6. Equivalence of two virtual concepts of category.

I want to elaborate on the natural consequence of the presence of FOLDS equivalence for an *arbitrary* FOLDS signature regarding the comparison of two concepts specified in distinct FOLDS signatures. The basic idea is very simple, and Tom Leinster already sketched it out in his recent contribution [L] to this discussion -- except that he expressed a skepticism about the ingredient we now have: the equivalence of structures of one and the same kind.

Suppose that we have two kinds, $\boldsymbol{K}_1$ (Mork's) and $\boldsymbol{K}_2$ (Mindy's), of categories, and we have the respective equivalences $\simeq_{L_1}$, $\simeq_{L_2}$ for categories of these kinds. To see that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are "equivalent", we should have constructions $\boldsymbol{K}_1 \longrightarrow \boldsymbol{K}_2 : \boldsymbol{X} \longmapsto \boldsymbol{X}^*$ and $\boldsymbol{K}_2 \longrightarrow \boldsymbol{K}_1 : \boldsymbol{A} \longmapsto \boldsymbol{A}^\#$, giving the $\boldsymbol{K}_2$-type category $\boldsymbol{X}^*$ from any $\boldsymbol{X}$ of type $\boldsymbol{K}_1$, and vice versa for $\boldsymbol{A}^\#$ from $\boldsymbol{A}$. Moreover, we should have that doing these constructions twice, we should get back to the original -- up to equivalence, to be reasonable. That is, we should have $\boldsymbol{X}^{*\#} \simeq_{L_1} \boldsymbol{X}$, $\boldsymbol{A}^{\#*} \simeq_{L_2} \boldsymbol{A}$.

It seems reasonable to insist that the maps $\boldsymbol{X} \longmapsto \boldsymbol{X}^*$, $\boldsymbol{A} \longmapsto \boldsymbol{A}^\#$ be "canonical" in some sense. But then, thinking of the fact that the equivalences $\simeq_{L_1}$, $\simeq_{L_2}$ are ascertained by the presence of certain data, it seems reasonable to insist that these data should also be canonically available, that is, explicitly definable, from the data for $\boldsymbol{X}$ in one case, from those for $\boldsymbol{A}$ in the other.

I want to propose a definite way of codifying a notion of equivalence of kinds of categories that conforms to the requirements set out above. I want to write this down in detail, since this is the point where I am making my most direct contribution to the subject at hand: comparing different definitions of "category".

The first time I talked about this was the 1998 June CMS meeting in St John, New Brunswick. I had occasions to talk about it at other times too, but I have not published anything about it.

The *first ingredient* is a concept called *regular FOLDS specification*. The word "regular" refers to the fact that the "regular" fragment of (categorical) logic is involved here -- but we do not have to worry about this since the description will be purely categorical.

Given a FOLDS signature L (remember, a special kind of 1-category), we pass to $\mathbf{B}[L]=\boldsymbol{B}=(\mathrm{Set}^{L}_{\mathrm{fin}})^{\mathrm{op}}$, the subcategory of $(\mathrm{Set}^{L})^{\mathrm{op}}$ consisting of the *finite* functors, where $F:L\to\mathrm{Set}$ being finite means that the set $\coprod_{K\in\mathrm{Ob}(L)} F(K)$ is finite. Because of the assumed properties of L, the Yoneda functor $L\longrightarrow(\mathrm{Set}^{L})^{\mathrm{op}}$ lands in $\boldsymbol{B}$; moreover, $\boldsymbol{B}$ is, via the resulting functor $y:L\longrightarrow\boldsymbol{B}$, the *lex completion* of L: $\boldsymbol{B}$ has all finite limits (terminal object and pullbacks), and for any category $\boldsymbol{S}$ with finite limits (and we are mainly, but not exclusively, thinking of $\boldsymbol{S}=\mathrm{Set}$), the functors $L\longrightarrow\boldsymbol{S}$ are in a natural up-to-isomorphism bijective correspondence with the lex (finite-limit-preserving) functors $\boldsymbol{B}\xrightarrow{\mathrm{lex}}\boldsymbol{S}$. In fact, I will use the same letter M in $M:L\longrightarrow\boldsymbol{S}$ and $M:\boldsymbol{B}\xrightarrow{\mathrm{lex}}\boldsymbol{S}$ when referring to the corresponding entities.

We put ourselves into the category $\boldsymbol{B}$. Let $\boldsymbol{Q}_0$ be any set of epimorphisms of $\boldsymbol{B}$ (an epimorphism in $\boldsymbol{B}$ is the same thing as a monomorphism of finite functors). $\boldsymbol{Q}$ denotes the closure of $\boldsymbol{Q}_0$ under composition, pullback along an arbitrary morphism, and under the conditions: $fg\in\boldsymbol{Q}$ implies $f\in\boldsymbol{Q}$, and $\boldsymbol{Q}$ is to contain all isomorphism. ($\boldsymbol{Q}$ is essentially the same as the Grothendieck topology generated by the single-arrow covering sieves $\{q\}$ for $q\in\boldsymbol{Q}_0$).

The pair $T=(L,\boldsymbol{Q}_0)$ is a typical *regular FOLDS specification*. For $\boldsymbol{S}$ any 1-category with finite limits in which a reasonable notion of "surjective arrow" is present -- a regular category will do, in which case "surjective arrow" is "regular epi" -- , we have the notion of an *$\boldsymbol{S}$-valued model of* T: it is any $M:\boldsymbol{B}\xrightarrow{\mathrm{lex}}\boldsymbol{S}$ for which $M(q)$ is a surjective arrow in $\boldsymbol{S}$ for any $q\in\boldsymbol{Q}_0$ (equivalently, for any $q\in\boldsymbol{Q}$). Of course, the main example for $\boldsymbol{S}$ is Set.

It would be nice to be able to spend some time with pointing out the regular FOLDS specifications for *all* of the concepts discussed earlier. The first example would be that of " 1-category". The signature L, of course, is now $L=\mathrm{L}_{1\text{-}\mathrm{cat}}$ given above. The present point is that the category axioms can all be written down by the surjectivity of specific epis $q\in\boldsymbol{Q}_0$, for an appropriate (small finite) set $\boldsymbol{Q}_0\subset\mathrm{Epi}(\boldsymbol{B})$. More precisely, an L-structure M is a model of $(L,\boldsymbol{Q}_0)$ iff M is L-equivalent to $\boldsymbol{C}^{\#}$, where $\boldsymbol{C}$ is a 1-category in the ordinary sense, and $\boldsymbol{C}^{\#}$ is obtained from $\boldsymbol{C}$ in the way we saw above.

