§4. The category of sheaves over a cHa

4.1. Sheaves

Let L be a cHa. We have explained how L is considered a category and in fact, a site, in §3. In brief, the topology is the canonical one: $\{x_i : i\in I\}$ is a covering of x if $\bigvee_{i\in I} x_i = x$.

A presheaf is a functor $F: L^{op} \longrightarrow SET$; i.e., a family $\langle F(U): U\in L\rangle$ of sets, and a family $\langle F^V_U : U \le V \in L\rangle$ of functions $F^V_U : F(V) \longrightarrow F(U)$ such that for $U \le V \le W$, we have $F^V_U \circ F^W_V = F^W_U$, and also, $F^U_U = Id_{F(U)}$. As a standard abuse of notation, we write $s\restriction U$ for $F^V_U(s)$ (for motivation, see below). A compatible family $\langle s_i : i\in I\rangle$ of sections (elements) $s_i \in F(U_i)$ is one that satisfies the following: for any $i,j \in I$, $s_i\restriction(U_i\wedge U_j) = s_j\restriction(U_i\wedge U_j)$. F is a sheaf if for every compatible family $\langle s_i : i\in I\rangle$, $s_i \in F(U_i)$, there is a unique $s \in \bigvee_{i\in I} U_i$ such that $s\restriction U_i = s_i$ for all $i\in I$.

A morphism $h: F \longrightarrow G$ of (pre-)sheaves is (of course) a family $(h_U)_{U\in L}$ of functions $h_U : F(U) \longrightarrow G(U)$ such that for any $U \le V$ and $s \in F(V)$ we have $h_U(s\restriction U) = h_V(s)\restriction U$ (of course, '$\restriction$' refers to F^V_U on the left, and G^V_U on the right).

The category of sheaves on L is denoted $Sh(L)$.

Here is an important kind of sheaf. Let $p: E' \to E$ be a continuous function between spaces. The sheaf F of continuous sections of p, a sheaf on $\mathcal{O}(E)$, is defined as follows: $F(U)$, for $U \in \mathcal{O}(E)$, is the set of continuous $s: U \longrightarrow E'$ such that

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "incl"] & E' \arrow[dl, "p"] \\ E & \end{tikzcd}$$

commutes; for $s \in F(V)$, $U \leq V$, $s \restriction U$ is the ordinary restriction of s.

Why is this a sheaf? On the one hand, the so-called separatedness condition (the uniqueness part of the condition in the definition of sheaf):

"if $s,t \in F(\bigvee_{i\in I} U_i)$ and $s\restriction U_i = t\restriction U_i$ for all $i\in I$, then $s=t$" - is true now simply because $\bigvee_{i\in I} U_i = U$ means $\bigcup_{i\in I} U_i = U$ and because functions are determined by their values. On the other hand, the 'completeness' part: "if $\langle s_i \in U_i : i\in I\rangle$ is a compatible family, then there is $s \in F(\bigvee_{i\in I} U_i)$ with $s\restriction U_i = s_i$ for all $i\in I$" - is true because, first of all, the compatibility implies that there is a function s with domain $\bigcup_i U_i$ with restrictions $s\restriction U_i = s_i$ (the union of the graphs of the s_i is functional with domain $\bigcup_i U_i$ by the compatibility conditions), and secondly, because continuity is a local condition: s is continuous iff for every $x \in \operatorname{dom} s$ there is a neighborhood V of x such that $s\restriction V$ is continuous.

Every sheaf on E (that is, on $\mathcal{O}(E)$) is the sheaf of continuous sections of some continuous $s: E' \to E$ (see e.g. TT).

4.2. L-sets

Here we present an alternative description of $Sh(L)$. Sources are: [D. Higgs, A category approach to Boolean valued set-theory, 1973], and FS.

L is a cHa. An L-set X is a pair $\langle |X|, \delta\rangle$ of a set $|X|$ and a function $\delta: |X|\times|X| \to L$, satisfying the conditions given below. For writing these conditions down, we introduce the following notation: we write $[\![x = x']\!]$, or $[\![x =_X x']\!]$, for $\delta(x,x')$ ($\in L$) (read: 'the truth value of x,x' being equal'); also, Ex for $[\![x = x]\!]$ (Ex is 'the truth value of the existence of x'). Here are the conditions:

$$[\![x = x']\!] = [\![x' = x]\!]$$

$$[\![x = x']\!] \cdot [\![x' = x'']\!] \leq [\![x = x'']\!] \quad (\cdot \text{ stands for } \wedge).$$

Of course, these are required for all $x, x', x'' \in X$.

A morphism of L-sets, $f\colon X \to Y$, is a function $F\colon |X| \times |Y| \to L$, satisfying the conditions given below. We introduce the notation $[\![x = f(y)]\!]$, or $[\![x \ominus f(y)]\!]$, for $F(x,y) \in L$. The conditions:

$$[\![x =_X x']\!][\![y \ominus f(x)]\!] \leq [\![y \ominus f(x')]\!] \tag{1}$$

$$[\![y =_Y y']\!][\![y \ominus f(x)]\!] \leq [\![y' \ominus f(x)]\!] \tag{2}$$

$$[\![y \ominus f(x)]\!][\![y' \ominus f(x)]\!] \leq [\![y =_Y y']\!] \tag{3}$$

$$Ex = \bigvee_{y \in Y} [\![y \ominus f(x)]\!] \tag{4}$$

Of course, x, x' range over $|X|$, y, y' over $|Y|$.

Composition of two morphisms $f\colon X \to Y$, $g\colon Y \to Z$ is defined as follows: we define $h = g \circ f\colon X \to Z$ by defining

$$[\![z \ominus h(x)]\!] \underset{\text{df}}{=} \bigvee_{y \in Y} [\![y \ominus f(x)]\!] \cdot [\![z \ominus g(y)]\!].$$

It is left as an exercise to show that we have thus defined a category, the category of L-sets, L-sets. We note only that the identity map $Id_X\colon X \to X$ is given by

$$[\![x \ominus Id_X(x)]\!] = Ex.$$

The verification of our claim uses, besides the definitions involved, some simple 'propositional logic' in L, actually involving only $\wedge$ and $\bigvee$.

Given a sheaf F on L, we can deduce the following L-set $[F]$: we have $|[F]| =_{df} \{<s,U>: s\in F(U),\ U\in L\}$, $[\![<s,U> =_{[F]} <t,V>]\!] =_{df} \bigvee\{W \le U\wedge V: s|W = t|W\}$. (In the case of a sheaf on $\mathcal{O}(E)$, this is the interior of the set of elements $\in E$ on which the two sections agree.) To verify that this is an L-set, one uses only that F is a presheaf. Furthermore, if $h: F \to G$ is a morphism of sheaves, we define $[h]: [F] \to [G]$ by putting

$$\underset{(t\in G(V),\ s\in F(U))}{[\![<t,V> \ominus [h](<s,U>)]\!]} =_{df} [\![<t,V> =_{[G]} <h_U(s),\ U>]\!].$$

$[-]$ so defined becomes a functor $Sh(L) \to$ L-sets; but before stating this, and some more surprising facts, we single out special properties of L-sets of the form $[F]$ for F a sheaf.

Definition 4.2.1 (i) The L-set X is called complete if the following holds: for any family $<x_i: i\in I>$ of elements of $|X|$, and any family $<U_i: i\in I>$ of elements of L, if $U_i \wedge U_j \le [\![x_i = x_j]\!]$ ('x_i and x_j agree on $U_i \wedge U_j$') for all $i,j \in I$ (compatibility), then there is at least one x with $Ex = \bigvee_{i\in I} U_i$ such that $U_i \le [\![x = x_i]\!]$. For this x we write: $x = \sum_{i\in I} x_i|U_i$, even though x may not be unique.

(ii) The L-set X is separated if $Ex = Ey \le [\![x = y]\!]$ implies that $x = y$.

Notice that for any sheaf F, $[F]$ is complete and separated. Namely, first of all, notice that for $X = <s,U> \in |[F]|$ we have $Ex = U$. Suppose the compatibility condition in 4.2.1 (i), with $x_i = <s_i,U_i'>$. The condition (with $i=j$) implies that $U_i \le U_i'$. Also, with $t_i = s_i|U_i$, we obtain that $<t_i: i\in I>$ is compatible in the original sense: since $U_i \wedge U_j \le [\![x_i = x_j]\!] = \bigvee\{W \le U_i' \wedge U_j': s_i|W = s_j|W\}$, we have that, for $U' = U_i \wedge U_j$,

$U' = \bigvee\{W \leq U' \colon (t_i \restriction U') \restriction W = (t_j \restriction U') \restriction W\}$, hence by the 'separatedness' of the sheaf F, $t_i \restriction U' = t_j \restriction U'$, as required. So, with $U = \bigvee_{i \in I} U_i$, there is $s \in F(U)$ with $s \restriction U_i = t_i$ for all $i \in I$; $x = \langle s, U\rangle$ will satisfy $Ex = U$ and $U_i \leq [\![x = x_i]\!]$. This shows that $[F]$ is complete. On the other hand, if $Ex = Ey \leq [\![x = y]\!]$, then $x = \langle s,U\rangle$, $y = \langle t,U\rangle$, $Ex = Ey = U$, and by the "separatedness" of F we find, similarly as above, that $s|U = t|U$, i.e. $s = t$, and $x = y$; so $[F]$ is separated.

Let X and Y be L-sets, and f_0 a function $f_0 \colon |X| \to |Y|$ with the following two properties:

$$\left.\begin{aligned} [\![x =_X x']\!] \leq [\![f_0(x) =_Y f_0(x')]\!] \quad (x, x' \in |X|) \\ E(f_0(x)) = Ex. \end{aligned}\right\} \tag{5}$$

Then the following definition

$$[\![y \in f(x)]\!] \underset{\mathrm{df}}{=} [\![y =_Y f_0(x)]\!] \quad (x \in |X|,\ y \in |Y|)$$

defines a morphism $f \colon X \to Y$ (exercise). We say in this case that f_0 strongly represents f. In particular, for a sheaf-morphism $h\colon\ F \to G$, $[h]\colon [F] \to [G]$ is strongly represented by the function $f_0 \colon \langle s,U\rangle \longmapsto \langle h_U(s), U\rangle$ $(s \in F(U),\ U \in L)$. This now tells us that $[h]$ is indeed an L-set map. Moreover, it is clear that a composite of functions (between the appropriate underlying sets) strongly represents the composite of the morphisms strongly represented by the original functions; and also that identity strongly represents identity; from which it follows that $[-]$ is indeed a functor

$$[-]\colon \mathrm{Sh}(L) \to \text{L-sets}.$$

Theorem 4.2.2 (D. Higgs; see also FS): the functor $[-]$ is an equivalence of categories

$$[\ ]: \mathrm{Sh}(L) \xrightarrow{\sim} L\text{-sets}.$$

Before the proof, some lemmas.

