On comparing definitions of "weak n-category"

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August 31, 2001; revised: September 7, 2001

1. My approach is "foundational". On the one hand, I am motivated by the problem of the foundations of mathematics (an unsolved problem as far as I am concerned). On the other hand -- and this is more relevant here --, I start "from scratch", and thus what I say can be understood with little technical knowledge. I only assume a modest amount of category theory as background.

I will talk informally about technical matters that are written down formally elsewhere, where they can be studied further.

[The text in square brackets [-] is either some technical explanation, or a digression.]

2. Terminology

First, some terminological conventions. I will use the word "category" in its most general sense: weak ω-category. This is completely inclusive: all sorts of "categories" are categories now.

There are two extensions of the original meaning: "weak", and "omega-dimensional".

"Weak" signifies an *indeterminate* notion; there are several different specific versions of weak category. It can also be used as a *vague* notion, when one is merely looking at what one *would like* to have. There are *specific* kinds of category, such as "Batanin category" [B1], "multitopic category" [HMP1,2,3,4,5], [M8]. When one wants to talk about the "ordinary, strict" version of the notion, one says "strict". Thus, "strict category" is my term for an ordinary, strict ω-category.

The main reason for the terminology is the desire to banish the word "weak".

n-categories, for finite n, are "truncated" categories; they *are* particular kinds of, possibly weak, categories (namely, ω-categories) in fact.

One slightly unpleasant thing with this convention is that one should say "let $\boldsymbol{C}$ be a 1-category" in place of "let $\boldsymbol{C}$ be a category". On the other hand, "the category of groups" is a perfectly good name for the (ordinary) category of groups.

It can safely be assumed that all versions of 1-category are essentially the same; thus " 1-category" is specific. But, " 2-category" is an indeterminate notion; "strict 2-category" is one specific version; "bicategory" (which can also be called "Benabou 2-category"), "Baez-Dolan, or opetopic, 2-category" [BD2], "Batanin 2-category" [B1], "multitopic 2-category" (see below) are further specific concepts.

The best thing about this terminology is that it makes good sense of talking about "the category of all categories"; see below.

3. Virtual vs honest operations

I want to make a distinction in the typology of the existing concepts of "category", one that is basic for the present purposes. I cannot make it *entirely* precise in general; but it will get gradually more and more precise.

Some existing concepts are *honest-algebraic*: they are made up of honest, univalued, algebraic operations. Some others are *virtual-algebraic*: they are made up of *virtual operations*. And then there are ones that mix the two types of operation.

Of course, the honest-algebraic type is the well-known one: it is in fact the one that one automatically expects when a new concept of "category" is brought up. For instance, "bicategory" is pure-algebraic; it is a concept that is (even) monadic over the category of 2-graphs (2-dimensional globular sets). Of course, a morphism of algebras of the monad is not the same as the intended notion of morphism (homomorphism of bicategories); the former is a special case, the *strict* case, of the latter. The Gordon/Power/Street "tricategory" [GPS] is also honest-algebraic, and so is Michael Batanin's concept of category [B1].

Virtual operations, an expression that I learned from John Baez and James Dolan [BD2], are in fact almost as well-known to category theorists as algebraic operations, even if the expression may be new. The operations defined by *universal properties* are virtual. There will be virtual operations here that are not given via universal properties, but universal properties will remain the main and preferred source for them.

[My work [M1,2,3,4,5,7] on "virtual operations" predates [BD2], even the famous announcement [BD1]; of course, I will say more about the work below.]

A definition of a (virtual) operation via a universal property is good because an *equivalence of categories automatically respects/preserves the operation*. For instance, a (standard) equivalence of 1-categories takes a product diagram into a product diagram; there is no need for a separate notion of equivalence of 1-categories-with-products; 1-category equivalence will do.

Since we are interested here in the concept of equivalence (which, of course, is something vague in the context of arbitrary categories), this is important: if we have defined a concept of category in such a way that a certain ingredient is defined by a universal property in a basic structure, the notion of equivalence for the whole concept can be taken to be identical to that of the basic structure. We will see how this will become operative in the definition of multitopic category, for instance.