The general idea is that we take our selected concept of "category" to be the $\mathtt{Set}$-valued models of a particular regular FOLDS specification.

Note that we have a good notion of a *morphism* $F:(L, \boldsymbol{Q}_0)\longrightarrow(L', \boldsymbol{Q}'_0)$ of regular FOLDS specifications. This is a lex functor $F:\boldsymbol{B}\to\boldsymbol{B}'$ that takes any $\boldsymbol{Q}_0$-arrow (equivalently, any $\boldsymbol{Q}$-arrow) into a $\boldsymbol{Q}'$-arrow ($\boldsymbol{Q}'$ is the (Grothendieck) closure of $\boldsymbol{Q}'_0$ as $\boldsymbol{Q}$ is of $\boldsymbol{Q}_0$).

The *second ingredient* of our proposed notion is an extension of L-equivalence for $\boldsymbol{S}$-valued functors models $L\longrightarrow\boldsymbol{S}$, for $\boldsymbol{S}$ more general than $\mathtt{Set}$.

Let us fix the FOLDS signature L . Recall the functors $\dot{K}$, $\hat{K}$ for any $K\in\mathrm{Ob}(L)$ from above. Suppose $L \overset{M}{\underset{P}{\rightrightarrows}} \boldsymbol{S}$ and $m:P\longrightarrow M$. Pick any $K\in\mathrm{Ob}(L)$, and form, in $\boldsymbol{S}$, the commutative diagram

$$\begin{tikzcd}
P\dot{K} \arrow[rr, "P\iota"] \arrow[dr] \arrow[dd, "m_{\dot{K}}"'] & & P\hat{K} \arrow[dd, "m_{\hat{K}}"] \\
& S \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\square"] & \\
M\dot{P} \arrow[rr, "M\iota"'] & & M\hat{K}
\end{tikzcd}$$

in which the quadrangle marked with $\square$ is a pullback. We say that m is fiberwise surjective (fs) if for every $K\in\mathrm{Ob}(L)$, the arrow $P\dot{K}\longrightarrow S$ in the above diagram is surjective. It will be seen immediately that this definition is equivalent to the earlier one when $\boldsymbol{S}=\mathtt{Set}$.

Note that this definition makes sense when we take, for $\boldsymbol{S}$, the category $\boldsymbol{B}'=\mathbf{B}[L']$, for a regular FOLDS specification $(L', \boldsymbol{Q}'_0)$, and we understand by "surjective arrow" to be an element of $\boldsymbol{Q}'=\mathbf{Q}[\boldsymbol{Q}_0]$.

For $M, N:L\longrightarrow\boldsymbol{S}$, the concept of L-equivalence $(P, m, n):M\simeq N$ is now defined as before. Of course, this is now meaningful when $M, N:\mathbf{B}\xrightarrow{\mathtt{lex}}\boldsymbol{S}$.

We are ready to make the main definition.

We say that the regular FOLDS specifications $T=(L, \boldsymbol{Q}_0)$, $T'=(L', \boldsymbol{Q}'_0)$ are *equivalent* if there are morphisms $F:T\longrightarrow T'$ and $G:T'\longrightarrow T$ such that $GF\simeq_L \mathrm{Id}_{\boldsymbol{B}}$, $FG\simeq_{L'} \mathrm{Id}_{\boldsymbol{B}'}$.

The latter expressions are meaningful since, e.g., both GF and $\mathrm{Id}_{\boldsymbol{B}}$ are functors $\boldsymbol{B}\longrightarrow\boldsymbol{B}$; that is, we take $\boldsymbol{S}$ to be $\boldsymbol{B}$ itself, with $\boldsymbol{Q}$ as the concept of surjectivity.

The full data for an equivalence of two specifications T and T' consist of F and G as above, and some $(P, m, n) : GF \simeq_L \mathrm{Id}_{\boldsymbol{B}}$, $(P', m', n') : FG \simeq_{L'} \mathrm{Id}_{\boldsymbol{B}'}$.

It is immediate that for any model M of T, and N of T', when writing M^* for $M\circ G$, $N^{\#}$ for $N\circ F$, we have that M^* is a model of T', N^* is a model of T, and $M^{*\#}\simeq_L M$, $N^{\#*}\simeq_{L'} N$, with equivalence-spans induced by the given (P, m, n) and (P', m', n').

The just-described concept is, in fact, just a pedantic formulation of something that one would immediately think of when the question of the equivalence of any two specific concepts of "category" arose -- except for one thing: only canonical constructions fit into the framework; cleavage does not.

Take, for instance, the three concepts of 2-category we encountered above: anabicategory, B/D-2-category, and truncated B/D-2-category. Each one will be seen to be specifiable by a regular FOLDS specification, with suitable signatures. The assertion is that all three specifications are equivalent in the technical sense described above. The way one starts thinking about this does not use, consciously at least, the formal definition; one thinks of how one gets a structure of one type from another one of another type. A "natural" way of doing this ends up giving the data for an equivalence of the FOLDS specifications in question.

There is an implicit vague claim here; namely, that the notion of equivalence of FOLDS specifications is the *right* one. (One has to keep in mind, however, that the notion is something for concepts with virtual operations only.) The claim is based on two things: one is that the concept of equivalence of structures of the same signature is right (another vague claim that was discussed in the previous section); the other is the purely internal or canonical nature the definition.

7. Multitopic sets and cellular sets

As I said before, the Baez/Dolan announcement [BD1] made a great impression on me. In 1996, having seen the basic ideas of [BD1], we (C. Hermida, J.Power and myself) put [BD1] aside, and worked out a new formulation of "weak n-category" (at first, for $n \in \mathbb{N}$) that I will call "multitopic".

Although the concept of multitopic category was inspired by the Baez/Dolan opetopic category, as a matter of fact, the exact relationship of the two concepts is still obscure to me (despite Eugenia Cheng's work [C]). I believe that the multitopic concept is now worked out in sufficient detail (see also below) to stand on its own feet, and therefore I am not actively trying to relate it to the opetopic approach -- although, of course, I am still interested in what the connections are.