Lemma 4.2.3. Let X, Y be L-sets, and assume that Y is complete. Then every morphism $f: X \to Y$ is strongly represented by a function $f_0: |X| \to |Y|$. If Y is also separated, f_0 is unique.

Proof: Define $f_0(x) = \sum_{y \in Y} y | [\![y \ominus f(x)]\!]$ (see the notation in 4.$\cancel{1}$2(i)) or more precisely, take for $f_0(x)$ any value that answers the description of $\sum_{y \in Y} y | [\![y \ominus f(x)]\!]$ (it being not necessarily uniquely determined). Note that $[\![y \ominus f(x)]\!] \cdot [\![y' \ominus f(x)]\!] \le [\![y = y']\!]$ (see (3)), so the necessary compatibility holds. We have that

$$[\![y \ominus f(x)]\!] \le [\![y =_Y f_0(x)]\!] \tag{6}$$

by the definition of $\Sigma \cdots$.

Next, we claim that

$$Ex \le [\![f_0(x) \ominus f(x)]\!] \ . \tag{7}$$

Indeed,

$$Ex = \bigvee_{y \in Y} [\![y \ominus f(x)]\!] \tag{8}$$

by (4);

$$[\![y \ominus f(x)]\!] [\![y = f_0(x)]\!] \le [\![f_0(x) \ominus f(x)]\!] \quad \text{by} \quad (3),$$

hence

$$[\![y \ominus f(x)]\!] \le [\![f_0(x) \ominus f(x)]\!] \text{ by } (6)$$

and we obtain (7) by (8). By the definition of $\Sigma\cdots$,

$E(f_0(x)) = [\![f_0(x) = f_0(x)]\!] = \bigvee_{y \in Y} [\![y \ominus f(x)]\!] = Ex$; so

$[\![y = f_0(x)]\!] = [\![y = f_0(x)]\!]$. $Ex \underbrace{\le}_{\text{by } (7)} [\![y = f_0(x)]\!][\![f_0(x) \ominus f(x)]\!]$

$\underbrace{\le}_{\text{by } (2)} [\![y \ominus f(x)]\!]$, which together with (6) gives

$$[\![y \ominus f(x)]\!] = [\![y = f_0(x)]\!]$$

showing that, indeed, f is strongly represented by f_0. - The uniqueness of f_0, in case Y is separated, is left as an exercise. □

<u>Remark to Lemma 4.3.</u> Assume that Y is complete and let $f: X \to Y$ be a morphism. Then $\max\limits_{y \in |Y|} [\![y \ominus f(x)]\!]$ exists, and in fact, $[\![f_0(x) \ominus f(x)]\!]$ is the maximum value of $[\![y \ominus f(x)]\!]$, with $f_0(x)$ defined in the proof. This is clear, since we have

$$[\![y \ominus f(x)]\!] \underset{\text{always}}{\le} Ex \underset{\text{by } (7)}{\le} [\![f_0(x) \ominus f(x)]\!].$$

This, in fact, shows that

$$\max_y [\![y \ominus f(x)]\!] = [\![f_0(x) \ominus f(x)]\!] = Ex = E(f_0(x)).$$

Proof of 4.2.2. Now we can show that $[-]$ is full and faithful. Let $f\colon [F] \to [G]$ be an L-set morphism. By 4.2.3, there is $f_0\colon |[F]| \to |[G]|$ representing f. In particular, (5) holds for f_0. If $x = \langle s,U\rangle \in |[F]|$, then by (5) we have $f_0(x) = \langle t,U\rangle$ for some t and the same U. Define $h_U(s) = t$. We verify that h so defined is a (pre-)sheaf morphism, i.e. for $s \in F(V)$ and $U \leq V$, we have

$$h_U(s\restriction U) = (h_V(s))\restriction U. \tag{9}$$

Let $x = \langle s,V\rangle \in |[F]|$, $f_0(x) = \langle t,V\rangle \in |[G]|$, $x' = \langle s\restriction U, U\rangle \in |[F]|$, $f_0(x') = \langle t', U\rangle \in |[G]|$. The left-hand side of (9) is t', the right-hand side is $t\restriction U$. By the definition of $=_{[G]}$, $[\![\langle t',U\rangle = \langle t\restriction U, U\rangle]\!] = [\![\langle t,V\rangle = \langle t',U\rangle]\!] = [\![f_0(x) = f_0(x')]\!]$, and the latter is $\geq [\![x = x']\!] = U$ by (5). By the separatedness of $[G]$(!), it follows that $t' = t\restriction U$, i.e. we have (9).

Going back to the definition of $[h]$, now we see that $[h] = f$, since $[\![(\cdot) \ominus f(\cdot)]\!] = [\![(\cdot) =_{[G]} f_0(\cdot)]\!] = [\![(\cdot) \ominus [h](\cdot)]\!]$. It is easy to see that the definition of h is forced, i.e. that $[-]$ is faithful.

Before continuing the proof of 4.2, we state two lemmas.

Lemma 4.4. Every L-set is isomorphic to a complete L-set.

Proof. Let X be an L-set. Define the L-set $\tilde{X}$ (the completion of X) as follows. The elements of $|\tilde{X}|$ are the formal sums $\sum^{(f)}_{i\in I} x_i|U_i$ of compatible families: $x_i \in |X|$, $U_i \in L$ and $U_i \wedge U_j \leq [\![x_i = x_j]\!]$ for $i,j \in I$ (such a formal sum is just a set $\{\langle x_i,U_i\rangle\colon i\in I\}$ satisfying the compatibility condition; in particular, $U_i \leq Ex_i$). We define

$$[\![\sum_{i\in I}^{(f)} x_i|U_i \underset{X}{\simeq} \sum_{j\in J}^{(f)} y_j|V_j]\!] = \bigvee_{\langle i,j\rangle\in I\times J} [\![x_i = y_j]\!]\cdot(U_i\wedge V_j) \quad .$$

The 'symmetry' condition for L-sets is clearly satisfied. With $\bar{x}$ and $\bar{y}$ the two formal sums shown, together with $\bar{z} = \sum_{k\in K}^{(f)} z_h|W_k$, we have

$$[\![\bar{x} = \bar{y}]\!][\![\bar{y} = \bar{z}]\!] = (\bigvee_{i,j}[\![x_i = y_j]\!]\cdot(U_i\wedge V_j)) \wedge (\bigvee_{j,k}[\![y_j = z_k]\!]\cdot(V_j\wedge W_k))$$

$$= \bigvee_{i,j,j',k} [\![x_i = y_j]\!][\![y_{j'} = z_k]\!]\cdot(U_i\wedge V_j\wedge V_{j'}\wedge W_k);$$

we have $V_j\wedge V_{j'} \le [\![y_j = y_{j'}]\!]$ by the compatibility of the y_j's; hence $(V_j\wedge V_{j'})[\![x_i = y_j]\!][\![y_{j'} = z_k]\!] \le [\![x_i = z_k]\!]$; so the last sup is $\le \bigvee_{i,k} [\![x_i = z_k]\!]\cdot(U_i\wedge W_k) = [\![\bar{x} = \bar{z}]\!]$. This shows the "transitivity" condition for $\underset{X}{\simeq}$. Before verifying the completeness of $\tilde{X}$, we note that "every formal sum is actually a real sum": in other words, if $\bar{x} = \sum_{i\in I}^{(f)} x_i|U_i$, then, first of all, <u>every individual</u> $x_i|U_i$ is considered an element of $\tilde{X}$ as a one element formal sum, and then, in this sense of $x_i|U_i \in \tilde{X}$, we claim that $\bar{x} = \sum_{i\in I} x_i|U_i$. This means: $E\bar{x} = \bigvee_{i\in I} U_i$, and $U_i \le [\![\bar{x} = x_i|U_i]\!]$. To see these relations, write:

$$E\bar{x} = [\![\bar{x} = \bar{x}]\!] = \bigvee_{i,j\in I} [\![x_i = x_j]\!](U_i\wedge U_j) =$$

$$= \bigvee_{i,j}(U_i\wedge U_j)(\text{since } U_i\wedge U_j \le [\![x_i = x_j]\!]) = \bigvee_i U_i \;;$$

$$[\![\bar{x} = x_i|U_i]\!] = \bigvee_{i'}[\![x_{i'} = x_i]\!](U_{i'}\wedge U_i) = \bigvee_{i'} (U_{i'}\wedge U_i) = U_i \;.$$

Once we know this, we note the following general fact in any L-set: if, for all $j\in J$, $\bar{x}_j = \sum_{i\in I_j} x_{ij}|U_{ij}$, the family $\overline{x_j}/V_j$ $(j\in J)$ is compatible, then, the family $x_{ij}/U_{ij}\wedge V_j$ $(i\in I_j,\ j\in J)$ is compatible and if $\bar{x} = \sum_{i,j} x_{ij}|U_{ij}\wedge V_j$ exists, then $\sum_{j\in J}\overline{x_j}|V_j$ exists and equals $\bar{x}$; this we leave as an <u>exercise</u> to verify. Knowing this general fact, and that in $\tilde{X}$, formal sums are actual sums, the completeness of $\tilde{X}$ is immediate.

Consider the morphism $\alpha: X \to \tilde{X}$ strongly represented by the function $\alpha_0: |X| \to |\tilde{X}|$: $\alpha_0(x) = x|Ex$. Clearly, the conditions (5) are satisfied for α_0 to define α with $[\![\bar{x} \ominus \alpha(x)]\!] = [\![\bar{x} = \alpha_0(x)]\!]$. We claim that α is an <u>isomorphism</u>. First of all, we note that for any $f: X \to Y$, <u>if</u> f is "1-1 and onto" in the sense that

$$[\![y \ominus f(x)]\!] \cdot [\![y \ominus f(x')]\!] \leq [\![x = x']\!],$$

$$Ey = \bigvee_x [\![y \ominus f(x)]\!]$$

for all $x, x' \in |X|$, $y \in |Y|$, <u>then</u> f is an isomorphism, with inverse $f': X \to Y$ defined by

$$[\![x \ominus f'(y)]\!] \underset{\mathrm{df}}{=} [\![y \ominus f(x)]\!]$$

(<u>exercise</u>). Finally, it is easy to see that our α is "1-1 and onto". □

<u>Lemma 4.2.5.</u> Every complete L-set is isomorphic to a complete and separated L-set.