A general feature of virtual operations is that any such determines its value, at each legitimate argument-complex, up to "isomorphism" only. I have put "isomorphism" in quotes because it may have to be replaced by something else, such as "equivalence", in a higher dimensional context. This is what happens with the operation of binary product in a 1-category having binary products, to take an example.

Another example for virtual operation is in the notion of (Grothendieck) fibration.

[Here, we have two categories $\boldsymbol{B}$ and $\boldsymbol{E}$, the base category and the total category, respectively, plus a functor $\begin{array}{c}\boldsymbol{E}\\ \downarrow p\\ \boldsymbol{B}\end{array}$ between them; we require, for each $X\xrightarrow{u}Y$ in $\boldsymbol{B}$ and each B in $\boldsymbol{E}$ over Y (meaning $p(B)=Y$), the existence of some A over X, together with an arrow $A\xrightarrow{f}B$ over u, with the universal property of a so-called Cartesian arrow. A here is denoted by $u^{*}(B)$ (and f as c_{u}^{B}), indicating that we are looking at A as the result of an operation (on u and B), when in fact, A is only determined up to isomorphism (in the part of E over X): $(u, B)\longmapsto u^{*}(B)$ is a *virtual operation*.]

This latter example is significant because it comes with a parallel honest-algebraic concept, that of pseudofunctor; and a good occasion arises to relate the two types (virtual-algebraic and honest-algebraic).

We (I mean: category theorists commonly, if not universally) consider the two notions (fibration and pseudofunctor) as two forms of essentially the same concept. One can pass from a fibration to the corresponding pseudofunctor and vice versa; and there is no loss of information in the process. Having said that, we must point out the asymmetry in this process.

Let's use the letter p for a fibration (because of the notation above); and F for a pseudofunctor.

[The latter is a "non-strict" (pseudo) version of a functor $\boldsymbol{B}^{\mathrm{op}} \longrightarrow \mathtt{Cat}$, into the category of all (small) categories (I use $\boldsymbol{B}^{op}$ because of the $\boldsymbol{B}$ in p). Because of the 2-dimensional structure of $\mathtt{Cat}$, one can make preservation of composites in $\boldsymbol{B}^{\mathrm{op}}$ to hold up to specified isomorphisms (only); the latter are to satisfy *coherence conditions* (which are the "bad guys" of our story); this is what takes place in the notion of "pseudofunctor".]

Let's write $p \longmapsto p^!$, $F \longmapsto F^\#$ for the two transitions in question. I will call the first *cleavage*, the second *saturation*.

[Cleavage starts by making simultaneous choices of the object/arrow pairs $(u^* B, c^B_u : u^* B \longrightarrow B)$, one for each B in $\mathrm{Ob}(\boldsymbol{B})$. The rest of the construction of the pseudofunctor $p^!$ is canonical. The process $F \longmapsto F^\#$ is known as the Grothendieck construction; it is entirely canonical.]

The asymmetry lies in the fact that cleavage is non-canonical, involves arbitrary choices; whereas saturation is canonical. In fact, the notation $p^!$ is an abuse; it is the same kind of abuse, only worse, as when we write $A \times B$ for "the" product of objects A, B.

It is pretty clear that I am heading to a conclusion to the effect that "fibration is good, pseudofunctor is bad", and more generally, "virtual-algebraic concepts are good, pure-algebraic ones are bad".

What I really want to say is that every time you relate a fibration/pseudofunctor to the larger world around it, you should use the fibration form; when you work inside the thing (fibration/pseudofunctor), you may be better off using the pseudofunctor form. Since now I am more interested in the global "super-"structures "relating everything" than in the practicalities of computing in individual structures, I now prefer the fibration form.

Whether or not we prefer one form to the other, the question of the *equivalence* of the two forms remains interesting.

The fact that the two forms of the fibration/pseudofunctor concept are *equivalent* (and now, we are coming to Tom Leinster's Mork and Mindy in his [L]) is that we have *equivalences* $\xi:(p^!)^\# \xrightarrow{\simeq} p$, $\zeta:(F^\#)^! \xrightarrow{\simeq} F$, and in fact, ξ *canonically* depends on (explicitly defined in terms of) p and $p^!$, ζ on F and $(F^\#)^!$.