The Baez/Dolan concept, published in [BD2], has two main ingredients: *opetopic sets*, and *universal arrows*. The multitopic concept has its own versions of these: the notion of *multitopic set*, and a particular notion of *virtual composition* in a multitopic set. The latter ingredient will be discussed in the next section.

The three of us came up, in 1997, with the paper [HMP4] whose first two parts have appeared as [HMP1,2] (the third and final part had its proofs returned to the publisher some time ago; so it should appear soon). This work deals with multitopic sets; it does not contain the part on virtual composition.

At this point of time, we have a simple definition of "multitopic set", thanks to [HMZ] (results of this paper were announced at the Toronto meeting of the AMS in September 2000); the papers just mentioned contain a complicated one; you'll see that the complicated definition is not at all superfluous though.

The simple definition in [HMZ] relies on Ross Street's notion of *computad* [S1], [S2].

I will now use " ω-category" for what I may call "strict category"; this is ordinary strict ω-category. (Small) ω-categories form a nice 1-category $\omega\mathtt{Cat}$. Given $\boldsymbol{A} \in \omega\mathtt{Cat}$, and a family $X=\{x_i : d_i \longrightarrow c_i\}_{i \in I}$ of indeterminates x_i with prescribed domain d_i and codomain c_i which are given cells of $\boldsymbol{X}$, required to be parallel, we have $\boldsymbol{B} = \boldsymbol{A}[X]$, the result of simultaneously and freely adjoining all x_i , $i \in I$, to $\boldsymbol{A}$. There is a canonical injection $\iota : \boldsymbol{A} \to \boldsymbol{B}$, in terms of which one can write down a familiar-looking universal property defining $\boldsymbol{B}$. A *computad* is any ω-category $\boldsymbol{X}$ of the form

$$\operatorname*{colim}_{n\in\mathbb{N}} (\boldsymbol{A}_0 \xrightarrow{\iota_0} \boldsymbol{A}_1 \xrightarrow{\iota_1} \ldots \xrightarrow{\iota_{n-1}} \boldsymbol{A}_n \xrightarrow{\iota_n} \ldots)$$

where $\boldsymbol{A}_0=\emptyset$ (empty ω-category), $\boldsymbol{A}_{n+1}=\boldsymbol{A}_n[X_n]$ for suitable sets X_n of indeterminates, and the ι_n are the corresponding canonical injections. It is an easy fact that the indeterminates, that is, the elements of $X=\dot{\bigcup}_{n\in\mathbb{N}} \varphi_n X_n$, with $\varphi_n:\boldsymbol{A}_n \longrightarrow \boldsymbol{X}$ the colimit coprojections, can be recaptured from the computad as an ω-category; this is important, since we want to refer to *the indeterminates* when talking about the computad.

A *morphism* of computads $\boldsymbol{X}\longrightarrow\boldsymbol{Y}$ is an arrow in $\omega\mathtt{Cat}$ which also takes indeterminates in $\boldsymbol{X}$ to indeterminates in $\boldsymbol{Y}$. $\mathtt{Comp}$ is the resulting 1-category of computads; $\mathtt{Comp}$ is a non-full subcategory of $\omega\mathtt{Cat}$.

A *many-to-one computad* is one in which the codomain of every indeterminate of positive dimension is again an indeterminate. $\mathtt{Comp}_{\mathtt{m/1}}$ is the full subcategory of $\mathtt{Comp}$ on the many-to-one computads as objects.

Multitopic sets, whatever they are, form a category $\mathtt{MltSet}$ which is equivalent to $\mathtt{Comp}_{\mathtt{m/1}}$. Thus, this is *one possible definition* of "multitopic set": it is a many-to-one computad.

I remember that we (Hermida, Power and me) were speculating about some such *result*; but we did not, with good reason, want to adopt the above as a *definition*. The "good reason" is that the above definition is "very non-constructive". Of course, it is good to know a simple, conceptual, non-combinatorial description like this one, but you need something constructive that you can really work with for the rest of the definition of "multitopic category". Just think of the two-dimensional case. In this case, we should end up with something that looks like the Baez/Dolan 2-category described in section 4, take-away the universals. In the final analysis, we do get that with the abstract definition too; but it takes some thought to see that we do.

The "constructive" definition of [HMP1,2,3,4] is a recursive one -- just as the definition of computad is recursive (essentially), but more complicated.

The main ingredient is the concept of *multicategory*. Multicategories are closely related to operads; but there is a difference. The main difference can perhaps be expressed by saying that in multicategories, there are permutations (therefore, they are akin to operads with permutations), but, there is no action of arbitrary permutations; there is a fixed permutation called up in each instance of a composition. A less important, because essentially only formal, difference is that in a multicategory, composition is a binary-plus operation. One composes *two* multiarrows at a time; however, one can choose any "fitting" place in the source of the multiarrow into which the composition takes place.

The definition of multicategory is almost fully present in the definition of "Baez/Dolan 2-category" (which, by the way, is so called here to honor these two gentlemen whose ideas began it all; they did not themselves promulgate (as far as I know) this particular, already heavily "multicategorical", definition) in section 4, except for the role of the permutations in the "possibly non-standard amalgamation", which becomes essential in higher dimensions.

The main theorem of [HMP1,2,3,4] is that $\mathtt{MltSet}$ is a presheaf category; in fact, there is a *FOLDS signature* $L=\mathtt{Mlt}^{\mathrm{op}}$, for which $\mathtt{MltSet}\simeq\mathtt{Set}^{L}$. (In the published sources, we wrote $\mathtt{Mlt}$ for what I would now like to write as $\mathtt{Mlt}^{\mathrm{op}}$.) $\mathtt{Mlt}$ is called the category of *multitopes*. If one wants to define it directly, the definition does not get simpler than the definition of "multitopic set" itself. In fact, we derive $\mathtt{Mlt}$ from the terminal multitopic set T; the objects of $\mathtt{Mlt}$ are the cells of T.

Although the definition is definitely "combinatorial" and complicated, one can get used to it, I think, because the intuitions are natural. The difficulty is to put these intuitions in a mathematically meaningful, "closed" form, valid in all dimensions.