<u>Proof</u>. Let X be a complete L-set. For any fixed $U \in L$, define the equivalence relation $\sim_U = \sim$ on $X(U) \underset{\mathrm{df}}{=} \{x \in |X|: Ex = U\}$ as follows: for $x, y \in X(U)$, $x \sim y \Leftrightarrow [\![x = y]\!] = U$. It is easy to see that $\sim_U$ is an equivalence relation. For any $x \in |X|$, let $\dot{x} = \{y: y \sim_{Ex} x\}$, let $|\dot{X}| = \{\dot{x}: x \in |X|\}$, and define $[\![\dot{x} = \dot{y}]\!] = [\![x = y]\!]$ (<u>exercise</u>: show that this definition by representatives is correct). We leave it to the reader to verify that $\dot{X}$ so defined is separated as well as complete (using that X is), and also that the morphism $\alpha: X \to \dot{X}$ strongly represented by $\alpha_0: x \longmapsto \dot{x}$ is an isomorphism. □

Conclusion of the proof of 4.2.2. It is left to verify that every L-set X is isomorphic to $[F]$ for a sheaf F. By 4.2.4 and 4.2.5, we can assume that X is complete and separated. Define F as follows: $F(U) = \{x \in |X| : Ex = U\}$; for $U \le V$, $x \in F(V)$, $x|U = x|U$ (one-element formal sum). It is immediate that F is a sheaf, by the very formulations of completeness and separatedness for X. It is also clear that $[F]$ 'is' X, more precisely, the function $x \longmapsto \langle x, Ex\rangle$, strongly represents an isomorphism $X \to [F]$. This completes the proof of 4.2.

□

4.3 L-sets as a topos.

We now know that L-sets is a Grothendieck topos; nonetheless, we reprove this fact by identifying the topos structure in L-sets. For some details we omit here see §3 of [D. Higgs].

A technical remark first. Suppose X, Y are L-sets: $f_0 : |X| \to |Y|$ a function satisfying the first of the two conditions under (5): $[\![x = x']\!] \le [\![f_0(x) = f_0(x')]\!]$ for $x, x' \in |X|$. Then we can define $f: X \to Y$ by putting $[\![y \ominus f(x)]\!] = Ex . [\![y =_Y f_0(x)]\!]$; this is easy to check. In this case, we say f_0 represents f; if, in addition, $Ex = E(f_0(x))$ (hence, the full (5) is true), then, of course, we have $[\![y \ominus f(x)]\!] = [\![y =_Y f_0(x)]\!]$, hence f is strongly represented by f_0.

4.3.1. The finite left-limit structure.

The terminal object is 1, an L-set such that $|1| = \{*\}$, $[\![* = *]\!] = 1_L$. The verification is left as an exercise - [A general remark, helpful here and below, is that the subcategory of complete and separated L-sets, with morphisms that are strongly represented by some (unique) function, is equivalent, by the inclusion as equivalence, to the whole category L-sets itself. As a consequence, 1 being the terminal object is equivalent to saying that for every complete and separated X, there is a unique $|X| \to \{*\}$ satisfying conditions (5) above;

and this is trivial].

Let X_1, X_2 be L-sets. Define $Y = X_1 \times X_2$ as follows: $|Y| = |X_1| \times |X_2|$; $[\![\langle x_1, x_2 \rangle = \langle x_1', x_2' \rangle]\!] \underset{\mathrm{df}}{=} [\![x_1 = x_1']\!] \cdot [\![x_2 = x_2']\!]$. The projection $\pi_1 \colon Y \to X_1$ is the morphism represented by the function $\langle x_1, x_2 \rangle \longmapsto x_1$, similarly for $\pi_2 \colon Y \to X_2$. - The verification that in this way we indeed have a product is straightforward, though tedious.

Let

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] & Y \\ Z \arrow[u, "g_1"] \arrow[r, "g_2"'] & X_2 \arrow[u, "f_2"'] \end{tikzcd}$$

be a diagram of L-sets and morphisms. The following are necessary and (jointly) sufficient for this diagram to be a pullback diagram:

$$[\![x_1 \ominus g_1(z)]\!] [\![x_2 \ominus g_2(z)]\!] [\![x_1 \ominus g_1(z')]\!] [\![x_2 \ominus g_2(z')]\!] \le [\![z = z']\!]$$

$$\bigvee_{y \in |Y|} [\![y \ominus f_1(x_1)]\!] [\![y \ominus f_2(x_2)]\!] = \bigvee_{z \in |Z|} [\![x_1 \ominus g_1(z)]\!] [\![x_2 \ominus g_2(z)]\!]$$

for all appropriate values of the variables.
(Of course, this is a natural generalization of the description of pullbacks in SET.) Again, the verification is direct but tedious.

In particular, assume that Y is complete, and let $f_{1,0}$, $f_{2,0}$ strongly represent f_1, f_2, resp. Then the left-hand side of the last equation equals $[\![f_{1,0}(x_1) =_Y f_{2,0}(x_2)]\!]$ (exercise), so the requirement becomes

$$[\![f_{1,0}(x_1) = f_{2,0}(x_2)]\!] = \bigvee_{z \in |Z|} [\![x_1 \ominus g_1(z)]\!] [\![x_2 \ominus g_2(z)]\!]$$

4.3.2. Subobjects.

Let X be an L-set. Let $A\colon |X| \to L$ be a function satisfying: $A(x) \leq Ex$; $[\![x = x']\!]. A(x) \leq A(x')$ for all $x, x' \in |X|$. Such an A is called an (L-valued) (one-place) predicate on X. Define an L-set $\bar{A}$ by putting: $|\bar{A}| = |X|$, $[\![x =_{\bar{A}} x']\!] = [\![x =_X x']\!]. A(x).A(x')$. We leave it to the reader to check that $\bar{A}$ is an L-set. Moreover, the function $x \longmapsto x$ (the identity on $|\bar{A}| = |X|$) represents a morphism $\bar{A} \xrightarrow{\alpha} X$, which is, in addition, a monomorphism.

Now, let $B \xrightarrow{f} X$ be a morphism. f is a monomorphism if and only if f is '1-1', in the sense that $[\![x \ominus f(y)]\!] \cdot [\![x \ominus f(y')]\!] \leq [\![y =_B y']\!]$ holds for all $x \in |X|$, $y, y' \in |B|$ (exercise). Assume that f is a monomorphism and define the predicate A on X by $A(x) = \bigvee_{y \in B} [\![x \ominus f(y)]\!]$, and form the monomorphism $\bar{A} \xrightarrow{\alpha} X$ as above. Let $g\colon B \to \bar{A}$ be defined by $[\![x \ominus g(y)]\!] = [\![x \ominus f(y)]\!] \cdot A(x)$. We claim that g is an isomorphism that makes the diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "g"'] & & X \\ & \bar{A} \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commute (exercise). This means that every subobject of X is represented by a monomorphism $\bar{A} \xrightarrow{\alpha} X$, for a predicate A on X and α the canonical map as above. It is also easy to see that two distinct predicates define distinct subobjects, moreover that $\bar{A} \leq \bar{B}$ (in the sense of subobjects iff $A(x) \leq B(x)$ for all $x \in |X|$.

In summary, subobjects of X are the same as (1-place) predicates on X, where a predicate is a function $A\colon |X| \to L$ satisfying

$$A(x) \leq Ex$$

$$[\![x = x']\!]\cdot A(x) \leq A(x').$$

[A hint to the exercise concerning the characterization of monomorphisms. To show that "f is 1-1" implies that f is a mono is easy. For the converse, define what is the natural equivalent of the kernel-pair of f [and what does turn out to be the kernel pair, though we don't need that fact]. Given $B \xrightarrow{f} X$, define the predicate C on the L-set $B\times B$ by

$C(b_1,b_2) \underset{df}{=} \bigvee_{x\in X} [\![x \ominus f(b_1)]\!][\![x \ominus f(b_2)]\!] \cdot C$

defines a subobject $C \hookrightarrow B\times B$. Denoting the domain of this subobject by C too, now define the morphisms $C \xrightarrow{g_1} B$, $C \xrightarrow{g_2} B$ as the composites $C \hookrightarrow B\times B \xrightarrow{\pi_1} B$, $C \hookrightarrow B\times B \xrightarrow{\pi_2} B$. Finally, check that $f\circ g_1 = f\circ g_2$; since f was assumed to be a mono, it follows that $g_1 = g_2$; now show that this means that f is "one-to-one".]

4.3.3. The subobject classifier

Let $\tilde{\Omega}$ denote the following L-set. $|\tilde{\Omega}|$ = the set of all pairs $\langle U,U'\rangle \in L\times L$ with $U \leq U'$, and $[\![\langle U_1,U_1'\rangle =_{\tilde{\Omega}} \langle U_2,U_2'\rangle]\!] = (U_1 \longleftrightarrow U_2)\cdot U_1'\cdot U_2'$. Note that $E(\langle U,U'\rangle) = U'$. We will first show that $\tilde{\Omega}$ is a complete and separated L-set, then point out a simpler L-set Ω which is however isomorphic to $\tilde{\Omega}$; and finally, show that Ω (or $\tilde{\Omega}$) can serve as a subobject classifier.

Let $\langle U_i,U_i'\rangle | V_i$ $(i\in I)$ be a compatible family; this means $V_i \wedge V_j \leq (U_i \longleftrightarrow U_j)\cdot(U_i'\wedge U_j')$. We claim that $x = \langle \bigvee_{i\in I} U_i\cdot V_i, \bigvee_{i\in I} V_i\rangle$ serves as

$\sum_{i\in I} \langle U_i, U_i'\rangle | V_i . \bigvee U_i \cdot V_i \le \bigvee V_i$ holds, making sure that x is an element of $|\tilde{\Omega}|$. $Ex = \bigvee_{i\in I} V_i$ as required. Finally, to show $V_j \le (U_j \leftrightarrow \bigvee_{i\in I} U_i \cdot V_i) \cdot U_j' \cdot \bigvee_{i\in I} V_i$ (as required) note that $V_j \le U_j' = E(U_j, U_j')$, as a consequence of compatibility, $V_j \le \bigvee_{i\in I} V_i$ is trivial, so we are left with showing $V_j \le U_j \leftrightarrow \bigvee_{i\in I} U_i \cdot V_i$ or <u>equivalently</u>: $V_j \le U_j \rightarrow \bigvee_i U_i \cdot V_i$ and $V_j \le (\bigvee_i U_i \cdot V_i) \rightarrow U_j$. Now remember (or verify) that $U \le U' \rightarrow U''$ is equivalent to $U.U' \le U''$. Hence, the first inequality is obvious. To show the second, note that because of $V_i . V_j \le U_i \leftrightarrow U_j$ (compatibility), we have $V_j . U_i . V_i \le U_j$, and since here i is arbitrary, $V_j \wedge (\bigvee_i U_i V_i) \le U_j$, hence $V_j \le \bigvee_i U_i V_i \rightarrow U_j$ as required. This completes the proof of completeness of $\tilde{\Omega}$.