As Tom Leinster says, we need a notion of morphism of fibrations, another one of pseudofunctors, and more in the way of "natural transformations", to be able to say what these equivalences ξ and ζ are. These notions are all available. For instance, one has "pseudonatural transformations", etc.

Without going into detail, let me say that those notions for fibrations are *simpler* than the corresponding ones for pseudofunctors. The ones for pseudofunctors involve (further) coherence structures and conditions; the ones for fibrations do not. In particular, ξ is simpler than ζ.

4. New virtual operations

I want to mention certain further, and lesser-known, virtual-algebraic concepts. For these, the "equivalent" pure-algebraic versions are very well known indeed (unlike "pseudofunctor").

The first is the virtual-algebraic counterpart of the notion of functor of (ordinary) 1-categories; let's call it, with John Baez and James Dolan, "virtual functor"; I had called it "anafunctor", or more fully, "saturated anafunctor", before; see [M4].

How does this concept arise?

"There is no equality of objects of a category; only isomorphism": this adage appears repeatedly in categorical writings; and it is in fact one of the starting points for the foundational view that I am trying to elaborate in my work (here I will (mostly) spare you the "idle thoughts" of foundations; but you may want to see [M6], [M7]). In view of the adage, a functor $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ is doing something bad: for a given $X\in\mathrm{Ob}(\boldsymbol{X})$, it picks out a definite object $F(X)$ in $\boldsymbol{A}$, instead of determining a value-object up to isomorphism only.

Surprisingly, this can be remedied. One can introduce a concept of "virtual functor" that determines its value *exactly* up to isomorphism, *and*, this concept of "virtual functor" is not so far from the ordinary concept of functor as to destroy, or even alter seriously, the usual manipulations and uses of functors one is used to. In fact, virtual functors are *better* than ordinary functors, because of the fact that we can construct them canonically in situations when the corresponding functor needs arbitrary choices. The simplest example for this is the product functor $\boldsymbol{C}\times\boldsymbol{C}\to\boldsymbol{C}$, mapping (A, B) to $A\times B$; its virtual version is canonical; whereas the $A\times B$ are not really there before we have made a system of simultaneous choices!

For all this, and for the relevant history as far as I know it, see [M4].

The second is "virtual monoidal category". This actually occurred to me before "anafunctor"; I used it in [M1,2,3]. The idea is (now) obvious: one wants $A\otimes B$ to be determined up to isomorphism only -- as it should be according to the adage. Of course, one also wants to hold onto the original concept in its essentials. It is possible to do this.

The best thing about it is that the concept of morphism of monoidal category changes, from the somewhat complicated (ad hoc?) original (which Saunders Mac Lane decided not to include in the 1971 edition of his book "Categories for the Working Mathematician", although the concept of (not necessarily strict) monoidal category is discussed in detail in the book) to the notion which is the straight-forward notion of *structure preserving mapping*. You can see the virtual monoidal categories and even the virtual bicategories (anabicategories) in [M4].

Another good thing is that the usual examples become canonical, rather than depending on arbitrary choices as they do in their common forms. Take, for instance, tensor product of Abelian groups. The definition depends on the *arbitrary choice* of a universal bilinear arrow $(A, B)\to A\otimes B$. In the virtual concept, you do not have to make any choice!

It should be pointed out that the concepts of "virtual functor" and "functor" (of 1-categories, for now) on the one hand, and the concepts of "virtual monoidal category" and "monoidal category" on the other, are *equivalent*, in the very same way as "fibration" and "pseudofunctor" were described to be equivalent above.

(One should keep this in mind while weighing the relative merits of the notions. There are some important *canonical* functors, such as Yoneda; if their virtual versions, their *saturations*, were not canonical, things would be bad; but no, "saturation" is a canonical process, unlike "cleavage".)

Both concepts discussed above can be improved on: we can arrange that the virtual operations are defined by *universal properties*. The welcome effect is the disappearance of coherence (structure and conditions) (which, by the way, are still there in "anafunctor" and "anamonoidal category"). In both cases, the negative effect is the need for *more entities* to be included in the structures than there were before (a kind of opposite of Occam's razor is operative here). Let me explain.