Multitopes are the "*shapes of cells*" of all dimensions in a many-to-one computad. Ross Street pointed out that the elements of the terminal computad are the shapes of cells.

By the way, once we know (as we do) that $\mathtt{Comp}_{\mathtt{m/1}}$ is a presheaf category, and in fact, that the exponent category is one-way (first part of the definition of "FOLDS signature"), we can recover L^{op} such that $\mathtt{Comp}_{\mathtt{m/1}}\simeq\mathtt{Set}^{L}$ as a subcategory of $\mathtt{Comp}_{\mathtt{m/1}}$ by an abstract condition on the objects (this is easy). Thus, we do have a conceptual definition for multitopes too.

Looking at the above, we see the corollary that *there is a one-way category* $\mathtt{Mlt}$ *such that* $\mathtt{Comp}_{m/1} \simeq \mathtt{Set}^{\mathtt{Mlt}^{op}}$.

I do not know how to prove this result, even with "one-way" removed, in a way that avoids talking about multitopic sets in the original, constructive, sense. In [B2], Michael Batanin stated a general result in this connection; but the proof, as he himself pointed out to Marek Zawadowski, is incorrect. As a matter of fact, the total category $\mathtt{Comp}$ is *not* a presheaf category.

The work [HMZ] by Victor Harnik, Marek Zawadowski and myself, proving the equivalence of the two definitions of "multitopic set", proceeds by setting up a pair of adjoint functors

$$\mathtt{MltSet} \underset{\langle - \rangle}{\overset{[-]}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \omega\mathtt{Cat}\ ; \qquad (*)$$

$[-]$, the *multitopic nerve functor*, is the right adjoint to $\langle - \rangle$, the "realization" functor, giving the free ω-category $\langle M \rangle$ on any multitopic set M . We show that $\langle - \rangle$ is faithful and full on isomorphisms, and its image is equivalent to $\mathtt{Comp}_{m/1}$.

Let us point out that if we replace $\mathtt{MltSet}$ with the equivalent category $\mathtt{Comp}_{m/1}$, these functors become easily described: $\langle - \rangle$ is just inclusion (because of the way we defined computads); also, $[-]$ can be given by a formula familiar from other "nerves". However, we found that proving that $\mathtt{MltSet}$ was equivalent to $\mathtt{Comp}_{m/1}$ took, essentially, all the work that goes into establishing the adjunction (*) directly.

The nerve functor $[-]:\omega\mathtt{Cat} \longrightarrow \mathtt{MltSet}$ is important because it tells us how a strict category $\boldsymbol{A}$ "is" a multitopic category: it "is" the multitopic set $[\boldsymbol{A}]$ which is in fact a multitopic category.

Let me mention that the paper [HMZ] clarifies a point that has seemed to cause misunderstandings concerning the notion of multicategory. In [HMP1,2,3,4], the *places* in the source of a multiarrow were defined to be a finite initial segment of the positive natural numbers, thereby causing the impression that the implied linear order of the places was important, or in other words, that the multicategory *linearizes* places that may have been without such a linear order in their natural state.

As a matter of fact, the linear order of the integers plays no role whatsoever in the definition of multicategory, which fact will be clear by a careful inspection of the definitions in [HMP1,2,3,4]. In [HMZ], any such misunderstanding is removed by allowing completely arbitrary places instead of just the integers. The axioms in [HMP1,2,3,4] governing the so-called "non-standard amalgamation" simply become the natural conditions for a calculus of abstract places. Thus, we may say that the notion of multicategory in [HMP1,2,3,4] in essence, and in [HMP] explicitly, is *the* concept of *multicategory with abstract places*, as opposed to J. Lambek's original notion (quoted in [HMP2]) which is the notion of multicategory with concrete places and with standard amalgamation. I may add that the change to explicitly abstract places is no real change: any multicategory in the sense of [HMZ] is isomorphic to one according to [HMP1,2,3,4].

In [J], dating from the Fall of 1997, Andre Joyal defined "theta category" as a proposal for (weak, omega-)category.

In many, maybe even most, ways, theta categories may be the best of the competing notions (except that I am not really familiar with the Simpson/Tamsamani proposal, and possibly with other important ones). First of all, the *full* definition is simple ("multitopic category" still has its second ingredient undescribed so far -- but see below). But also, several further elements of the theory are simpler than the corresponding parts for multitopic categories. And simply, theta categories are nice. (Although, multitopic categories are nice too.)

Furthermore, there is a conceptual relatedness of the two notions. For instance, both are based on a concept of a kind of "set", a concept embodied by a presheaf category: multitopic sets in one case, and the so-called *cellular sets* in the other. And both concepts use a set of horn-filling type conditions, although in the multitopic case these are not so easy to state directly.

Then, why bother with the multitopics?

The short, and somewhat imprecise, answer is this. In a multitopic category, we have an explicit framework for *all possible* (virtual) *compositions*; in a theta category, we have a (very) judiciously chosen few of the possible (mostly virtual, but sometimes honest) compositions *that should be sufficient* -- and by the last few words hangs the tale.

If we are at all serious, we should ask ourselves what it should mean that we have enough compositions accounted for in a theta category. And a reasonable way to approach this is to try and see (1) what a sufficiently general idea of composition precisely is, and (2) that this general idea is *de facto* (although not *per definitionem*) incorporated in the concept of theta category.

This is the same thought that dictates that it is not enough to define the notion of monoidal category as it is in fact being defined, but, also, one has to prove a *coherence theorem* for it, one of the kind that Saunders Mac Lane did in fact prove.

I am proposing that the multitopic concept is a kind of "complete" standard to which others should be compared to, and be found satisfactory or wanting as the case may be. Of course, "satisfactory" here means "satisfactory as a *general* notion of category"; a special notion that is not at all equivalent to the multitopic concept may very well be "satisfactory" for the special purpose at hand. I am proposing that one should, if at all possible, *prove* that any given proposal for a *general* concept of "category" is *equivalent* to the multitopic notion.

Of course, this is a proposal for a research program, not an ideological statement. I am giving my motivations and hunches which may or may not be born out eventually by the results of a careful scrutiny. As a research program, these ideas seem, in view of the available evidence, quite reasonable to me.

