rangle_1^n$ such that the level of K_{y_i} is non-increasing, and put $\mathcal{X}_i=\mathcal{X}\cup\{x_1,\ldots x_i\}$. This shows that every inclusion $i:\mathcal{X}\to\mathcal{Y}$ is the composite of arrows in $\mathcal{Q}_1$; since every 1-1 arrow is isomorphic to an inclusion, the assertion follows.

Without talking about syntax, $[\mathcal{Q}_0] = \{[q] : q\in\mathcal{Q}_0\}$ may be described as the class of arrows of the form $q: iK\to[K]$, where $K\in\boldsymbol{K}$, $i:\boldsymbol{K}\to\boldsymbol{B}=(\mathrm{Fin}(\mathrm{Set}^{\boldsymbol{K}}))^{\mathrm{op}}$ is induced by Yoneda, and $[K]$ is the limit of the composite $K\downarrow(\boldsymbol{K}-\{1_K\})\xrightarrow{\Phi}\boldsymbol{K}\xrightarrow{i}\boldsymbol{B}$. $[\mathcal{Q}_2]$ is the closure of $[\mathcal{Q}_0]$ under pullback. $[\mathcal{Q}_3]$ is the class of all epimorphisms; also, it is the closure of $[\mathcal{Q}_2]$ under composition.

For the purposes for logic without equality, we let the class $\mathcal{Q}^{\neq}$ of arrows in $\boldsymbol{B}$ be either $[\mathcal{Q}_2]$ ($=\{[q]:q\in\mathcal{Q}_2\}$ or $[\mathcal{Q}_3]$; both $[\mathcal{Q}_2]$ and $[\mathcal{Q}_3]$ are closed under pullback, and the second class is the closure of the first under composition. (According the remarks at the end of the last section, the two possible choices are essentially equivalent).

Corresponding to logic with equality, we have $\mathcal{Q}^{=}$, which is obtained by adding to $\mathcal{Q}^{\neq}$ all isomorphic copies of arrows of the form $[p]$ for p of the form $p:\mathcal{X}\dot{\cup}\{x, y\}\to\mathcal{X}\dot{\cup}\{x\}$ such that x and y are distinct variables of the same type, their kind is a maximal one, and p is defined so that $p\restriction\mathcal{X}$ is the identity and $p(x)=p(y)=x$. Categorically, if we put $A=[\mathcal{X}]$, $B=[\mathcal{X}\dot{\cup}\{x\}]$, and $q:B\to A$, $q=[\mathrm{incl}]$, we have $[p]=\delta=B\to B\times_A B$, the diagonal.

If $s:\mathcal{X}\dot{\cup}\{x, y\}\longrightarrow\mathcal{Y}$, then for $x'=s(x)$, $y'=s(y)$ and $\mathcal{X}'=\mathcal{Y}-\{x', y'\}$, $\mathcal{X}'$ is a context, since no variable z can have $x'\in\mathrm{Dep}(z)$ or $y'\in\mathrm{Dep}(z)$, by the maximality assumption on the kind of x and y; $\mathcal{Y}=\mathcal{X}'\dot{\cup}\{x', y'\}$. We have a pushout

$$\begin{array}{ccc}
\mathcal{X}\dot{\cup}\{x, y\} & \xrightarrow{\;p\;} & \mathcal{X}\dot{\cup}\{x\} \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\
\mathcal{X}'\dot{\cup}\{x', y'\} & \xrightarrow[p']{} & \mathcal{X}'\dot{\cup}\{x'\}
\end{array}$$

with the evident p' and t. It follows that all pullbacks of the additional arrows in $\mathcal{Q}^{=}$ are again of the same form, thus $\mathcal{Q}^{=}$ is closed under pullback. Also, all the additional arrows in $\mathcal{Q}^{=}$ are pullbacks of the specific ones $[p_K]$ where K is a maximal kind, $p_K:\mathcal{X}_K\dot{\cup}\{x_K, y\}\longrightarrow\mathcal{X}_K\dot{\cup}\{x_K\}$; here, $\mathcal{X}_K\dot{\cup}\{x_K\}=\mathcal{X}_K^{*}$ defined above, *etc.*

Suppose $T=(\boldsymbol{L},\Sigma)$ is a theory; there are six possibilities for the logic: coherent, intuitionistic, or classical, each with or without equality. We define a fibration

$\mathcal{C} = [T] = \mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}}$, with a set $\mathcal{Q}=\mathcal{Q}_{\mathcal{C}}$ of distinguished (quantifiable) arrows in $\boldsymbol{B}$. $\boldsymbol{B}$ has been given in the foregoing; we use $\mathcal{Q}^{\neq}$ when we exclude equality, $\mathcal{Q}^{=}$ otherwise, as $\mathcal{Q}$.