My personal background here is my reading of the announcement [BD1] that John Baez and James Dolan wrote to Ross Street about their n-categories at the end of 1995. This acted as a revelation on me. Although I did not understand everything in detail at first, I right away understood enough to see that here was, at least the essence of, *the* definition of n-category ($n\in\mathbb{N}$; I was not thinking of ω-categories yet; neither were Baez and Dolan at that time, apparently) that suited my purposes. And right away I understood two elements of the picture: the Baez/Dolan 2-category, and the specific form that my saturated anafunctor (virtual functor) should be presented in (resulting in the exact same notion mathematically, mind you).

Let me start with the Baez/Dolan 2-category (same as multitopic 2-category). This is a very simple and intuitive notion; and the proof of the fact that it is *equivalent* to "bicategory" is fundamental to see. A B/D 2-cat has, as you would expect, 0-, 1- and 2-cells. There is nothing new with the 0- and 1-cells, except that we do *not* define composition of 1-cells. A 2-cell a has a domain da which is a composable string of 1-cells, possibly empty (in which case the 0-cell which is both the start and the end of it is still there), and a codomain ca which is a single 1-cell but one which that matches the domain da as far as the start- and end-0-cells are concerned.

Next, we have *identity* 2-cells, and an *honest* (for now!) *composition* of 2-cells. I skip identities. The composite $c=a\circ_3 b$ for a and b from the picture

$$
\begin{tikzcd}
 & & & Y \arrow[dd, "g_1"] & \\
 & & X_2 \arrow[ur, "g_0"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\swarrow b"] \arrow[dd, phantom, "\Downarrow a"] & & \\
 & X_1 \arrow[ur, "f_1"] & & X_3 \arrow[dr, "f_3"] & \\
X_0 \arrow[ur, "f_0"] \arrow[rrrr, "h"'] & & & & X_4
\end{tikzcd}
$$

is of the form

$$
\begin{tikzcd}
 & & & Y \arrow[dd, "g_1"] & \\
 & & X_2 \arrow[ur, "g_0"] \arrow[dd, phantom, "a\circ_2 b \ \swarrow c"] & & \\
 & X_1 \arrow[ur, "f_1"] & & X_3 \arrow[dr, "f_3"] & \\
X_0 \arrow[ur, "f_0"] \arrow[rrrr, "h"'] & & & & X_4
\end{tikzcd}
$$

Composition is a three-argument operation; it is a *placed* composition of two 2-cells, the place being 2, picking out f_2, in the example.

Formally, $a\circ_p b$ is well-defined iff $(\mathrm{d}a)(p)=\mathrm{c}b$; $\mathrm{d}(a\circ_p b)$ is the string obtained by replacing the single term $(\mathrm{d}a)(p)$ at p in the string $\mathrm{d}a$ by the string $\mathrm{d}b$; $\mathrm{c}(a\circ_p b) = \mathrm{c}a$.

There are four laws, two of them concerning identities, the third an associative law, and the fourth a commutative law. These might as well be called the law for serial composition, and the law for parallel composition, respectively. I think, you will immediately see what these laws should be. The simple idea is, of course, that when you see a "composable" diagram of interlocking 2-cells, the composite should be independent of the order in which you perform the compositions: the notion is *purely geometric: what you see is what you get.*

Incidentally, the composition structure of 2-cells is what we call a *multicategory*; this will become important in the notion of general multitopic category; see below.

That's all as far as the *data* for the B/D 2-category are concerned. Next, there is a definition, and an imposed condition.

The definition is that of a universal 2-cell. We say that a (refer to the above picture) is *universal* if for all c as above, there is a unique b as above such that $c=a\circ_2 b$ -- except that when here I say "for all c as above", I do not just mean the one particular shape of c as in the second picture, but all possible shapes "extending" the given shape for a .

Finally, the condition: for every diagram as

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "f_2"] & & \\
 & X_1 \arrow[ur, "f_1"] & & X_3 \arrow[dr, "f_3"] & \\
X_0 \arrow[ur, "f_0"] & & & & X_4
\end{tikzcd}$$

(*horn!*) there is h and a (not necessarily unique) *universal* a as in

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "f_2"] \arrow[dd, phantom, "\Downarrow a"] & & \\
 & X_1 \arrow[ur, "f_1"] & & X_3 \arrow[dr, "f_3"] & \\
X_0 \arrow[ur, "f_0"] \arrow[rrrr, "h"'] & & & & X_4
\end{tikzcd}$$

Of course, all possible shapes for a are meant; the empty-domain 2-cells are especially important: they give the virtual identity 1-cells.