Assume $x_1 = \langle U_1, U_1'\rangle$, $x_2 = \langle U_2, U_2'\rangle \in |\tilde{\Omega}|$, $Ex_1 = Ex_2 \le [\![x_1 = x_2]\!]$. Then $U_1' = U_2' \underset{df}{=} U'$, and $U' \le U_1 \leftrightarrow U_2$, hence $U_1 . U' = U_1 \le U_2$, and similarly, $U_2 \le U_1$, hence $U_1 = U_2$; so, $x_1 = x_2$ indeed.

Now, consider the simpler L set Ω defined as follows: $|\Omega| = L$ $(= |L|)$, $[\![U_1 = U_2]\!] \underset{df}{=} U_1 \leftrightarrow U_2$. Now $EU = 1$ for all $U \in |\Omega|$. Let $f\colon \Omega \to \tilde{\Omega}$ be strongly represented by the function $U \longmapsto \langle U, 1\rangle$; this function does satisfy conditions (5). We show that f is '1-1 and onto'. $[\![\langle V,V'\rangle \ominus f(U)]\!] = [\![\langle V,V'\rangle = \langle U,1\rangle]\!] = (U \leftrightarrow V).V'$. So, $[\![\langle V,V'\rangle \ominus f(U_1)]\!] . [\![\langle V,V'\rangle \ominus f(U_2)]\!] = (U_1 \leftrightarrow V)(U_2 \leftrightarrow V).V'$ $\le U_1 \leftrightarrow U_2$ (since $\varphi_1 \leftrightarrow \psi \cdot\wedge\cdot \varphi_2 \leftrightarrow \psi \vdash \varphi_1 \leftrightarrow \varphi_2$ is valid) $= [\![U_1 =_\Omega U_2]\!]$, showing that f is '1-1'. On the other hand, $EU = 1 = (U \leftrightarrow U).1 = \vee\{(U \leftrightarrow V).V' : \langle V,V'\rangle \in \tilde{\Omega}\}$, so f is 'onto'.

Let $1 \xrightarrow{\tau} \tilde{\Omega}$ be the morphism represented by the function $* \longmapsto \langle 1_L, 1_L\rangle$. We show that τ is a 'generic monomorphism', i.e. $\tilde{\Omega}$ is a subobject classifier via τ i.e. that for any subobject $A \xhookrightarrow{\alpha} X$ there is a unique $X \xrightarrow{\gamma} \tilde{\Omega}$ such that

$$\begin{tikzcd} 1 \arrow[r, "\tau"] & \tilde{\Omega} \\ A \arrow[u, "p"] \arrow[r, "\alpha"] & X \arrow[u, "\gamma"'] \end{tikzcd}$$

is a pullback. Let $A(x)$ be the predicate defining/defined by the subobject $A \overset{\alpha}{\hookrightarrow} X$; we can assume $|A| = |X|$, $[\![x =_A x']\!] = [\![x =_X x']\!].A(x).A(x')$, and $[\![y \in \alpha(x)]\!] = [\![y =_X x]\!].A(x)$ (see on subobjects above). Recall that $[\![* = p(x)]\!] = E_{(A)}(x) = A(x)$, $[\![<U,U'> = \tau(*)]\!] = [\![<U,U'> =_{\tilde{\Omega}} 1]\!] = (U \leftrightarrow 1).U' = U.U' = U.$ One checks easily the first condition for the diagram to be a pullback is true (regardless what γ is). Since $\tilde{\Omega}$ is complete, γ (if exists) is strongly represented by a function $\gamma_0 \colon |X| \to |\tilde{\Omega}|$; we'll denote this by $x \longmapsto <U(x),U'(x)>$. Now, the second condition, in the form for a complete Y, becomes

$$[\![<U(x),U'(x)> =_{\tilde{\Omega}} 1]\!] = \bigvee_{y \in |X|} A(x)[\![y =_X x]\!]A(x) .$$

The left-hand side, as we noted, equals $U(x)$; the right hand side clearly equals $A(x)$ since $A(x) \le [\![x = x]\!]$. We have obtained that the condition is equivalent to saying that $U(x) = A(x)$. Since for γ_0 to strongly represent a function it is necessary that $E(\gamma_0(x)) = Ex$ for all $x \in |X|$, we also have that $U'(x) = Ex$ must be the case. If we put $\gamma_0(x) = <A(x),Ex>$ then we indeed have a function strongly representing a morphism, since (5) holds, as a consequence of $A(\ )$ being a predicate (check!). This completes the verification that $\tilde{\Omega}$ is a subobject classifier.

Using the isomorphism $f \colon \Omega \to \tilde{\Omega}$ introduced above, now it follows that Ω is a subobject classifier, with $1 \xrightarrow{\text{True}} \Omega$ represented by $* \longmapsto 1$; in fact, given any predicate $A(\cdot)$ on X, the characteristic morphism $X \to \Omega$ of $A \hookrightarrow X$ is represented (but not strongly represented) by the function $x \longmapsto A(x)$; i.e. A itself; just carry things over by the isomorphism f (exercise). [It is interesting that checking the last statement directly, without knowing about $\tilde{\Omega}$, is quite hard; see the next sub-subsection 4.3.4.]

This last Ω is our 'official' subobject classifier. Note that we have shown (though Ω is not a sheaf) that every map $X \to \Omega$ is represented by a unique function which is a predicate on X.

4.3.4. Exponentials, power-sets.

Recall (see T.T.) that exponentials B^A can be defined by saying that $(\ )^A$ should be a right adjoint to the functor $A \times (\ )$. Explicitly, we have a morphism

$$A \times B^A \xrightarrow[\epsilon_{B,A}]{} B$$

such that for any $A \times C \xrightarrow[h]{} B$ there is a unique $h^*: C \to B^A$ such that $h = \epsilon_{B,A} \circ (1_A \times h^*)$.

We denote Ω^X by $P(X)$. It turns out that $P(X)$ is much easier to describe (and verify) than B^A in general. We define $P(X)$ as follows. $|P(X)|$ = the set of all predicates on X; $[\![A =_{P(X)} B]\!] \underset{df}{=} \bigwedge_{x \in |X|} (A(x) \leftrightarrow B(x))$. Again, we have that $EA = 1$ for all $A \in |P(X)|$. $\epsilon_X = \epsilon_{\Omega,X}: X \times P(X) \to \Omega$ is defined to be represented by $\langle x, A\rangle \longmapsto A(x)$ (check!). Next we show

<u>Lemma</u> Every morphism of the form $Y \xrightarrow{g} \mathcal{P}(X)$ is represented by some function $g_0 : |Y| \to |\mathcal{P}(X)|$.

Of course, this is a generalization of the corresponding statement about Ω since $\Omega = \mathcal{P}(1)$. We could proceed as before by pointing out what the 'sheaf'-completion of $\mathcal{P}(X)$ is; here we give the direct proof.

<u>Proof of L.</u> Let $g: Y \to \mathcal{P}(X)$ be given. For a given $y \in |Y|$, what should be the '<u>value</u>' of g? It should be the predicate A for which $A(x)$ is the proposition that "there exists B such that $B = g(y)$ and $B(x)$ holds". Define accordingly, for any fixed $y \in |Y|$, the predicate A_y as follows:

$$A_y(x) = \bigvee_{B \in |\mathcal{P}(X)|} [\![B \ominus g(y)]\!] \cdot B(x).$$

It is easy to verify that A_y is indeed a predicate for each $y \in |Y|$ (<u>exercise</u>). Define the function

$$g_0 : |Y| \to |\mathcal{P}(X)| \quad \text{by} \quad g_0(y) \underset{\text{df}}{=} A_y .$$

We claim that g_0 represents g. This means the truth of the following identity:

$$[\![A \ominus g(y)]\!] = Ey. \ (\bigwedge_{x \in |X|} A(x) \rightleftarrows \bigvee_{B \in |\mathcal{P}(X)|} B(x) [\![B \ominus g(y)]\!])$$

for all $y \in |Y|$, $A \in |\mathcal{P}(X)|$.

First we show that the left-hand side $\leq$ the right hand side. First, an informal argument, in infinitary propositional logic (more-or-less).

"Assume $[\![A \ominus g(y)]\!]$. Then Ey. To show the other conjunct of the r.h.s., first assume $A(x)$ and show: there is B s.t. $B(x)$, $[\![B \ominus g(y)]\!]$. But the assumptions mean that $B \underset{df}{=} A$ works. Second, assume there is B with $B(x)$ and $[\![B \ominus g(y)]\!]$. Since $[\![A \ominus g(y)]\!]$ is true, it follows that $[\![A = B]\!]$ (by requirement (3) on morphisms). By the definition of $[\![A = B]\!]$, this together with $B(x)$ implies $A(x)$, and this was what we wanted."

Here is the formal argument that essentially reverses the steps. First note

$$[\![A = B]\!] \cdot B(x) \leq A(x) \tag{10}$$

(by the definition of $[\![A = B]\!] = \bigwedge_x (A(x) \longleftrightarrow B(x))$, and the fact that $\varphi \vdash \psi \longrightarrow \sigma$ iff $\varphi \wedge \psi \vdash \sigma$);

$$[\![A \ominus g(y)]\!][\![B \ominus g(y)]\!] \leq [\![A = B]\!] \tag{11}$$

by (3) for the morphism g; (10) and (11) combined give:

$$[\![A \ominus g(y)]\!] \cdot B(x) \cdot [\![B \ominus g(y)]\!] \leq A(x). \tag{12}$$

Take the sup of the l.h.s. for all B. Obtain

$$[\![A \ominus g(y)]\!] \bigvee_B B(x) \cdot [\![B \ominus g(y)]\!] \leq A(x) . \tag{13}$$

Hence

$$[\![A \ominus g(y)]\!] \leq \bigvee_B B(x)[\![B \ominus g(y)]\!] \longrightarrow A(x). \tag{14}$$

Obviously, $[\![A \ominus g(y)]\!] . A(x) \leq \bigvee_B B(x)[\![B \ominus g(y)]\!]$

hence

$$[\![A \ominus g(y)]\!] \leq A(x) \longrightarrow \bigvee_B B(x)[\![B \ominus g(y)]\!]. \tag{15}$$

Taking the meet of the r.h.s's of (14) and (15), we get that $[\![A \ominus g(y)]\!]$ is $\le$ the second conjunct on the required identity; $[\![A \ominus g(y)]\!] \le Ey$ is obvious, completing the (one-sided) proof.