Let me give some idea of the evidence -- and of the difficulties. I will start with the difficulties.

To illustrate a point, I now return to the concept of virtual functor (of 1-categories; this phrase will be suppressed, but understood, below); this was discussed in section 4.

As I mentioned, I originally named "virtual functor" as "saturated anafunctor". There is a notion of "anafunctor", and "saturated" is a property of anafunctors. Saturation of the anafunctor $F:\boldsymbol{X}\to\boldsymbol{A}$ is a property that ensures that if $A\in\mathrm{Ob}(\boldsymbol{A})$ is a virtual value of F at $X\in\mathrm{Ob}(\boldsymbol{X})$, then any other $B\in\mathrm{Ob}(\boldsymbol{A})$ which is isomorphic to A is also a virtual value of F at the same X (however, saturation is *not the same* as the consequence just described); without saturation, this does not necessarily hold.

An ordinary functor is, essentially as it is, an anafunctor, albeit (usually) not a saturated one.

Since "saturated" is a property definable in FOLDS, it is preserved by the concept of FOLDS-equivalence of anafunctors; the concept of FOLDS-equivalence for saturated anafunctors is identical to the one for anafunctors is general. The correspondences $F \longmapsto F^{\#}$, $F \longmapsto F^{!}$ of saturation, respectively, cleavage, extend to arbitrary anafunctors F, from being defined for ordinary functors F, respectively, for saturated anafunctors F.

Now, let's take an ordinary functor F, and regard it as an anafunctor, and compare it to its saturation, $F^{\#}$. They are both anafunctors, but they are not FOLDS-equivalent, despite the fact that they both give the same cleavage, namely F itself. Is there something wrong with the notion of FOLDS equivalence?

Yes and no. "No", I think, because the concept of anafunctor, without the requirement of being saturated, is not really the notion I want; it is an *incomplete notion*; and the FOLDS equivalence detects this incompleteness. "Yes" because, complete or incomplete, "anafunctor" in general is a good notion (actually, very useful for the work in [M4]), and it is a good notion of equivalence to say that two (general) anafunctors give the same (equivalent) ordinary functors.

It may very well be that it is here that Daniel Quillen's model categories should enter the picture, and the equivalence of "anafunctors" (or, *mutatis mutandis*, of theta categories, etc) in the more general sense will be provided for by the existence of suitable weak-equivalence maps. But I do not want to speculate more about this, since I believe that the FOLDS equivalence is perfectly good as long as one sticks to the saturated entities; moreover, saturation is a canonical process.

Now, to return to multitopic versus theta, all multitopic categories and other multitopic things are automatically saturated, but theta categories are not.

Recall my description of the multitopic virtual functor at the end of section 4. This is automatically saturated; it is in fact completely equivalent to the notion of saturated anafunctor.

On the other hand, in the case of a theta category, the lack of saturation can be seen, for instance, in the "lack of (automatic) communication" between cells of the two respective forms

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\circ\ a"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} U \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow b"] & V \end{tikzcd}$$

(both forms are legitimate types of 2-cell in a theta category).

To see this, consider the fact that, for any strict category $\boldsymbol{A}$, there are two different natural cellular nerves $[\boldsymbol{A}]_1$, $[\boldsymbol{A}]_2$. In both, the 2-cells of the second kind (as b) are the same, namely, the 2-cells of $\boldsymbol{A}$. However, in $[\boldsymbol{A}]_1$, for any given X, Y, Z, f, g, h, there is at most one a as shown: there is one just in case $gf=h$; but in $[\boldsymbol{A}]_2$, a may be an arbitrary isomorphism 2-cell from gf to h. This shows that, in a theta category, cells of type b do not always give rise to their natural counterparts of the form a.

While we are on the "negatives" concerning "theta category", let me "complain" that this concept is a hybrid, in as much its compositions are mostly virtual, but its identities are given by honest operations. The composition of f and g as in the first diagram above is a virtual operation, and it is provided for by the Kan-condition that says that for any f and g as shown, there is at least one pair (h, a) filling the first diagram; h is the virtual composite of f and g. On the other hand, we have an identity arrow

$$\begin{tikzcd} U \arrow[r, bend left, "i"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow 1_i"] & V \end{tikzcd}$$

by a suitable degeneracy-operation $i \longmapsto 1_i$, an honest operation.

This is only a problem when I would like to use my framework for comparing FOLDS specified concepts -- but then, it is a problem.

Another, more vague, "complaint" is that, although theta categories look very natural from the point of view of simplicial homotopy theory, they are less natural when we come to the "ordinary" view of what a category should be like. Take 2-categories, for instance. The Baez/Dolan 2-category, described in section 4, is, essentially, the multitopic notion of 2-category. The theta-version of 2-category would look less "familiar". It would, for instance, have the strange thing that its composition of 1-cells is virtual, but identity 1-cells are given honestly. Also, in other ways, the definition, when completely spelled out, would look more complicated than that of the Baez/Dolan 2-category (which we saw), despite the fact that in the case of the theta-concept, we have a finite signature, whereas for the multitopic 2-category, the signature is infinite.

One can also say that the two concepts are sufficiently different for their essential equivalence to be an interesting question.

One more thing. I think that, broadly speaking, there are *two* application areas HDC's. One is homotopy theory; the other is, I can almost say, the rest. The latter of course is mainly quantum group theory ("higher dimensional algebra", if I want to use a more general term, although it sounds too general now). But any consideration of "the category [of course, in our general sense!] of such and such structures" is automatically in need of the concept of "category". My feeling is that for such applications, i.e., for more or less all except homotopy theory, the style of the multitopic definition is more appropriate than the theta. Of course, this leaves open the possibility that there is a third one that is better than both; I am not passing judgment on that now.

I believe that multitopic categories and theta categories are in fact essentially equivalent. This involves a concept of "saturated theta category", which is given by a regular FOLDS specification, and by another FOLDS specification for multitopic categories; and these two specifications are equivalent in the technical sense described in section 6.

These assertions are, at this time, somewhat conjectural, although many relevant computations have been made. The completion of the idea just sketched out is one of the projects of the present research proposal.

At the June 1998 CMS conference in St John, New Brunswick, I was talking about "protocategories". This was in ignorance of the connections with Joyal's earlier work [J]. But these protocategories were what I above called saturated theta categories.