That is the end of the definition of "Baez/Dolan 2-category".

The obvious thing is that, in the B/D-2-category, instead of having an ordinary algebraic composition of 1-cells, we have a *virtual* such: h above is *a* composite of (f_0, f_1, f_2, f_3) , the one *via* a .

It is a delight to see how a B/D-2-category has all the structure of a bicategory -- once you have made a simultaneous arbitrary choice of a universal a for each horn as above (but of course, of an arbitrary size) (cleavage). It is even better to work without cleavage, and get an anabicategory [M4] briefly alluded to above. While doing so, one observes that one only needs

k-ary 2-cells for k with $0 \leq k \leq 4$ and no more. This of course means that there is an essentially obvious "truncated" version of the B/D-2-cat *that should be as good as the complete notion*; this turns out to be right, in a suitable, new, and precise, sense; see below.

The reader may try his/her hand at defining the multitopic 2-category of 2-sided modules over variable base-rings. (The rings are the 0-cells, the modules are the 1-cells; the 2-cells are multilinear maps. Composition of 1-cells should turn out to be "multi-"tensor product. Everything is canonical. The proofs of the laws are interesting, but, of course, standard.)

It is just as nice to see how a virtual functor $F: \boldsymbol{X} \to \boldsymbol{A}$ should look like. This will have the added beauty that it will be visible that the notion should straightforwardly generalize to a concept of "functor" for n-categories $\boldsymbol{X}$, $\boldsymbol{A}$, for arbitrary n. But for now, let $\boldsymbol{X}$, $\boldsymbol{A}$ be 1-categories; X, Y are objects of $\boldsymbol{X}$; A, B those of $\boldsymbol{A}$.

F is a 1-category (!) whose 0-cells are those for $\boldsymbol{X}$ and those of $\boldsymbol{A}$ (disjoint union). F has three distinct types of 1-cell: type-$(0,0)$, which is of the form $X \to Y$, type-$(0,1)$: $A \to B$ and type-$(1,1)$: $X \to A$. *There is no arrow of type*-$(1,0)$, i.e. $A \to X$. The $(0,0)$-arrows are exactly those of $\boldsymbol{X}$, the $(1,1)$'s those of $\boldsymbol{A}$. Definition: a $(0,1)$-arrow $u: X \to A$ is *universal* if for all $x: X \to B$ (type-$(0,1)$), there is a unique [this uniqueness is removed for higher dimensions!] $a: A \to B$ (type-$(1,1)$) such that $x = a \circ u$. Requirement: for all X, there are at least one A and a universal $u: X \to A$. End of definition of virtual functor.

5. The general concept of equivalence for virtual–algebraic structures

In the Summer of 1995 I presented my then-new theory of First Order Logic with Dependent Sorts (FOLDS) at two conferences, and I submitted a text of it, [M5], for publication in the Springer Lecture Notes in Logic (there is such a thing) in the Fall of the same year. In 1996, it was accepted for publication. However, I withheld it, pending revisions, not because of the errors (I think, there are only minor ones), but rather because I wanted to do some things in a more elegant manner. It is still unpublished. A detailed announcement is contained in [M7].

One of the two main ingredients of this theory is the thing in the title of the present section; it is called *FOLDS-equivalence*. *This notion, in itself, has nothing to do with logic,* although I will keep calling it FOLDS-equivalence, for its *relation* to logic that will be explained later. Here is the definition.

First of all, there is a concept of *FOLDS signature*. This is any 1-category L with the following properties: for all $K\in\mathrm{Ob}(L)$,

- **(i)** $\mathrm{end}(K)$ $(=\mathrm{hom}(K,K))=\{1_K\}$;
- **(ii)** the set $\{f\in\mathrm{Arr}(L):\mathrm{dom}(f)=K\}$ is finite;
- **(iii)** L is skeletal.