Second, we show the reverse inequality.

"Assume Ey and for all $x \in |X|$, $A(x)$ if and only if there is B such that $B(x)$ and $[\![B \ominus g(y)]\!]$. By Ey, there is C such that $[\![C \ominus g(y)]\!]$. We claim $[\![A = C]\!]$. By definition, it suffices to show that $A(x)$ if $C(x)$. Assume $A(x)$. By assumption, there is B such that $B(x)$ and $[\![B \ominus g(y)]\!]$. But since we also have $[\![C \ominus g(y)]\!]$, it follows that $[\![B = C]\!]$, hence by $B(x)$ we conclude $C(x)$. This shows $A(x) \to C(x)$. Conversely, assume $C(x)$. But then $C(x)$ and $[\![C \ominus g(y)]\!]$ both hold, hence by assumption $A(x)$ as required."

The formal proof:

$$\underbrace{[\bigwedge_{x \in |X|} (A(x) \longleftrightarrow \bigvee_B B(x)[\![B \ominus g(y)]\!])]}_{\text{call this } U \text{ temporarily}} \cdot C(x) \cdot [\![C \ominus g(y)]\!] \le A(x)$$

(this is clear)

$$\therefore \qquad U \cdot [\![C \ominus g(y)]\!] \le C(x) \rightharpoonup A(x) \tag{16}$$

$$[\![C \ominus g(y)]\!] \cdot B(x) \cdot [\![B \ominus g(y)]\!] \le C(x)$$

(this is a somewhat condensed step) (exercise).

$$\therefore \qquad [\![C \ominus g(y)]\!] \underbrace{\bigvee_B (B(x) \cdot [\![B \ominus g(y)]\!])}_{\text{call this } V;} \le C(x)$$

Clearly, $U \cdot A(x) \le V$, hence

$$[\![C \ominus g(y)]\!] \cdot U \cdot A(x) \cdot \le C(x)$$

$$\therefore \qquad U \cdot [\![C \ominus g(y)]\!] \le A(x) \rightharpoonup C(x) \tag{17}$$

(16) and (17) combined:

$$U.[\![C \in g(y)]\!] \le A(x) \rightleftharpoons C(x).$$

$$\therefore \qquad U.[\![C \in g(y)]\!] \le \bigwedge_{x\in|X|} (A(x) \rightleftharpoons C(x))$$

$$\therefore \qquad U.[\![C \in g(y)]\!] \le [\![A = C]\!].$$

But $[\![C \in g(y)]\!]\cdot[\![A = C]\!] \le [\![A \in g(y)]\!]$, hence $U.[\![C \in g(y)]\!] \le [\![A \in g(y)]\!]$

$$\therefore \qquad U.\underbrace{\bigvee_{C} [\![C \in g(y)]\!]}_{= Ey \text{ by condition (4) for } g;} \le [\![A \in g(y)]\!]$$

$$\therefore \qquad U.Ey \le [\![A \in g(y)]\!], \text{ which was to be shown.}$$

This completes the proof of the Lemma.

Now, we complete the verification of $P(X)$ as follows. We have to show that for any $h: X\times Y \to \Omega$ there is a unique $h^*: Y \to P(X)$ such that $h = \epsilon_X \circ (1_X \times h^*)$. As we know (in two different ways), h is represented by some $h_0: |X| \times |Y| \to \Omega$; actually h_0 is unique if it is required to be a predicate on $X\times Y$. Now, the following is a simple general fact: if $f_0: |A| \to |B|$, $g_0: |B| \to |C|$ represent the morphisms $f: A \to B$, $g: B \to C$, respectively, then $g_0 \circ f_0$ represents $g \circ f$ (exercise). To find h^*, note that we know (Lemma) that it must be represented by some $h_0^*: |Y| \to |P(X)|$; so $h = \epsilon_X \circ (1_X \times h^*)$ will be represented by

$$\langle x,y\rangle \xmapsto{(1_X\times h^*)_0} \langle x,h_0^*(y)\rangle \xmapsto{(\epsilon_X)_0} (h_0^*(y))(x).$$

So, if $h_0(x,y) = (h_0^*(y))(x)$, then indeed, h^* will be as required; to this end, we have to define

$$h_0^* = \text{the predicate } A_y \text{ such that } A_y(x) = h_0(x,y).$$
$$= \lambda x\, A(x,y).$$

A_y is indeed a predicate for all y; h_0^* represents a function h^*, and by the above, h^* satisfies the required identity. But also, h^* is unique: the above argument essentially shows that the definition of h^* was forced (<u>exercise</u>).

<u>Summary:</u> $P(X)$ is defined as the L-set whose underlying set is the set of all predicates A on X ($[\![x = x']\!] \le A(x) \rightharpoonup A(x')$; $A(x) \le Ex$), $[\![A =_{P(X)} B]\!] = \bigwedge_x (A(x) \leftrightharpoons B(x))$; $\epsilon_X\colon X\times P(X) \to \Omega$ is represented by the function $\langle x,A\rangle \longmapsto A(x)$; given $X\times Y \xrightarrow{h} \Omega$ represented by $h_0\colon |X| \times |Y| \to L$, then its 'exponential transpose' $Y \xrightarrow{h^*} P(X)$ is represented by the function $y \longmapsto \lambda x h_0(x,y)$. In other words, everything is as natural as possibly can be!

4.3.5. <u>The natural number object.</u>

We postpone the discussion of this until a bit later.

4.4. <u>Logic in L-sets.</u>

Now we not only know that <u>L-sets</u> is a topos; we know the topos structure on it exactly. This enables us to compute the interpretations of formulas. For simplicity, we restrict attention to the canonical frame of reference associated with ξ = <u>L-sets</u>; actually, this is no loss of generality (why?).

Recall that we denoted by

$$[\vec{x} \mid \varphi]$$

the interpretation of φ, an arbitrary formula of L_1; we have

$$[\vec{x} \mid \varphi]: X \to Y$$

if φ is a term of type Y, $\vec{x} = (x_1,\dots,x_n)$, $X = X_1\times\dots\times X_n$, x_i is of type X_i $(i = 1,\dots,n)$.

If, in particular, φ is of type Ω (it is a formula), then $[\vec{x} \mid \varphi]: X \to \Omega$. As we know, $|\Omega| = L$, and any morphism $X \to \Omega$ is represented by a unique predicate on X. We will denote this predicate by $[\![\varphi(\vec{x})]\!]$ or just $[\![\varphi]\!]$. Furthermore, the value of $[\![\varphi(\vec{x})]\!](\vec{a})$ at $\vec{a} \in |X| = |X_1|\times\dots\times|X_n|$ we will mostly denote by directly writing the arguments in the places occupied by the corresponding free variables. E.g. let $\varphi \cdot{=}\cdot \forall_{y\in Y}(Rxy \to x\in A)$; here the free variables are: x of sort X, and A of sort $P(X)$; R denotes a relation on $X\times Y$, i.e. a subobject of $X\times Y$. Then

$$[\![\forall_{y\in Y}(Rxy \to x\in A)]\!]$$

will denote the predicate on $X\times P(X)$ representing the morphism $[x,A|\forall_{y\in Y}(Rxy \to x\in A)]: X\times P(X) \to \Omega$; and even more, whenever x is a concrete element of $|X|$, $A \in |P(X)|$ (that is, A is a predicate on X), then

$$[\![\forall_{y\in Y}(Rxy \to x\in A)]\!]$$

means the value $\in \Omega$ of the above predicate at the said arguments. Of course, this kind of abuse of notation is common in logic and mathematics.

Some care has to be exercised if the free variables are not exactly the ones occuring in the formula. We have the following the general rule. Let $\vec{x} = x_1, \ldots, x_n$; suppose it is exactly $x_1, \ldots, x_k$ that occur in $\varphi (k < n)$; $x_{k+1}, \ldots, x_n$ do not. Let $[\![\varphi]\!]$ denote what we denoted by $[\![\varphi(x_1, \ldots, x_k)]\!]$ above. Then $[\![\varphi(x_1, \ldots, x_n)]\!] = [\![\varphi]\!] \wedge Ex_{k+1} \wedge \ldots \wedge Ex_n$; i.e., the effect of the 'vacuous' variables is their contribution of their existence-values to the formula. This assertion can be easily proved by using the fact that the following diagram is a pullback:

$$\begin{tikzcd}
{[\vec{x} \mid \varphi]} \arrow[r, hook] \arrow[d] & X_1 \times \ldots \times X_n \arrow[d, "\text{proj.}"] \\
{[x_1, \ldots, x_k \mid \varphi]} \arrow[r, hook] & X_1 \times \ldots \times X_k
\end{tikzcd} .$$

E.g., this tells us to compute

$$[\![\varphi(x) \vee \psi(y)]\!]$$

(with exactly the free variables as indicated) as follows:

$[\![\varphi(x) \vee \psi(y)]\!] = [\![\varphi(x,y)]\!] \vee [\![\psi(x,y]\!] = [\![\varphi]\!] \cdot Ey \vee [\![\psi]\!] \cdot Ex$ (for the logical connectives, see also below).

We discuss the interpretation of $L_=$ first; $L_=$ was a part of L_1, but (at least as far as the finitary part of L_1 was concerned) $L_=$ was sufficient to express everything in L_1. We postpone the natural number object until later.

The morphism $[\vec{x}: *]: X \to 1$ is, of course, the one represented by the constant $*$-valued function $|X| \to \{*\}$.

Given a,b terms of type A and B, respectively, then $[\vec{x}|\langle a,b\rangle]$ is the morphism $(X \to A\times B) = \langle[\vec{x}|a],[\vec{x}|b]\rangle$. Let $[\vec{x}|a] = \alpha: X \to A$, $[\vec{x}|b] = \beta: X \to B$. Then, as the computations concerning products (given as an exercise) show, $\langle\alpha,\beta\rangle$ is the function for which

$$[\![\langle u,v\rangle \ominus \langle\alpha,\beta\rangle(x)]\!] = [\![u \ominus \alpha(x)]\!]\cdot[\![v \ominus \beta(x)]\!].$$

Now we come to the three important operations: $=_A$, $\in$, and $\{\ \}$.

