§2. Formal systems

In this section, a vocabulary $\boldsymbol{L}$ for logic with dependent sorts is assumed fixed. Relations, formulas, *etc.*, are all from/over $\boldsymbol{L}$.

For a formula φ, $\mathrm{Var}^*(\varphi)$ is "the set of all variables in φ, free or bound". More precisely, $\mathrm{Var}^*(\varphi)=\mathrm{Var}(\varphi)$ for atomic φ; $\mathrm{Var}^*(\varphi\wedge\psi)=\mathrm{Var}^*(\varphi)\cup\mathrm{Var}^*(\psi)$, and similarly for the other connectives;

$$\mathrm{Var}^*(\forall x\varphi)=\mathrm{Var}^*(\exists x\varphi)=\{x\}\cup\mathrm{Dep}(x)\cup\mathrm{Var}^*(\varphi).$$

Let $\mathcal{X}$, $\mathcal{Y}$ be contexts. A map $s:\mathcal{X}\to\mathcal{Y}$ is called a *specialization* if whenever $x\in\mathcal{X}$, $x:K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$, we have $X=K(\langle s(x_p)\rangle_{p\in K|\boldsymbol{L}})$ is a sort, and $s(x):X$. The identity map $\mathcal{X}\to\mathcal{X}$ is a specialization, the composite of specializations is a specialization. Moreover, if a specialization is a bijection, then its inverse is also a specialization, and the restriction of a specialization to a subset of its domain which is a context is also a specialization. A notation such as $s:\mathcal{X}\to\mathcal{Y}$ will refer to a specialization.

For a sort X, resp. a formula φ, and a specialization $s:\mathcal{X}\to\mathcal{Y}$ such that $\mathrm{Var}(X)\subset\mathcal{X}$, resp. $\mathrm{Var}(\varphi)\subset\mathcal{X}$, we define $X|s$, resp. $\varphi|s$, "the result of substituting $s(x)$ for all free occurrences of x in X, resp. in φ, simultaneously for all $x\in\mathcal{X}$".

If X is the sort $K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$, and if φ is the atomic formula $R(\langle x_p\rangle_{p\in R|\boldsymbol{L}})$, we put

$$X|s \underset{\mathrm{def}}{=} K(\langle s(x_p)\rangle_{p\in K|\boldsymbol{L}}),\quad \varphi|s \underset{\mathrm{def}}{=} R(\langle s(x_p)\rangle_{p\in R|\boldsymbol{L}}).$$

For the equality formula $\varphi:=:x=_X y$, $\varphi|s:=:s(x)=_{X|s}s(y)$. The property of s being a specialization ensures that $X|s$ is a sort, and $\varphi|s$ is a(n atomic) formula in both cases.

$$(\varphi\wedge\psi)|s \underset{\mathrm{def}}{=} (\varphi|s)\wedge(\psi|s),$$

and similarly for the other connectives.

Suppose $\varphi = \forall x\psi$. Let us first assume that $\mathcal{X}=\mathrm{Var}(\forall x\varphi)$. Consider the sort X of x, $x{:}X$; let y be a variable of sort $X|s$ which is *new* in the sense that $y\notin\mathrm{Var}^*(\psi)\cup\mathcal{X}\cup\mathcal{Y}$. Define t to be the function $t{:}\mathcal{X}\cup\{x\}\longrightarrow\mathcal{Y}\cup\{y\}$ for which $t\restriction\mathcal{X} = s$, and $t(x) = y$ (note that $x\notin\mathcal{X}$). Notice that $\mathrm{Var}(X)\subset\mathcal{X}$, $\mathrm{Var}(X|s)\subset\mathcal{Y}$, thus $\mathcal{X}\cup\{x\}$ and $\mathcal{Y}\cup\{y\}$ are contexts, and t is a specialization. We put $(\forall x\psi)|s \underset{\mathrm{def}}{=} \forall y(\psi|t)$. For a general $s{:}\mathcal{X}\to\mathcal{Y}$, $(\forall x\psi)|s$ is defined as $(\forall x\psi)|s'$, with $s' = s\restriction\mathrm{Var}(\forall x\psi)$. We make a similar definition for $\exists$ in place of $\forall$.

Since in the above description, y was not uniquely determined by the conditions given, substitution is not quite well-defined. We may correct this by making a particular, but artificial, choice of y. A better procedure is to *identify* the formulas obtained by different choices of y; this we do by defining the equivalence relation on formulas of one being an *alphabetic variant* of the other. However, for defining "alphabetic variant" it is convenient to use substitution. As long as substitution is not "well-defined", what we have is a relation " $\varphi|s = \theta$ " of three variables φ, s, θ rather than an operation $(\varphi, s)\longmapsto\varphi|s$.