A FOLDS signature L has its object-set graded: $\mathrm{Ob}(L)=\dot{\bigcup\limits_{n\in\mathbb{N}}}L_n$ such that $K\in L_n$ iff for all $K\xrightarrow{f}K'$, $f\neq 1_K$, we have $K'\in\dot{\bigcup\limits_{k<n}}L_k$. An object in L_n has *dimension* n.

An *L-structure* is a Set-valued functor $M:L\to\mathrm{Set}$.

An example is the category that I denote by $(\Delta^{\uparrow})^{\mathrm{op}}$; it is the subcategory of the simplicial category Δ^{op} (Δ being the skeletal category of non-empty finite orders) with all the objects of Δ, but with arrows only the *injective* ones of Δ. That is: from the simplicial category, keep the face operators, throw away all degeneracies. Thus, every simplicial set $S:\Delta^{\mathrm{op}}\to\mathrm{Set}$ has an *underlying* $(\Delta^{\uparrow})^{\mathrm{op}}$-structure that I will denote by $S^{\uparrow}$. (Peter May told me that $(\Delta^{\uparrow})^{\mathrm{op}}$-structures are called Δ-sets, and they have an extended literature; unfortunately, I am not familiar with that literature yet.)

Let L be a FOLDS signature. We place ourselves into the category $\boldsymbol{A}=\mathrm{Set}^L$ of all L-structures. M, N, P are objects of $\boldsymbol{A}$.

A morphism (in $\boldsymbol{A}$) is *fiberwise surjective* (fs) if it has the Right Lifting Property (familiar from D. Quillen's model categories) with respect to all injective morphisms. Thus, fs is like "trivial fibration", except that we are not contemplating any other ingredient of a Quillen model category. One gets an equivalent definition (as expected) when one takes the class of arrows $\dot{K}\xrightarrow{\iota}\hat{K}$, for all $K\in\mathrm{Ob}(L)$, in place of all injective arrows; here $\hat{K}=\mathrm{hom}_L(K,-)$ and $\dot{K}$ is the subfunctor of $\hat{K}$ that misses just one element, 1_K, of $\hat{K}$; ι is the inclusion.

An *equivalence* $E:M\simeq N$ is a span $(P,m,n):M\xleftarrow{m}P\xrightarrow{n}N$ of fs morphisms m and n. M and N are *equivalent,* $M\simeq N$, if there is $E:M\simeq N$. For emphasis, we may write $M\simeq_L N$ in place of $M \simeq N$.

Here is a fact, which should be well-known (is it?). For simplicial sets S and T satisfying the Kan condition (S and T are fibrant), S and T are homotopically equivalent iff $S^{\uparrow} \simeq T^{\uparrow}$. A slightly unfamiliar feature of this fact might be that, having started with simplicial sets, we drop degeneracies, and compare the face-structures only.

A large part of the monograph [M5] is devoted to showing that the FOLDS equivalence captures the various existing notions of "equivalence" in category theory. One has to present the category, or categorical structure (possibly consisting of two categories and a functor between them, for instance), call it $\boldsymbol{C}$, as an L-structure, for a suitable FOLDS signature L. This invariably means taking the "natural" virtual-algebraic version, or saturation, written $\boldsymbol{C}^{\#}$, of the given $\boldsymbol{C}$, and see that there is an "obvious" FOLDS-signature L for which $\boldsymbol{C}^{\#}$ is (naturally) an L-structure. We obtain that $\boldsymbol{C} \simeq \boldsymbol{D}$ (meaning the operative categorical equivalence) iff $\boldsymbol{C}^{\#} \simeq_L \boldsymbol{D}^{\#}$. To see this, one has to make some calculations that are sometimes quite extensive -- still, the facts are natural enough.