As expected, it turns out that the value $[\![a =_A a']\!]$ for $a,a' \in |A|$ in the sense introduced in this subsection is the same as the one that goes into the very specification of A as an L-set. The verification is easy; one has to look at the diagonal $A \to A\times A$, etc.

In general, if a and a' are <u>terms</u> of type A, then
$$[\![a(\vec{x}) =_A a'(\vec{x})]\!] = \bigvee_{u\in|A|} [\![u \ominus [\vec{x}\,|\,a](\vec{x})]\!]\cdot[\![u \ominus [\vec{x}\,|\,a'](\vec{x})]\!] .$$
This comes from the characterization of pullbacks. E.g., if we have $f: B \to A$, $g: C \to A$, then

$$[\![f(b) =_A g(c)]\!] = \bigvee_{a\in|A|} [\![a \ominus f(b)]\!]\cdot[\![a \ominus g(c)]\!]$$

for any $b \in |B|$, $c \in |C|$.

The morphism $\in_X$: $X\times\mathcal{P}(X) \to \Omega$ was given above. As a consequence

$$[\![x \in_X A]\!] = [\![x\in A]\!] = A(x)$$

for any $x\in|X|$ and predicate $A\in|\mathcal{P}(X)|$. For <u>terms</u> in place of x and A, there is a formula involving some sups that can be figured out by pullbacks; but also, once we have more logic, we can handle this in a more straightforward way; see below.

Now, let $\varphi(x,\vec{y})$ be a formula, x of sort X, the product of sorts of $\vec{y}$ being Y. So, $[x,\vec{y}\,|\,\varphi]: X\times Y \xrightarrow{h} \Omega$. The term $\{x\in X\,|\,\varphi(x,\vec{y})\}$ should be interpreted as an arrow $Y \longrightarrow P(X)$; and indeed, we have that $[\vec{y}\,|\,\{x\in X\,|\,\varphi(x,\vec{y})\}]$ is the exponential adjoint h^* of $h \cdot\!=\!\cdot [x,\vec{y}\,|\,\varphi]$. Therefore, as we learned, $[\vec{y}\,|\,\{x\in X\,|\,\varphi(x,\vec{y})\}]$ is represented by the function

$$\vec{y}(\in |Y|) \longmapsto \lambda x[\![\varphi(x,\vec{y})]\!] .$$

Finally, we state the effect of the connectives and quantifiers. As we know, these could be computed on the basis of the above things; it is an instructive <u>exercise</u> to show that, indeed, the rules given below are true. Let us write $E\vec{x}$ for $Ex_1 \wedge\ldots\wedge Ex_n$; $E\vec{x} = 1$ if $n = 0$.

$$[\![(\varphi\wedge\psi)(\vec{x})]\!] = [\![\varphi(\vec{x})]\!] \wedge [\![\psi(\vec{x})]\!]$$

denotes the meet $\wedge$ in L

$$[\![(\varphi\vee\psi)(\vec{x})]\!] = [\![\varphi(\vec{x})]\!] \vee [\![\psi(\vec{x})]\!]$$

these are of course not given by $L_=$, rather by the additional rules for infinitary logic.

inf in L

$$[\![\bigwedge_{i\in I}\varphi_i(\vec{x})]\!] = E\vec{x}\cdot \bigwedge_{i\in I}[\![\varphi_i(\vec{x})]\!]$$

$$= \begin{cases} \bigwedge_{i\in I}[\![\varphi_i(\vec{x})]\!] & \underline{\text{if } I \text{ is nonempty}} \\ E\vec{x} & \text{if } I = \emptyset \end{cases}$$

$$[\![\bigvee_{i\in I}\varphi_i(\vec{x})]\!] = \bigvee_{i\in I}[\![\varphi_i(\vec{x})]\!]$$

sup in L.

$$[\![(\varphi \to \psi)(\vec{x})]\!] = E\vec{x} \cdot ([\![\varphi(\vec{x})]\!] \rightharpoonup [\![\psi(\vec{x})]\!])$$

$$[\![(\neg\varphi)(\vec{x})]\!] = E\vec{x} \cdot - [\![\varphi(\vec{x})]\!]$$

$$[\![(\forall_y \varphi)(\vec{x})]\!] = E\vec{x} \cdot \bigwedge_{y \in |Y|} (Ey \rightharpoonup [\![\varphi(y,\vec{x})]\!])$$

(y of sort Y)

$$[\![(\exists_y \varphi)(\vec{x})]\!] = \bigvee_{y \in |Y|} [\![\varphi(y,\vec{x})]\!] \ .$$

Finally, the atomic formulas that we added to L_1. If e.g. a binary relation R '$\hookrightarrow$' $X \times Y$ is given, this is the same as a predicate R on $X \times Y$, and in fact, $[\![R(x,y)]\!] = R(x,y)$. If e.g. $f\colon X \times Y \to Z$ is a morphism as shown, then, as it is expected, we have

$$[\![z =_Z f(x,y)]\!] = [\![z \ominus f(x,y)]\!] \ .$$

the way f is given

<u>Example</u>. The definition of equality on $P(X)$ was given so that we have the bi-entailment

$$A = B \equiv \forall_{x \in X} (A(x) \leftrightarrow B(x)).$$

Indeed, this is true; $[\![\forall_{x \in X} (A(x) \leftrightarrow B(x))]\!] = \bigwedge_{x \in |X|} Ex \rightharpoonup ([\![A(x)]\!] \leftrightharpoons [\![B(x)]\!])$ $= \bigwedge_{x \in |X|} Ex \rightharpoonup (A(x) \leftrightharpoons B(x)) = \bigwedge_{x \in |X|} (A(x) \leftrightharpoons B(x)) = [\![A = B]\!]$.

The following is a way of dealing with atomic formulas that have terms in them that are more complicated than variables. E.g., if $f\colon B \to A$, $g\colon C \to P(A)$, then $[\![f(b) \in g(c)]\!] = [\![\exists_{x \in A,\ X \in P(A)} (x = f(b) \wedge X = g(c) \wedge x \in X)]\!]$ $= \bigvee_{\substack{x \in |A| \\ X \in |P(A)|}} [\![x \ominus f(b)]\!] \cdot [\![X \ominus g(c)]\!] \cdot X(x)$.

4.5. The effect of morphisms between cHa's

4.5.1. Let $u: L \to H$ be a cHa-morphism. Then u is a continuous functor between L and H as sites. Therefore, it induces a pair of adjoint functors

$$\begin{tikzcd} Sh(L) \arrow[r, shift left, "u^*"] & Sh(H) \arrow[l, shift left, "u_*"] \end{tikzcd}$$

$$u^* \dashv u_*, \quad u^* \text{ left exact;}$$

such that the diagram

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "\varepsilon_L"] & H \arrow[d, "\varepsilon_H"] \\ Sh(L) \arrow[r, "u^*"] & Sh(H) \end{tikzcd}$$

canonical functor: Yoneda, followed by 'associated sheaf'

commutes; also u^* (and u_*) are unique up to a unique isomorphism; for these matters, consult Section 2.

Since L-sets is equivalent to Sh(L), in the above we can replace Sh(L) and Sh(H) by L-sets and H-sets, respectively. One of the advantages of using L-sets instead of (besides) Sh(L) is that with the former, the description of the functor u^* is very simple.

Let's denote the composite $L \xrightarrow{\varepsilon_L} Sh(L) \xrightarrow[{[-]}]{\sim}$ L-sets, with $[-]$ the equivalence constructed before, also by ε_L. This is the effect of ε_L: $L \to$ L-sets, as one can easily check: note that every $U \in L$ is a subobject of 1_L, the terminal object (maximal element) of L; since ε_L is left exact, it maps 1_L to 1, the terminal object of $\mathcal{E}$ = L-sets; and it should map $U \leq 1_L$ to a subobject of 1; now, the subobjects of 1 are in 1-1 correspondence with the predicates on 1, and a predicate on 1, $|1|$ being the singleton $\{*\}$, is simply a 'truthvalue' $U \in L$; not surprisingly $U \in L$ is mapped onto (the) U (-valued-predicate-determined subobject of) 1.

Next, let $u: L \to H$ be a morphism of cHa's. If X is an L-set, then let $u^*(X)$ be the following H-set: $|u^*(X)| = |X|$;

$$\llbracket x =_{u^*(X)} x' \rrbracket \underset{\mathrm{df}}{=} u(\llbracket x =_X x' \rrbracket).$$

It is left an exercise to check that $u^*(X)$ is an H-set. Given a morphism $f: X \to Y$ of L-sets, then we define $u^*(f): u^*(X) \to u^*(Y)$, an H-sets morphism, as follows:

$$\llbracket y \in (u^*(f))(x) \rrbracket \underset{\mathrm{df}}{=} u(\llbracket y \in f(x) \rrbracket).$$

It is left as a further exercise to check that u^* so defined is a functor u^*: L-sets $\to$ H-sets. Furthermore, the reader should verify that

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "\varepsilon_L"'] & H \arrow[d, "\varepsilon_H"] \\ \text{L-set} \arrow[r, "u^*"] & \text{H-sets} \end{tikzcd}$$

commutes. - Finally, we claim that u^* is continuous, i.e., it is left exact and has a right adjoint. As we said in § 2, to this end it suffices to show that u^* preserves the validity of infinitary coherent entailments $\varphi \vdash \psi$, formulated in the canonical frame of reference associated with L. For the benefit of the fastidious, we give some details of the proof; those who have a well established trust in the existence of order in things should ignore this.

Recall that $\llbracket \varphi(\vec{x}) \rrbracket$ is the predicate (on X, where X is the L-set associated with the variables $\vec{x}$ as usual) that represents $[\vec{x}: \varphi(\vec{x})]: X \to \Omega$.

Now, $(u^*)_{\vec{x}}(\varphi)$ is $\begin{Bmatrix} \text{a subobject of } u^*(X) \\ \text{a morphism } u^*(X) \to \Omega \end{Bmatrix}$ in H-sets; let us denote the predicate representing $(u^*)_{\vec{x}}(\varphi)$ by $[\![\varphi(\vec{x})]\!]$. Note that φ is still in the language of L, but 'interpreted' in H-sets: $[\![\varphi(\vec{x})]\!]^*$ is a function on $|u^*(X)| \underset{(!)}{=} |X|$ with values in $\Omega_H = H$. We are using here u^* as an interpretation of (the canonical frame of reference associated with) L in H-sets; we had a notation $M_{\vec{x}}(\varphi)$ for $M: F_L \to H$; now u^* takes the place of M.

