§3. Quantificational fibrations

The notation and terminology of [M3] is used. The particular kinds of fibrations introduced here do not appear in *loc.cit.*, but most of the needed ingredients do.

Let $\underset{\boldsymbol{B}}{\overset{\boldsymbol{E}}{\mathcal{C}\downarrow}} = \underset{\mathbf{B}_{\mathcal{C}}}{\overset{\mathbf{E}_{\mathcal{C}}}{\mathcal{C}\downarrow}}$ be a fibration; let $\mathcal{Q}$ be a class of arrows in $\boldsymbol{B}$. Assume:

$\boldsymbol{B}$ has a terminal object, and pullbacks ($\boldsymbol{B}$ is left exact) .
$\mathcal{Q}$ is closed under pullbacks: when

$$\begin{array}{ccc} A & \xrightarrow{\quad q \quad} & B \\ \uparrow & \square & \uparrow \\ A' & \xrightarrow{\quad q' \quad} & B' \end{array} \tag{1}$$

is a pullback, then $q\in\mathcal{Q}$ implies $q'\in\mathcal{Q}$.

Each fiber $\mathcal{C}^A$ ($A\in\boldsymbol{B}$) is a poset; in fact, it is a lattice (with top and bottom elements, denoted $\mathbf{t}_A$, $\mathbf{f}_A$; the meet and join operations are written as $\wedge_A$, $\vee_A$, or more simply as $\wedge$, $\vee$ if no confusion may arise).

For each $(q\colon A\to B)\in\mathcal{Q}$, $q^*\colon\mathcal{C}^B\longrightarrow\mathcal{C}^A$ has a left adjoint $\exists_q\colon\mathcal{C}^A\longrightarrow\mathcal{C}^B$, which satisfies the Beck-Chevalley condition with respect to *all* pullback squares (1), and which satisfies Frobenius reciprocity (see pp. 342 and 343 in [M3]).

(Note that a fibration with posetal fibers (the only ones we are interested in here) is the same as a functor

$$\boldsymbol{B}^{\mathrm{op}}\longrightarrow\mathtt{Poset} : A\xrightarrow{f}B \longmapsto \mathcal{C}^B\xrightarrow{f^*}\mathcal{C}^A$$

to the category $\mathtt{Poset}$ of posets and order-preserving maps.)

The data $\mathcal{C}$, $\mathcal{Q}$ as described make the pair $(\mathcal{C}, \mathcal{Q})$ a $\wedge\vee\exists$-*fibration*. We may denote $(\mathcal{C}, \mathcal{Q})$ by $\mathcal{C}$; we may write $\mathcal{Q}_{\mathcal{C}}$ for $\mathcal{Q}$. Dropping the references to $\mathbf{f}_A$ and $\vee_A$ results in the notion of $\wedge\exists$-*fibration*.

A *morphism* $M:\mathcal{C}\to\mathcal{D}$ *of* $\wedge\vee\exists$-*fibrations* is a morphism of fibrations (among others,

$M=(M_1, M_2)$, $M_1:\mathbf{B}_{\mathcal{C}}\to\mathbf{B}_{\mathcal{D}}$, $M_2:\mathbf{E}_{\mathcal{C}}\to\mathbf{E}_{\mathcal{C}}$, $\begin{array}{ccc}\mathbf{E}_{\mathcal{C}} & \longrightarrow & \mathbf{E}_{\mathcal{D}}\\ \downarrow & \circ & \downarrow\\ \mathbf{B}_{\mathcal{C}} & \longrightarrow & \mathbf{B}_{\mathcal{C}}\end{array}$; in practice, we omit the subscripts 1 and 2, and write $M(A)$ for $M_1(A)$, *etc.*) that takes $\mathcal{Q}_{\mathcal{C}}$-arrows to $\mathcal{Q}_{\mathcal{D}}$-arrows, induces lattice homomorphisms on the fibers, and preserves all instances of each $\exists_q$ ($q\in\mathcal{Q}_{\mathcal{C}}$). M is *conservative with respect to* a pair (X, Y) of predicates over the same base-object A if $MX\le_{MA}MY$ implies $X\le_A Y$; M is conservative if it is *conservative* for all such (X, Y) .