Let φ be a formula, x and u variables of the same sort (for which we write $x\simeq u$), and assume that for all $v\in\mathrm{Var}(\varphi)$, $x\notin\mathrm{Dep}(v)$ (that is, either $x\notin\mathrm{Var}(\varphi)$ or it is a "top" element in $\mathrm{Var}(\varphi)$). Then the mapping

$$s{:}\mathrm{Var}(\varphi)\cup\{x\}\to\mathrm{Var}(\varphi)\cup\{u\},$$

defined by $s(v)=v$ for $v\in\mathrm{Var}(\varphi)-\{x\}$ and $s(x)=u$, is a specialization. Under these conditions, we put $\varphi|(x\mapsto u) \underset{\mathrm{def}}{=} \varphi|s$ [more precisely, " $\varphi|(x\mapsto u) = \theta$ " iff " $\varphi|s = \theta$ "].

The relation $\varphi\sim\psi$, " φ is an alphabetic variant of ψ ", is defined as follows.

If φ is atomic, then $\varphi\sim\psi$ iff $\varphi = \psi$.

$\varphi_1\wedge\varphi_2 \sim \psi$ iff $\psi = \psi_1\wedge\psi_2$ for some ψ_i with $\varphi_i\sim\psi_i$ $(i=1, 2)$; and similarly for the other connectives.

$\forall x\varphi \sim \psi$ iff $\psi = \forall x'\varphi'$ for some $x'\simeq x$ and φ' such that, for some u for which $u\simeq x\simeq x'$ and $u\notin\mathrm{Var}^*(\varphi)\cup\mathrm{Var}^*(\varphi')$, we have that $\varphi|(x\mapsto u) \sim \varphi'|(x'\mapsto u)$. Similarly for $\exists$ in place of $\forall$. [More precisely, we should say, in place of $\varphi|(x\mapsto u) \sim \varphi'|(x'\mapsto u)$, that for some σ and τ such that " $\varphi|(x\mapsto u) = \sigma$ " and

" $\varphi|(x'\mapsto u) = \tau$ ", we have $\sigma\sim\tau$.]

One shows in a routine manner that $\sim$ is an equivalence relation, $\varphi\sim\psi$ implies that $\mathtt{Var}(\varphi)=\mathtt{Var}(\psi)$, and $\sim$ is compatible with substitution: if $\varphi\sim\psi$, " $\varphi|s=\varphi'$ ", and " $\psi|s=\psi'$ " imply that $\varphi'\sim\psi'$. In particular, substitution $(\ )|s$ is an operation on equivalence classes of $\sim$. Note that the logical operations are compatible with $\sim$; $\varphi\sim\psi$ implies that $\forall x\varphi\sim\forall x\psi$, etc. Also, the semantics of alphabetic variants are identical. Henceforth, we *identify* alphabetic variants. In other words, a formula is, strictly speaking, an equivalence class of the "alphabetic variant" relation $\sim$.

When $s:\mathcal{X}\to\mathcal{Y}$, $\mathtt{Var}(\varphi)\subset\mathcal{X}$, we have $\mathtt{Var}(\varphi|s)\subset\mathcal{Y}$. If, in addition, $t:\mathcal{Y}\to\mathcal{Z}$, then $(\varphi|s)|t=\varphi|(ts)$. Also, $\varphi|1_{\mathcal{X}}=\varphi$.

An *entailment* is an entity of the form $\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi$, where φ , ψ are formulas, $\mathcal{X}$ is a context, and $\mathtt{Var}(\varphi)$, $\mathtt{Var}(\psi)\subset\mathcal{X}$. We formulate *rules of inference* involving entailments. Each rule is a relation $\mathcal{R}(\varepsilon_0,\ldots,\varepsilon_{n-1};\varepsilon_n)$ between entailments $\varepsilon_0,\ldots,\varepsilon_{n-1},\varepsilon_n$; $\varepsilon_0,\ldots,\varepsilon_{n-1}$ are the *premises*, ε_n is the *conclusion* of the respective instance of $\mathcal{R}$. We display instances of $\mathcal{R}$ in the form

$$\mathcal{R}\quad\frac{\varepsilon_0\quad\varepsilon_1\qquad\qquad\varepsilon_{n-1}}{\varepsilon_n}\ .$$

n may be 0 , in which case we have a rule with no premises, an *axiom schema*.