A *formula-in-a-context* is an ordered pair $(\mathcal{X},\varphi)$, written as $[\mathcal{X}:\varphi]$, such that $\mathcal{X}$ is a context, and φ is a formula with $\mathrm{Var}(\varphi)\subset\mathcal{X}$. With a given $\mathcal{X}$, $[\mathcal{X}:\varphi]$ is called a *formula-over* $\mathcal{X}$.

To define $\mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}}$, for $[\mathcal{X}]\in\boldsymbol{B}$, the fiber $\mathcal{C}^{[\mathcal{X}]}$ is given as the set of equivalence classes $[\mathcal{X}:\varphi]/\sim_{\mathcal{X}}$ of formulas-over $\mathcal{X}$ under the equivalence relation

$$[\mathcal{X}:\varphi]\sim_{\mathcal{X}}[\mathcal{X}:\psi] \quad\Longleftrightarrow\quad T\vdash\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi \text{ and } T\vdash\psi\underset{\mathcal{X}}{\Longrightarrow}\varphi$$

(the range of the formulas φ, ψ, and the deducibility relation $\vdash$ is understood according to the logic in question). In what follows, we will write $[\mathcal{X}:\varphi]$ for $[\mathcal{X}:\varphi]/\sim_{\mathcal{X}}$. $\mathcal{C}^{[\mathcal{X}]}$ is partially ordered by

$$[\mathcal{X}:\varphi]\leq_{\mathcal{X}}[\mathcal{X}:\psi] \quad\Longleftrightarrow\quad T\vdash\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi\,;$$

by the rules (Taut) and (Cut) this is well-defined and it is a partial order. Finally, for $s:\mathcal{X}\to\mathcal{Y}$ in $\mathrm{Con}[\boldsymbol{K}]$, that is, $[s]:[\mathcal{Y}]\longrightarrow[\mathcal{X}]$, $[s]^{*}([\mathcal{X}:\varphi])\underset{\text{def}}{=}[\mathcal{Y}:\varphi|s]$. By the rule (Subst), $[s]^{*}:\mathcal{C}^{\mathcal{X}}\longrightarrow\mathcal{C}^{\mathcal{Y}}$ is a map of posets.

Since $(\varphi|s)|t = \varphi|ts$, and $\varphi|\mathrm{id}=\varphi$, we have a (pseudo)functor $\mathcal{X}\mapsto\mathcal{C}^{X}$, $([\mathcal{X}]\xrightarrow{[s]}[\mathcal{Y}])\mapsto[s]^{*}$; thus, we have a fibration. The rules for connectives (not counting the last two) make sure that each fiber has the necessary (propositional) structure, where each operation is given by the corresponding syntactic operation on formulas; e.g., $[\mathcal{X}:\varphi]\wedge_{[\mathcal{X}]}[\mathcal{X}:\psi]=[\mathcal{X}:\varphi\wedge\psi]$.

For $[i]:[\mathcal{X}\dot{\cup}\{x\}]\longrightarrow[\mathcal{X}]$ ($i:\mathcal{X}\longrightarrow\mathcal{X}\dot{\cup}\{x\}$ the inclusion) in $[\mathcal{Q}_1]$ and

$[\mathcal{X}\dot\cup\{x\}:\varphi]\in\mathcal{C}^{[\mathcal{X}\dot\cup\{x\}]}$, we have $\exists_{[i]}([\mathcal{X}\dot\cup\{x\}:\varphi]) = [\mathcal{X}:\exists x\varphi]\in\mathcal{C}^{[\mathcal{X}]}$, and similarly for $\forall$ in place of $\exists$. This follows from the rules $(\exists)$ and $(\forall)$. As we pointed out in Section 2 , if $\mathrm{Var}(\varphi)\subset\mathcal{X}\dot\cup\{x\}$, then $\forall x\varphi$, $\exists x\varphi$ are well-formed. Since every arrow q in $[\mathcal{Q}_3]$ is an isomorphic copy of a composite of arrows in $[\mathcal{Q}_1]$, the operation $\exists_q$, or $\forall_q$, will be well-defined, and can be expressed in terms of $\exists_r$, or $\forall_r$, for $r\in[\mathcal{Q}_1]$.