The $\wedge\vee\exists$-fibrations and their morphisms form a category $\wedge\vee\exists$. In fact, we can make $\wedge\vee\exists$ into a 2-category, by making $\wedge\vee\exists(\mathcal{C},\mathcal{D})$ into a category; the latter is a full subcategory of $[\mathcal{C}, \mathcal{D}]$ (see p. 348 in [M3]). An arrow

$$\mathcal{C}\ \underset{N}{\overset{M}{\rightrightarrows}}\ \mathcal{D}\qquad (\downarrow h)$$

is a natural transformation $h:M_1\longrightarrow N_1$ satisfying $MP\le_{h_A} NP$ for all $A\in\mathbf{B}_{\mathcal{C}}$, $P\in\mathcal{C}^A$ (for the notation $X\le_f Y$, see p. 349 in [M3]; $X\le_f Y \Longleftrightarrow X\le f^* Y$).

For a category $\boldsymbol{C}$ with pullbacks, $\mathcal{P}(\boldsymbol{C})$, the *fibration of predicates of* $\boldsymbol{C}$, is the fibration $\mathcal{C}$ with base-category $\boldsymbol{C}$ for which $\mathcal{C}^A=\mathrm{S}(A)$, the $\wedge$-semi-lattice of subobjects of A, and for $f:A\to B$, $f^*:\mathrm{S}(B)\to\mathrm{S}(A)$ is the usual pullback-mapping. To say that $\mathcal{P}(\boldsymbol{C})$ is a $\wedge\vee\exists$-fibration, with $\mathcal{Q}$ the class of all arrows in $\boldsymbol{C}$, is the same as to say that $\boldsymbol{C}$ is a coherent category (see, *e.g.*, [MR2]).

Consider $\mathcal{P}(\mathrm{Set})$ as a $\wedge\vee\exists$-fibration, with $\mathcal{Q}$ the class of all arrows in Set . A *model* of $\mathcal{C}$ is a morphism $\mathcal{C}\to\mathcal{P}(\mathrm{Set})$. $\mathrm{Mod}(\mathcal{C})$ is the *category of models* of $\mathcal{C}$; $\mathrm{Mod}(\mathcal{C})$ = $\wedge\vee\exists(\mathcal{C},\mathcal{P}(\mathrm{Set}))$. More generally, let us write $\mathrm{Mod}_{\mathcal{D}}(\mathcal{C})$ for $\wedge\vee\exists(\mathcal{C},\mathcal{D})$.

Until further notice, fix $\mathcal{C}$ = $\left(\mathcal{C}\begin{array}{c}\boldsymbol{E}\\ \downarrow\\ \boldsymbol{B}\end{array}, \mathcal{Q}\right)$, a small $\wedge\vee\exists$-fibration. Proposition (5) below is the *completeness theorem* for $\wedge\vee\exists$-fibrations, the fact that there are enough models of $\mathcal{C}$ to distinguish between any pair of different predicates in a fiber. The ones preceding (5) are used for the proof of (5).

Let us write $\mathbf{1}$ for $1_{\mathbf{B}}$, the terminal object of $\mathbf{B}$; and $\mathbf{t}$ for $\mathbf{t}_{\mathbf{1}}$, $\mathbf{f}$ for $\mathbf{f}_{\mathbf{1}}$. $\mathcal{C}$ has the *disjunction property* if for any $X, Y \in \mathcal{C}^{\mathbf{1}}$, if $X \vee Y = \mathbf{t}$, then either $X = \mathbf{t}$, or $Y = \mathbf{t}$. $\mathcal{C}$ has the *existence property* if whenever $(!_A : A \to \mathbf{1}) \in \mathcal{Q}$ and $X \in \mathcal{C}^A$, we have that $\exists_{!_A}(X) = \mathbf{t}$ implies the existence of some $c : \mathbf{1} \to A$ such that $c^*(X) = \mathbf{t}$.

(1) Suppose $\mathcal{C}$ has the disjunction and the existence properties, and that $\mathbf{t} \neq \mathbf{f}$ (consistency). Then $\mathrm{Mod}(\mathcal{C})$ has an initial object ; in fact, $M = (M_1, M_2)$ given by $M_1 = \hom_{\mathbf{B}}(\mathbf{1}, -)$ and for $X \in \mathcal{C}^A$, $M_2(X) = \{c : \mathbf{1} \to A : c^*(X) = \mathbf{t}\}$ is an initial object.