I. Structural rules:

$$(\mathtt{Taut})\qquad\frac{}{\varphi\underset{\mathcal{X}}{\Longrightarrow}\varphi}$$

$$(\mathtt{Cut})\qquad\frac{\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi\qquad\psi\underset{\mathcal{X}}{\Longrightarrow}\sigma}{\varphi\underset{\mathcal{X}}{\Longrightarrow}\sigma}$$

$$\text{(Subst)}\qquad \frac{\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi}{\varphi|s \underset{\mathcal{Y}}{\Longrightarrow} \psi|s} \qquad\qquad (\, s\colon \mathcal{X} \longrightarrow \mathcal{Y}\,)$$

II. Rules for the connectives

$$(\mathbf{t})\qquad \frac{}{\psi \underset{\mathcal{X}}{\Longrightarrow} \mathbf{t}}$$

$$(\mathbf{f})\qquad \frac{}{\mathbf{f} \underset{\mathcal{X}}{\Longrightarrow} \psi}$$

$$(\wedge)\qquad \frac{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi \qquad \theta \underset{\mathcal{X}}{\Longrightarrow} \psi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi\wedge\psi}$$

$$(\vee)\qquad \frac{\varphi \underset{\mathcal{X}}{\Longrightarrow} \theta \qquad \psi \underset{\mathcal{X}}{\Longrightarrow} \theta}{\varphi\vee\psi \underset{\mathcal{X}}{\Longrightarrow} \theta}$$

$$(\rightarrow)\qquad \frac{\theta\wedge\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi\rightarrow\psi}$$

$$(\neg)\qquad \frac{}{\mathbf{t} \underset{\mathcal{X}}{\Longrightarrow} \theta\vee\neg\theta} \qquad\qquad (\,\neg\theta \text{ abbreviates } \theta\rightarrow\mathbf{f}\,)$$

$$(\wedge\vee)\qquad \frac{}{(\varphi\vee\psi)\wedge\theta \underset{\mathcal{X}}{\Longrightarrow} (\varphi\wedge\theta)\vee(\psi\wedge\theta)}$$

III. Quantifier rules

$$(\forall)\qquad \frac{\theta \underset{\mathcal{X}\dot{\cup}\{x\}}{\Longrightarrow} \varphi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \forall x\varphi} \qquad\qquad (\, x\notin\mathcal{X}\,)$$

$$(\exists)\qquad \frac{\varphi \xRightarrow[\mathcal{X}\dot{\cup}\{x\}]{} \theta}{\exists x\varphi \xRightarrow[\mathcal{X}]{} \theta}\ \text{(double line)}\qquad (x\notin\mathcal{X})$$

$$(\wedge\exists)\qquad \frac{}{\theta\wedge\exists x\varphi \xRightarrow[\mathcal{X}]{} \exists x(\theta\wedge\varphi)}\qquad (x\notin\mathtt{Var}(\theta))$$

IV. Equality axioms

$$(\mathrm{E}_1)\qquad \frac{}{\mathbf{t} \xRightarrow[\mathcal{X}]{} x=_X x}$$

$$(\mathrm{E}_2)\qquad \frac{}{x=_X y \xRightarrow[\mathcal{X}]{} y=_X x}$$

$$(\mathrm{E}_3)\qquad \frac{}{x=_X y\ \wedge\ \varphi \xRightarrow[\mathcal{X}]{} \varphi|(x\mapsto y)}$$

In the rules, φ, ψ, θ and σ ranges over formulas, x over variables, $\mathcal{X}$ and $\mathcal{Y}$ over finite contexts. An implicit condition is that each entailment shown has to be well-formed. *E.g.*, in $(\mathbf{t})$ and $(\mathbf{f})$, $\mathtt{Var}(\sigma)\subset\mathcal{X}$. In $(\forall)$ and $(\exists)$, $\mathtt{Var}(\theta)\subset\mathcal{X}$; since $x\notin\mathcal{X}$ is explicitly assumed, it follows that $x\notin\mathtt{Var}(\theta)$. Note that, in the same rules, the condition for the well-formedness of $\forall x\varphi$, $\exists x\varphi$ is satisfied as a consequence of the other provisos. More precisely, if $\mathcal{X}$ is a context, $\mathtt{Var}(\varphi)\subset\mathcal{X}\dot{\cup}\{x\}$ (in particular $x\notin\mathcal{X}$), then $\forall x\varphi$, $\exists x\varphi$ are well-formed. Namely, for $y\in\mathtt{Var}(\varphi)$, if $y\neq x$, then $y\in\mathcal{X}$, hence $\mathtt{Dep}(y)\subset\mathcal{X}$, and thus $x\notin\mathtt{Dep}(y)$; and if $y=x$, then $x\notin\mathtt{Dep}(x)$ anyway.

For (E_3), note that since X is a sort of a maximal kind, $\varphi|(x\mapsto y)$ is well-defined.

The double-lined "rules" contain more than one rule. The double line indicates that inference can proceed in both directions. E.g., in $(\vee)$, three rules are contained: the one that infers the entailment below $=\!=$ from the two above $=\!=$, and the ones allowing to infer either of the two entailments above $=\!=$ from the one below $=\!=$.