The Joyal concept contains a fundamental discovery: his identification of the category of certain (so called, by us in [MZ], "simple") ω-categories with the opposite of a new, and very nice, combinatorial object: Θ, the category of finite disks. We (with Marek Zawadowski) rediscovered the fact of those two categories being equivalent, but only after having seen Joyal's Θ itself. We published the proof in [MZ]. Joyal said at the time when we reported on our proof to him that he did not have a proof, although he had suspected the result was true.

The "simple" categories of the previous paragraph have an important connection with Michael Batanin's and Ross Street's work in [B1], [BS], [S]. In particular, Batanin's construction in [B1] of the free ω-category on a globular set plays an important role in [MZ].

[**Digression.** I would like to mention the encouraging fact that, it seems, the concept of theta category is almost FOLDS-specified. Let me explain.

One has a natural notion of homotopy of cellular sets, in the original style for topological spaces and then also for simplicial sets, except that we have to use (homotopy) "invertible" 1-cells, instead of arbitrary ones (since in theta categories, 1-cells are not any more necessarily "invertible").

{Cellular sets have the structure of a simplicial set, and more. In a cellular set, a 1-simplex $X \xrightarrow{a} Y$ is *invertible* if there are: $X \xleftarrow{a'} Y$ and 2-simplices b, b' as in

$$
\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[d, phantom, "b"] & \\
Y & {} & Y \arrow[ul, "a'"'] \arrow[ll, "s_0 Y"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "s_0 X"] \arrow[dr, "a"'] & \arrow[d, phantom, "b'"] & X \\
 & Y \arrow[ur, "a'"'] &
\end{tikzcd}
.
$$

For maps $T \overset{f}{\underset{g}{\rightrightarrows}} S$ of cellular sets, a homotopy $h: f \xrightarrow{\cdot} g$ is $h: \Delta[1]\times T \longrightarrow S$ such that, with $\Delta[1] = \langle 0 \xrightarrow{\lambda} 1 \rangle$, for all $X \in T(\Delta_0)$, $h(\lambda, s_0 X)\Big\downarrow$ from $(0, X)$ to $(1, X)$ is invertible, and, as usual,

$$
\begin{tikzcd}
T \arrow[r, "\cong"] \arrow[rrr, bend left, "f"] \arrow[rrr, bend right, "g"'] & \Delta[0]\times T \arrow[r, shift left, "\delta_1 \times T"] \arrow[r, shift right, "\delta_0 \times T"'] & \Delta[1]\times T \arrow[r] & S
\end{tikzcd}
$$

commutes. f and g are *homotopic*, $f \sim g$, if there is a homotopy $h: f \xrightarrow{\cdot} g$. S and

T, assumed to be theta categories (satisfy Joyal's version of the restricted Kan condition), are *homotopically equivalent*, $S \sim T$, if there are $S \underset{\ell}{\overset{k}{\rightleftarrows}} T$ such that $\ell k \sim \mathrm{id}_S$, $k\ell \sim \mathrm{id}_T$. }

Cellular sets are functors $\Theta \longrightarrow \mathtt{Set}$, with Joyal's category Θ of finite disks. Take the epimorphisms in Θ only, with all the objects of Θ, to get a non-full subcategory $\Theta^{\uparrow}$. $\Theta^{\uparrow}$ is a FOLDS signature; every cellular set S has an underlying $\Theta^{\uparrow}$-structure, written $S^{\uparrow}$. ($\Theta^{\uparrow}$ is the face structure of Θ, with degeneracies removed.)

Proposition **(i)** For theta categories S and T, $S \sim T$ iff $S^{\uparrow} \underset{\Theta^{\uparrow}}{\simeq} T^{\uparrow}$.

(ii) For $T \underset{g}{\overset{f}{\rightrightarrows}} S$, where S and T are theta categories, $f \sim g$ iff there f and g are *FOLDS homotopic*, which, by definition, means that there exists, in $\mathtt{Set}^{\Theta^{\uparrow}}$, a commutative diagram

$$\begin{tikzcd}
& S^{\uparrow} \arrow[dl, "\mathrm{id}"'] \arrow[d, "p"] \arrow[dr, "\mathrm{id}"] & \\
S^{\uparrow} & P \arrow[l, "m"'] \arrow[r, "n"] & S^{\uparrow} \\
& T^{\uparrow} \arrow[ul, "g^{\uparrow}"] \arrow[u] \arrow[ur, "f^{\uparrow}"'] &
\end{tikzcd}$$

In other words, the degeneracy structure in a theta category is important only as far as its existence is concerned. Also note that a theta category is defined as a cellular set S with the restricted Kan condition, which latter property is a property of $S^{\uparrow}$! **Digression ends**]

8. The universe

In [M8], I do two things: one is the precise definition of virtual composition in multitopic sets; the other is the description of the multitopic category of all small multitopic categories.

Both things are very combinatorial. This, of course, will put you off. I will now try to defend the thing.

Let us talk about composition. The great discovery of Baez and Dolan is that one can define the composite of a composable diagram of cells by a universal property. The main consequence is the total lack of a need for coherence conditions (it is a common categorical experience that the coherence structure/conditions are a consequence of the definition by a universal property: witness the tensor product of Abelian groups via the universal bilinear map).

They expressed this by introducing the concept of *universal cell*. A universal cell is like a coprojection in a colimit, or the universal bilinear map in the definition of tensor product of Abelian groups. The complication arising here is that the universal property of a universal k-cell involves cells of all dimensions $\geq k$ present in the multitopic set.

What I do in [M8] is find a formulation of the universal property in the style which is the one customarily used for the definition of adjoint functors. I mean the definition that talks about the *natural bijection* between $\mathrm{hom}(FX, A)$ and $\mathrm{hom}(X, GA)$; this does not talk directly about "(co)projections", that is, about the unit and the counit maps. This "adjoint" style definition moves around a lot more entities than the "(co)projection" style definition does; but it is also, somehow, more natural (?).

Notice the word "bijection" in the previous paragraph; I could have said "isomorphism"; and of course, in higher dimensions, such as 2-categories, it becomes "equivalence" of hom-categories, etc. This is what I am equipped to deal with, by using the FOLDS equivalence. I define composition in a multitopic set by using (many) FOLDS equivalences.