(M may be called the *global-sections model* $\mathcal{C} \to \mathcal{P}(\mathrm{Set})$; we say $c : \mathbf{1} \to A$ *belongs to* X over A if $c^*(X) = \mathbf{t}$.)

The proof is identical to that of 2.2, p. 351 in [M3], although the statement of the latter does not include that of the present proposition.

For a fibration $\mathcal{C}$, $X \in \mathbf{B}$ and $X \in \mathcal{C}^A$, the "slice" fibration $\mathcal{C}/(A, X)$ was described in [M3]. The base-category of $\mathcal{C}/(A, X)$ is $\mathbf{B}/A$; the fiber over $(B \xrightarrow{f} A) \in \mathbf{B}/A$ is $\{Y \in \mathcal{C}^B : Y \leq_f X\}$, ordered as $\mathcal{C}^B$ is. We have a canonical morphism $\delta = \delta_{A,X} : \mathcal{C} \to \mathcal{C}/(A, X)$ that takes $B \in \mathbf{B}$ to $(B \times A \xrightarrow{\pi'} A)$, and $Y \in \mathcal{C}^B$ to $Y \dot{\wedge} X \underset{\text{def}}{=} \pi^* Y \wedge \pi'^* X$ ($\leq_{\pi'} X$; $\pi : B \times A \longrightarrow B$ is the other projection).

For a $\wedge\vee\exists$-fibration $\mathcal{C}$, we define the $\wedge\vee\exists$-fibration $\mathcal{D} = \mathcal{C}/(A, X)$ by also putting

$$\left(\begin{array}{c} B \xrightarrow{f} C \\ \searrow \circ \swarrow \\ A \end{array}\right) \in \mathcal{Q}_{\mathcal{D}} \underset{\text{def}}{\Longleftrightarrow} f \in \mathcal{Q} .$$

(2) $\mathcal{C}/(A,X)$ is a $\wedge\vee\exists$-fibration, and $\delta_{A,X}:\mathcal{C}\to\mathcal{C}/(A,X)$ is a map of $\wedge\vee\exists$-fibrations.

The proof is essentially contained in Section 2 of [M3]. It is helpful to add to 2.4(i) and (ii) of [M3] that the forgetful functor $\mathbf{B}/A\to\mathbf{B}$ creates pullbacks; with this, the required instances of the Beck-Chevalley and Frobenius reciprocity conditions become clear.

(3) If $(!_A:A\to\mathbf{1})\in\mathcal{Q}$ and $X\in\mathcal{C}^A$ such that $\exists_{!_A}(X)=\mathbf{t}$, then $\delta_{A,X}$ is conservative. If $X_1\vee X_2=_{\mathbf{t}}\mathbf{t}$, and $Y, Z\in\mathcal{C}^B$, then either $\delta_{\mathbf{t},X_1}$ or $\delta_{\mathbf{t},X_2}$ is conservative with respect to (Y,Z).

See 2.7 in [M3].

By a straightforward transfinite iteration of the construction of $\mathcal{C}/(A,X)$ (compare 2.8 in [M3]), we conclude from (2) and (3) that

(4) For any given $A\in\mathbf{B}$, $X, Y\in\mathcal{C}^A$, there are a $\wedge\vee\exists$-fibration $\mathcal{C}^*$ having the disjunction and existence properties, and a map $\mathcal{C}\to\mathcal{C}^*$ of $\wedge\vee\exists$-fibrations which is conservative with respect to (X,Y).

(5) For any given $A\in\mathbf{B}$, $X, Y\in\mathcal{C}^A$, there is $M:\mathcal{C}\to\mathcal{P}(\mathrm{Set})$, a map of $\wedge\vee\exists$-fibrations, which is conservative with respect to (X,Y).