Lemma. For any infinitary coherent φ,

$$[\![\varphi(\vec{x})]\!]^*(\vec{x}) = u([\![\varphi(\vec{x})]\!](\vec{x}))$$

for any $\vec{x} \in |X|$.

The proof is a trivial induction on the complexity on φ. Note that it suffices to consider the only kind of atomic formula $x = f(y)$, since by the trick mentioned at the end of 4.4 all others can be eliminated [Now we are in the basic first order language, i.e. only unary operations are present; no relation symbols other than =].

So, let $\varphi :=: x = f(y)$; $\vec{x} = \langle xy \rangle$ (for simplicity), $X = X_1 \times X_2$, x of sort X_1, y of sort X_2, f an arrow $X_2 \to X_1$ in L-sets. - What is $[\![x = f(y)]\!]^*$?

By definition, it is the representing predicate of the arrow $(u^*)_{x,y}(x = f(y)): u^*(X) \to \Omega_H$. Now, $(u^*)_{x,y}(x = f(y))$ is obtained by passing to the interpretation $u^*(f)$ of the op. symbol f, and then computing $[x,y \mid x = u^*(f)(y)]$ in the topos H-sets;

i.e. $$(u^*)_{x,y}(x = f(y)) = \underbrace{[x,y \mid x = (u^*(f))(y)]}_{\substack{\text{interpretation of the formula}\\ x = (u^*(f))(y) \text{ in the canonical}\\ \text{frame of reference associated}\\ \text{with } H.}}$$

But we know, from 4.4, that $[x,y \mid x = (u^*(f))(y)]$ is represented by the predicate $[\![x \ominus (u^*(f))(y)]\!]: \ |X| \to H$. By the definition of $[\![x = f(y)]\!]^*$, we have therefore

$$[\![\varphi(\vec{x})]\!]^*(\vec{x}) = [\![x = f(y)]\!]^* = [\![x \ominus (u^*(f))(y)]\!] \underset{\substack{\text{def. of } u^*(f)\\ \text{see above}}}{=}$$

$$= u([\![x \ominus f(y)]\!]) = u([\![\varphi(\vec{x})]\!](\vec{x})),$$

as was to be shown.

Remembering the definition of (infinitary) coherent formulas, we are left with showing the equality for $\varphi := \text{True}$; as well as with an inductive argument showing that the equality is preserved upon applying the logical operations $\wedge$, $\bigvee$ and $\exists y$. We deal with $\exists y$ only.

Let $R \hookrightarrow u^*(Y) \times u^*(X)$ be the interpretation $(u^*)_{y,\vec{x}}(\varphi(y,\vec{x}))$; then $(u^*)_{\vec{x}}(\exists y \ \varphi(y,\vec{x})) = [\vec{x} \mid \exists y \ Ry\vec{x}]^{\underline{(H\text{-sets})}}$, hence

$$\underset{\substack{\uparrow\\ \text{elements from } X}}{[\![\exists y \ \varphi(y,\vec{x})]\!]^*(\vec{x})} = \underbrace{[\![\exists y \ Ry\vec{x}]\!](\vec{x})}_{\text{in } H\text{-sets as in 4.4}}$$

We know: $[\![\exists y\ Ry\vec{x}]\!](\vec{x}) = [\![\exists y\ Ry\vec{x}]\!] \underset{4.4}{=}$

$$= \bigvee_{y\in|Y|\,(=u^*(Y))} [\![Ry\vec{x}]\!] = \bigvee_{y\in|Y|} \underbrace{[\![\varphi(y,\vec{x})]\!]^* =}_{\text{ind. hyp.}}$$

$$= \bigvee_{y\in|Y|} u([\![\varphi(y,\vec{x})]\!]) \underbrace{=}_{\substack{\text{since } u \text{ is} \\ \text{a cHa-morphism}}} u\Big(\bigvee_{y\in|Y|}[\![\varphi(y,\vec{x})]\!]\Big) =$$

$= u([\![\exists y\ \varphi(y,\vec{x})]\!])$, and this establishes what we wanted (right?).

Using the lemma, now we prove that u^* preserves the validity of coherent entailments. Suppose $\overset{\text{L-sets}}{\models} (\varphi \vdash \psi)$, i.e. $[\vec{x}|\varphi] \underbrace{\leq}_{\substack{\text{order in}\\ \text{Sub}(X)}} [\vec{x}\mid\psi]$

i.e. $[\![\varphi(\vec{x})]\!](\vec{x}) \underbrace{\leq}_{\text{order in } L} [\![\psi(\vec{x})]\!](\vec{x})$ for all $\vec{x} \in |X|$. ($\vec{x}$: all free variables in φ or ψ). Since u preserves order, $u([\![\varphi(\vec{x})]\!](\vec{x})) \leq u([\![\psi(\vec{x})]\!](\vec{x}))$.
By the Lemma, we get $[\![\varphi(\vec{x})]\!]^*(\vec{x}) \leq [\![\psi(\vec{x})]\!]^*(\vec{x})$; $(\vec{x} \in |X|)$

i.e.

$$(u^*)_{\vec{x}}(\varphi) \underbrace{\leq}_{\text{order in } \text{Sub}_{\text{H-sets}}(u^*(X))} (u^*)_{\vec{x}}(\psi), \text{ i.e.}$$

$$u^* \underset{\text{H-sets}}{\models} (\varphi \vdash \psi)$$

as was to be shown.

We have established that u^*: L-sets $\to$ H-sets is the (up to isomorphism, unique) canonical continuous lifting of u: $L \to H$.

4.5.2. We now consider the following important special case. Let $H = \mathbb{P}(\mathbb{1})$, (classically) the 2-element Boolean algebra (cHa); the unique cHa morphism u_0: $\mathbb{P}(\mathbb{1}) \to L$ maps $\emptyset$ into 0 of L, $\mathbb{1}$ into 1 of L. [Sorry, we have reversed the direction $L \xrightarrow{u} H$ to $H \to L$].

$\mathbb{P}(\mathbb{1})$-sets is the category of sets, SET (why?). The functor u_0^*: SET $\to$ L-sets is denoted by $(\check{\ })$. For a set S, $\check{S}$ is the L-valued set with underlying set S itself and with equality:

$$[\![x =_S y]\!] = \begin{cases} 1 \ \text{(of L)} & \text{if} \ \ x = y \\ 0 \ \text{(of L)} & \text{if} \ \ x \neq y \end{cases} .$$

For a function $f\colon S_1 \to S_2$, $\check{f}$ is the morphism $\check{S}_1 \to \check{S}_2$ represented by f.

As an application, we identify the natural number object in L-sets. As it is known, continuous functors between preserve N; see e.g. Prop. 6.16, p. 170 in [T.T.]. As a consequence, the natural number object in L-sets is $\check{\mathbb{N}}$, with $\mathbb{N}$ the set or ordinary natural numbers; $0\colon 1 \to N$ is the map $1 \to \check{\mathbb{N}}$ represented by $* \longmapsto 0 \in \mathbb{N}$; $\check{N} \xrightarrow{\check{S}} \check{N}$ is $\mathbb{N} \xrightarrow{\mathbf{S}} \mathbb{N}$, with $\mathbf{S}\colon \mathbb{N} \longrightarrow \mathbb{N}$ the ordinary successor function.

4.6. Restating matters in the sheaf-language.

We start with the general remark that since $Sh(L) \simeq$ L-sets, once we have proved something about L-sets, we have proved "the same" about $Sh(L)$. It takes some effort to make the precise statements, however. - It is worth the effort since the idea of sheaf, and the sheaf theoretic language, seems $\left\{\begin{matrix}\text{'conceptually'} \\ \text{geometrically}\end{matrix}\right\}$ superior to the L-sets-language; the latter is needed only because it makes it possible to use 'logic'.

4.6.1. The global section functor

For any category $\mathcal{C}$ with a terminal object 1, the global section functor $\Gamma: \mathcal{C} \to SET$ is the covariant functor represented by 1: $\Gamma = Hom_{\mathcal{C}}(1,-)$.
- In case of $\mathcal{C} = Sh(L)$ (see 4.1), there is the following essentially equivalent but more convenient, definition. $\Gamma: Sh(L) \to SET$ is defined by:

$$X \overset{\Gamma}{\longmapsto} X(1_{(L)})$$

nat. transf.

$$\begin{tikzcd} X \arrow[d, "h"'] & \overset{\Gamma}{\longmapsto} & X(1_{(L)}) \arrow[d, "h_1"] \\ Y & & Y(1) \end{tikzcd}$$

- More generally, we define

$$\Gamma_U: Sh(L) \to SET$$

by:

$$\begin{tikzcd} X \arrow[d, "h"] & \longmapsto & h_U \quad X(U) \arrow[d] \\ Y & & Y(U) \end{tikzcd} .$$

Also, let's introduce the notation $L|U$ $(U \in L)$ for the poset $\{V \in L: V \leq U\}$ with $\leq_L$ restricted to it. Clearly, $L|U$ is a cHa; $1_{L|U} = U$; $x \wedge_{L|U} y = x \wedge_L y$; $\bigvee^{L|U}\{x_i: i \in I\} = \bigvee^{L}\{x_i: i \in I\}$. Now, given any presheaf X on L, we can define the presheaf $X|U$ on $L|U$ by simple restriction:

$$(X|U)(V) = X(V) \quad \text{for} \quad V \le U.$$

Clearly, $(-)|U$ is a functor

$$Sh(L) \to Sh(L|U);$$

and Γ defined above is the composite

$$\begin{tikzcd} Sh(L) \arrow[r, "(-)|U"'] & Sh(L|U) \arrow[r] & SET. \\ & & \text{'global' } \Gamma \text{ of } Sh(L|U). \arrow[u] \end{tikzcd}$$

4.6.2. <u>Exponentials in Sh(L).</u>

Let X, Y be presheaves on L. Define the presheaf Y^X as follows:

$Y^X(U) =$ the set of all natural transformations

$$X|U \longrightarrow Y|U = \mathrm{Hom}_{Presh(L|U)}(X|U, Y|U);$$

for $V \le U$, $h \in Y^X(U)$, define

$$h \restriction V = \text{'restriction'} = \Gamma_V^{(Sh(L|U))}(h).$$

It turns out that if Y is a sheaf, so is Y^X (see below). So for sheaves X,Y, we have $Y^X \in Sh(L)$. Also, define $ev\colon Y^X \times X \to Y$ in the obvious way:

$$ev_U\colon \; Y^X(U) \times X(U) \to Y(U):$$

$$\langle h, s\rangle \longmapsto h_U(s) \quad .$$

<u>Proposition 4.6.2.1.</u> Y^X, together with ev, qualifies as the exponential in $Sh(L)$: for every morphism $Z\times X \to Y$ there is a unique $Z \xrightarrow{f} Y^X$ such that $Z\times X \to Y$ equals the composite $Z\times X \xrightarrow{f\times 1_Y} Y^X\times X \xrightarrow[ev]{} Y$.