In the case of logic with equality, we have, for

$$\delta:[\mathcal{X}\dot\cup\{x\}]\xrightarrow{[p]}[\mathcal{X}\dot\cup\{x,y\}] ,$$

an additional arrow in $\mathcal{Q}^{=}$, $\exists_\delta(\mathbf{t}_{[\mathcal{X}\dot\cup\{x\}]}) = [\mathcal{X}\dot\cup\{x,y\}:x=_{\mathcal{X}}y]$, and more generally, $\exists_{[\delta]}([\mathcal{X}\dot\cup\{x\}:\varphi]) = [\mathcal{X}\dot\cup\{x,y\}:\ x=_{\mathcal{X}}y\wedge\varphi]$. This is F. W. Lawvere's observation on the definition of equality in hyperdoctrines [L2]. The claimed equality can be deduced by using the rules of equality. We also have that

$$\forall_{[\delta]}([\mathcal{X}\dot\cup\{x\}:\varphi]) = [\mathcal{X}\dot\cup\{x,y\}:\ x=_{\mathcal{X}}y\longrightarrow\varphi] .$$

The fact that substitution is compatible with the logical operations gives that for any specialization $s:\mathcal{Y}\to\mathcal{X}$, $[s]^*:\mathcal{C}^{\mathcal{X}}\longrightarrow\mathcal{C}^{\mathcal{Y}}$ preserves the (propositional) structure, and that the Beck-Chevalley conditions are fulfilled. We obtain a $\wedge\vee\exists$-fibration, a $\wedge\vee\to\exists\forall$-fibration and a $\wedge\vee\neg\exists$-fibration in the respective cases of coherent logic, intuitionistic logic and classical logic; the presence of the rules $(\wedge\vee)$, $(\wedge\exists)$ ensures this in the coherent case, and that of $(\neg)$ in the classical case.

The construction $[T]$ has the universal property of the fibration of the appropriate kind that is freely generated by T . In what follows, we describe this universal property in a somewhat incomplete way, namely, for "target" fibrations of the form $\mathcal{P}(\boldsymbol{C})$, rather than arbitrary (suitably structured) fibrations.

For a relation $R\in\mathrm{Rel}(\boldsymbol{L})$, we make a definition of the context $\mathcal{X}_R$ analogously to $\mathcal{X}_K$ in (2) : $\mathcal{X}_R \overset{=}{\mathrm{def}} \{x^R_p : p\in R|\boldsymbol{L}\}$ such that $\bar R \overset{=}{\mathrm{def}} R(\langle x^R_p\rangle_{p\in R|\boldsymbol{L}})$ is a well-formed atomic formula, and $\alpha(X)=\langle p\rangle_{p\in R|\boldsymbol{L}}$. $\bar R$ is the "most general" atomic formula using the relation

R. Moreover, for $p \in R|\boldsymbol{L}$, we let

$$s_p : \mathcal{X}^*_{K_p} \to \mathcal{X}^*_R : \quad x^{K_p}_q \longmapsto x^R_{qp} \quad (q: K_p \to K_q) \,, \quad x_{K_p} \longmapsto x^R_p \,.$$

Changing the meaning of the symbol $\mathrm{Mod}_{\mathcal{D}}(\mathcal{C})$, let us use the notation now in the variable sense of either of $\wedge\vee\exists(\mathcal{C},\mathcal{D})$, $\wedge\vee\to\exists\forall(\mathcal{C},\mathcal{D})$, $\wedge\vee\neg\exists(\mathcal{C},\mathcal{D})$ as the context requires it, according to which logic we are dealing with. In what follows, $\boldsymbol{C}$ is a category having enough structure for the logic at hand: it is a coherent, a Heyting or a Boolean category in the three respective cases.