Proof. In $\mathcal{C}/(A,X)$, with $\mathbf{1}=\mathbf{1}_{\mathcal{C}/(A,X)}$ and $\delta=\delta_{A,X}$, we have the global element

$$d_A:\mathbf{1}\longrightarrow\delta(A)\;:\;\begin{array}{ccc} A & \longrightarrow & A\times A \\ & {\scriptstyle 1_A}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & A & \end{array}$$

that belongs to $\delta(X)$; moreover, d_A belongs to $\delta(Y)$ over A iff $X \leq Y$. Now, start with X, Y over A in $\mathcal{C}$ such that $X \nleq Y$; pass to $\mathcal{C}' = \mathcal{C}/(A, X)$; in $\mathcal{C}'$, $\mathbf{t} = d_A^* \delta(X) \nleq d_A^* \delta(Y) = Y'$. By (4), there is $\Phi : \mathcal{C}' \to \mathcal{C}^*$ which is conservative with respect to $(\mathbf{t}, Y')$ such that $\mathcal{C}^*$ has the disjunction and existence properties. By (1), we have the global-sections model $N : \mathcal{C}^* \to \mathcal{P}(\mathrm{Set})$. The global-sections model is automatically conservative with respect to any pair $(\mathbf{t}, Z)$ over $\mathbf{1}$ in its domain. We conclude that, for $P = N \circ \Phi : \mathcal{C}' \to \mathcal{P}(\mathrm{Set})$, P is conservative with respect to $(\mathbf{t}, Y')$, that is,

$$P(d_A^* \delta(X)) \nleq P(d_A^* \delta(Y)) \ .$$

It follows that

$$P(\delta(X)) \nleq P(\delta(Y)) \ .$$

For $M = P \circ \delta : \mathcal{C} \to \mathcal{P}(\mathrm{Set})$, this means that $M(X) \nleq M(Y)$.

A $\wedge\vee\rightarrow\exists\forall$*-fibration* is a $\wedge\vee\exists$-fibration $\mathcal{C}$ such that

every fiber $\mathcal{C}^A$ is a Heyting algebra, and for all $f : A \to B$, $f^* : \mathcal{C}^B \to \mathcal{C}^A$ is a homomorphism of Heyting algebra; and

for each $q \in \mathcal{Q}_{\mathcal{C}}$, q^* (also) has a right adjoint which satisfies the Beck-Chevalley condition with respect to all (relevant) pullback squares.

For a category $\boldsymbol{C}$ with pullbacks, to say that $\mathcal{P}(\boldsymbol{C})$ is a $\wedge\vee\rightarrow\exists\forall$-fibration, with $\mathcal{Q}$ the class of all arrows in $\boldsymbol{C}$, is the same as what we usually express by saying that $\boldsymbol{C}$ is a Heyting category (see [MR2]). Of course, Set is a Heyting category; more, for any (not necessarily small) category $\boldsymbol{A}$, $\mathrm{Set}^{\boldsymbol{A}}$ is a Heyting category. See e.g. [MR2]. The coherent structure in $\mathrm{Set}^{\boldsymbol{A}}$ (the $\wedge\vee\exists$-fibration structure in $\mathcal{P}(\mathrm{Set}^{\boldsymbol{A}})$), although not the full Heyting-structure, is "computed pointwise"; that is, the projections $\pi_A : \mathcal{P}(\mathrm{Set}^{\boldsymbol{A}}) \to \mathcal{P}(\mathrm{Set})$ ($A \in \boldsymbol{A}$) are morphisms of $\wedge\vee\exists$-fibrations.

Given any small $\wedge\vee\exists$-fibration $\begin{array}{c}\boldsymbol{E}\\ \mathcal{C}\downarrow\\ \boldsymbol{B}\end{array}$, we may form $\mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})})$, and we have the evaluation morphism

$$\begin{array}{llll}
e_{\mathcal{C}} : \mathcal{C} & \longrightarrow & \mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})}) & \\
\quad\underline{X} & \longmapsto & \underline{[M\longmapsto M(X)]} & \\
\quad\ \downarrow & & \qquad\quad\downarrow & \\
\quad A & \longmapsto & [M\longmapsto M(A)] &
\end{array}$$

of $\wedge\vee\exists$-fibrations.

(6) For a small $\wedge\vee\rightarrow\exists\forall$-fibration $\mathcal{C}$, $e_{\mathcal{C}}:\mathcal{C}\longrightarrow\mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})})$ is a morphism of $\wedge\vee\rightarrow\exists\forall$-fibrations.