This requires the use of a system of auxiliary signatures. The whole thing becomes a calculus of signatures. Since the signatures are basically embodying shapes of diagrams, multitopes in this case, the definitions I am talking about are using a calculus of multitopes. There is, for instance, an operation of substitution, or insertion, of a multitope into another multitope. I find it all very natural -- but there is no denying that the thing is *very combinatorial*.

Each of the FOLDS equivalences used here can be regarded as (*winning*) *strategies* in a two-person, infinite game of perfect information; "infinite" because there is a first, second, .. nth,... move, one for all $n \in \mathbb{N}$.

This all makes sense intuitively. Look at the case of composition of two 1-cells, the simplest example:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h \;\text{"="}\; g\circ f"'] & & Z \end{tikzcd} .$$

In a multitopic category, getting h is not an easy matter since it has to satisfy some complicated universal property that refers to (many) cells of all dimensions. The basic idea can be put in this way. h does qualify as $g\circ f$ if " h can be replaced in any situation by the pair (f, g) , and vice versa". This is quite imprecise though. Imagine we have the situation

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[drr, "j"'] & Z \arrow[d, phantom, "\swarrow v"] \arrow[dr, "i"] & \\ & {} & W \end{tikzcd}$$

involving our h . Replacing h by (f, g) means considering the shape:

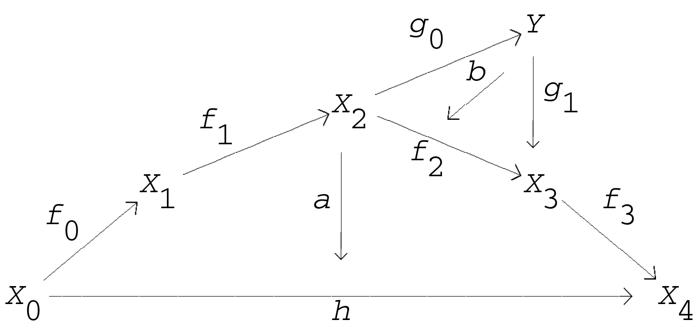
$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \hat{v}"] & & \\ X \arrow[ur, "f"] \arrow[drrr, "j"'] & {} & Z \arrow[dr, "i"] & \\ & & & W \end{tikzcd}$$

One should have the *existence* of $\hat{v}$ as part of the replacebility of h by (f, g) in the present situation. But, not just the existence of $\hat{v}$ is what matters. Imagine now two bigger situations each involving one place for a 3-cell, one of the new situations extending the situation with v , the other that with $\hat{v}$. If one can be filled with a 3-cell, the other should also be possible to fill.

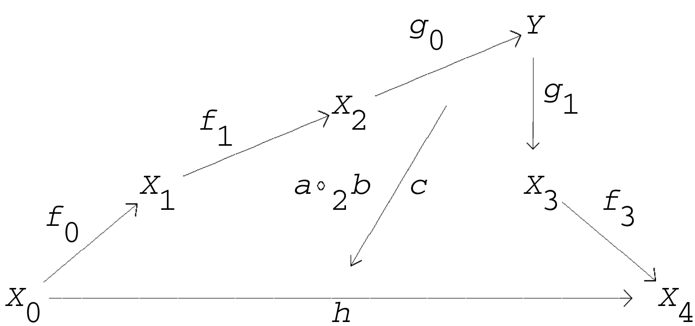
We have kept both v and $\hat{v}$. We play a game in which the Challenger takes further and further cells fitting in with either h or (f, g) , *and* with the other cells previously chosen in the course of the game. The Hero has to answer each move of the Challenger (they alternate) with a cell of the same dimension -- but not of the same shape -- in the other situation. The moves are cumulative: they do not forget anything. If Hero can keep it up forever, he wins; otherwise he loses.

Such games are familiar in foundations, in a very different context ("axiom of determinacy" in Set Theory; maybe the hottest thing in Set Theory nowadays). The Set Theory games do not have the varied syntactical/combinatorial forms the FOLDS games possess here, but the two (the set theory games and the FOLDS games) share one very important feature. This is that *strategies can be composed canonically.* Let me indicate what this means.

Let us put ourselves into a multitopic set M . Let's look at the shapes that were shown in section 4, but now from a different perspective:



and



Now, we have 0- and 1-cells in M; but here, a , b and c signify not 2-cells, but *strategies*: a is the strategy, that is, FOLDS equivalence (in a suitably fancy auxiliary signature), that exhibits h as a (virtual) composite of (f_0, f_1, f_2, f_3) . Similarly for b

as well as for c. Now, the thing is that there is a canonical, honest, composition of strategies that, once we have a and b, gives $a\circ_2 b$ as one particular suitable c.

There is no such composition, directly at least, of "universal arrows", although there is a *virtual* composition for them: if in the above, a and b signify universal 2-cells, then no matter how, but virtually, we compose them into $a\circ_2 b$, we get another universal 2-cell.

One can sense that this (the honest composability of strategies) should be right. a tells us *completely* why h is a composite of the f_i's. b tells us *completely* why f_2 is a composite of the g_j's. Therefore, *I should be able to exhibit directly a complete reason why* h *is a composite of* $(f_0, f_1, g_0, g_1, f_3)$. And in fact, I can: the resulting complete reason is $a\circ_2 b$.

This phenomenon of canonical composites of strategies goes further. Imagine, for instance, that we are looking at the "associativity isomorphism". This involves four 1-cells, and several compositions of them. The "associativity isomorphism" α appears as a higher order strategy: "arrow" between composites of strategies. The point is that α is canonically/honestly given from the strategies for the compositions.

The upshot is that, having a multitopic category, which is nothing but a multitopic set in which all compositions can be performed, we can, by a single act of a (big) simultaneous cleavage, choosing one particular strategy for each composition, obtain a "category" in which all operations are honest.

The latter thing is not written in [M8]: it is one of those things that still have to be put into "closed form". But this promises to bring us closer to the *equivalence of the multitopic categories with an honest-operational concept* such as Michael Batanin's one.

Next, the category $\mathtt{MltCat}$ of all categories.