We have a "forgetful" functor

$$(\ )^- : \mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T]) \longrightarrow \mathrm{Mod}_{\boldsymbol{C}}(T) \tag{5}$$

defined as follows. Given $P=(P_1, P_2) \in \mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T])$, we define $P^- : \boldsymbol{L} \to \boldsymbol{C}$, $P^- \in \mathrm{Mod}_{\boldsymbol{C}}(T)$, by $P^-(K) = P_1([\mathcal{X}^*_K])$; for $p: K \to K_p$, $P^-(p) = P_1([s_p])$ (see (3)) (more briefly, $P^- \restriction \boldsymbol{K} = P_1 \circ i$, for the canonical embedding $i : \boldsymbol{K} \to \boldsymbol{B}$); for $R \in \mathrm{Rel}(\boldsymbol{L})$, $P^-(R)$ the domain of a monomorphism m representing the subobject $P_2([\mathcal{X}_R : \bar{R}])$ of $P_1([\mathcal{X}_R])$; and for $p: R \to K_p$, $P^-(p) = P_1([s_p]) \circ m$.

For $h: P \to Q$ in $\mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T])$ (that is, $h: P_1 \to Q_1$ with properties), $h^- = h \circ i$; it is easy to see that h^- is an arrow $P^- \to Q^-$.

In the case of coherent logic, the functor (5) is full and faithful, and in the case of intuitionistic and classical logics,

$$(\ )^- : \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}([T]) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \ , \tag{6}$$

with both categories restricted to have only isomorphisms as arrows (thus, they are groupoids), is full and faithful. The faithfulness is obvious; the fullness requires an easy proof by induction on the complexity of formulas.

In fact, in the case of coherent logic, (5), and in the other two cases, (6), is an equivalence of categories. Indeed, if $M: \boldsymbol{L} \to \boldsymbol{C}$ is a model of T, we define

$$[M] : [T] \longrightarrow \mathcal{P}(\boldsymbol{C})$$

by $[M]_1([\mathcal{X}]) = M[\mathcal{X}]$, $[M]_2([\mathcal{X}:\varphi]) = M[\mathcal{X}:\varphi]$. The fact that M is a model ensures that $[M]$ is well-defined (on equivalence classes); the rules of the logic, built into the definition of $[T]$, ensure that $[M]$ is a morphism of fibrations with the appropriate preservation properties. Finally, we have $j_M: [M]^{-}\restriction \boldsymbol{K} \cong M\restriction \boldsymbol{K}$ whose components are canonical isomorphisms $M([\mathcal{X}^*_K]) \cong M(K)$, and j_M is in fact an isomorphism $j_M: [M]^{-} \cong M$.

The completeness theorem

$$T \vdash \varepsilon \iff T \vDash_{\mathrm{Set}} \varepsilon$$

for coherent logic with dependent sorts, with or without equality, is now an immediate consequence of 3.(5). Indeed,

$T \vdash \varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$ $\iff$ $[\mathcal{X}:\varphi] \leq_{\mathcal{X}} [\mathcal{X}:\psi]$ in $[T]$

by the construction of $[T]$;

$\iff$ for all $P: [T] \to \mathcal{P}(\mathrm{Set})$, $P[\mathcal{X}:\varphi] \leq P[\mathcal{X}:\psi]$

by 3.(5) ;

$\iff$ for all $M \vDash T$, $M \vDash \varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$

by the above description of the equivalence $\mathrm{Mod}_{\boldsymbol{C}}(T) \simeq \mathrm{Mod}_{\mathcal{P}(\mathcal{C})}([T])$,

$\iff$ $T \vDash_{\mathrm{Set}} \varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$

by definition.

3.(6) gives a proof of the completeness theorem for intuitionistic logic. 3.(6) says that there is a category $\boldsymbol{K}$, namely $\mathrm{Mod}(T)$, such that T has a conservative Heyting morphism into $\mathrm{Set}^{\boldsymbol{K}}$; changing here $\boldsymbol{K}$ into a small category, and then into a poset is an easy matter; see [MR2], [M3].

As it is well-known, completeness for classical logic follows from that for coherent logic directly.

In summary, it is worth emphasizing that the study of first-order logic with dependent sorts without equality is the same as the study of "quantificational" fibrations $(\mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}}, \mathcal{Q})$ where the base category is $\boldsymbol{B}=((\mathrm{Set}^{\boldsymbol{K}})_{\mathrm{fin}})^{\mathrm{op}}$ for a simple category $\boldsymbol{K}$, with $\mathcal{Q}$ being the class of all epimorphisms in $\boldsymbol{B}$. This is a remarkably simple algebraic description of the objects of our interest, even though it is not one that is conjured up immediately by the idea of "first-order logic with dependent sorts".