For ordinary 1-categories $\boldsymbol{C}$, $\boldsymbol{C}^{\#}$ is just $\boldsymbol{C}$, essentially; but, one has to see what L is the right one. Here it is, $\mathrm{L}_{1\text{-cat}}$, given by generators and relations:

$$\begin{tikzcd}
 & \mathrm{T} \arrow[d, shift right=3, "t_0"'] \arrow[d, "t_1"] \arrow[d, shift left=3, "t_2"] & \\
\mathrm{E} \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & \mathrm{A} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & \mathrm{I} \arrow[l, "i"'] \\
 & \mathrm{O} &
\end{tikzcd}$$

$$\begin{array}{l}
dt_1=ct_0 \text{ , } dt_2=ct_1 \text{ , } \\
dt_2=dt_0 \text{ , } \\
di \ =ci \quad , \\
de_0=de_1 \text{ , } ce_0=ce_1 \text{ . }
\end{array}$$

When $\boldsymbol{C}$ is regarded as being an $\mathrm{L}_{1\text{-cat}}$-structure $\boldsymbol{C}^{\#}$, $\boldsymbol{C}^{\#}(\mathrm{O})$ is the set of all objects of $\boldsymbol{C}$, $\boldsymbol{C}^{\#}(\mathrm{A})$ is the set of arrows, $\boldsymbol{C}^{\#}(\mathrm{T})$ is the set of commutative triangles, $\boldsymbol{C}^{\#}(\mathrm{I})$ is the set of identity arrows, $\boldsymbol{C}^{\#}(\mathrm{E}) \rightrightarrows \boldsymbol{C}^{\#}(\mathrm{A})$ is the equality relation on arrows. Notice that $\mathrm{L}_{1\text{-cat}}$ is 2-dimensional (the largest dimension of an object in it is 2). Not all $\mathrm{L}_{1\text{-cat}}$-structures are, or come from, 1-categories; appropriate additional conditions are needed for this. Thus, $\mathrm{L}_{1\text{-cat}}$-equivalence is something *more primitive* than the ordinary notion of category equivalence. To repeat, for 1-categories $\boldsymbol{C}$, $\boldsymbol{D}$, we have $\boldsymbol{C} \simeq \boldsymbol{D}$ iff

$C^{\#} \simeq_{L_{1\text{-cat}}} D^{\#}$.

When we talk about bicategories and their equivalence, $\simeq$, which is called biequivalence, for a bicategory C , $C^{\#}$ may be taken to be an associated anabicategory. There is a suitable, 3-dimensional, finite FOLDS-signature L_{anabicat} for anabicategories, and we have $C \simeq D$ iff $C^{\#} \simeq D^{\#}$.

Let me describe the simplest, and most important, element of the definition of $C^{\#}$ in this last case. In an anabicategory, instead of having a straight composite $X \xrightarrow{h} Z$ of a pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have, for any f, g and h with domains and codomains as shown, a set $T(f, g, h)$ of *specifications* "for h being the composite of f and g in a definite way". Formally, there is a part of L_{anabicat} which looks just like a corresponding part of $L_{1\text{-cat}}$:

$$\begin{tikzcd} T \arrow[d, shift right=4, "t_0"'] \arrow[d] \arrow[d, shift left=4, "t_2"] \arrow[d, phantom, shift left=2, "t_1"] \\ A \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ O \end{tikzcd} \qquad \begin{array}{l} dt_1=ct_0\ ,\ dt_2=ct_1\ , \\ dt_2=dt_0\ , \end{array} \qquad .$$

Now, for an ordinary bicategory C , $C^{\#}(T)$ will be the set of all diagrams

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\cong \Downarrow a"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & \, & Z \end{tikzcd}$$

with a an arbitrary isomorphism 2-cell. (Recall that in the 1-category example, in this place we had the set of commutative triangles.)

Speaking now generally, recall that $C^{\#}$ is obtained canonically/uniformly from C . Usually, the categorical equivalence $C \simeq D$ involves morphisms $C \rightleftarrows D$ and further ingredients. We again have the contrasting facts that from the data of a categorical equivalence, one gets those for the FOLDS-equivalence in a canonical manner, whereas in the other direction there is a process of cleavage.

There are other examples, examples of composite "categories", such as fibrations, that are worked out in [M5] to show that a suitable FOLDS equivalence gives the accepted notion of equivalence.

But we are mainly interested in the situations when we do not have an established notion of equivalence, or if we do, it is too complicated, as for instance it is in the case of "tricategory" [GPS]. What evidence do we have the FOLDS-equivalence will serve well?