The starting point is the description at the end of section 4 of "virtual functor". We are talking about a 1-cell $\boldsymbol{X}\xrightarrow{F}\boldsymbol{A}$ in $\mathtt{MltCat}$. The definition is almost completely given as at the place in section 4; note though that there we only had the case when $\boldsymbol{X}$ and $\boldsymbol{A}$ were 1-categories; here they are arbitrary ((small) multitopic, ω-dimensional) categories. But the definition does not really change; of course, now F *is* a category (!), not a 1-category.

The remarkable thing is that one does not have to worry separately about the effect of the "functor" F on 1- and higher-dimensional cells; the explicit dispensation about the effect on 0-cells, and the fact that F is a category (which, of course, is a lot) is enough.

I really think that this notion of 1-cell in $\mathtt{MltCat}$ is *natural*. I do not think it will be delivered by some abstract principle or consideration. One just have to take it as it is.

The rest of the definition of $\mathtt{MltCat}$: the cells of all dimensions and all shapes in $\mathtt{MltCat}$, is a kind of guessing game, trying to generalize the idea in the 1-cell. The basis of any given cell in $\mathtt{MltCat}$ is a *colored multitopic set.*

The first thing is to describe, for an arbitrary multitope π , what the *π-colored multitopes* are. The *colors* are as many as there are cells in π . In the case of the 1-cell $\alpha \xrightarrow{\beta} \gamma$ as π , our only example, the colors are $\alpha=(0, 0)$, $\beta=(0, 1)$ and $\gamma=(1, 1)$. Of course, the colors are an added structure on the multitopic set.

But I will not carry on.

What the definition does is the complete description of a large multitopic set $\mathtt{MltCat}$. It is a "theorem" that this multitopic set is in fact a multitopic category, i.e., "all possible composites exist in it". This "theorem" is not proved in [M8]; and I have not completely written out the details yet. But I have done many particular calculations, and these fall into suggestive-enough patterns. I am fairly certain that the "theorem" is correct.

Of course, it is not enough to have this theorem. Further tests are needed.

Another "theorem" is that the 1-cells of $\mathtt{MltCat}$ as defined here correspond "essentially bijectively" to the morphisms that Baez and Dolan originally suggested: these are the morphisms of multitopic sets that preserves composites (in a sense that is analogous to the notion of a functor preserving binary products).

Furthermore, the maps $\boldsymbol{A} \longmapsto [\boldsymbol{A}]$ from a strict category $\boldsymbol{A}$ to the corresponding multitopic category (the nerve) should be extended to a map from all the cells of the strict ω-category of all small strict ω-categories to corresponding cells in $\mathtt{MltCat}$; this has not been done yet (although I do not see it as a difficult thing).

Let me note that MltCat contains, for each n, the category of all small n-categories. For a finite n, an n-category is a multitopic category which is, in a specific sense, truncated (but not as trivially as simply throwing away all cells of dimension higher than n. I prefer to use "truncated" as a property of a multitopic category, rather than referring to something else obtained from the multitopic category by an act of truncation.) The n-categories form a "full subcategory" of MltCat, called $\mathtt{MltCat}_n$. $\mathtt{MltCat}_n$ is also truncated: it is an $(n+1)$-category, the $(n+1)$-category of all small n-categories.

From my point of view, the most important thing to do, after the ones indicated above, is to establish the *exactness properties* of MltCat. The underlying foundational goal is to formulate formal axioms that talk about MltCat; these formal axioms will be first order, in fact FOLDS (!), statements of the exactness properties.

Of course, the original forms of the exactness properties, such as Cartesian closure, have to be altered to fit the new, virtual, ω-dimensional, context. But that is not a problem at all. The FOLDS equivalences, or strategies, delivering equivalences of hom-categories will be very useful in this.

Perhaps, here I mean by "exactness property" something more general than traditionally done. For instance, saying that the category of sets, or any Grothendieck topos, is an elementary topos is, for me, entirely within saying things about the exactness properties of Set, or the Grothendieck topos.

MltCat is just as "unique" an entity as Set is (assuming that I am right, and I did not make a mistake in my guessing of what MltCat actually is!). It is worth our attention.

MltCat also has infinitary exactness properties; the Giraud definition of "Grothendieck topos" is entirely, except for the small-generated part, infinitary exactness conditions.

We can speculate further on *concepts of universes* in general.

Small theta-categories should also form a theta category Theta.

Let me point out that if we knew what is the *equivalence of specifications* of multitopics and saturated theta's, without talking about any higher-than-zero dimensional cells in $\mathtt{MltCat}$ or $\mathtt{Theta}$, in the style of section 6, then we would know how to get $\mathtt{Theta}$ from $\mathtt{MltCat}$, and vice versa.

Using the notation of section 6, assume $T=(L, Q_0)$ and $T'=(L', Q_0')$ are the specifications of the concept of multitopic category, and of the concept of saturated theta category (or, protocategory), respectively; and assume that these two specifications are equivalent as in section 6, and we have F, G, (P, m, n) and (P', m', n') witnessing these facts.

We have that $\mathtt{MltCat}$ is a functor $\mathtt{MltCat}:\boldsymbol{B}\xrightarrow{\mathrm{lex}}\mathrm{SET}$ into the category of large sets. $\mathtt{Theta}$ can be taken to be the composite

$$\mathtt{MltCat}^{*} \underset{\mathrm{def}}{=} \mathtt{MltCat}\circ G\ .$$

And vice versa: if $\mathtt{Theta}$ is given, $\mathtt{MltCat}$ can be taken to be

$$\mathtt{Theta}^{\#} \underset{\mathrm{def}}{=} \mathtt{Theta}\circ F\,.$$

We will have that $\mathtt{MltCat}^{*\#}$ is L-equivalent to $\mathtt{MltCat}$, by the equivalence-span $(\mathtt{MltCat}\circ P, \mathtt{MltCat}\circ m, \mathtt{MltCat}\circ n)$, and similarly, that $\mathtt{Theta}^{\#*}$ is L'-equivalent to $\mathtt{Theta}$.

If both $\mathtt{MltCat}$ and $\mathtt{Theta}$ are given in advance, then *we'd better have* that $\mathtt{Theta}^{\#}$ is L-equivalent to $\mathtt{MltCat}$, and that $\mathtt{MltCat}^{*}$ is L'-equivalent to $\mathtt{Theta}$ (either of these two statements implies the other).