Proof. The proof is a variant of that of 5.1 in [M3]. The fact that e is conservative follows from (5). We need to show that $e_{\mathcal{C}}$ preserves Heyting implications in the fibers, and $\forall_f$'s; we limit ourselves to the second task. By using the way the $\forall_f$'s are computed in any $\mathcal{P}(\mathrm{Set}^{\boldsymbol{A}})$, our task is as follows.

Assume $M:\mathcal{C}\rightarrow\mathrm{Set}$, a morphism of $\wedge\vee\exists$-fibrations; $(f:A\rightarrow B)\in\mathcal{Q}$, $X\in\mathcal{C}^A$, $\forall_f X\in\mathcal{C}^B$ and $b\in M(B)-M(\forall_f X)$. We want the existence of $N\in\mathrm{Mod}(\mathcal{C})$, a homomorphism $h:M\rightarrow N$ and $a\in N(A)-N(X)$ such that $h_A(b) = (Nf)(a)$.

Let us use ordinary multisorted first-order logic to talk about models of $\mathcal{C}$ and homomorphisms between them. Consider the language $L=\mathrm{L}(\mathcal{C})$ whose sorts are the objects of $\boldsymbol{B}$, operation-symbols are the arrows of $\boldsymbol{B}$, and relation-symbols are all unary, and they correspond to the predicates $P\in\mathcal{C}^A$; P is sorted $P\subset A$. It is clear that every $M\in\mathrm{Mod}(\mathcal{C})$ may be regarded an L-structure; morphisms in $\mathrm{Mod}(\mathcal{C})$ are exactly the morphisms of L-structures. Moreover, there is a (coherent) theory $T=\mathrm{T}(\mathcal{C})$ over L such that $\mathrm{Mod}(\mathcal{C}) = \mathrm{Mod}(T)$.

For a given L-structure M, homomorphisms $h:M\rightarrow N$ with varying N are in a 1-1 correspondence with models of $\mathrm{Diag}^+(M)$, the positive diagram of M, which is a set of atomic sentences in the diagram language $L(|M|)$ in which an individual constant $\underline{a}$ of sort A has been added to L for each sort and $a\in M(A)$; the elements $\mathrm{Diag}^+(M)$ are those atomic sentences that are true in $(M,\underline{a})_{a\in|M|}$. We may also define $\mathrm{Diag}^+(M)$ as $\mathrm{D_b}(M)\cup\mathrm{D_p}(M)$, where $\mathrm{D_b}(M)$ contains all $f(\underline{a})=_B\underline{b}$ for which $f:A\to B$ in $\boldsymbol{B}$, $a\in M(A)$, $b\in M(B)$ and $(Mf)(a)=(b)$; and $\mathrm{D_p}(M)$ contains all $P(\underline{a})$ where $A\in\boldsymbol{B}$, $P\in\mathcal{C}^A$ and $a\in M(P)\subset M(A)$.

Returning to our task, let $\underset{\sim}{a}$ be a new individual constant of sort A; under the assumptions, we need the satisfiability of the set

$$T\cup\mathrm{D_b}(M)\cup\mathrm{D_p}(M)\cup\{\neg X(\underset{\sim}{a})\}\cup\{\underline{b}=_B f(\underset{\sim}{a})\}\ .$$

Assume this fails. By compactness, there are finite subsets $D\subset\mathrm{D_b}(M)$, $D'\subset\mathrm{D_p}(M)$ such that

$$T\cup D\cup D'\ \vDash\ \underline{b}=_B f(\underset{\sim}{a})\longrightarrow X(\underset{\sim}{a})\ .$$

Let $\langle c_i\rangle_{i<n}$ be distinct elements of M, $c_i\in M(C_i)$, each distinct from b, such that every $\underline{c}$ that occurs in $D\cup D'$ is one of the $\underline{c}_i$, or is $\underline{b}$. Let z_i be distinct variables, z_i of sort C_i; y a variable of sort B, x one of sort A, all distinct. Let us replace c_i by z_i, b by y; we obtain $\bar{D}$ from D, $\bar{D}'$ from D', and we get that

$$T\vDash\forall\langle z_i\rangle_{i<n}\forall y\forall x(\bigwedge\bar{D}\wedge\bigwedge\bar{D}'\wedge y=_B f(x)\longrightarrow X(x))\ . \tag{7}$$