This is where *logic proper* comes in. The work on FOLDS [M5], [M7] develops the syntax and the semantics of a new logical language, which is described quite well by its name "First Order Logic with Dependent Sorts" (the dependent sorts are like the dependent types in Per Martin-Lof's higher order theory [M-L]). It is shown to have close ties with what we called FOLDS equivalence above, in the following sense: for any given FOLDS signature L, we have, **first,** that every statement written down in FOLDS using the vocabulary L is invariant under FOLDS-L-equivalence; and **second**, for any general first order statement Φ that is formulated in some language possibly extending L, if Φ is (universally) invariant under FOLDS-L-equivalence, then there is a statement Ψ written down in FOLDS using the vocabulary L which is equivalent, for all structures under consideration, to the original Φ.

I think the formulations I just gave are descriptive enough to convey the idea so that I may skip a formal statement of this **Invariance Theorem**. In fact, it is important that we have a more complete statement that is relative to the models of a given first order theory. See [M5], [M7].

If one can say with some confidence that the FOLDS language is the "right" one to express relevant properties of a given kind of structure, and of diagrams of elements in such a structure, *then* one is supported in the view that the FOLDS equivalence is the "right" one for the given kind of structure.

Let's take the case of (simple) 1-categories. One feature of FOLDS in general is that, unlike in classical first order logic, there is no *equality* as a *logical primitive* at all. This is all right for objects (remember the "adage") -- but what about arrows? There is a *kind* E of entities (see the FOLDS signature $L_{1\text{-cat}}$ above) that serves, because of the axioms that I have not shown, as a surrogate for equality, *but only of arrows already assumed to be parallel.* In other words, one can ask of two arrows if they are equal only if they are already assumed to be parallel. Now, I say, this is a reasonable restriction on the use of equality of arrows, one that a category theorist instinctively follows. Note that the usual statements of the form "there is a unique arrow of such and such description" do obey said restriction.

There are several further *restrictions* in FOLDS on logical manipulation, the most important one being the restriction on *quantification*. The important *discovery* is that all these restrictions can be summarized succinctly in the *uniform definition* of the FOLDS language; and that this uniform syntax seems to "work", that is, give the right results, for all the categorical concepts that come up. Of course, the close links between the syntax of FOLDS and the concept of FOLDS equivalence help support both.

The vague claim here that FOLDS equivalence is the right notion has two aspects: first, the notion is not too weak, and second, it is not too strong. If I took some n-category, considered its 1-collapse, or "homotopy category", in which the 1-arrows are appropriate equivalence classes of the original 1-arrows, and then I said that equivalence means the 1-equivalence of 1-collapses, this would be too weak. If on the other hand, I took ordinary isomorphism for equivalence, the notion would be too strong. How do I recognize these facts? In the first case, certain cherished higher dimensional properties will get lost: they will not be preserved by the proposed "equivalences". In the second case, there will be properties (such as the cardinality of the set of 0-cells) that will be preserved, and which we do not care for. The Invariance Theorem clarifies exactly what (first order) properties are respected by FOLDS equivalence. Now, we can examine whether these, the ones written in FOLDS, are the ones we really want, no more and no less; and the answer seems to be "yes".

Remember that everything is relative to a given signature.

There is an interesting example to consider in this context.

Fibrations are structures consisting two 1-categories and a functor between them. There are other categorical structures consisting of two categories and a functor between them; in [MR], such were used to do categorical modal logic; for instance, we had so-called S4-categories in there. Now, the point is that the natural signatures for fibrations on the one hand and for S4-categories on the other, are different. The difference lies in the fact that in a fibration $\begin{smallmatrix}\boldsymbol{E}\\ \downarrow p\\ \boldsymbol{B}\end{smallmatrix}$, "one should not talk about an object A in $\boldsymbol{E}$ without first having introduced the object X in $\boldsymbol{B}$ over which A is". This adage is enforced by the signature for fibrations; the *kind* (object of the signature category) for the objects of $\boldsymbol{B}$ is of dimension 0, the one for objects of $\boldsymbol{E}$ of dimension 1. On the other hand, the signature for S4-categories does not introduce *dependence* of objects of one category on objects of the other; both kinds of objects are of dimension 0.

This difference in the logics of fibrations and S4-categories is reflected in the difference of their respective notions of *equivalence*. This is a case where we have both the FOLDS equivalence and a classical notion; so we can confirm our intuition by ascertaining that the concepts that should coincide do coincide.