We have coherent, classical and intuitionistic logic with dependent sorts, each with or without equality. *Coherent* logic involves the (*coherent*) operators $\mathbf{t}$, $\mathbf{f}$, $\wedge$, $\vee$, $\exists$; classical and intuitionistic logics involve the remaining two, $\rightarrow$ and $\forall$. Coherent logic without equality has the rules all those not mentioning $\rightarrow$ and $\forall$ in their names; intuitionistic logic also has the additional rules $(\rightarrow)$ and $(\forall)$ (and then $(\wedge\vee)$, $(\wedge\exists)$ become superfluous); classical logic has also the remaining rule $(\neg)$. The versions with equality also have the rules (E_1) , (E_2) and (E_3) .

A *coherent formula* is one built up by the coherent operators starting with the atomic formulas; an entailment $\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$ is *coherent* if both φ , ψ are coherent formulas. A *coherent theory in logic with dependent logic* is a pair $T=(\boldsymbol{L}, \Sigma)$ of a DS vocabulary $\boldsymbol{L}$ and a set Σ of coherent entailments over $\boldsymbol{L}$. $\mathrm{Cons}_{\mathrm{coh}}(T)$ is the least set of coherent $\boldsymbol{L}$-entailments that contains Σ as a subset, and is closed under the rules for coherent logic; we write $T \vdash \varepsilon$, or $T \vdash_{\mathrm{coh}} \varepsilon$, and say that ε is *deducible from* T *in coherent logic with* dependent sorts, for $\varepsilon \in \mathrm{Cons}_{\mathrm{coh}}(T)$. Again, we have the versions with or without equality.

A theory in intuitionistic logic , or in classical logic (with dependent sorts) is defined similarly, *mutatis mutandis*. Again, we have logic with or without equality. Aside the exclusion of equality in the logics without equality, all formulas are used, in contrast to coherent logic. We have the concept $T \vdash \varepsilon$ of deducibility for each of these logics with dependent sorts.

We have completeness theorems for the various logics (coherent, intuitionistic, classical) with dependent sorts. What these completeness theorems show is that logic with dependent sorts is "self-contained". The initial view of logic with dependent sorts is that it is a fragment of ordinary multi-sorted logic. The fact that truths in the fragment can be deduced by deductions using only formulas also in the fragment is a sign, indeed, a necessary sign, that the fragment deserves the designation "logic".

To formulate completeness, let us fix a semantic category $\boldsymbol{C}$ (in the first instance, $\boldsymbol{C} = \mathrm{Set}$). Let M be a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure. Let us write $M \models \varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$ for $M[\mathcal{X}:\varphi] \leq_{M[\mathcal{X}]} M[\mathcal{X}:\psi]$, and say that M *satisfies* the entailment $\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$. A *model* of a theory $T=(\boldsymbol{L}, \Sigma)$ is a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure that satisfies all entailments in Σ . For a theory T , and an entailment ε , let us write $T \models_{\boldsymbol{C}} \varepsilon$, and say that the entailment ε is a $\boldsymbol{C}$-*consequence* of T , to mean that all $\boldsymbol{C}$-valued models M of T satisfy ε . For a class $\mathcal{C}$ of categories, $T \models_{\mathcal{C}} \varepsilon$ means that $T \models_{\boldsymbol{C}} \varepsilon$ for all $\boldsymbol{C} \in \mathcal{C}$.

$\mathrm{Mod}_{\boldsymbol{C}}(T)$ is the category of all $\boldsymbol{C}$-valued models of T; it is a full subcategory of $\mathrm{Fun}(\boldsymbol{L},\boldsymbol{C})$. We write $\mathrm{Mod}(T)$ when $\boldsymbol{C} = \mathrm{Set}$.

The completeness theorem for coherent logic, as well as for classical logic, with dependent sorts, with or without equality, is expressed by the equivalence

$$T \vdash \varepsilon \iff T \models_{\mathrm{Set}} \varepsilon$$

(of course, the symbol $\vdash$ is to be taken in any one of the four distinct senses corresponding to the four logics listed; ε accordingly ranges over the entailments of the corresponding logic). The completeness theorem for intuitionistic logic with dependent sorts, with or without equality, is

$$T \vdash \varepsilon \iff T \models_{\mathrm{Kr}} \varepsilon \ ,$$

where Kr (for Kripke) denotes the class of categories of the form Set^{P}, with P any poset.

As usual (see *e.g.* [MR2]), the completeness theorem for intuitionistic logic with dependent sorts may be formulated in the style of Kripke's semantics.

We will prove of the completeness theorems in §4.