<u>Proof</u>: Of course, this can be proved directly; see [1.12, p. 24 in T.T.]. But it is worth pointing out the connection with another natural definition of exponentials, this time in L-sets. - Let now X, Y be L-sets; with a variable R of sort $\mathcal{P}(X\times Y)$, let us define the formula $Func(R)$ to be

$\forall_{x\in X}\ \exists!_{y\in Y}(\langle x,y\rangle \in R)$; define the subobject $Y^X \hookrightarrow P(X\times Y)$ as the extension of $\mathrm{Func}(R)$

$$Y^X = \ker\{R \in P(X\times Y):\ \mathrm{Func}(R)\}$$

and also denote the domain of this subobject by Y^X. In other words, Y^X is the L-set such that

$$|Y^X| = |P(X\times Y)| = \mathrm{Pred}(X\times Y)$$

and

$$[\![R =_{Y^X} S]\!] = [\![R =_{P(X\times Y)} S]\!]\cdot[\![\mathrm{Func}(R)]\!]\cdot[\![\mathrm{Func}(S)]\!]$$

Note that $E_{(Y^X)}(R) = [\![\mathrm{Func}(R)]\!]$.

Let $ev\colon Y^X\times X \to Y$ be defined by

$$[\![y \ominus ev(R,x)]\!] = \underbrace{[\![\mathrm{Func}(R)]\!]}_{E_{(Y^X)}(R)}\cdot[\![\langle x,y\rangle \in R]\!].$$

What was stated in [Lecture 12, J. Lambek, Fall 1979] shows that the above qualifies as exponential in L-sets.

Now, we compare $Sh(L)$ and L-sets by the equivalence functor $Sh(L) \xrightarrow[{[\cdot]}]{} $ L-sets. Let X, Y be sheaves $\in |Sh(L)|$. Recall the definition of the 'sheaves' (separated and complete L-sets) $[X]$, $[Y]$. We exhibit an isomorphism

$$\underbrace{[Y^X]}_{\begin{array}{l}\text{exponential taken in}\\ \text{Sh(L) as described}\\ \text{in the statement of}\\ \text{the Proposition;}\\ \text{then [ ] applied}\end{array}} \xrightarrow{\quad\quad i \quad\quad} \underbrace{[Y]^{[X]}}_{\begin{array}{l}\text{first [ ] applied, then}\\ \text{exponential in L-sets taken}\\ \text{as described above.}\end{array}}$$

Let $\langle h,U\rangle$ be an element of $|[Y^X]|$; $h \in (Y^X)(U)$. Define the predicate $R = i(\langle h, U\rangle)$ on $[X]\times[Y]$ as follows: let $x \in X(U)$, $y \in Y(V)$; $W \underset{df}{=} U \cap V$; we have that $h_W(x \upharpoonright W) \in Y(W)$; put

$$[\![\langle\langle x,U\rangle, \langle y,V\rangle\rangle \in R]\!] \underset{df}{=} [\![\langle y,V\rangle =_{[Y]} \langle h_W(x \upharpoonright W, W)\rangle]\!].$$

We leave it an <u>exercise</u> to check that i is an isomorphism; and in fact that the diagram

$$\begin{tikzcd}
{[Y^X]\times[X]} \arrow[r, "{[ev]}"] \arrow[d, "{i \times Id_{[X]}}"] & {[Y]} \\
{[Y]^{[X]}\times[X]} \arrow[ur, "{ev_{(L\text{-sets})}}"'] &
\end{tikzcd}$$

commutes. - What this says is that the equivalence functor $[-]$ carries the specification (Y^X, ev) into something that qualifies as the exponential; hence the assertion of the Proposition follows.

$\square$

4.6.3. Subsheaves.

Given a sheaf $X \in |Sh(L)|$, a subsheaf of X is a sheaf A such that $A(u) \subset X(u)$ for all $U \in L$, and for $a \in A(u) \subset X(u)$, $a \upharpoonright_{(A)} U = a \upharpoonright_{(X)} U$. The inclusion maps $A(u) \xhookrightarrow[i_u]{} X(u)$ combine into a monomorphism $A \xhookrightarrow[i]{} X$ (check).

Proposition 4.6.3.1. The subobjects of X are 'faithfully represented' by the subsheaves: every subobject of X is represented by exactly one subsheaf with its canonical inclusion.

The proof, of course, is almost trivial. But the reader might want to compare, via the equivalence $[-]$: $Sh(L) \to S$-sets, the two definitions of 'canonical subobjects'. □

Here is a piece of general nonsense concerning the interpretation of formulas.

Proposition 4.6.3.2. Let $X \in Sh(L)$; $c \in \Gamma(X) = X(1_L)$; define $\underline{c}: 1_{Sh(L)} \to X$ by $\underline{c}_u(*) = c \upharpoonright u$ [$1_{Sh(L)}$ = the sheaf for which $(1_{Sh(L)})(U) = \{*\}$]. Then, for any formula $M(x)$ (x: var. of sort X)

$$\underbrace{\models_{Sh(L)} M(\underline{c})}_{\substack{\text{substitute } \underline{c} \\ \text{(a term) for} \\ \text{the variable } x}} \Leftrightarrow c \in \Gamma(\underbrace{[x: M(x)]}_{\substack{\text{a subsheaf} \\ \text{of } X}})$$

Proof: $M(\underline{c})$ is a grammatically correct formula; $[M(\underline{c})]$ is a subobject of 1 $(= 1_{Sh(L)})$. One has a general substitution theorem for almost arbitrary categories: see e.g. [3.2.3, MR, p.103]. According to that, we have a pull-back diagram

$$\begin{tikzcd}
{[M(\underline{c})]} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & 1 \arrow[d, "\underline{c}"] \\
{[x\colon M(x)]} \arrow[r, hook] & M .
\end{tikzcd} \qquad (1)$$

$\models M(\underline{c}) \Leftrightarrow M(\underline{c}) \longrightarrow 1$ is an iso. $\underset{\text{by (1)}}{\Leftrightarrow}$ $\underline{c}$ factors through $[x\colon M(x)] \hookrightarrow X \underset{\text{clearly}}{\Leftrightarrow}$ $c \in \Gamma([x \in M(x)]$. $\square$

The functor $Sh(L) \xrightarrow[(-)|U]{} Sh(L|U)$ is logical, as well as geometric; this is a special case of the general results [1.42 p.35 & 1.46 p. 37, TT]. We leave it to the reader to make the connection; now we want to point out a consequence of this fact.

Let $M(x)$ be a formula in the frame of ref. associated with $Sh(L)$, x of sort X; replacing symbols in $M(x)$ by their images under the functor $(-)|U$, we get a formula denoted $M_{|U}(x)$; now x denotes a variable of sort $X|U$ in $Sh(L|U)$; $M_{|U}(x)$ is a formula in the frame of ref. associated with $L|U$. We can form, on the one hand, the interpretation

$$[x \in X\colon M(x)]_{Sh(L)} \underset{\text{in short}}{=} [M(x)]$$

in $Sh(L)$; and also,

$$[x \in X|U\colon M_{|U}(x)]_{Sh(L|U)} \underset{\text{in short}}{=} [M_{|U}(x)] ,$$

the interpretation of $M_{|U}(x)$ in $Sh(L|U)$. The first of these is a subsheaf of X, the second is a subsheaf of $X|U$. Now, what is said above about the functor $(-)|U$ means

<u>Proposition 4.6.3.3.</u> $[M(x)]|U = [M_{|U}(x)]$.

<u>Note</u> that this is a literal equality; both terms of the equality are subsheaves of $X|U$. - In particular, applying the global section functor of $Sh(L|U)$ to both sides, we get

$$\Gamma_U[M(x)] = \Gamma^{(Sh(L|U))}([M_{|U}(x)]).$$

- · -

We can generalize 4.6.3.2. as follows. If $c \in \Gamma_U(x) = X(U)$, then $c \in \Gamma^{(L|U)}(X|U)$ (and vice versa). Let $M(x)$ be a formula over the frame of ref. associated with $Sh(L)$; then $M_{|U}(x)$ is one over $Sh(L|U)$; we have

$$c \in \Gamma_U[x\colon M(x)] \Leftrightarrow c \in \Gamma^{(L|U)}([x\colon M_{|U}(x)])$$

see 4.6.3.3.

$$\Leftrightarrow \models_{Sh(L|U)} (M_{|U})(\underline{c}) \ .$$

by 4.6.3.2.

4.6.4. <u>Ω and powersheaves.</u>

The 'sheaf'-definition of Ω clearly gives us the following description of Ω in $Sh(L)$:

$$\Omega(U) = L|U = \{V \in L\colon V \le U\}$$

$W \le U,\ V \in \Omega(U)$:

$$V \restriction W = V \cap W$$

(also compare [1.12, TT]). - Given a subsheaf A of X, the characteristic morphism of A

$$\gamma: X \to \Omega$$

is defined as follows:

$$\gamma_U: X(U) \to L|U$$
$$x \longmapsto \bigvee\{V \leq U: x \restriction V \in A(V)\}$$
$$= \text{the maximal } V \leq U \text{ such that } x \restriction V \in A(V).$$

It is left to the reader to convince himself of the correctness of the above and make the comparison with our specifications in L-sets.

The power sheaf $\Omega^X = \mathcal{P}(X)$ is, consequently, described as follows:

$(\Omega^X)(U)$ = the set of nat. transf.'s $X|U \to \Omega|U$

= the set of those $h = (h_U)_{U \in L}$ s.t. $h_U: X(U) \to L|U$ with the property that for $W \leq U$ and $x \in X(U)$,

$$h_W(x \restriction W) = h_U(x) \cap W.$$