Working inside the category $\boldsymbol{B}$ with finite limits, we can construct as an appropriate finite limit an object C together with morphisms $\pi_i:C\to C_i$, $\pi:C\to B$ such that for any L-structure N, $N\vDash(\bigwedge\bar{D})[\langle\dot{c}_i\rangle_{i<n}\dot{b}/\langle z_i\rangle_{i<n}y]$ iff there is $\dot{c}\in N(C)$ with $N(\pi_i)(\dot{c})=\dot{c}_i$, $N(\pi)(\dot{c})=\dot{b}$ (actually, $\dot{c}$ is then uniquely given). In particular, there is an element $c\in M(C)$ such that $M(\pi_i)(c)=c_i$, $M(\pi)(c)=b$. For any $\alpha\in\bar{D}'$, let α^* be the element of the fiber over C given as follows: if $\alpha:=:P(z_i)$, $\alpha^*\underset{\mathrm{def}}{=}\pi_i^*(P)$; if $\alpha:=:P(y)$, $\alpha^*\underset{\mathrm{def}}{=}\pi^*(P)$. Let $Q=\bigwedge\{\alpha^*:\alpha\in\bar{D}'\}\in\mathcal{C}^C$. Notice that $c\in M(Q)$. Consider the pullback-square

$$\begin{array}{ccc} A\times_B C & \xrightarrow{\rho} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\pi]{} & B \end{array}$$

We claim that

$$g^*(Q) \le a^*(X) . \qquad (8)$$

By (5), it suffices to check that this holds in each model $N \in \mathrm{Mod}(\mathcal{C})$. Assume $N \in \mathrm{Mod}(\mathcal{C}) = \mathrm{Mod}(T)$, $\dot{d} \in N(g^*(Q))$, $\dot{c} = (Ng)\dot{d}$, $\dot{a} = (N\rho)\dot{d}$, $\dot{c}_i = (N\pi_i)\dot{c}$, $\dot{b} = (N\pi)\dot{c}$; we have $\dot{b} = (Nf)\dot{a}$, $N \models (\bigwedge \bar{D})[\langle \dot{c}_i \rangle_{i<n} \dot{b} / \langle z_i \rangle_{i<n} y]$ by the defining property of $(C, \langle \pi_i \rangle_i, \pi)$ and $N \models (\bigwedge \bar{D}')[\langle \dot{c}_i \rangle_{i<n} \dot{b} / \langle z_i \rangle_{i<n} y]$ by the definition of Q. Since N satisfies the sentence in (7), it follows that $\dot{a} \in NX$, and thus $\dot{d} \in N(a^*(X))$, which shows the claim.

Since $f \in \mathcal{Q}$, also $g \in \mathcal{Q}$. By (8), $Q \le \forall_g \rho^*(X) = \pi^* \forall_f (X)$. However, in M, $c \in M(Q)$, but $c \notin \pi^* \forall_f (X)$, since $b \notin \forall_f (X)$; this is a contradiction.

A $\wedge\vee\neg\exists$-*fibration* is a $\wedge\vee\exists$-fibration in which every fiber is a Boolean algebra. Every $\wedge\vee\neg\exists$-fibration is a $\wedge\vee\to\exists\forall$-fibration.

Without essentially changing the concepts, in each of the various kinds of fibrations introduced above, the class $\mathcal{Q}$ of "quantifiable" arrows may be required, in addition, to be closed under composition. If $(\mathcal{C}, \mathcal{Q})$ is a "quantificational" fibration (of one of the four kinds introduced above), then, with $\mathcal{Q}^\circ$ the closure of $\mathcal{Q}$ under composition, $(\mathcal{C}, \mathcal{Q}^\circ)$ is again one of the same kind as the reader will readily see. Also, any morphism $f: (\mathcal{C}, \mathcal{Q}) \to (\mathcal{C}', \mathcal{Q}')$ of one of the four kinds is a morphism $f: (\mathcal{C}, \mathcal{Q}^\circ) \to (\mathcal{C}', \mathcal{Q}'^\circ)$ of the same kind.